Exercise 6.R

Answer 1CC.

(a)

A function *f* is called a one-to-one function if it never takes on the same value twice;

That is when $x_1 \neq x_2$

then $f(x_1) \neq f(x_2)$

$\left[(\text{OR}) f(x_1) = f(x_2) \Rightarrow x_1 = x_2\right]$

A function is on-to-one if and only if no horizontal line inter sects its graph more than once.

(b)

Let *f* be a one-to-one function with domain *A* and range *B*.

Then its inverse function f^{-1} has domain B and range *A* and is defined by

$\boxed{f^{-1}(y) = x \Leftrightarrow f(x) = y}$ for *y* in *B*.

The graph of f^{-1} is obtained by reflecting the graph of *f* about the line $y = x$.

(c)

Given that *f* is one-to-one $f'\left(f^{-1}(a)\right) \neq 0$

Then $\boxed{\left(f^{-1}\right)'(a) = \dfrac{1}{f'\left(f^{-1}(a)\right)}}$

Answer 1E.

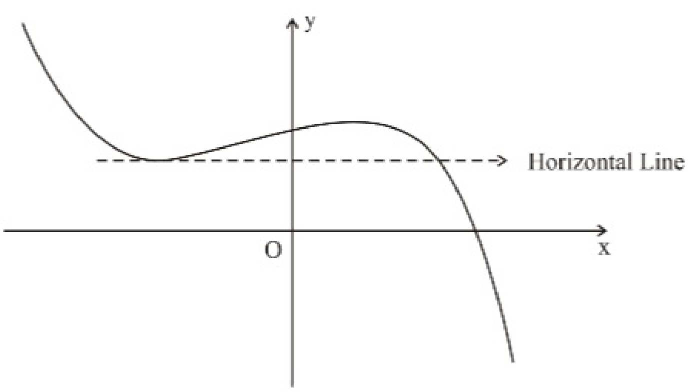


Fig. 1

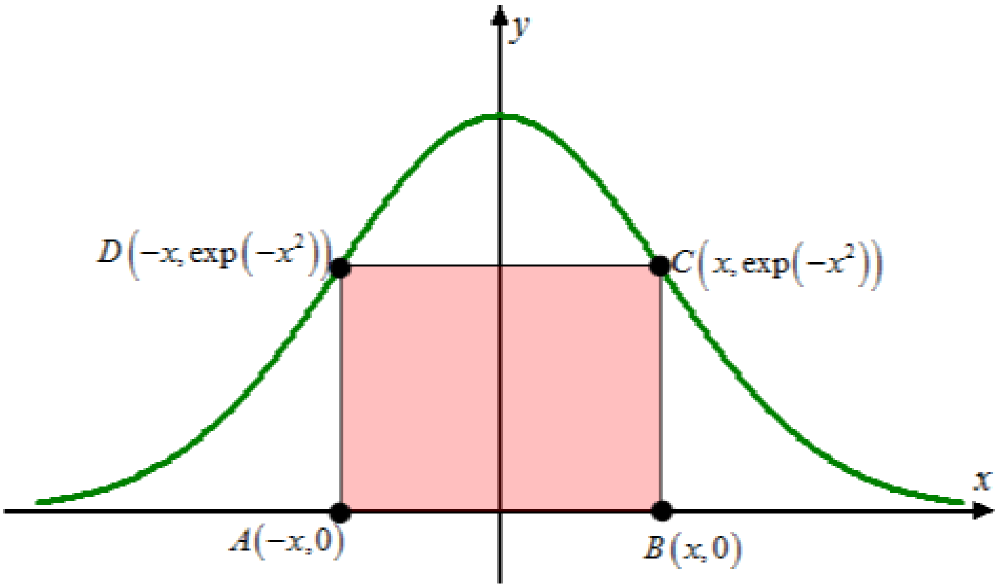
If we draw a horizontal line, we see that this horizontal line intersects the graph at more than one point. Here horizontal line test is failed. So f is **not one to one**

Answer 1P.

Let the base vertices of the rectangle be $A(-x,0)\ B(x,0)$.

The other two vertices of the rectangle are lies on the curve $y=\exp(-x^2)$ this follows that the other two vertices are $C(x,\exp(-x^2))$ and $D(-x,\exp(-x^2))$.

If we sketch the rectangle then it will be as shown in the following diagram.



From the above diagram it is obvious that the length of the rectangle is $2x$ and the width is $\exp(-x^2)$ this implies the area of the rectangle is $2x\exp(-x^2)$.

Let the area of the rectangle be a function of *x*.

That is $A(x)=2x\exp(-x^2)$.

Find the derivative of the function $A(x)=2x\exp(-x^2)$.

$$\begin{aligned} A'(x) &= 2\exp(-x^2)+2x\exp(-x^2)(-2x) \\ &= 2\exp(-x^2)(1-2x^2) \end{aligned}$$

Find the critical points of the function.

$$\begin{aligned} A'(x) &= 0 \\ 2\exp(-x^2)(1-2x^2) &= 0 \\ 1-2x^2 &= 0 \quad \text{Since } 2\exp(-x^2)\neq 0 \text{ for all } x \\ x &= \frac{\pm 1}{\sqrt{2}} \approx \pm 0.71 \end{aligned}$$

Find the double derivative of the function $A(x)=2x\exp(-x^2)$.

$$\begin{aligned} A''(x) &= \left(2\exp(-x^2)(1-2x^2)\right)' \\ &= 2\exp(-x^2)(-4x)+2(1-2x^2)\exp(-x^2)(-2x) \\ &= -8x\exp(-x^2)+(-4x+8x^3)\exp(-x^2) \\ &= -4x\exp(-x^2)(3-2x^2) \end{aligned}$$

Next find the sign of the double derivative at the critical points.

$$A''(0.71) = -4(0.71)\exp(-0.71^2)\left(3-2(0.71)^2\right) \approx -3.14 < 0$$

This implies **the function has maximum value at** $x=\boxed{0.71}$. (by the second derivative test)

This follows that the area of the rectangle is maximum when $x=\boxed{0.71}$.

$$A''(-0.71) = -4(-0.71)\exp(0.71^2)\left(3-2(-0.71)^2\right) \approx 9.36 > 0$$

This implies **the function has minimum value at** $x=\boxed{-0.71}$. (by the second derivative test)

This follows that the area of the rectangle is minimum when $x=\boxed{-0.71}$.

Consider the equation $y=\exp(-x^2)$ and find its points of inflection.

Firstly find its double derivative.

$$\begin{aligned} & y=\exp(-x^2) \\ & \Rightarrow y' = -2x\exp(-x^2) \\ & \Rightarrow y'' = -2\exp(-x^2)-2x\left(-2x\exp(-x^2)\right) \\ & \quad = -2\exp(-x^2)+4x^2\exp(-x^2) \\ & \quad = (-2+4x^2)\exp(-x^2) \end{aligned}$$

Solve the equation $y''=0$ for *x* to obtain the points of inflection.

$$\begin{aligned} y'' &= 0 \\ \exp(-x^2)(-2+4x^2) &= 0 \\ -2+4x^2 &= 0 \quad \text{Since } 2\exp(-x^2)\neq 0 \text{ for all } x \\ x &= \frac{\pm 1}{\sqrt{2}} \approx \pm 0.71 \end{aligned}$$

Hence the points of inflection are $x=0.71$ and $x=-0.71$ which are same as maxima and minima of the area function $A(x)=2x\exp(-x^2)$.

This shows that the rectangle has maximum or minimum area at the points of inflections of the curve $y=\exp(-x^2)$.

Answer 1TFQ.

Given that f is one-to-one with domain $\mathbb{R}$, then $f^{-1}(f(6)) = 6$

Since every element has unique image and no two elements have the same image.

$\boxed{f^{-1}(f(6)) = 6 \text{ is true}}$

Answer 2CC.

(a)

Given function is $f(x) = e^x$

The domain of natural exponential function $f(x) = e^x$ is R

The range of natural exponential function $f(x) = e^x$ is $(0, \infty)$

(b)

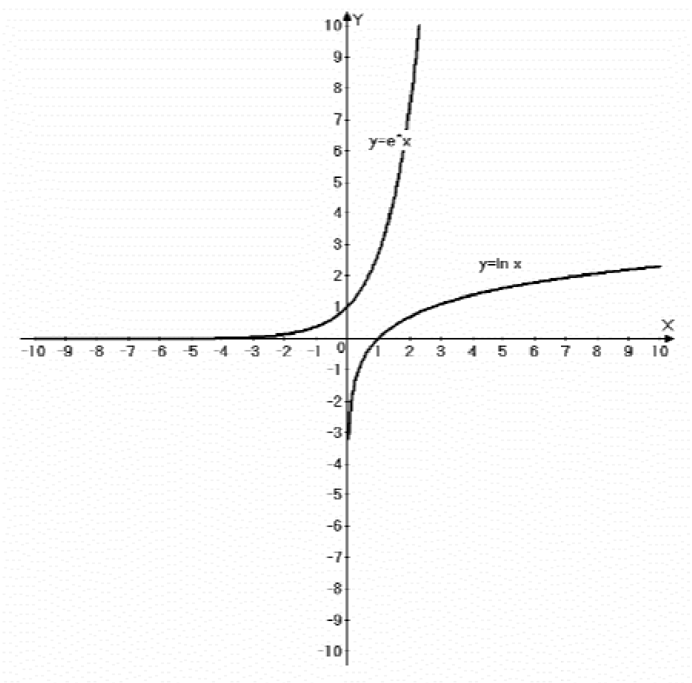
Given function is $f(x) = \ln x$

The domain of natural logorithmic function $f(x) = \ln x$ is $(0, \infty)$

The range of natural logorithmic function $f(x) = \ln x$ is R

(c)

The graphs of $f(x) = e^x; f(x) = \ln x$ are



Answer 2E.

(A) If we draw a horizontal and a vertical line any where on the graph we see that these lines intersect the graph at only one point so g is one to one

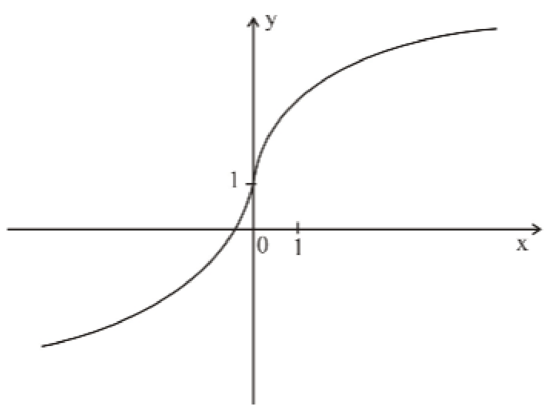


Fig. 1

(B) Here we see from the graph $g(0.25)=2$

So $g^{-1}(2)\approx 0.25$

(C) Domain of the g is R and range of the function is R

So domain of g^{-1} is R (a set of real number)

(D)

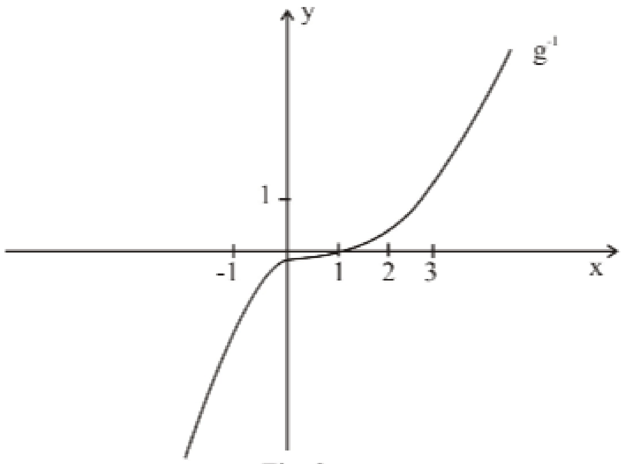


Fig. 2

Answer 2P.

Show that $\log_2 5$ is an irrational number.

Consider $\log_2 5$ to be a rational number.

It means, $\log_2 5=\frac{a}{b}$ for some integers a and b, where $b\neq 0$.

$\log_2 5=\frac{a}{b}$

$5=2^{\frac{a}{b}}$ Use the definition of the logarithm

$5^b=\left(2^{\frac{a}{b}}\right)^b$ Raise the power b on both the sides

$5^b=2^a$ Simplify

Observe the equation $5^b=2^a$, where 2^a is an even number for every integer a and 5^b is an odd number for every non-zero integer b.

Moreover, an even number can never be equal to an odd number.

So this is a contradiction.

Hence, $\log_2 5$ is an irrational number.

Answer 3CC.

(a)

The inverse sine function a are sin function is

Defined as the inverse of the function $f(x)=\sin x$,

$$\sin^{-1}x=y \Leftrightarrow x=\sin y,\quad -\frac{\pi}{2}\leq x\leq\frac{\pi}{2}$$

Domain of $f(x)=\sin^{-1}x$ is [-1,1]

and range of $f(x)=\sin^{-1}x$ is $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

(b)

The inverse cosine function or arc cosine function is defined as the inverse of the function $f(x) = \cos x$

$\boxed{\cos^{-1} x = y \iff x = \cos y \text{ and } 0 \le y \le \pi}$

$\boxed{\text{Domain of } f(x) = \cos^{-1} x \text{ is } [-1, 1]}$

$\boxed{\text{Range of } f(x) = \cos^{-1} x \text{ is } [0, \pi]}$

(c)

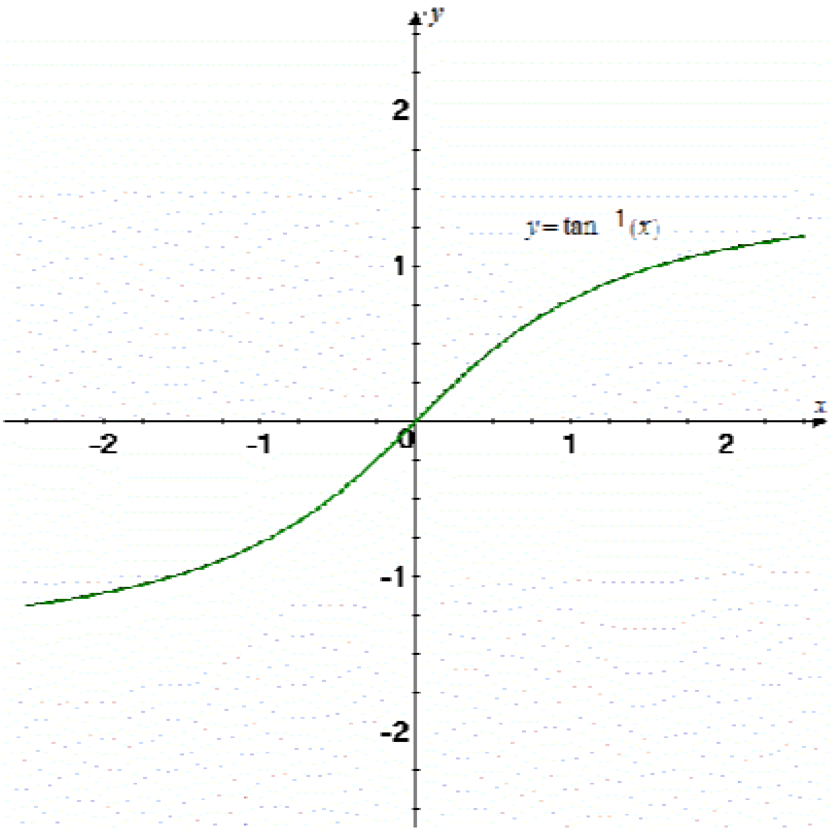
The inverse tangent function a arc tangent function is defined as the inverse of the function $f(x) = \tan x$

$\boxed{\tan^{-1} x = y \iff \tan y = x \text{ and } -\frac{\pi}{2} < \pi < \frac{\pi}{2}}$

$\boxed{\text{Damain of } f(x) = \tan^{-1} \text{ is R}}$

$\boxed{\text{Range of } f(x) = \tan^{-1} x \text{ is } \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)}$

The graph of $f(x) = \tan^{-1} x$ is



Answer 3E.

(A). We know that if f is a one-to-one function with domain A and Range B. Then its inverse function f^{-1} has domain B and range A and is defined by

$f^{-1}(y) = x \iff f(x) = y$ for any y in B.

Now by the definition of inverse function $f^{-1}(3) = 7$.

Hence, $\boxed{f^{-1}(3) = 7}$.

(B) We know that if f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$.

Given that $f'(7) = 8$.

Therefore by the above stated theorem,

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))}$$

$$= \frac{1}{f'(7)} \qquad [\because f^{-1}(3) = 7]$$

$$= \frac{1}{8}$$

So $\boxed{(f^{-1})'(3) = \frac{1}{8}}$

Answer 3P.

Consider the following function:

$$f(x) = e^{10|x-2|-x^2}$$

To find the function $f(x)$ is absolute maximum.

Differentiate the function $f(x)$ with respect to *x*.

$$f'(x) = e^{10|x-2|-x^2}\left(10\left(\frac{|x-2|}{x-2}\right) - 2x\right)$$

The function $f(x)$ is absolute maximum when $f'(x) = 0$

$$e^{10|x-2|-x^2}\left(10\left(\frac{|x-2|}{x-2}\right) - 2x\right) = 0$$

$$\pm(10 - 2x) = 0$$

$$x = \pm 5$$

$$x = +5 \text{ or } -5$$

Substitute the values $x = 5$ and $x = -5$ in $f(x)$.

$$f(5) = e^{10|5-2|-25}$$

$$= e^5$$

$$= 148.41$$

$$f(-5) = e^{10|-5-2|-25}$$

$$= e^5$$

$$= 148.41$$

Therefore, the function $f(x)$ is absolute maximum at $x = 5$ and $x = -5$.

There is no absolute minimum because exponential function is always positive.

Answer 3TFQ.

Consider the statement,

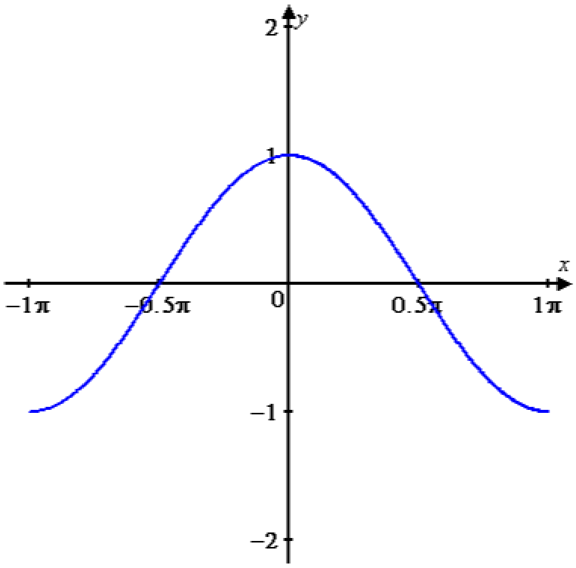
The function $f(x)=\cos x, \frac{-\pi}{2} \le x \le \frac{\pi}{2}$, is one-to-one.

Determine whether the statement is true or false.

Recall that, if some horizontal line intersects the graph of the function more than once, then the function is not one-to-one.

If no horizontal line intersects the graph of the function more than once, then the function is one-to-one.

The sketch of the function $f(x)=\cos x, \frac{-\pi}{2} \le x \le \frac{\pi}{2}$ is shown below:



Use horizontal line test, draw a horizontal line on the graph of the function, the horizontal line meet the graph at more than one point, which is shown in the below figure 1.

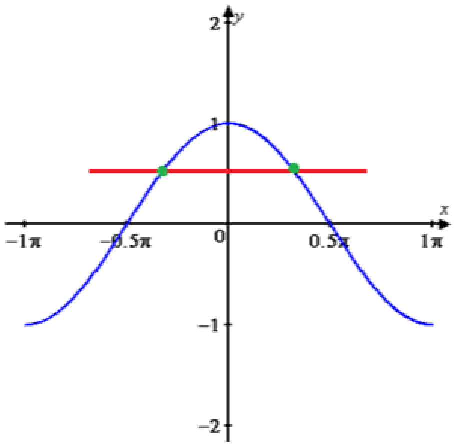


Figure 1

Observe the figure, it confirms that the horizontal line meet the graph at more than one point.

So the function $f(x)=\cos x, \frac{-\pi}{2} \le x \le \frac{\pi}{2}$ is not one-to-one.

Hence, the statement is false.

Answer 4CC.

$$\sin hx = \frac{e^x - e^{-x}}{2}$$

$$\cos hx = \frac{e^x + e^{-x}}{2}$$

$$\tan hx = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Answer 4E.

We have $f(x)=\frac{x+1}{2x+1}$

We write $y=\frac{x+1}{2x+1}$

Solving for x

$$(2x+1)y=x+1$$
$$\Rightarrow 2xy+y=x+1$$
$$\Rightarrow 2xy-x=1-y$$
$$\Rightarrow (2y-1)x=1-y$$
$$\Rightarrow x=\frac{(1-y)}{(2y-1)}$$

Replacing x and y the inverse function is

$$\boxed{f^{-1}(x)=\frac{1-x}{2x-1}}$$

Answer 4P.

Consider the following value is,

$$\int_0^4 e^{(x-2)^4}dx=k$$

The objective is to find the value of $\int_0^4 xe^{(x-2)^4}dx$.

$$\int_0^4 xe^{(x-2)^4}dx=\int_0^4 xkdx \quad \text{since} \left(\int_0^4 e^{(x-2)^4}dx=k\right)$$
$$=k\int_0^4 xdx \quad (\text{since } k \text{ is a constant})$$
$$=k\left[\frac{x^2}{2}\right]_0^4 \quad \left(\text{use formula } \int x^n dx=\frac{x^{n+1}}{n+1}\right)$$
$$=8k$$

Therefore, the value is $\boxed{8k}$

Answer 4TFQ.

$$\text{Consider } \tan^{-1}(-1)=\tan^{-1}\left(\tan\left(\pi-\frac{\pi}{4}\right)\right)$$
$$=\pi-\frac{\pi}{4}$$
$$=\frac{3\pi}{4}$$

Therefore $\boxed{\tan^{-1}(-1)=\frac{3\pi}{4} \text{ is true}}$

Answer 5CC.

(a) $$\frac{dy}{dx}=\frac{d}{dx}\left(e^x\right)$$
$$=e^x$$

(b) $$\frac{dy}{dx}=\frac{d}{dx}\left(a^x\right)$$
$$=a^x \ln a$$

(c)
$$\frac{dy}{dx} = \frac{d}{dx}(\ln x)$$
$$= \frac{1}{x}$$

(d)
$$\frac{dy}{dx} = \frac{d}{dx}(\log a^x)$$
$$= \frac{1}{x \ln a}$$

(e)
$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1} x)$$
$$= \frac{1}{\sqrt{1-x^2}}$$

(f)
$$\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1} x)$$
$$= \frac{-1}{\sqrt{1-x^2}}$$

(g)
$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1} x)$$
$$= \frac{1}{1+x^2}$$

(h)
$$\frac{dy}{dx} = \frac{d}{dx}(\sin hx)$$
$$= \cos hx$$

(i)
$$\frac{dy}{dx} = \frac{d}{dx}(\cos hx)$$
$$= \sin hx$$

(j)
$$\frac{dy}{dx} = \frac{d}{dx}(\tan hx)$$
$$= \sec h^2 x$$

(k)
$$\frac{dy}{dx} = \frac{d}{dx}(\sin h^{-1} x)$$
$$= \frac{1}{\sqrt{1+x^2}}$$

(l)
$$\frac{dy}{dx} = \frac{d}{dx}(\cos h^{-1} x)$$
$$= \frac{1}{\sqrt{x^2-1}}$$

(m)
$$\frac{dy}{dx} = \frac{d}{dx}(\tan h^{-1} x)$$
$$= \frac{1}{1-x^2}$$

Answer 5E.

We have the graph of 5^x as follows

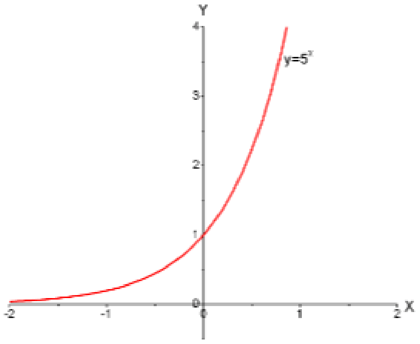


Fig.1

We can get the graph of $y = 5^x - 1$ by shifting the graph of 5^x, 1 unit downward

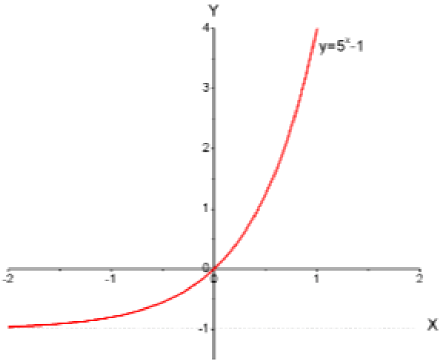


Fig.2

Answer 5P.

We know that $\sin bx =$ imaginary part of re^{ibx} where $e^{ibx} = \cos bx + i\sin bx$ also, by Demovrie's therom , we have $(\cos bx + i\sin bx) = \cos nbx + i\sin nbx$

$\therefore\ e^{ax} * e^{ibx} = e^{(a+ib)x}(a+ib)$

$\frac{d^2}{dx^2} f(x) = e^{(a+ib)x}(a+ib)^2$

$\frac{d^n}{dx^n} f(x) = e^{(a+ib)x}(a+ib)^n$ where $f(x) = e^{(a+ib)x}$

$= e^{ax}\{\cos bx + i\sin bx\}(a+ib)^n$

Now, putting $a = r\cos\theta, b = r\sin\theta$, we get

$\frac{d^n}{dx^n} e^{(a+ib)x} = e^{ax}\{\cos bx + i\sin bx\}\{r^n\cos n\theta + ir^n\sin\theta\}$

Now, imaginary part of

$\frac{d^n}{dx^n} e^{(a+ib)x} = r^n e^{ax}\{\cos bx\sin n\theta + \sin bx\cos n\theta\}$

$= r^n e^{ax}\{\sin(n\theta + bx)\}$

where $\frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} \Rightarrow \theta = \tan^{-1}\frac{b}{a}$ and $r^2\sin^2\theta + r^2\cos^2\theta = a^2 + b^2$

Answer 5TFQ.

Given that $0 < a < b$

Then $0 < a < e^{\ln b}$

$\Rightarrow\ a < e^{\ln b}$

$\Rightarrow\ e^{\ln a} < e^{\ln b}$

$\Rightarrow\ \boxed{\ln a < \ln b \text{ is true}}$ for $0 < a < b$

Answer 6CC.

(a)

The number e is such that

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

(b)

Given e as a limit

$$\boxed{e=\lim_{x\to 0}(1+x)^{\frac{1}{x}}}$$

(c)

When $y=a^x$

Then $\frac{dy}{dx}=a^x\ln a$

But when $a=e$,

$$\frac{dy}{dx}=e^x\cdot\ln e$$
$$=e^x$$

$$y=e^x \quad\Rightarrow\frac{dy}{dx}=e^x$$

(d)

When $y=\log_a x$

Then $y=\frac{\log_e x}{\log_e a}$

Then $y'=\frac{dy}{dx}$

$$=\frac{1}{\log_e a}\cdot\frac{1}{x}$$

$$=\frac{1}{x(\ln a)}$$

When $a=e$, then $y'=\frac{1}{x}$ simple

Since $e^x\neq 0$ for every $x\in R$

So we can divide by e^x

Hence $\boxed{\text{True}}$

Answer 6E.

We have the curve of e^x.

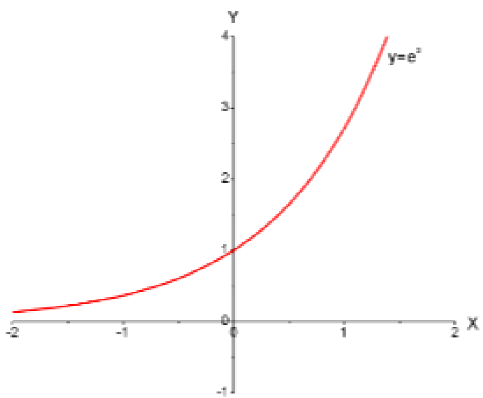


Fig.1

For getting the graph of e^{-x} reflect the curve of e^{x} about y-axis

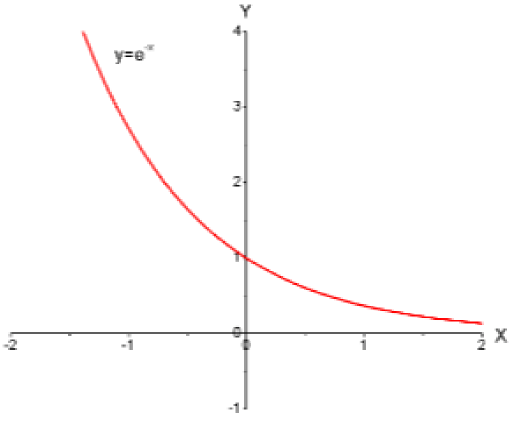


Fig.2

Now we can get the graph of $y=-e^{-x}$ by reflecting the curve of $y=e^{-x}$ about x – axis.

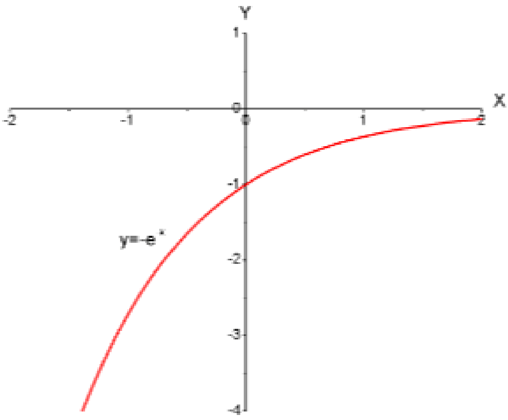


Fig.3

Answer 6P.

Show that $\sin^{-1}(\tanh x)=\tan^{-1}(\sinh x)$.

Here we use formulas of inverse trigonometric

Let $\sin^{-1}x=y$ then $x=\sin y$.

So

$$\cos y=\sqrt{1-\sin^2 y}$$
$$=\sqrt{1-x^2}$$

And also

$$\frac{\sin y}{\cos y}=\frac{x}{\sqrt{1-x^2}}$$

$$\tan y=\frac{x}{\sqrt{1-x^2}}$$

And $y=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

So $\sin^{-1}x=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

To prove the result, we use this formula as follows.

$$\sin^{-1} x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\sin^{-1}(\tanh x) = \tan^{-1}\left(\frac{\tanh x}{\sqrt{1-\tanh^2 x}}\right) \quad \text{Write } \tanh x \text{ inplace of } x$$

$$= \tan^{-1}\left(\frac{\tanh x}{\sqrt{\operatorname{sech}^2 x}}\right) \quad \text{Using the identity } 1-\tanh^2 x = \operatorname{sech}^2 x$$

$$= \tan^{-1}\left(\frac{\tanh x}{\operatorname{sech} x}\right) \quad \text{Since } \sqrt{\operatorname{sech}^2 x} = \operatorname{sech} x$$

$$= \tan^{-1}\left(\frac{\left(\frac{\sinh x}{\cosh x}\right)}{\left(\frac{1}{\cosh x}\right)}\right) \quad \text{Since } \tanh x = \frac{\sinh x}{\cosh x} \text{ and } \operatorname{sech} x = \frac{1}{\cosh x}$$

$$= \tan^{-1}(\sinh x)$$

Hence the result is $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$.

Answer 6TFQ.

Let $x = \pi^{\sqrt{5}}$

Then $\ln x = \ln \pi^{\sqrt{5}}$

$= \sqrt{5}\ln \pi$

$\Rightarrow \quad x = e^{\sqrt{5}\ln \pi}$ (Because by definition of logarithm)

$\therefore$ $\boxed{\pi^{\sqrt{5}} = e^{\sqrt{5}\ln \pi} \text{ is true}}$

Answer 7CC.

(a)

Differential equation that expresses the law of natural growth is,

$$\frac{dy}{dt} = ky, k > 0$$

The relative growth rate, $\frac{1}{y}\frac{dy}{dt}$, is constant.

(b)

The equation $\frac{dy}{dt} = ky$; is an appropriate model for population growth, assuming that there is enough room and nutrition to support the growth.

(c)

If $y(0) = y_0$, then the solution is $\boxed{y(t) = y_0 e^{kt}}$.

Answer 7E.

We have the graph of $y = \ln x$ as follows

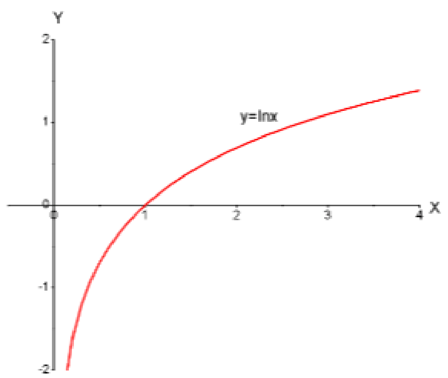


Fig.1

We can find the graph of $y = -\ln x$ by reflecting the graph of $y = \ln x$ about x-axis.

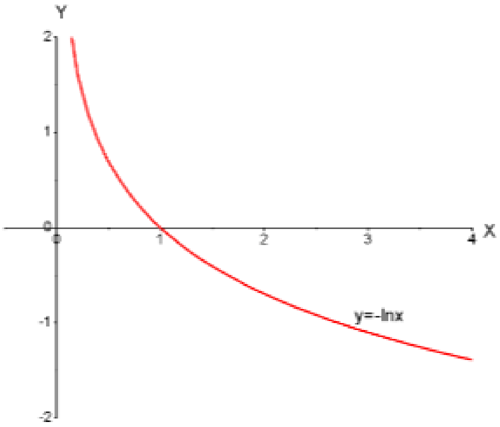


Fig.2

Answer 7P.

Let $f(x) = \tan^{-1}(x),\ x > 0.$

Apply the Lagrange's Mean value theorem to this function.

Obviously, this function is continuous on $[0, x]$ and differentiable on $(0, x)$.

Then, there is a number $0 < c < x$ such that, $f'(c) = \dfrac{f(x) - f(0)}{x - 0}$.

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$
$$\left(\tan^{-1}(c)\right)' = \frac{\tan^{-1}(x) - \tan^{-1}(0)}{x - 0}$$
$$\frac{1}{1 + c^2} = \frac{\tan^{-1}(x)}{x}$$

Consider $0 < c < x.$

Simplify as shown below:

$$0 < c < x$$
$$0 < c^2 < x^2$$
$$1 < 1 + c^2 < x^2 + 1$$
$$1 > \frac{1}{1 + c^2} > \frac{1}{x^2 + 1}$$
$$1 > \frac{\tan^{-1}(x)}{x} > \frac{1}{x^2 + 1} \quad \text{Since } \frac{1}{1 + c^2} = \frac{\tan^{-1}(x)}{x}$$
$$x > \tan^{-1}(x) > \frac{x}{x^2 + 1}$$

Hence, $\boxed{\dfrac{x}{1 + x^2} < \tan^{-1} x < x}$ for $x > 0$.

Answer 7TFQ.

Given statement is you can always divide by e^x.
Since $e^x \neq 0$ for every $x \in \mathbb{R}$
So we can divide by e^x.
Hence $\boxed{\text{True}}$

Answer 8CC.

(a)

L'Hospital's rule states that if $f(x), g(x)$ are differentiable functions and $g'(x) \neq 0$ with

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty} \text{ then } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right hand side exist or $\pm\infty$

(b)

If $f(x) \to 0, g(x) \to \infty$ then we can apply L'Hospital's rule for the product $f(x)g(x)$

By writing $f(x)g(x) = \dfrac{f(x)}{1/g(x)}$ or $\dfrac{g(x)}{1/f(x)}$

(c)

If $f(x) \to \infty, g(x) \to \infty$ then we apply L'Hospital's rule for the difference $f(x) - g(x)$ as below

Convert the difference into a quotient using a common denominator, rationalizing, factoring, or some other method

(d)

If $f(x) \to 0, g(x) \to 0$ as $x \to a$ then we apply L'Hospital's rule for the power $[f(x)]^{g(x)}$ as below

Convert the power to a product by taking the natural logarithm of both sides of $y = f^g$ by writing $f^g = e^{g \ln f}$

Answer 8E.

We have the graph of $y = \ln x$

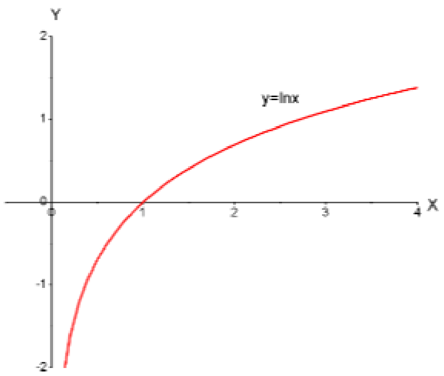


Fig.1

We can get the graph of $y = \ln(x-1)$ by shifting the graph of $y = \ln x$ a distance 1 unit to the right

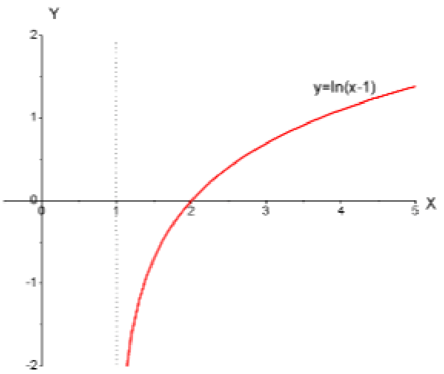


Fig.2

Answer 8P.

Suppose *f* is continuous, $f(0)=0,\ f(1)=1,\ f'(x)>0,$ and $\int_0^1 f(x)dx=\frac{1}{3}$

The objective is to find the value of the integral $\int_0^1 f^{-1}(y)dy$

Take the function $f(x)=x^2$ which is same as $f(x)=y$.

Where $y=x^2$

$$x=\sqrt{y},\ x>0, y>0$$

The function is one-one function .

Apply inverse function on both sides, the function is

$$f(x)=y$$
$$f^{-1}(y)=x$$

Now the integral is,

$$\int_0^1 f^{-1}(y)dy=\int_0^1 xdy$$
$$=\int_0^1 \sqrt{y}dy$$
$$=\int_0^1 (y)^{\frac{1}{2}}dy$$

Use integral formula $\int x^n dx=\frac{x^{n+1}}{n+1}$

$$\int_0^1 (y)^{\frac{1}{2}}=\frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1}$$
$$=\left(\frac{2}{3}y^{\frac{3}{2}}\right)_0^1$$
$$=\frac{2}{3}\left((1)^{\frac{3}{2}}-0\right)$$
$$=\frac{2}{3}$$

Therefore, the value of the integral $\int_0^1 f^{-1}(y)dy$ is $\boxed{\frac{2}{3}}$.

Answer 8TFQ.

Given that $a>0$ and $b>0$,

then $\ln(a+b)=\ln a+\ln b$

This is not true, because

$$\ln(2+3)=\ln 5\neq \ln 2+\ln 3$$

Where $\ln 5=1.6094379$

$\ln 2=0.693147$

$\ln 3=1.098612$

$$\boxed{\ln(a+b)=\ln a+\ln b \text{ is false}}$$

Answer 9E.

We have the graph of the $y = \arctan x$

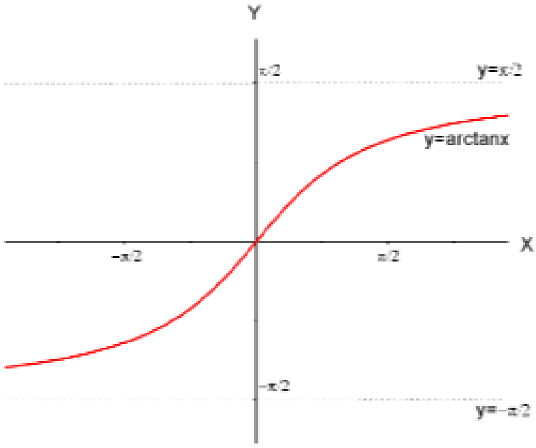


Fig.1

We can get the graph of $y = 2\arctan x$ by streaking the graph of $y = \arctan x$ vertically by a factor of 2

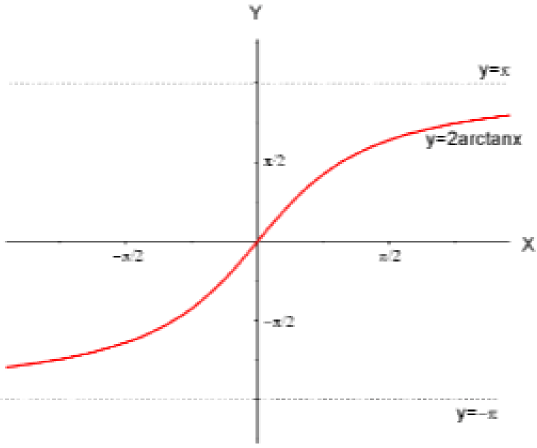


Fig.2

Answer 9P.

Show that $f(x) = \int_1^x \sqrt{1+t^3}\,dt$ is one-one and find $(f^{-1})'(0)$.

Obviously the integrand $g(t) = \sqrt{1+t^3}$ is continuous on $[a,b]$ where a and b are some positive numbers.

Now apply the Fundamental theorem of calculus part-1.

So the function $f(x) = \int_1^x \sqrt{1+t^3}\,dt;\ a \le x \le b$ is continuous on $[a,b]$ and differentiable on (a,b) and $f'(x) = g(x)$.

This implies $f'(x) = g(x) = \sqrt{1+x^3}$.

Show that the function $f(x)=\int_1^x \sqrt{1+t^3}\,dt$ is one-one.

If a function is one-one if $f(x_1)=f(x_2)$ implies $x_1=x_2$.

Let x_1, x_2 be two points in $[a,b]$.

Now consider that $f(x_1)=f(x_2)$ that is $\int_1^{x_1}\sqrt{1+t^3}\,dt=\int_1^{x_2}\sqrt{1+t^3}\,dt.$

$$\int_1^{x_1}\sqrt{1+t^3}\,dt=\int_1^{x_2}\sqrt{1+t^3}\,dt$$

$$\left(\int_1^{x_1}\sqrt{1+t^3}\,dt\right)'=\left(\int_1^{x_2}\sqrt{1+t^3}\,dt\right)'$$

$$\sqrt{1+x_1^3}=\sqrt{1+x_2^3} \quad \text{By fundamental theorem.}$$

$$1+x_1^3=1+x_2^3$$

$$x_1^3=x_2^3$$

$$x_1=x_2$$

This shows that $f(x)=\int_1^x \sqrt{1+t^3}\,dt$ is one-one.

Find the inverse of the function $f'(x)=\sqrt{1+x^3}$.

Let $f'(x)=y \Rightarrow x=\left(f'\right)^{-1}(y)$

$$f'(x)=y$$

$$\sqrt{1+x^3}=y$$

$$1+x^3=y^2$$

$$x^3=y^2-1$$

$$x=\sqrt[3]{y^2-1}$$

$$\left(f'\right)^{-1}(y)=\sqrt[3]{y^2-1}$$

$$\left(f^{-1}\right)'(y)=\sqrt[3]{y^2-1}$$

Hence $\left(f^{-1}\right)'(x)=\sqrt[3]{x^2-1}.$

This implies

$$\left(f^{-1}\right)'(0)=\sqrt[3]{0^2-1}$$

$$=\sqrt[3]{-1}$$

$$=-1$$

Hence $\left(f^{-1}\right)'(0)=\boxed{-1}$.

Answer 9TFQ.

Given if $x>0$, then $(\ln x)^6=6\ln x$

$\boxed{\text{This is not true}}$

When $x=e$,

Then $(\ln x)^6=(\ln e)^6$

$$=1$$

And $6\ln e=6$

Therefore $(\ln e)^6\neq 6\ln e$

Answer 10E.

If $a > 1$, then for large values of x, the function $y = a^x$ will have the largest values. And the function $y = \log_a x$ will have the smallest values.

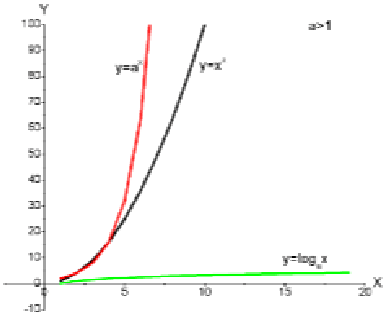


Fig.1

Answer 10P.

Consider the function

$$y = \frac{x}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \arctan\left(\frac{\sin x}{a+\sqrt{a^2-1}+\cos x}\right)$$

Find the derivative of the function with respect to *x*.

$$y' = \left(\frac{x}{\sqrt{a^2-1}}\right)' - \left(\frac{2}{\sqrt{a^2-1}} \arctan\left(\frac{\sin x}{a+\sqrt{a^2-1}+\cos x}\right)\right)'$$

$$= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}}\left(\arctan\left(\frac{\sin x}{a+\sqrt{a^2-1}+\cos x}\right)\right)'$$

$$= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{1}{1+\left(\frac{\sin x}{a+\sqrt{a^2-1}+\cos x}\right)^2}\left(\frac{\sin x}{a+\sqrt{a^2-1}+\cos x}\right)'$$

Continue the above step.

$$= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{1}{1+\left(\frac{\sin x}{a+\sqrt{a^2-1}+\cos x}\right)^2}\left(\frac{\left(a+\sqrt{a^2-1}+\cos x\right)(\sin x)' - \left(a+\sqrt{a^2-1}+\cos x\right)'(\sin x)}{\left(a+\sqrt{a^2-1}+\cos x\right)^2}\right)$$

$$= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{\left(a+\sqrt{a^2-1}+\cos x\right)^2}{\left(a+\sqrt{a^2-1}+\cos x\right)^2+(\sin x)^2}\left(\frac{\left(a+\sqrt{a^2-1}+\cos x\right)\cos x + \sin^2 x}{\left(a+\sqrt{a^2-1}+\cos x\right)^2}\right)$$

Continue the above step.

$$= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{a\cos x+\sqrt{a^2-1}\cos x+\cos^2 x+\sin^2 x}{\left(a+\sqrt{a^2-1}+\cos x\right)^2+(\sin x)^2}$$

$$= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{\left(a+\sqrt{a^2-1}\right)\cos x+1}{\left(a+\sqrt{a^2-1}\right)^2+2\left(a+\sqrt{a^2-1}\right)\cos x+(\cos x)^2+(\sin x)^2}$$

$$= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{\left(a+\sqrt{a^2-1}\right)\cos x+1}{2a^2-1+2a\sqrt{a^2-1}+2a\cos x+2\sqrt{a^2-1}\cos x+1}$$

Continue the above step.

$$=\frac{1}{\sqrt{a^2-1}}-\frac{2}{\sqrt{a^2-1}}\cdot\frac{\left(a+\sqrt{a^2-1}\right)\cos x+1}{2a^2+2a\sqrt{a^2-1}+2a\cos x+2\sqrt{a^2-1}\cos x}$$

$$=\frac{1}{\sqrt{a^2-1}}-\frac{2}{\sqrt{a^2-1}}\cdot\frac{\left(a+\sqrt{a^2-1}\right)\cos x+1}{2a\left(a+\sqrt{a^2-1}\right)+2\cos x\left(a+\sqrt{a^2-1}\right)}$$

$$=\frac{1}{\sqrt{a^2-1}}-\frac{2}{\sqrt{a^2-1}}\cdot\frac{\left(a+\sqrt{a^2-1}\right)\cos x+1}{2\left(a+\sqrt{a^2-1}\right)\left(a+\cos x\right)}$$

$$=\frac{1}{\sqrt{a^2-1}}-\frac{1}{\sqrt{a^2-1}}\cdot\frac{\left(a+\sqrt{a^2-1}\right)\cos x+1}{\left(a+\sqrt{a^2-1}\right)\left(a+\cos x\right)}$$

$$=\frac{1}{\sqrt{a^2-1}}\left(\frac{\left(a+\sqrt{a^2-1}\right)\left(a+\cos x\right)-\left(a+\sqrt{a^2-1}\right)\cos x-1}{\left(a+\sqrt{a^2-1}\right)\left(a+\cos x\right)}\right)$$

$$=\frac{a^2+a\sqrt{a^2-1}-1}{\left(a\sqrt{a^2-1}+a^2-1\right)\left(a+\cos x\right)}$$

$$=\frac{1}{a+\cos x}$$

Hence $y'=\boxed{\frac{1}{a+\cos x}}$.

Answer 10TFQ.

Given that $\frac{d}{dx}\left(10^x\right)=x10^{x-1}$

$\boxed{\text{This is not true}}$

Since $\frac{d}{dx}\left(10^x\right)=10^x\ln 10$

Therefore $\boxed{\frac{d}{dx}\left(10^x\right)\neq x10^{x-1}}$

Answer 11E.

(A). We have to evaluate the value of $e^{2\ln 3}$

Here $e^{2\ln 3}=e^{\ln(3)^2}$ Since $\ln x^y=y\ln x$

$=e^{\ln 9}$

$=9$ Since $e^{\ln x}=x$

Hence

$\boxed{e^{2\ln 3}=9}$

(B) We have to evaluate the value of

$\log_{10}25+\log_{10}4$

Here $\log_{10}25+\log_{10}4=\log_{10}\left(25\times 4\right)$ Since $\log_{10}a+\log_{10}b=\log_{10}\left(ab\right)$

$=\log_{10}100$

$=\log_{10}10^2$

$=2\log_{10}10$ Since $\ln x^y=y\ln x$.

$=2$ Since $\log_a a=1$

Hence $\boxed{\log_{10}25+\log_{10}4=2}$

Answer 11P.

Solve the following equation for *a*.

$$\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x = e \quad ......(1)$$

Consider the left hand side of the above equation.

$$\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x = \lim_{x\to\infty}\left(\frac{x\left(1+\frac{a}{x}\right)}{x\left(1-\frac{a}{x}\right)}\right)^x$$

$$= \frac{\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x}{\lim_{x\to\infty}\left(1+\frac{-a}{x}\right)^x}$$

$$= \frac{e^a}{e^{-a}} \qquad \text{Use the formula } \lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k$$

$$= e^{2a}$$

Substitute e^{2a} for $\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x$ in (1) and solve it for *a*.

$$e^{2a} = e$$
$$e^{2a} = e^1$$
$$\Rightarrow 2a = 1$$
$$\Rightarrow a = \frac{1}{2}$$

Hence the required value for *a* is $\boxed{\frac{1}{2}}$.

Answer 11TFQ.

Given $\frac{d}{dx}(\ln 10) = \frac{1}{10}$

$\boxed{\text{This is false}}$

Since $\ln 10$ is a constant, and hence $\frac{d}{dx}(\ln 10) = 0$

Therefore $\boxed{\frac{d}{dx}(\ln 10) \neq \frac{1}{10}}$

Answer 12E.

(A). $\ln e^{\pi} = \pi \ln e$ $\qquad [\because \ln a^b = b\ln a]$

$= \pi\,(1)$ $\qquad [\because \ln e = 1]$

$= \pi.$

(B). We have to evaluate $\tan\left(\arcsin\frac{1}{2}\right)$

Let $\arcsin\left(\frac{1}{2}\right) = y$, then $\sin y = \frac{1}{2}$ where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\Rightarrow$ $\operatorname{Cosec}(y) = 2$ $\qquad$ Since $\operatorname{Cosec}(y) = \frac{1}{\sin y}$.

Now from trigonometric identities,

$$1+\cot^2 y = cosec^2 y$$

$\Rightarrow \quad \cot^2 y = cosec^2 y - 1$

$\Rightarrow \quad \cot^2 y = (2)^2 - 1 \qquad$ *Since* $\quad \text{Cosec}(y) = 2$

$\Rightarrow \quad \cot^2 y = 3$

$\Rightarrow \quad \tan^2 y = \dfrac{1}{3} \qquad$ Since $\text{Cot}(y) = \dfrac{1}{Tany}$

$\Rightarrow \quad \tan y = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad y = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

$\Rightarrow \quad \sin^{-1}\left(\dfrac{1}{2}\right) = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) \qquad$ *Since* $\sin y = \dfrac{1}{2} \Rightarrow \sin^{-1}\left(\dfrac{1}{2}\right) = y$

Therefore

$$\tan\left(\arcsin\frac{1}{2}\right) = \tan\left[\sin^{-1}\left(\frac{1}{2}\right)\right]$$

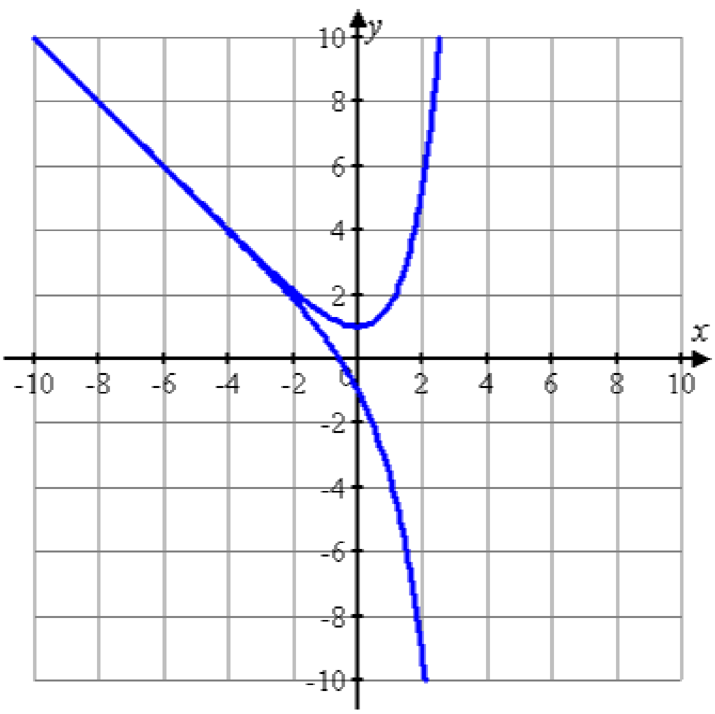
$$= \tan\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right] \quad \left[\because \sin^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{\sqrt{3}}$$

Hence $\boxed{\tan\left(\arcsin\frac{1}{2}\right) = \frac{1}{\sqrt{3}}}$.

Answer 12P.

Firstly sketch the function $f(x) = |x+y| - e^x$ using graphing calculator.

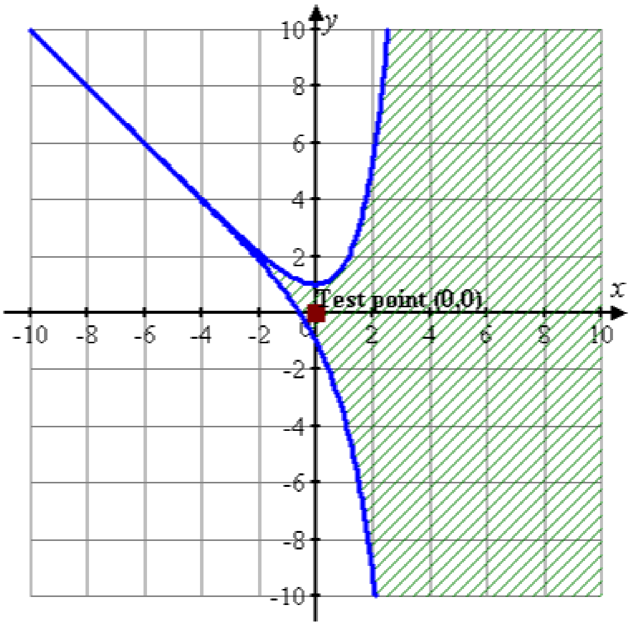


Take the test point $O(0,0)$ and substitute 0 for *x* and 0 for *y* in $|x+y|-e^x \le 0$.

$$\begin{aligned} |x+y|-e^x &\le 0 \\ |0+0|-e^0 &\le 0 \\ 0-1 &\le 0 \\ -1 &\le 0 \quad \text{True} \end{aligned}$$

This means the test point $O(0,0)$ is in the region represented by the inequality $|x+y|-e^x \le 0$.

So shade the half plane containing $O(0,0)$ and bounded by the curve $f(x)=|x+y|-e^x$ then it will be as shown in the following figure.



Answer 12TFQ.

Given that $y=e^{3x}$

Taking logarithm, on both sides,

$$\begin{aligned} \ln y &= \ln e^{3x} \\ &= 3x \ln e \\ &= 3x \end{aligned}$$

$$\Rightarrow \quad x = \frac{1}{3}\ln y$$

Therefore $\boxed{\text{The inverse function of } y=e^{3x} \text{ is } y=\frac{1}{3}\ln x \text{ is true}}$

Answer 13E.

We have to solve the equation $\ln x = \frac{1}{3}$

The given equation is

$$\ln x = \frac{1}{3}$$

Applying exponential function both sides, we get,

$$e^{\ln x} = e^{1/3}$$

$$\Rightarrow \quad x = e^{1/3} \qquad \text{Since } e^{\ln x} = x$$

Hence

$$\boxed{\begin{array}{l}\text{The solution is} \\ x = e^{1/3}\end{array}}$$

Answer 13P.

Show that $\cosh(\sinh(x)) < \sinh(\cosh(x))$ for all x.

Consider the function $f(x) = \cosh(\sinh(x)) - \sinh(\cosh(x))$.

Find the Taylor series for this function using Maple.

Type the following command in Maple and then press ENTER to obtain the result.

$$evalf(taylor(\cosh(\sinh(x)) - \sinh(\cosh(x)), x = 0, 5));$$

$$-0.175201194 - 0.2715403172\,x^2 - 0.0028618423\,x^4 + O(x^6)$$

From the above output it is observed that,

$$\cosh(\sinh(x)) - \sinh(\cosh(x))$$
$$= -0.17501194 - 0.2715403172x^2 - 0.0028618423x^4 - ... - \infty$$
$$= -(0.17501194 + 0.2715403172x^2 + 0.0028618423x^4 + ... + \infty)$$

Obviously the series $0.17501194 + 0.2715403172x^2 + 0.0028618423x^4 + ... + \infty$ is positive as it contains positive coefficients and even powered *x* terms.

So $-(0.17501194 + 0.2715403172x^2 + 0.0028618423x^4 + ... + \infty) < 0$ for all *x*.

This follows that $\cosh(\sinh(x)) - \sinh(\cosh(x)) < 0$, for all x.

Answer 13TFQ.

The statement $\cos^{-1} x = \dfrac{1}{\cos x}$ is false

For example when

$$x = 0, \cos^{-1} x = \cos^{-1}(0)$$
$$= 1.5708$$
$$\frac{1}{\cos x} = \frac{1}{\cos 0}$$
$$= \frac{1}{1}$$
$$= 1$$
$$\therefore \cos^{-1}(0) \neq \frac{1}{\cos(0)}$$

Therefore $\boxed{\cos^{-1} x = \dfrac{1}{\cos x} \text{ is false}}$

Answer 14E.

The given equation is

$$e^x = \frac{1}{3}$$

Applying natural logarithms on both sides of the equation, we get,

$$\ln e^x = \ln \frac{1}{3}$$

$$\Rightarrow \quad x = \ln 1 - \ln 3$$

Since $\ln e^x = x$ And $\ln \dfrac{x}{y} = \ln x - \ln y$

$$\Rightarrow \quad x = -\ln 3$$
$$\Rightarrow \quad x \approx -1.0986$$

Hence

$$\boxed{\begin{array}{l}\text{The solution is } x = -\ln 3 \\ or \quad x \approx -1.0986\end{array}}$$

Answer 14P.

Show that $\frac{e^{x+y}}{xy} \geq e^2$ for all positive values of x.

Firstly find minimum value of the function $f(x)=\frac{\exp(x)}{x}$;$x>0$.

Find the derivative and the critical points of the function $f(x)=\frac{\exp(x)}{x}$.

$$f'(x)=\frac{\exp(x)}{x}-\frac{\exp(x)}{x^2}$$

Solve the equation $f'(x)=0$ for x.

$$\begin{aligned} f'(x)&=0 \\ \frac{\exp(x)}{x}-\frac{\exp(x)}{x^2}&=0 \\ x\exp(x)-\exp(x)&=0 \\ \exp(x)(x-1)&=0 \\ x&=1 \text{ Since } \exp(x)\neq 0 \end{aligned}$$

Hence the critical point of the function $f(x)=\frac{\exp(x)}{x}$ is $x=1$.

Find the second derivative of the function $f(x)=\frac{\exp(x)}{x}$.

$$\begin{aligned} f''(x)&=\left(\frac{\exp(x)}{x}-\frac{\exp(x)}{x^2}\right)-\left(\frac{\exp(x)}{x^2}-\frac{2\exp(x)}{x^3}\right) \\ &=\exp(x)\left(\frac{1}{x}-\frac{2}{x^2}+\frac{2}{x^3}\right) \end{aligned}$$

Put $x=1$ in $f''(x)=\exp(x)\left(\frac{1}{x}-\frac{2}{x^2}+\frac{2}{x^3}\right)$ to find $f''(1)$.

$$f''(1)=\exp(1)\left(\frac{1}{1}-\frac{2}{1}+\frac{2}{1}\right)=e>0$$

Hence $f''(1)>0$ this follows that the function has minimum value at $x=1$.

(By the second derivative test)

Substitute 1 for x in $f(x)=\frac{\exp(x)}{x}$ to find the minimum value of the function.

$$f(1)=\frac{\exp(1)}{1}=e$$

Hence the minimum value of the function $f(x)=\frac{\exp(x)}{x}$ is e.

This implies $f(x)=\frac{\exp(x)}{x}>e$ for all $x>0$.

Now use this fact to show the given inequality $\frac{e^{x+y}}{xy}\geq e^2$.

$$\frac{\exp(x)}{x}>e \text{ for all } x>0 \quad \text{......}(1)$$
$$\frac{\exp(y)}{y}>e \text{ for all } y>0 \quad \text{......}(2)$$

Multiply (1) and (2) to obtain the desired result.

$$\frac{\exp(x)}{x}\cdot\frac{\exp(y)}{y}>e\cdot e \text{ for all } x,y>0$$
$$\frac{\exp(x+y)}{xy}>e^2 \text{ for all } x,y>0$$

This proves the desired result.

Answer 14TFQ.

Given that $\tan^{-1}x=\frac{\sin^{-1}x}{\cos^{-1}x}$

When $x=1$,

$$\tan^{-1}x=\tan^{-1}(1)$$
$$=0.7854$$
$$\sin^{-1}x=\sin^{-1}(1)$$
$$=1.5708$$
$$\cos^{-1}x=\cos^{-1}(1)$$
$$=0$$

Therefore $\tan^{-1}(1)\neq\frac{\sin^{-1}(1)}{\cos^{-1}(1)}$

Therefore $\boxed{\text{The given statement } \tan^{-1}x=\frac{\sin^{-1}x}{\cos^{-1}x} \text{ is false}}$

Answer 15E.

The given equation is

$$e^{e^x}=17$$

Applying natural logarithms on both sides, we get

$$\ln e^{e^x}=\ln(17)$$

$\Rightarrow \quad e^x=\ln(17)$ Since $\ln e^x=x$

Again applying natural logarithms on both sides, we get,

$$\ln e^x=\ln\ln(17)$$

$\Rightarrow \quad x=\ln\ln(17)$ Since $\ln e^x=x$

Hence,

$\boxed{\text{The solution is } x=\ln\ln(17)}$

Answer 15P.

Consider the equation $e^{2x} = k\sqrt{x}$.

This equation will have a unique solution when the curves $y = \exp(2x)$ and $y = k\sqrt{x}$ meets at single point. Here the slope of the tangents for the both curves is same.

Suppose that both the curves meet at $x = a$ then it follows that,

$$\exp(2a) = k\sqrt{a}$$

And also implies that the slopes of the tangents at $x = a$ are equal. That means the derivatives of the functions $y = \exp(2x)$ and $y = k\sqrt{x}$ are equal at $x = a$.

$$\exp(2a)\cdot 2 = k\frac{1}{2\sqrt{a}}$$

$$2k\sqrt{a} = \frac{k}{2\sqrt{a}} \quad \text{Since } \exp(2x) = k\sqrt{x}$$

$$a = \frac{1}{4}$$

Substitute $\frac{1}{4}$ by *a* in the equation $\exp(2a) = k\sqrt{a}$ to obtain the value of *k*.

$$\exp\left(2\cdot\frac{1}{4}\right) = k\sqrt{\frac{1}{4}}$$

$$\Rightarrow k = 2\sqrt{e}$$

Hence the curves $y = \exp(2x)$ and $y = k\sqrt{x}$ meet at single point for $k = 2\sqrt{e}$.

Therefore the equation $e^{2x} = k\sqrt{x}$ will have a single solution when $k = \boxed{2\sqrt{e}}$.

Answer 15TFQ.

Given statement is $\cosh x \geq 1$ for all x.

For any $x \in R$ $\left(e^x - 1\right)^2 \geq 0$

$$\Rightarrow \quad e^{2x} - 2e^x + 1 \geq 0$$

$$\Rightarrow \quad e^{2x} + \geq 2e^x$$

$$\Rightarrow \quad \frac{e^x + e^{-x}}{2} \geq 1$$

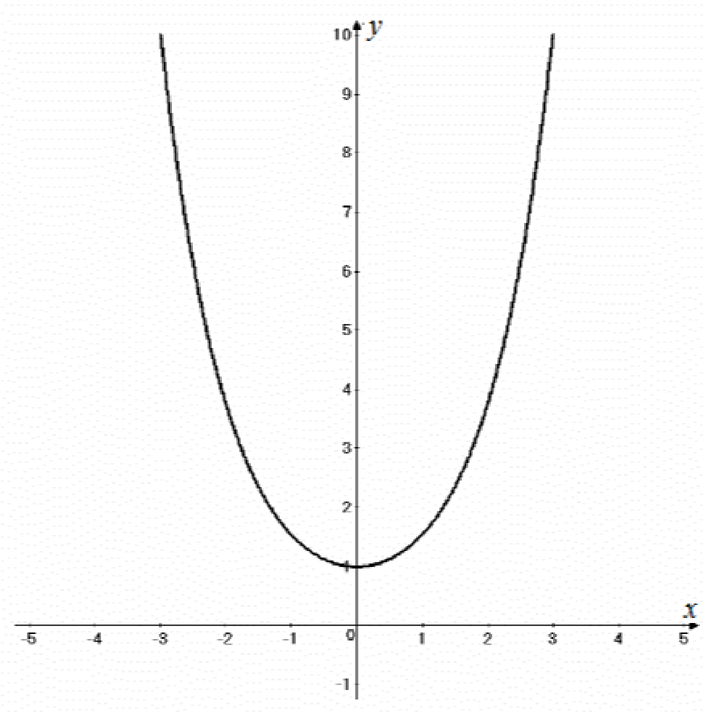
$$\Rightarrow \quad \cosh x \geq 1 \forall x$$

$\cosh x \geq 1 \forall x$ is $\boxed{\text{True}}$

From the graph of $\cosh x$, we have

$$\cosh x \geq 1 \forall x$$

The graph of $f(x) = \cosh x$



Answer 16E.

We have $\ln\left(1+e^{-x}\right)=3$

$$\Rightarrow 1+e^{-x}=e^{3} \qquad \left[\ln x = y \Leftrightarrow x = e^{y}\right]$$

$$\Rightarrow e^{-x}=e^{3}-1$$

Taking logarithms of both sides

$$\ln e^{-x} = \ln\left(e^{3}-1\right)$$

$$\Rightarrow -x = \ln\left(e^{3}-1\right) \qquad \left[\ln e^{x} = x \text{ for all } x\right]$$

$$\Rightarrow \boxed{x = -\ln\left(e^{3}-1\right)}$$

Answer 16P.

Consider the following inequality

$a^{x} \geq 1+x$

Find the positive set of all positive numbers of a such that the following condition holds true:

$a^{x} \geq 1+x$, for all x

Consider the function for fixed positive number a

$f_{a}(x) = a^{x} - 1 - x$

to find value of a such that $a^{x} - 1 - x \geq 0, \forall x$

the minimum value of $f_{a}(x)$ is greater than or equal to zero

$$\lim_{x\to-\infty} f_{a}(x) = \lim_{x\to-\infty}\left(a^{x}-1-x\right)$$

$$= a^{-\infty} - 1 + \infty$$

$$= 0 - 1 + \infty$$

$$\lim_{x\to-\infty} f_{a}(x) \to \infty$$

$$\lim_{x\to\infty} f_{a}(x) = \lim_{x\to\infty}\left(a^{x}-1-x\right)$$

since $a^{x} \gg x$

$$\lim_{x\to\infty} f_{a}(x) = \lim_{x\to\infty} a^{x} \to \infty$$

To find the minimum value of $f_a(x)$ consider $f_a'(x)=0$

$$f_a(x)=a^x-1-x$$

$$f_a'(x)=a^x\ln a-1$$

$$f_a'(x_0)=0$$

$$a^{x_0}\ln a-1=0$$

$$a^{x_0}=\frac{1}{\ln a} \quad \text{......(1)}$$

$$x_0\ln a=\ln\left(\frac{1}{\ln a}\right)$$

$$x_0=\frac{\ln\left(\frac{1}{\ln a}\right)}{\ln a} \quad \text{......(2)}$$

The minimum value of $f_a(x)$ is $f_a(x_0)$

$$f_a(x_0)=a^{x_0}-1-x_0$$

$$f_a(x_0)=\frac{1}{\ln a}-1-\frac{\ln\left(\frac{1}{\ln a}\right)}{\ln a} \quad (\text{ from (1) and (2)})$$

$$f_a(x_0)=\frac{1-\ln a-\ln\left(\frac{1}{\ln a}\right)}{\ln a}\geq 0$$

$$1-\ln a-\ln\left(\frac{1}{\ln a}\right)\geq 0$$

$$\ln(e)-\ln a-\ln\left(\frac{1}{\ln a}\right)\geq 0$$

$$\ln\left(\frac{e}{a}\times\ln a\right)\geq \ln 1$$

$$\frac{e}{a}\times\ln a\geq 1$$

$$\frac{\ln a}{a}\geq\frac{1}{e}$$

$$\ln(a)^{\frac{1}{a}}\geq\frac{1}{e}$$

Hence, the required answer is $\boxed{(a)^{\frac{1}{a}}\geq e^{\frac{1}{e}}}$

Answer 16TFQ.

Given $\ln\frac{1}{10}=-\int_1^{10}\frac{dx}{x}$

Consider $-\int_1^{10}\frac{dx}{x}=-(\ln x)_1^{10}$

$$=-(\ln 10-\ln 1)$$

$$=\ln 1-\ln 10$$

$$=\ln(1/10)$$

Therefore $\boxed{\text{The given statement } \ln(1/10)=-\int_1^{10}\frac{dx}{x} \text{ is true}}$

Answer 17E.

We have $\ln(x+1)+\ln(x-1)=1$

$$\Rightarrow \ln[(x+1).(x-1)]=1 \qquad [\ln m+\ln n=\ln(m.n)]$$

$$\Rightarrow \ln(x^2-1)=1 \qquad [(x-1)(x+1)=x^2-1]$$

$$\Rightarrow x^2-1=e \qquad [\ln x=y \Leftrightarrow e^y=x]$$

$$\Rightarrow x^2=e+1$$

$$\Rightarrow \boxed{x=\sqrt{e+1}}$$

Answer 17P.

It is required to find for which positive numbers a does the curve $y=a^x$ intersects the line $y=x$.

Suppose that the curve $y=a^x$ intersects the line $y=x$, then $a^{x_0}=x_0$ for some $x_0>0$.

And therefore

$$a^{x_0}=x_0$$

$$\left(a^{x_0}\right)^{\frac{1}{x_0}}=x_0^{\frac{1}{x_0}}$$

$$a=x_0^{\frac{1}{x_0}}$$

Let $g(x)=x^{\frac{1}{x}}, x>0$

Find the maximum value of $g(x)=x^{\frac{1}{x}}, x>0$ because if a is larger than the maximum value of this function $g(x)$, then the curve $y=a^x$ does not intersect the line $y=x$.

Differentiate g with respect to x, to get

$$g'(x)=e^{\frac{1}{x}\ln x}\left(-\frac{1}{x^2}\ln x+\frac{1}{x}\cdot\frac{1}{x}\right)$$

$$=x^{\frac{1}{x}}\left(\frac{1}{x^2}\right)(1-\ln x)$$

It is clear that $g'(x)$ is 0 only where $x=e$ and for $0<x<e, g'(x)>0$ a while for $x>e$, $g'(x)<0,$

So g has an absolute maximum of $g(e)=e^{\frac{1}{e}}$. (Here replace e for x in $g(x)=x^{\frac{1}{x}}$,)

So if $y=a^x$ intersects the line $y=x$, must have $0<a\le e^{\frac{1}{e}}$.

Conversely, suppose that $0<a\le e^{\frac{1}{e}}$, then $a^e\le e$.

So the graph of $y=a^x$ lies above that of $y=x$ at $x=0$.

Therefore, by the intermediate value theorem, the graphs of $y=a^x$ and $y=x$ must intersect somewhere between $\boxed{x=0 \text{ and } x=e}$.

Answer 17TFQ.

Given $\int_2^{16}\frac{dx}{x}=3\ln 2$

Consider

$$\int_2^{16}\frac{dx}{x}=\int_2^{16}\frac{1}{x}\cdot dx$$

$$=[\ln x]_2^{16}$$

$$=\ln 16-\ln 2$$

$$= \ln 2^4 - \ln 2$$
$$= 4\ln 2 - \ln 2$$
$$= (4-1)\ln 2$$
$$= 3\ln 2$$

Therefore $\boxed{\text{The given statement } \int_2^{16} \frac{dx}{x} = 3\ln 2 \text{ is true}}$

Answer 18E.

We have $\log_5\left(c^x\right) = d$

Using $\quad \log_a x = \frac{\ln x}{\ln a}$

We have

$$\frac{\ln\left(c^x\right)}{\ln 5} = d$$
$$\Rightarrow \frac{x\ln c}{\ln 5} = d$$
$$\Rightarrow \boxed{x = d.\ln 5/\ln c} \text{ or } \boxed{x = \ln 5^d/\ln c}$$

Answer 18P.

Consider the curve

$y = cx^3 + e^x$

It is required to find the values of c does the given curve having inflection points.

Recollect the definition of inflection points:

A point P on a curve $y = f(x)$ is called an inflection point if the function f is continuous at a point P and the curve changes from concave upward to concave downward or from concave downward to concave upward at a point P.

Differentiate $y = cx^3 + e^x$ with respect to x, to obtain

$y' = 3cx^2 + e^x$

Again differentiate $y' = 3cx^2 + e^x$ with respect to x, to get

$y'' = 6cx + e^x$

Suppose $F(x) = y'' = 6cx + e^x$

To find the values of c such that given function has inflection point, it is sufficient to find the value c for which $F(x)$ has a root.

Now differentiate $F(x)$ with respect to x, to get

$F'(x) = 6c + e^x$

Again differentiate $F'(x)$ with respect to x, to get

$F''(x) = e^x$

Setting
$$F'(x) = 0$$
$$6c + e^x = 0$$
$$6c = -e^x$$
$$-6c = e^x$$

Apply $\ln$ on both sides, to get

$$\ln(-6c) = \ln(e^x)$$
$$x = \ln(-6c)$$

Substitute $x = \ln(-6c)$ in $F''(x) = e^x$, to obtain

$$F''(\ln(-6c)) = e^{\ln(-6c)}$$
$$F''(\ln(-6c)) = -6c < 0$$
$$c < 0$$

Therefore, F has minimum value where $c < 0$.

Now,
$$\begin{aligned} F(\ln(-6c)) &= 6c(\ln(-6c)) + e^{\ln(-6c)} \\ &= 6c(\ln(-6c)) - 6c \\ &= 6c(\ln(-6c) - 1) \end{aligned}$$

$$F(\ln(-6c)) < 0, c < 0$$
$$\ln(-6c) - 1 > 0$$
$$\ln(-6c) > 1$$
$$-6c > e$$
$$c > \frac{e}{-6}$$

Therefore, $\boxed{c \in \left(\frac{e}{6}, \infty\right)}$

F Has minimum and $F(\ln(-6c)) < 0$

$\Rightarrow F$ has a root for $c \in \left(\frac{e}{6}, \infty\right)$

Hence, the required values of c are $c \in \boxed{\left(\frac{e}{6}, \infty\right)}$.

Answer 18TFQ.

Consider
$$\begin{aligned} \underset{x\to\pi-}{Lt} \frac{\tan x}{1-\cos x} &= \frac{\tan \pi}{1-\cos \pi} \\ &= \frac{0}{1+1} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \underset{x\to\pi-}{Lt} \frac{\sec^2 x}{\sin x} &= \frac{\sec^2 \pi}{\sin \pi} \\ &= \frac{1}{0} \\ &= \infty \end{aligned}$$

Therefore $\boxed{\text{The given statement, } \underset{x\to\pi-}{Lt} \frac{\tan x}{1-\cos x} = \underset{x\to\pi-}{Lt} \frac{\sec^2 x}{\sin x} \text{ is false}}$

Answer 19E.

We have $\tan^{-1} x = 1$

We use the fact:-

$$\tan^{-1} x = y \Leftrightarrow \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Here $-\frac{\pi}{2} < 1 < \frac{\pi}{2}$

So $\tan^{-1} x = 1$

$\boxed{x = \tan 1}$

$\boxed{x \approx 1.5574}$

Answer 20E.

We have $\sin x = 0.3$

$$\Rightarrow \sin^{-1}(\sin x) = \sin^{-1}(0.3)$$

$$\Rightarrow x = \sin^{-1}(0.3) \qquad \left[\sin^{-1}(\sin x) = x \text{ for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}\right]$$

$$\Rightarrow \boxed{x \approx 0.3047}$$

Answer 21E.

We have to differentiate $f(t) = t^2 \ln t$.

Product rule is given by $\frac{d}{dt}[g(t)h(t)] = g(t)\frac{d}{dt}[h(t)] + h(t)\frac{d}{dt}[g(t)]$.

Let $g(t) = t^2$, $h(t) = \ln t$, now applying the product rule we get

$$f'(t) = t^2 \frac{d}{dt}(\ln t) + \ln t.\left(\frac{d}{dt}t^2\right)$$

$$= t^2.\frac{1}{t} + \ln t.(2t) \qquad \left[\because \frac{d}{dt}(\ln t) = \frac{1}{t}, \frac{d}{dt}(t^n) = nt^{n-1}\right]$$

$$= t + 2t\ln t$$

Or $\boxed{f'(t) = t(1 + 2\ln t)}$.

Answer 22E.

We have $g(t) = \frac{e^t}{1+e^t}$

Differentiating by Quotient rule with respect to t

$$g'(t) = \frac{(1+e^t).\frac{d}{dt}(e^t) - e^t\frac{d}{dt}(1+e^t)}{(1+e^t)^2}$$

$$\Rightarrow g'(t) = \frac{(1+e^t).e^t - e^t.e^t}{(1+e^t)^2} \qquad \left[\frac{d}{dt}e^t = e^t\right]$$

$$\Rightarrow g'(t) = \frac{e^t[1+e^t-e^t]}{(1+e^t)^2}$$

$$\Rightarrow \boxed{g'(t) = \frac{e^t}{(1+e^t)^2}}$$

Answer 23E.

We have $h(\theta) = e^{\tan 2\theta}$

Differentiating with respect to θ by chain rule

$$h'(\theta) = \frac{d}{d\theta}\left(e^{\tan 2\theta}\right)$$

$$= e^{\tan 2\theta}.\frac{d}{d\theta}(\tan 2\theta)$$

$$= e^{\tan 2\theta}.\sec^2(2\theta).\frac{d}{d\theta}(2\theta)$$

$$= 2\sec^2(2\theta)\, e^{\tan 2\theta}$$

So $\boxed{h'(\theta) = 2\sec^2(2\theta)\, e^{\tan 2\theta}}$

Answer 24E.

We have $h(u) = 10^{\sqrt{u}}$

Differentiating with respect to u, by chain rule an using the formula

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$h'(u) = 10^{\sqrt{u}}.\ln(10).\frac{d}{du}(\sqrt{u})$$

$$\Rightarrow \boxed{h'(u) = \frac{1}{2\sqrt{u}}.10^{\sqrt{u}}.\ln 10} \qquad \text{Since } \left[\frac{d}{du}\sqrt{u} = \frac{1}{2\sqrt{u}}\right]$$

Answer 25E.

Consider the following function:

$$y = \ln\left|\sec 5x + \tan 5x\right|$$

To differentiate the given function, first let:

$$t = \sec 5x + \tan 5x$$

Then

$$y = \ln|t|$$

Differentiate with respect to x:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt}.\frac{dt}{dx}\\ &= \frac{d}{dt}\left(\ln|t|\right).\frac{dt}{dx} \quad \text{...... (1)}\\ &= \frac{1}{t}.\frac{dt}{dx}\end{aligned}$$

Now, to determine $\frac{dt}{dx}$ differentiate $t = \sec 5x + \tan 5x$:

$$\frac{dt}{dx} = \frac{d}{dx}(\sec 5x) + \frac{d}{dx}(\tan 5x) \quad \text{...... (2)}$$

Find the derivatives of the sec and tan functions separately:

$$\begin{aligned}\frac{d}{dx}(\sec 5x) &= \frac{d}{dx}(\sec 5\text{x})\\ &= 5\sec 5x \tan 5x\end{aligned}$$

Also,

$$\begin{aligned}\frac{d}{dx}(\tan 5x) &= \frac{d}{dx}(\tan 5\text{x})\\ &= 5\sec^2 5x\end{aligned}$$

Substitute these values in (2):

$$\begin{aligned}\frac{dt}{dx} &= 5\sec 5x \tan 5x + 5\sec^2 5x\\ &= 5\sec 5x(\sec 5x + \tan 5x)\end{aligned}$$

Use this value of $\frac{dt}{dx}$ in (1) then:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{(\sec 5x + \tan 5x)}.5\sec 5x(\sec 5x + \tan 5x)\\ &= 5\sec 5x\end{aligned}$$

Since $t = \sec 5x + \tan 5x$

Therefore the derivative of the given function is:

$$\boxed{y' = 5\sec 5x}$$

Answer 26E.

Given $y = x\cos^{-1}x$

On differentiation

$$y' = \frac{dy}{dx}$$
$$= \frac{d}{dx}\left[x\cos^{-1}x\right]$$
$$= x\frac{d}{dx}\left(\cos^{-1}x\right) + \left(\cos^{-1}x\right)\frac{d}{dx}(x)$$
$$= x\left(\frac{-1}{\sqrt{1-x^2}}\right) + \cos^{-1}x$$
$$= \frac{-x}{\sqrt{1-x^2}} + \cos^{-1}x$$

Therefore $\boxed{y' = \frac{-x}{\sqrt{1-x^2}} + \cos^{-1}x}$

Answer 27E.

Given $y = x\tan^{-1}(4x)$

On differentiation,

$$y' = \frac{dy}{dx}$$
$$= \frac{d}{dx}\left(x\tan^{-1}(4x)\right)$$
$$= x\frac{d}{dx}\left(\tan^{-1}(4x)\right) + \tan^{-1}(4x)\cdot\frac{d}{dx}(x)$$
$$= x\cdot\frac{1}{1+16x^2}\times 4 + \left[\tan^{-1}(4x)\right]\cdot 1$$
$$= \frac{4x}{1+16x^2} + \tan^{-1}(4x)$$

Therefore $\boxed{y' = \frac{4x}{1+16x^2} + \tan^{-1}(4x)}$

Answer 28E.

Consider the function,

$$y = e^{mx}\cos nx$$

Product rule:

$$\frac{dy}{dx}(f\cdot g) = f\frac{dg}{dx} + g\frac{df}{dx}$$

Use product to rule and differentiate.

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{mx}\cos nx\right)$$
$$= e^{mx}\frac{d}{dx}(\cos nx) + \cos nx\frac{d}{dx}\left(e^{mx}\right)$$
$$= e^{mx}(-n\sin nx) + \cos nx\left(me^{mx}\right)$$
$$= -ne^{mx}\sin nx + me^{mx}\cos nx$$
$$= e^{mx}(m\cos nx - n\sin nx)$$

Therefore,

$$\frac{dy}{dx} = \boxed{e^{mx}(m\cos nx - n\sin nx)}$$

Answer 29E.

We have $y = \ln(\sec^2 x)$

Let $\sec^2 x = t$

Then $\frac{dt}{dx} = \frac{d}{dx}(\sec^2 x)$

$\Rightarrow \frac{dt}{dx} = 2\sec x \sec x . \tan x$ [By chain rule]

$\Rightarrow \frac{dt}{dx} = 2\sec^2 x . \tan x$

We write $y = \ln(t)$

Differentiating with respect to x by chain rule

$$\frac{dy}{dx} = \frac{d}{dx}(\ln(t))$$

$$= \frac{d}{dt}(\ln t) . \frac{dt}{dx} \quad \text{[By chain rule]}$$

$$= \frac{1}{t} . \frac{dt}{dx}$$

$$= \frac{1}{\sec^2 x} . 2\sec^2 x . \tan x$$

Or $\boxed{y' = 2\tan x}$

Answer 30E.

Given $y = \sqrt{t \cdot \ln(t^4)}$

$= \sqrt{t \cdot 4\ln t}$ $\left(\text{since } \ln a^m = m \cdot \ln a\right)$

On differentiation,

$$y' = \frac{dy}{dt}$$

$$= \frac{d}{dt}\left(\sqrt{4t \cdot \ln t}\right)$$

$$= 2 \cdot \frac{d}{dt}(t \ln t)^{1/2}$$

$$= 2 \cdot \frac{1}{2\sqrt{t \cdot \ln t}} \cdot \frac{d}{dt}(t \cdot \ln t)$$

$$= \frac{1}{\sqrt{t \ln t}}\left[t \cdot \frac{d}{dt}(\ln t) + (\ln t)\frac{d}{dt}(t)\right]$$

$$= \frac{1}{\sqrt{t \cdot \ln t}}\left[t \cdot \frac{1}{t} + (\ln t) \cdot 1\right]$$

$$= \frac{1}{\sqrt{t \cdot \ln t}}(1 + \ln t)$$

$$= \frac{1 + \ln t}{\sqrt{t \cdot \ln t}}$$

Therefore $\boxed{y' = \frac{1 + \ln t}{\sqrt{t \cdot \ln t}}}$

Answer 31E.

We have $y=\dfrac{e^{1/x}}{x^2}=e^{1/x}x^{-2}$

Differentiating with respect to x by product rule

$$\frac{dy}{dx}=x^{-2}.\frac{d}{dx}\left(e^{1/x}\right)+e^{1/x}.\frac{d}{dx}\left(x^{-2}\right)$$

$$=x^{-2}e^{1/x}.\frac{d}{dx}\left(\frac{1}{x}\right)+e^{1/x}.\left(-2x^{-3}\right)$$ [By chain rule]

$$=x^{-2}e^{1/x}\left(-\frac{1}{x^2}\right)-\frac{2}{x^3}e^{1/x}$$ $\left[\frac{d}{dx}\left(\frac{1}{x}\right)=-(x)^{-2}\right]$

$$=-\frac{1}{x^4}e^{1/x}-\frac{2}{x^3}e^{1/x}$$

$$=e^{1/x}\left(\frac{-1-2x}{x^4}\right)$$

So $\boxed{y'=e^{1/x}\left(\frac{-1-2x}{x^4}\right)}$

Answer 32E.

Consider the function $y=(\arcsin 2x)^2$

Differentiate both sides with respect to x

$$\frac{dy}{dx}=\frac{d}{dx}(\arcsin 2x)^2$$

$$=2\sin^{-1}(2x)\frac{d}{dx}\left(\sin^{-1}2x\right)$$ • $\frac{d}{dx}(f(x))^n=n(f(x))^{n-1}\cdot f'(x)$

$$=2\sin^{-1}(2x)\frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}(2x)$$ • $\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}$

$$=2\sin^{-1}(2x)\frac{1}{\sqrt{1-4x^2}}\cdot 2$$

$$=\frac{4\sin^{-1}(2x)}{\sqrt{1-4x^2}}$$

Therefore, $\dfrac{dy}{dx}=\boxed{\dfrac{4\sin^{-1}(2x)}{\sqrt{1-4x^2}}}$

Answer 33E.

Consider the function $y=3^{x\ln x}$

Differentiate both sides with respect to x

$$\frac{dy}{dx}=\frac{d}{dx}\left(3^{x\ln x}\right)$$

$$=3^{x\ln x}\cdot\ln 3\frac{d}{dx}(x\ln x)$$ • $\frac{d}{dx}\left(a^{f(x)}\right)=a^{f(x)}f'(x)$

$$=3^{x\ln x}\cdot\ln 3\left[x\cdot\frac{d}{dx}(\ln x)+\ln x\cdot\frac{d}{dx}(x)\right]$$

$$=3^{x\ln x}\ln 3\left[x\cdot\frac{1}{x}+\ln x\cdot 1\right]$$

$$=3^{x\ln x}\ln 3(1+\ln x)$$

Therefore, $\dfrac{dy}{dx}=\boxed{3^{x\ln x}\ln 3(1+\ln x)}$

Answer 34E.

We have $y = e^{\cos x} + \cos(e^x)$

Differentiating with respect to x by chain rule

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\cos x}\right) + \frac{d}{dx}\cos\left(e^x\right)$$

$$= e^{\cos x}.\frac{d}{dx}(\cos x) + \left(-\sin\left(e^x\right)\right).\frac{d}{dx}.e^x \qquad \text{[By chain rule]}$$

$$= e^{\cos x}(-\sin x) - e^x \sin\left(e^x\right)$$

So $\boxed{y' = -e^{\cos x}.\sin x - e^x.\sin\left(e^x\right)}$

Answer 35E.

We have $H(v) = v\tan^{-1} v$

Differentiating with respect to v by product rule

$$H'(v) = v.\frac{d}{dv}\left(\tan^{-1} v\right) + \tan^{-1} v.\frac{d}{dv}(v)$$

$$= v.\frac{1}{1+v^2} + \tan^{-1} v \qquad \text{Since } \left[\frac{d}{dx}\tan^{-1} x = \frac{1}{x^2+1}\right]$$

So $\boxed{H'(v) = \frac{v}{1+v^2} + \tan^{-1} v}$

Answer 36E.

We have $F(z) = \log_{10}\left(1+z^2\right)$

Differentiating by chain rule and using the formula $\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$

$$F'(z) = \frac{1}{\left(1+z^2\right).\ln 10}\frac{d}{dz}\left(1+z^2\right)$$

$$= \frac{1}{\left(1+z^2\right)\ln 10}.2z$$

Then $\boxed{F'(z) = \frac{2z}{\left(1+z^2\right)\ln 10}}$

Answer 37E.

We have $y = x\sin h\left(x^2\right)$

Differentiating by product rule with respect to x

$$\frac{dy}{dx} = x\frac{d}{dx}\sin h\left(x^2\right) + \sin h\left(x^2\right).\frac{d}{dx}.x \qquad \text{[Product rule]}$$

$$= x.\cos h\left(x^2\right).\frac{d}{dx}\left(x^2\right) + \sin h\left(x^2\right) \qquad \left[\frac{d}{dx}\sin hx = \cos hx\right]$$

[Using chain rule]

$$= x.\cos h\left(x^2\right).2x + \sin h\left(x^2\right)$$

$\Rightarrow \boxed{y' = 2x^2\cos h\left(x^2\right) + \sin h\left(x^2\right)}$

Answer 38E.

We have $y = (\cos x)^x$

Taking logarithm of both sides

$$\ln y = \ln(\cos x)^x$$

$$\Rightarrow \ln y = x\ln(\cos x) \qquad \left[\ln x^r = r\ln x\right]$$

Differentiating with respect to x by product rule

$$\frac{d}{dy}(\ln y).\frac{dy}{dx} = x.\frac{d}{dx}\ln(\cos x) + \ln(\cos x).\frac{d}{dx}(x) \qquad \text{[Chain rule]}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{\cos x}(-\sin x) + \ln(\cos x).1 \qquad \text{[By chain rule]}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = -x\tan x + \ln(\cos x) \qquad \left[\frac{\sin x}{\cos x} = \tan x\right]$$

$$\Rightarrow \frac{dy}{dx} = \left[-x\tan x + \ln(\cos x)\right]y$$

$$\Rightarrow \boxed{y' = \left[\ln(\cos x) - x\tan x\right](\cos x)^x} \qquad \left[y = (\cos x)^x\right]$$

Answer 39E.

We have $y = \ln\sin x - \frac{1}{2}\sin^2 x$

Let $t = \sin x$ then $\frac{dt}{dx} = \cos x$

Then $y = \ln t - \frac{1}{2}t^2$

Differentiating with respect to x by chain rule

$$\frac{dy}{dx} = \frac{d}{dt}\left(\ln t - \frac{1}{2}t^2\right)\frac{dt}{dx}$$

$$= \left(\frac{d}{dt}\ln t - \frac{1}{2}\frac{d}{dt}t^2\right)\frac{dt}{dx}$$

$$= \left(\frac{1}{t} - \frac{1}{2}.2t\right)\frac{dt}{dx}$$

$$= \left(\frac{1}{t} - t\right)\frac{dt}{dx}$$

$$= \left(\frac{1}{\sin x} - \sin x\right)(\cos x)$$

$$= \frac{\cos x}{\sin x} - \sin x.\cos x$$

Or $\boxed{y' = \cot x - \sin x\cos x}$ $\qquad \left[\cot x = \frac{\cos x}{\sin x}\right]$

Answer 40E.

We have $y = \arctan\left(\arcsin\sqrt{x}\right)$

Or $y = \tan^{-1}\left(\sin^{-1}\sqrt{x}\right)$

Let $\sin^{-1}\sqrt{x} = t$

then $\frac{dt}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}.\frac{d}{dx}\left(\sqrt{x}\right)$ [By chain rule]

Or $\frac{dt}{dx} = \frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}}$

Or $\frac{dt}{dx} = \frac{1}{2\sqrt{x(1-x)}}$

We have $y = \tan^{-1} t$

Differentiating with respect to x by chain rule

$$\frac{dy}{dx} = \left[\frac{d}{dt}\left(\tan^{-1} t\right)\right] \cdot \frac{dt}{dx}$$

$$= \left[\frac{1}{1+t^2}\right]\frac{dt}{dx}$$

$$= \frac{1}{1+\left[\sin^{-1}\left(\sqrt{x}\right)\right]^2} \cdot \frac{1}{\left(2\sqrt{x(1-x)}\right)}$$

Or $$\boxed{y' = \frac{1}{2\sqrt{x(1-x)}\left[1+\left(\sin^{-1}\sqrt{x}\right)^2\right]}}$$

Answer 41E.

We have $y = \ln\left(\frac{1}{x}\right) + \frac{1}{\ln x}$

$$\Rightarrow y = \ln\left(x^{-1}\right) + (\ln x)^{-1}$$

$$\Rightarrow y = -\ln x + (\ln x)^{-1} \qquad \left[\ln x^r = r\ln x\right]$$

Let $\ln x = t$ then $\frac{dt}{dx} = \frac{1}{x}$

So we have $y = -t + t^{-1}$

Differentiating with respect to x by chain rule

$$\frac{dy}{dx} = \frac{d}{dt}\left(-t+t^{-1}\right)\frac{dt}{dx}$$

$$= \left(-1-t^{-2}\right)\frac{dt}{dx}$$

$$= \left(-1-\frac{1}{t^2}\right)\frac{dt}{dx}$$

$$= \left(-1-\frac{1}{(\ln x)^2}\right) \cdot \frac{1}{x}$$

$$\Rightarrow \boxed{y' = -\frac{1}{x}\left(1+\frac{1}{(\ln x)^2}\right)}$$

Answer 42E.

Given that $xe^y = y - 1$.

Differentiating both sides of the equation with respect to x implicitly, we get

$$\frac{d}{dx}\left(xe^y\right) = \frac{d}{dx}(y-1)$$

$$x.\frac{d}{dx}\left(e^y\right) + e^y.\frac{d}{dx}(x) = \frac{d}{dx}(y) - \frac{d}{dx}(1) \qquad \text{[Using Product rule]}$$

$$\Rightarrow x.\frac{d}{dy}\left(e^y\right).\frac{dy}{dx} + e^y.1 = \frac{dy}{dx} - 0 \qquad \text{[Using Chain rule]}$$

$$\Rightarrow xe^y.\frac{dy}{dx} + e^y = \frac{dy}{dx}$$

Solving for $\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} - xe^y\frac{dy}{dx} = e^y$$

$$\Rightarrow \frac{dy}{dx}\left(1-xe^y\right) = e^y$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^y}{\left(1-xe^y\right)}$$

$$\therefore \boxed{y' = \frac{e^y}{\left(1-xe^y\right)}}.$$

Answer 43E.

Given that $y = \ln(\cos h(3x))$. Here outer function is logarithm, middle function is hyperbolic cosine and the inner function is 3x.

Let $t = \cos h(3x)$ then $y = \ln t \Rightarrow \frac{dy}{dt} = \frac{1}{t}$.

$t = \cos h(3x)$

$$\Rightarrow \frac{dt}{dx} = \frac{d}{dx}(\cos h(3x))$$

$$= \sin h(3x).\frac{d}{dx}(3x) \qquad \left[\text{By Chain Rule } \& \frac{d}{dx}(\cos h\, x) = \text{sinhx}\right]$$

$$= 3\sin h(3x) \qquad ...(1)$$

$y = \ln t$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} \qquad [\text{By Chain Rule}]$$

$$= \frac{1}{\cosh(3x)}\frac{d}{dx}(\cosh(3x)) \qquad \left[\because \frac{dy}{dt} = \frac{1}{t} \,\&\, t = \cos h(3x)\right]$$

$$= \frac{1}{\cosh(3x)}.3\sinh(3x) \qquad [By (1)]$$

$$= 3\tanh(3x)$$

$$\therefore \boxed{y' = 3\tan h(3x)}$$

Answer 44E.

We have $y = \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}$

Taking logarithm of both sides

$$\ln y = \ln\left[\frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\right]$$

$$= \ln(x^2+1)^4 - \ln(2x+1)^3(3x-1)^5 \qquad \left[\ln\frac{m}{n} = \ln m - \ln n\right]$$

$$= \ln(x^2+1)^4 - \ln(2x+1)^3 - \ln(3x-1)^5 \qquad [\ln m.n = \ln m + \ln n]$$

$$\Rightarrow \ln y = 4\ln(x^2+1) - 3\ln(2x+1) - 5\ln(3x-1) \qquad [\ln x^r = r\ln x]$$

Differentiating with respect to x by chain rule

$$\frac{1}{y}\frac{dy}{dx} = \frac{4}{(x^2+1)}(2x) - \frac{3}{(2x+1)}(2) - \frac{5}{(3x-1)}(3)$$

$$= \frac{8x}{(x^2+1)} - \frac{6}{(2x+1)} - \frac{15}{(3x-1)}$$

$$= \frac{8x(2x+1)(3x-1) - 6(x^2+1)(3x-1) - 15(x^2+1)(2x+1)}{(x^2+1)(2x+1)(3x-1)}$$

$$= \frac{8x(6x^2+x-1) - 6(3x^3-x^2+3x-1) - 15(2x^3+x^2+2x+1)}{(x^2+1)(2x+1)(3x-1)}$$

$$= \frac{48x^3+8x^2-8x-18x^3+6x^2-18x+6-30x^3-15x^2-30x-15}{(x^2+1)(2x+1)(3x-1)}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{-x^2-56x-9}{(x^2+1)(2x+1)(3x-1)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(x^2+56x+9)}{(x^2+1)(2x+1)(3x-1)}.y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(x^2+56x+9)(x^2+1)^4}{(x^2+1)(2x+1)(3x-1)(2x+1)^3(3x-1)^5}$$

$$\Rightarrow \boxed{y' = \frac{-(x^2+56x+9)(x^2+1)^3}{(2x+1)^4(3x-1)^6}}$$

Answer 45E.

Given $y = \cos h^{-1}(\sin hx)$

Let $t = \sin hx$ $\frac{dt}{dx} = \cos hx$

We have $y = \cos^{-1}(t)$

Differentiating with respect to x by chain rule, we have

$$\frac{dy}{dx} = \frac{1}{\sqrt{t^2-1}}.\frac{dt}{dx} \quad \frac{d}{dx}\cos h^{-1}x = \frac{1}{\sqrt{x^2-1}}$$

$$\Rightarrow y' = \frac{1}{\sqrt{\sin h^2 x - 1}}.\cosh x$$

$$\Rightarrow \boxed{y' = \frac{\cosh x}{\sqrt{(\sin h^2 x - 1)}}}$$

Answer 46E.

We have $y = x \tan h^{-1}\sqrt{x}$

Differentiating with respect to x by product rule

$$\frac{dy}{dx} = x.\frac{d}{dx}\tan h^{-1}\sqrt{x}. + \tan h^{-1}\sqrt{x}.\frac{d}{dx}.x$$

$$= x.\frac{1}{1-(\sqrt{x})^2}\frac{d}{dx}(\sqrt{x}) + \tan h^{-1}\sqrt{x} \qquad \left[\frac{d}{dx}\tan h^{-1}x = \frac{1}{1-x^2}\right]$$

$$= \frac{x}{(1-x)}.\frac{1}{2\sqrt{x}} + \tan h^{-1}\sqrt{x}$$

$$= \frac{\sqrt{x}}{2(1-x)} + \tan h^{-1}\sqrt{x}$$

Or $\boxed{y' = \frac{\sqrt{x}}{2(1-x)} + \tan h^{-1}\sqrt{x}}$

Answer 47E.

Evaluate the derivative of the function $y=\cos\left(e^{\sqrt{\tan 3x}}\right)$.

$$y'=\left(\cos\left(e^{\sqrt{\tan 3x}}\right)\right)'$$

$$=-\sin\left(e^{\sqrt{\tan 3x}}\right)\cdot\left(e^{\sqrt{\tan 3x}}\right)' \quad \begin{cases}\text{Use the chain rule and}\\ \text{the formula } (\cos x)'=-\sin x\end{cases}$$

$$=-\sin\left(e^{\sqrt{\tan 3x}}\right)\cdot e^{\sqrt{\tan 3x}}\left(\sqrt{\tan 3x}\right)' \quad \begin{cases}\text{Use the chain rule and}\\ \text{the formula } \left(e^x\right)'=e^x\end{cases}$$

$$=-\sin\left(e^{\sqrt{\tan 3x}}\right)\cdot e^{\sqrt{\tan 3x}}\cdot\frac{1}{2\sqrt{\tan 3x}}\cdot(\tan 3x)' \quad \begin{cases}\text{Use the chain rule and}\\ \text{the formula } \left(\sqrt{x}\right)'=\dfrac{1}{2\sqrt{x}}\end{cases}$$

$$=-\sin\left(e^{\sqrt{\tan 3x}}\right)\cdot e^{\sqrt{\tan 3x}}\cdot\frac{1}{2\sqrt{\tan 3x}}\cdot\sec^2 3x\cdot(3x)' \quad \begin{cases}\text{Use the chain rule and}\\ \text{the formula } (\tan x)'=\sec^2 x\end{cases}$$

$$=-\sin\left(e^{\sqrt{\tan 3x}}\right)\cdot e^{\sqrt{\tan 3x}}\cdot\frac{1}{2\sqrt{\tan 3x}}\cdot\sec^2 3x\cdot 3 \quad \begin{cases}\text{Use the chain rule and}\\ \text{the formula } (kx)'=k\end{cases}$$

$$=\frac{-3e^{\sqrt{\tan 3x}}\sec^2 3x\sin\left(e^{\sqrt{\tan 3x}}\right)}{2\sqrt{\tan 3x}} \quad \text{Simplify}$$

Hence the derivative of the given function $y=\cos\left(e^{\sqrt{\tan 3x}}\right)$ is

$$y'=\boxed{\frac{-3e^{\sqrt{\tan 3x}}\sec^2 3x\sin\left(e^{\sqrt{\tan 3x}}\right)}{2\sqrt{\tan 3x}}}.$$

Answer 48E.

Left hand side is

$$\frac{d}{dx}\left[\frac{1}{2}\tan^{-1}x+\frac{1}{4}\ln\frac{(x+1)^2}{\left(x^2+1\right)}\right]$$

$$=\frac{d}{dx}\left[\frac{1}{2}\tan^{-1}x+\frac{1}{4}\ln(x+1)^2-\frac{1}{4}\ln\left(x^2+1\right)\right]$$

$$=\frac{d}{dx}\left[\frac{1}{2}\tan^{-1}x+\frac{1}{2}\ln(x+1)-\frac{1}{4}\ln\left(x^2+1\right)\right]$$

$$=\frac{1}{2}\cdot\frac{1}{1+x^2}+\frac{1}{2}\cdot\frac{1}{x+1}-\frac{2x}{4\left(x^2+1\right)}$$

$$=\frac{1+x+1+x^2-x-x^2}{2\left(1+x^2\right)(1+x)}$$

$$=\frac{1}{(1+x)\left(1+x^2\right)}$$

$=$ Right hand side $\qquad$ hence proved

Answer 49E.

We have $f(x)=e^{g(x)}$

Differentiating with respect to x by chain rule

$$\boxed{f'(x)=e^{g(x)}.g'(x)}$$

Answer 50E.

We have $f(x)=g\left(e^x\right)$

Differentiating with respect to x by chain rule

$$f'(x)=g'\left(e^x\right).\frac{d}{dx}e^x \qquad \text{[By chain rule]}$$

$$\boxed{f'(x)=e^x.g'\left(e^x\right)}$$

Answer 51E.

We have $f(x)=\ln|g(x)|$

Differentiating with respect to x by chain rule

$$f'(x)=\frac{1}{g(x)}.g'(x)$$

$$\Rightarrow \boxed{f'(x)=\frac{g'(x)}{g(x)}}$$

Answer 52E.

We have $f(x)=g(\ln x)$

Differentiating with respect to x by chain rule

$$f'(x)=g'(\ln x).\frac{d}{dx}(\ln x) \qquad \text{[By chain rule]}$$

$$\Rightarrow \boxed{f'(x)=\frac{1}{x}g'(\ln x)}$$

Answer 53E.

We have $f(x)=2^x$.

Differentiating with respect to x

The first derivative $f'(x)=2^x.\ln 2$

Again differentiating with respect to x

Second derivative

$$\Rightarrow f''(x)=\ln 2(2^x.\ln 2)$$

$$\Rightarrow f''(x)=(\ln 2)^2\, 2^x \qquad \text{(Second derivative)}$$

Third derivative

$$\Rightarrow f'''(x)=(\ln 2)^2 .2^x.\ln 2$$

$$=(\ln 2)^3\, 2^x \qquad \text{(Third derivative)}$$

Similarly proceeding as above, finally for the nth derivative we get

$$f^{(n)}(x)=(\ln 2)^n .2^x \qquad \text{(nth derivative)}$$

$$\therefore \boxed{f^{(n)}(x)=(\ln 2)^n .2^x}$$

Answer 54E.

We have $f(x)=\ln(2x)$

Differentiating with respect to x by chain rule

The first derivative

$$\Rightarrow f'(x)=\frac{1}{2x}.2$$

$$=\frac{1}{x}$$

Again differentiating with respect to x

Second derivative

$$f''(x)=-\frac{1}{x^2}$$

Similarly third derivative

$$f'''(x)=-(-2)\frac{1}{x^3}$$

$$=\frac{2}{x^3}$$

Forth derivative

$$f^{(iv)}(x)=\frac{-2.(-3)}{x^4}$$
$$=\frac{-2.3}{x^4}$$

Fifth derivative

$$f^{(v)}(x)=\frac{-2.3.(-4)}{x^5}$$
$$=\frac{2.3.4}{x^5}$$

Sixth derivative

$$f^{(vi)}(x)=\frac{2.3.4.(-5)}{x^6}$$
$$=\frac{-2.3.4.5}{x^6}$$

After seeing the trend the n^{th} derivative of $f(x)$ is

$$\boxed{f^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{x^n}}$$

Answer 55E.

For $n=1$

First derivative of $f(x)=x.e^x$ is

$$f'(x)=x.e^x+e^x=(x+1)e^x \qquad \text{[By product rule]}$$

Let the statement be true for $n=k$.

So for $n=k$

$$f^{(k)}(x)=(x+k)e^x \qquad \text{--- (1)}$$

We show that it is true for $n=k+1$, for this we differentiate equation (1) with respect to x,

$$f^{(k+1)}(x)=(x+k)e^x+e^x$$
$$=(x+(k+1))e^x$$

Therefore by mathematical induction the statement is true for all n.

So we have

$$f^{(n)}(x)=(x+n)e^x$$

Answer 56E.

We have $y=x+\arctan y$

Or $\quad y=x+\tan^{-1}y \qquad$ --- (1)

Differentiating both sides of the equation (1) with respect to x by chain rule

$$\frac{dy}{dx}=1+\frac{1}{1+y^2}.\frac{dy}{dx} \qquad \left[\frac{d}{dy}\tan^{-1}y=\frac{1}{1+y^2}\right]$$

$$\Rightarrow y'=1+\frac{1}{1+y^2}.y'$$

Now we solve for y'

$$\Rightarrow\left(1-\frac{1}{1+y^2}\right)y'=1$$

$$\Rightarrow\frac{\left(1+y^2-1\right)}{\left(1+y^2\right)}y'=1$$

$$\Rightarrow\frac{y^2}{\left(1+y^2\right)}y'=1$$

$$\Rightarrow\boxed{y'=\frac{\left(1+y^2\right)}{y^2}}$$

Answer 57E.

We have the equation of curve

$$y=(2+x)e^{-x} \qquad \text{--- (1)}$$

The slope of the tangent line at any point of the curve $=\frac{dy}{dx}$

So we find $\frac{dy}{dx}$.

Differentiating (1) with respect to x by product rule

$$\frac{dy}{dx}=(2+x)\frac{d}{dx}\left(e^{-x}\right)+e^{-x}\frac{d}{dx}(2+x)$$

$$=(2+x)\left(-e^{-x}\right)+e^{-x}(1)$$

$$=(1-2-x)e^{-x}$$

$$=-(1+x)e^{-x}$$

Then slope of the tangent line at the point (0, 2) is

$$\left(\frac{dy}{dx}\right)_{(0,2)}=-(1+0)e^{0}$$

$$=-1.1$$

$$=-1$$

We know that equation of the tangent line to the given curve at the point (x_1,y_1) is

$$y-y_1=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x-x_1).$$

Here $(x_1,y_1)=(0,2)$, so the equation of tangent line passing though (0, 2) is

$$y-2=(-1).(x-0)$$

$$Or\ y-2=-x$$

$$Or\ y=2-x.$$

Or $\boxed{y=-x+2}$.

Answer 58E.

We have the equation of the curve $y=x\ln x$ --- (1)

The slope of the tangent line at any point of the curve $=\frac{dy}{dx}$

So we find $\frac{dy}{dx}$

Differentiating (1) with respect to x by product rule

$$\frac{dy}{dx}=x.\frac{d}{dx}\ln x+\ln x\frac{d}{dx}.x$$

$$=x\frac{1}{x}+\ln x.1$$

$$\Rightarrow\frac{dy}{dx}=1+\ln x$$

Then slope of the tangent line at the point (e, e) is

$$\left(\frac{dy}{dx}\right)_{(e,e)} = 1+\ln e = 2 \qquad (\ln e = 1)$$

The equation of tangent line passing though (e, e) is

$$y-e = 2.(x-e)$$
$$\Rightarrow y-e = 2x-2e$$
$$\Rightarrow y = 2x-2e+e$$
$$\Rightarrow \boxed{y = 2x-e}$$

Answer 59E.

We have the equation of the curve

$$y = [\ln(x+4)]^2$$

Let $\ln(x+4) = t \Rightarrow \frac{dt}{dx} = \frac{1}{(x+4)}$ [By chain rule]

Then $y = t^2$

Differentiating with respect to x by chain rule

$$\frac{dy}{dx} = 2t.\frac{dt}{dx}$$
$$\Rightarrow \frac{dy}{dx} = 2.\ln(x+4).\frac{1}{(x+4)}$$
$$\Rightarrow \frac{dy}{dx} = \frac{2\ln(x+4)}{(x+4)}$$

The given equation will have horizontal tangent when $\frac{dy}{dx} = 0$

So $\frac{2\ln(x+4)}{(x+4)} = 0$

$$\Rightarrow 2\ln(x+4) = 0$$
$$\Rightarrow \ln(x+4) = 0$$
$$\Rightarrow x+4 = e^0 \qquad [\ln x = y \Leftrightarrow x = e^y]$$
$$\Rightarrow x+4 = 1$$
$$\Rightarrow x = -3$$

When x = -3

Then $y = [\ln(-3+4)]^2$

$$= (\ln 1)^2$$
$$= 0 \qquad [\ln 1 = 0]$$

So the curve has horizontal tangent at the point $\boxed{(-3,0)}$

Answer 60E.

We have $f(x) = xe^{\sin x}$

Differentiating with respect to x

$$f'(x) = x.\frac{d}{dx}e^{\sin x} + e^{\sin x}\frac{d}{dx}.x \qquad \text{[Product rule]}$$
$$= x.e^{\sin x}\frac{d}{dx}(\sin x) + e^{\sin x} \qquad \text{[By chain rule]}$$
$$= x.e^{\sin x}.\cos x + e^{\sin x}$$
$$\Rightarrow \boxed{f'(x) = (x\cos x+1)e^{\sin x}}$$

Now we graph the functions $f(x)$ and $f'(x)$ on the same set of axis [Figure 1]

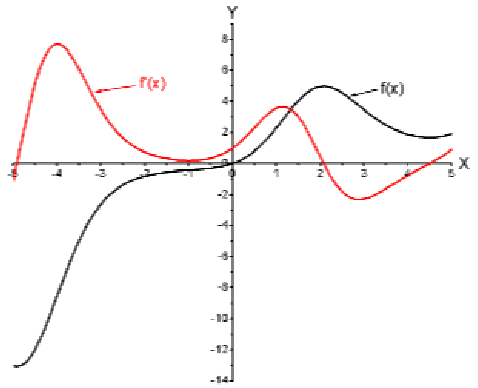


Figure-1

We see that $f'(x)$ is positive, $f(x)$ is increasing and where $f'(x)$ is negative $f(x)$ is decreasing and where $f(x)$ has horizontal tangent, $f'(x)=0$

So our answer in step 1 is reasonable

Answer 61E.

(A) We have the equation of the curve $y=e^x$ --- (1)

Then the slope of the tangent line at any point of the curve is $m_1=\frac{dy}{dx}$

Differentiating (1) with respect to x

$$\Rightarrow m_1=\frac{dy}{dx}=e^x$$

Since this tangent line is parallel to the line

$$x-4y=1$$

Or $y=\frac{1}{4}x-\frac{1}{4}$

Comparing with the equation of line $(y-y_1)=\frac{dy}{dx}(x-x_1)$

Slope of the line $x-4y=1$ $\quad m_2=\frac{dy}{dx}=\frac{1}{4}$

This line will be parallel to the tangent of the curve $y=e^x$

When $m_1=m_2=\frac{1}{4}$

So $e^x=\frac{1}{4}$

$$\Rightarrow x=\left(\ln\frac{1}{4}\right)$$

Then from (1)

$$y=e^{\ln\frac{1}{4}}=\frac{1}{4}$$

So the tangent at $\left(\ln\frac{1}{4},\frac{1}{4}\right)$ is parallel to line $x-4y=1$ with the slope $\frac{1}{4}$

So equation of the tangent line at the point $\left(\ln\frac{1}{4},\frac{1}{4}\right)$ is

$$\left(y-\frac{1}{4}\right)=\frac{1}{4}\left(x-\ln\frac{1}{4}\right)$$

$$=\frac{1}{4}(x+\ln 4) \qquad \left[\ln\frac{1}{4}=-\ln 4\right]$$

$$\Rightarrow \boxed{y=\frac{1}{4}x+\frac{1}{4}(\ln 4+1)}$$

(B) Let a point be (a, e^a) on the curve

Then slope of the tangent line at the point $(a, e^a) = e^a$ [From Part (A)]

Thus the equation of the tangent line is

$$y - e^a = e^a(x - a)$$

Since this tangent line is passing through the point (0, 0) then

$$0 - e^a = e^a(0 - a)$$
$$\Rightarrow e^a = ae^a \Rightarrow a = 1$$

Then the point is $(1, e)$ and the equation of the tangent line of the curve $y = e^x$, passing though the origin is

$$(y - e) = e(x - 1)$$
$$\Rightarrow \boxed{y = ex}$$

Answer 62E.

(A) We have $C(t) = K(e^{-at} - e^{-bt})$

Where a, b and K are positive constants and b > a

Taking limit as $t \to \infty$

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} K(e^{-at} - e^{-bt})$$
$$= K\left(\lim_{t\to\infty} e^{-at} - \lim_{t\to\infty} e^{-bt}\right)$$

Let $u = -at$ so $u \to -\infty$ as $t \to \infty$

And $v = -bt$ so $v \to -\infty$ as $t \to \infty$

Then $\lim_{t\to\infty} C(t) = K\left(\lim_{u\to-\infty} e^u - \lim_{v\to-\infty} e^v\right)$

$$= k(0 - 0) \qquad \left[\lim_{x\to-\infty} e^x = 0\right]$$
$$= 0$$

So $\boxed{\lim_{t\to\infty} c(t) = 0}$

(B) We have $C(t) = K(e^{-at} - e^{-bt})$

Differentiating with respect to t by chain rule

$$C'(t) = K\left(e^{-at}\frac{d}{dt}(-at) - e^{-bt}\frac{d}{dt}(-bt)\right)$$
$$\boxed{C'(t) = K(-ae^{-at} + be^{-bt})}$$

This is the rate at which the drug is cleared from circulation

(C) We have to find the time at which $C'(t) = 0$

So $K(-ae^{-at} + be^{-bt}) = 0$

$$\Rightarrow -ae^{-at} + be^{-bt} = 0$$
$$\Rightarrow ae^{-at} = be^{-bt}$$
$$\Rightarrow \frac{e^{-at}}{e^{-bt}} = \frac{b}{a}$$
$$\Rightarrow e^{(b-a)t} = \frac{b}{a}$$

Taking logarithm of both sides

$$\ln e^{(b-a)t} = \ln\left(\frac{b}{a}\right)$$
$$\Rightarrow (b - a)t = \ln\left(\frac{b}{a}\right) \qquad [\ln e^x = x]$$
$$\Rightarrow \boxed{t = \frac{1}{(b-a)}\ln\left(\frac{b}{a}\right)}$$ This is the time at which $C'(t) = 0$

Answer 63E.

We have to evaluate $\lim_{x\to\infty} e^{-3x}$

Let $y=-3x$ so $y\to-\infty$ as $x\to\infty$

So $\lim_{x\to\infty} e^{-3x} = \lim_{y\to-\infty} e^{y}$

$=0$ $\left[\lim_{x\to-\infty} e^{x}=0\right]$

So $\boxed{\lim_{x\to\infty} e^{-3x}=0}$

Answer 64E.

Find the following limit:

$\lim_{x\to10^-}\ln\left(100-x^2\right)$

Determine the limit as shown below:

$$\begin{aligned}\lim_{x\to10^-}\ln\left(100-x^2\right)&=\ln\left(\lim_{x\to10^-}\left(100-x^2\right)\right)\\&=\ln\left(100-\lim_{x\to10^-}\left(x^2\right)\right)\\&=\ln\left(100-\left(\lim_{x\to10^-}x\right)^2\right)\\&=\ln\left(100-10^2\right)\\&=\ln\left(100-100\right)\\&=-\infty\end{aligned}$$

Hence, $\lim_{x\to10^-}\ln\left(100-x^2\right)=\boxed{-\infty}$.

Answer 65E.

We have to evaluate $\lim_{x\to3^-} e^{2/(x-3)}$

Let $y=\dfrac{2}{x-3}$ so $y\to-\infty$ as $x\to3^-$

So $\lim_{x\to3^-} e^{2/(x-3)} = \lim_{y\to-\infty} e^{y}$

$=0$ $\left[\lim_{x\to-\infty} e^{x}=x\right]$

So $\boxed{\lim_{x\to3^-} e^{2/(x-3)}=0}$

Answer 66E.

We have to evaluate $\lim_{x\to\infty}\arctan\left(x^3-x\right)$ or $\lim_{x\to\infty}\tan^{-1}\left(x^3-x\right)$

Let $x^3-x=y$ so $y\to\infty$ as $x\to\infty$

So $\lim_{x\to\infty}\arctan\left(x^3-x\right)=\lim_{y\to\infty}\tan^{-1}y$

$=\dfrac{\pi}{2}$ $\left[\lim_{x\to\infty}\tan^{-1}x=\dfrac{\pi}{2}\right]$

So $\boxed{\lim_{x\to\infty}\arctan\left(x^3-x\right)=\dfrac{\pi}{2}}$

Answer 67E.

We have to evaluate $\lim_{x\to0^+}\ln\left(\sin hx\right)$

Let $y=\sin hx$ as $x\to0^+$, $y\to0^+$

So $\lim_{x\to0^+}\ln\left(\sin hx\right)=\lim_{y\to0^+}\ln y$

$=-\infty$ $\left[\lim_{x\to0^+}\ln x=-\infty\right]$

So $\boxed{\lim_{x\to0^+}\ln\left(\sin hx\right)=-\infty}$

Answer 68E.

We have to evaluate $\lim_{x\to\infty} e^{-x}\sin x$

$$\lim_{x\to\infty} e^{-x}\sin x = \left(\lim_{x\to\infty} e^{-x}\right).\left(\lim_{x\to\infty}\sin x\right)$$
$$= 0.\left(\lim_{x\to\infty}\sin x\right)$$
$$= 0$$

[Since as $x\to\infty$, $\sin x$ does not have a certain value, it oscillates between -1 and 1 but it is a finite value, and multiplying 0 with any finite value we get 0.]

So $\boxed{\lim_{x\to\infty} e^{-x}\sin x = 0}$

Answer 69E.

We have to evaluate $\lim_{x\to\infty}\frac{1+2^x}{1-2^x}$

Dividing numerator and denominator by 2^x

$$\lim_{x\to\infty}\frac{1+2^x}{1-2^x} = \lim_{x\to\infty}\frac{(1+2^x)/2^x}{(1-2^x)/2^x}$$
$$= \lim_{x\to\infty}\frac{2^{-x}+1}{2^{-x}-1}$$
$$= \frac{\left(\lim_{x\to\infty}2^{-x}+1\right)}{\left(\lim_{x\to\infty}2^{-x}-1\right)}$$
$$= \frac{(0+1)}{(0-1)} \qquad \left[\lim_{x\to\infty}a^{-x}=0\right]$$
$$= \frac{1}{-1}$$

Or $\boxed{\lim_{x\to\infty}\frac{1+2^x}{1-2^x} = -1}$

Answer 70E.

Consider the following limit,

$$\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x$$

The objective is to evaluate the limit.

The limit $\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x$ is of the form $\lim_{x\to\infty}\left[f(x)\right]^{g(x)}$.

First we notice that, as $x\to\infty$, then $\left(1+\frac{4}{x}\right)\to 1$. So the given limit is indeterminate form of the type 1^∞.

Let $y=\left(1+\frac{4}{x}\right)^x$.

Then, $\ln y = \ln\left(1+\frac{4}{x}\right)^x$

$= x\ln\left(1+\frac{4}{x}\right)$

So L'Hospital's Rule gives,

$$\begin{aligned}\lim_{x\to\infty}\ln y &= \lim_{x\to\infty} x\ln\left(1+\frac{4}{x}\right)\\ &= \lim_{x\to\infty}\frac{\ln\left(1+\frac{4}{x}\right)}{\frac{1}{x}}\\ &= \lim_{x\to\infty}\frac{\frac{d}{dx}\ln\left(1+\frac{4}{x}\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}\\ &= \lim_{x\to\infty}\frac{\frac{1}{\left(1+\frac{4}{x}\right)}\frac{d}{dx}\left(1+\frac{4}{x}\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}\end{aligned}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{\left(1+\frac{4}{x}\right)}\left(-\frac{4}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{4}{\left(1+\frac{4}{x}\right)}$$

$$= 4 \qquad \text{as } x\to\infty, \text{then } \frac{4}{x}\to 0$$

So far we have computed the limit of $\ln y$, but we want is the limit of y. To find this we use the fact that $y = e^{\ln y}$.

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x &= \lim_{x\to\infty} y\\ &= \lim_{x\to\infty} e^{\ln y}\\ &= e^{\lim_{x\to\infty}\ln y} && \text{By Limit laws}\\ &= e^4 && \text{Since } \lim_{x\to\infty}\ln y = 4\end{aligned}$$

Thus, the value of the $\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x$ is $\boxed{e^4}$.

Answer 71E.

The functions $e^x - 1 \to 0$ and $\tan x \to 0$ as $x \to 0$.
Using l' Hospitals rule,

$$\begin{aligned}\underset{x\to 0}{Lt}\frac{e^x-1}{\tan x} &= \underset{x\to 0}{Lt}\frac{e^x}{\sec^2 x}\\ &= \frac{e^0}{\sec^2(0)}\\ &= \frac{1}{1}\\ &= 1\end{aligned}$$

Therefore $\boxed{\underset{x\to 0}{Lt}\frac{e^x-1}{\tan x} = 1}$

Answer 72E.

We have to evaluate $\lim_{x\to 0}\frac{1-\cos x}{x^2+x}$

This is the form of $\frac{0}{0}$, so we can use L – hospital rule

$$\lim_{x\to 0}\frac{1-\cos x}{x^2+x}\overset{H}{=}\lim_{x\to 0}\frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(x^2+x)}$$

$$=\lim_{x\to 0}\frac{\sin x}{(2x+1)} \qquad \left[\frac{d}{dx}\cos x=-\sin x\right]$$

$$=\frac{\sin 0}{(0+1)}$$

$$=\frac{0}{1}=0$$

So $\boxed{\lim_{x\to 0}\frac{1-\cos x}{x^2+x}=0}$

Answer 73E.

We have to evaluate $\lim_{x\to 0}\frac{e^{4x}-1-4x}{x^2}$

This the form of $\frac{0}{0}$, so we use L – hospital rule

$$\lim_{x\to 0}\frac{e^{4x}-1-4x}{x^2}\overset{H}{=}\lim_{x\to 0}\frac{\frac{d}{dx}(e^{4x}-1-4x)}{\frac{d}{dx}x^2}$$

$$=\lim_{x\to 0}\frac{4e^{4x}-4}{2x}$$

$$=\lim_{x\to 0}\frac{2e^{4x}-2}{x}$$

It is still the form $\frac{0}{0}$ so again using L – hospital rule

$$\lim_{x\to 0}\frac{e^{4x}-1-4x}{x^2}\overset{H}{=}\lim_{x\to 0}\frac{\frac{d}{dx}(2e^{4x}-2)}{\frac{d}{dx}x}$$

$$=\lim_{x\to 0}8e^{4x}$$

$$=8e^0$$

$$=8\times 1 \qquad (e^0=1)$$

$$=8$$

So $\boxed{\lim_{x\to 0}\frac{e^{4x}-1-4x}{x^2}=8}$

Answer 74E.

We have to evaluate $\lim_{x\to\infty}\frac{e^{4x}-1-4x}{x^2}$

This the form of $\frac{\infty}{\infty}$, so we use L – hospital rule

$$\lim_{x\to\infty}\frac{e^{4x}-1-4x}{x^2}\overset{H}{=}\lim_{x\to\infty}\frac{\frac{d}{dx}(e^{4x}-1-4x)}{\frac{d}{dx}x^2}$$

$$=\lim_{x\to\infty}\frac{4e^{4x}-4}{2x}$$

$$=\lim_{x\to\infty}\frac{2e^{4x}-2}{x}$$

This is also the form $\frac{\infty}{\infty}$ so again by L – hospitals rule

$$\lim_{x\to\infty}\frac{e^{4x}-1-4x}{x^2}\overset{H}{=}\lim_{x\to\infty}\frac{\frac{d}{dx}\left(2e^{4x}-2\right)}{\frac{d}{dx}.x}$$

$$=\lim_{x\to\infty}8e^{4x}$$

$$=8\lim_{x\to\infty}e^{4x}$$

$$=8\lim_{t\to\infty}e^{t} \qquad \left[4x=t\,(let)\right]$$

$$=8.\infty \qquad \left[\lim_{x\to\infty}e^{x}=\infty\right]$$

$$=\infty$$

So $\boxed{\lim_{x\to\infty}\frac{e^{4x}-1-4x}{x^2}=\infty}$

Answer 75E.

Given $\underset{x\to-\infty}{Lt}\left(x^2-x^3\right)e^{2x}$

The given limit is indeterminate because, as $x\to-\infty$ the first factor $\left(x^2-x^3\right)$ approaches ∞ while the second factor e^{2x} approaches 0. Writing $e^{2x}=1/e^{-2x}$ we have $e^{-2x}\to\infty$ as $x\to-\infty$.

So by L' Hospitals rule

$$\underset{x\to-\infty}{Lt}\left(x^2-x^3\right)e^{2x}=\underset{x\to-\infty}{Lt}\frac{x^2-x^3}{e^{-2x}}$$

$$=\underset{x\to-\infty}{Lt}\frac{2x-3x^2}{(-2)e^{-2x}}$$

$$=\underset{x\to-\infty}{Lt}\frac{2-6x}{4e^{-2x}}$$

$$=\underset{x\to-\infty}{Lt}\frac{-6}{-8e^{-2x}}$$

$$=\frac{6}{8}\times\frac{1}{e^{\infty}}$$

$$=0$$

Therefore $\boxed{\underset{x\to-\infty}{Lt}\left(x^2-x^3\right)e^{2x}=0}$

Answer 76E.

We have to evaluate $\lim_{x\to0^+}x^2\ln x$

$$\lim_{x\to0^+}x^2\ln x=\lim_{x\to0^+}\frac{\ln x}{1/x^2}$$

Using L-Hospital's rule

$$\lim_{x\to0^+}\frac{\ln x}{1/x^2}\overset{H}{=}\lim_{x\to0^+}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}\left(1/x^2\right)}$$

$$=\lim_{x\to0^+}\frac{1/x}{\left(-2/x^3\right)}$$

$$=\lim_{x\to0^+}\left(-\frac{1}{2}\right)x^2$$

$$=0$$

So $\boxed{\lim_{x\to0^+}x^2\ln x=0}$

Answer 77E.

We have to evaluate $\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$

$$\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1^+}\left(\frac{x\ln x-(x-1)}{(x-1)\ln x}\right)$$
$$=\lim_{x\to 1^+}\left(\frac{x\ln x-x+1}{(x-1)\ln x}\right)$$

This is the form of $\frac{0}{0}$ so we use L-hospital's rule $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$

$$\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)\overset{H}{=}\lim_{x\to 1^+}\frac{1+\ln x-1}{\frac{(x-1)}{x}+\ln x}$$
$$=\lim_{x\to 1^+}\frac{\ln x}{\left(\frac{x-1}{x}+\ln x\right)}$$

This is also the form of $\frac{0}{0}$ so we use L-Hospital's rule again

$$\lim_{x\to 1^+}\frac{\ln x}{\left(\frac{x-1}{x}+\ln x\right)}\overset{H}{=}\lim_{x\to 1^+}\frac{1/x}{\left(\frac{x-(x-1)}{x^2}+\frac{1}{x}\right)}$$
$$=\lim_{x\to 1^+}\frac{1/x}{1/x^2+1/x}$$
$$=\lim_{x\to 1^+}\frac{x^2}{x(x+1)}$$
$$=\lim_{x\to 1^+}\frac{x}{(x+1)}$$
$$=\frac{1}{1+1}=\frac{1}{2}$$

So $$\boxed{\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\frac{1}{2}}$$

Answer 78E.

We have to evaluate $\lim_{x\to\left(\frac{\pi}{2}\right)^-}(\tan x)^{\cos x}$. First notice that as $x\to\left(\frac{\pi}{2}\right)^-$, we have

$\tan x\to\infty$ and $\cos x\to 0$, so the given limit is indeterminate.

Let $y=(\tan x)^{\cos x}$, taking logarithms both sides we get

$$\ln y=\ln(\tan x)^{\cos x}$$
$$=\cos x\ln(\tan x)\qquad\left[\because \ln x^n=n\ln x\right]$$
$$=\frac{\ln(\tan x)}{\sec x}\qquad\left[\because \cos x=\frac{1}{\sec x}\right]$$

Taking limit as $x \to \left(\frac{\pi}{2}\right)^-$

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} \ln y = \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\ln(\tan x)}{\sec x} \qquad \text{This is the form } \frac{\infty}{\infty}$$

So by l'Hospital's rule

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} \ln y = \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\frac{d}{dx}(\ln(\tan x))}{\frac{d}{dx}(\sec x)}$$

$$= \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{(1/\tan x).\sec^2 x}{\sec x.\tan x}$$

$$= \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\sec x}{\tan^2 x}$$

Again as $x \to \left(\frac{\pi}{2}\right)^-$, $\sec x \to \infty$ and $\tan x \to \infty$, so the given limit is indeterminate.

This is also the form of $\frac{\infty}{\infty}$ so by l'Hospital's rule

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} \ln y = \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\frac{d}{dx}(\sec x)}{\frac{d}{dx}(\tan^2 x)}$$

$$= \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\sec x \tan x}{2\tan x \sec^2 x}$$

$$= \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{1}{2\sec x}$$

$$= \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{(\cos x)}{2}$$

$$= 0$$

So far we have computed the limit of lny, but what we want is the limit of y. To find this we use the fact that $y = e^{\ln y}$:

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} (\tan x)^{\cos x} = \lim_{x\to\left(\frac{\pi}{2}\right)^-} y$$

$$= \lim_{x\to\left(\frac{\pi}{2}\right)^-} e^{\ln y}$$

$$= e^{\lim_{x\to\left(\frac{\pi}{2}\right)^-} \ln y}$$

$$= e^0$$

$$= 1$$

Therefore $\boxed{\lim_{x\to\left(\frac{\pi}{2}\right)^-} (\tan x)^{\cos x} = 1}$

Answer 79E.

Consider the following curve:

$y = e^x \sin x, \ -\pi \le x \le \pi$

(a)

Domain:

The function $e^x \sin x$ exists because the functions e^x and $\sin x$ are defined at every point in Real numbers system.

So, domain is $D = \mathbb{R}$

(b)

Intercepts:

The *x*- intercept is a point in the equation where the *y*-value is zero.

$y = 0$ for the *x*-intercept,

$$\begin{aligned} e^x \sin x &= 0 \\ x &= 0 \\ x &= (2n+1)\pi, \text{ for } n \in Z \end{aligned}$$

Therefore, the *x*-intercept is $(2n+1)\pi,$ for $n \in Z$.

The *y*- intercept is a point in the equation where the *x*-value is zero.

$x = 0$ for the *y*-intercept,

$$\begin{aligned} y &= e^0 \sin(0) \\ &= 0 \\ y &= 0 \end{aligned}$$

Therefore, the *y*- intercept is 0.

(c)

Symmetry:

$$\begin{aligned} \text{When the function } f(-x) &= e^{-x} \sin(-x) \\ &= -e^{-x} \sin x \\ &\neq f(x) \end{aligned}$$

So, the function is neither even nor odd.

If $f(x+p) = f(x)$ for all x in D, where *p* is a positive constant, then *f* is called a periodic function.

Here, $f(x+\pi) = f(x)$ for all *x* and so *f* is periodic and has period π.

Therefore, the interval is $-\pi \le x \le \pi$.

(d)

Asymptote:

Since the limit function does not exist.

Therefore, there is no asymptote.

(e)

Interval of increase or decrease:

$$f'(x) = e^x \cos x + e^x \sin x$$
$$= e^x(\cos x + \sin x)$$

Since the function is increasing when $f'(x) > 0$

$$e^x(\cos x + \sin x) > 0$$

Here, e^x is always positive, $e^x > 0$ and $\cos x + \sin x > 0$ only on the interval $\left(-\frac{1}{4}\pi, \frac{3}{4}\pi\right)$

Therefore, $f(x)$ is increasing on $\left(-\frac{1}{4}\pi, \frac{3}{4}\pi\right)$.

$$f'(x) = e^x \cos x + e^x \sin x$$
$$= e^x(\cos x + \sin x)$$

Since the function is decreasing when $f'(x) < 0$

$$e^x(\cos x + \sin x) < 0$$

Here, the function is $\cos x + \sin x < 0$ only on the interval $\left(-\frac{5}{4}\pi, -\frac{1}{4}\pi\right)$.

Therefore, $f(x)$ is increasing on $\left(-\frac{5}{4}\pi, -\frac{1}{4}\pi\right)$

(f)

Local maximum and minimum values:

$$f'(x) = 0$$

$$e^x(\cos x + \sin x) = 0$$

$$x = -\frac{1}{4}\pi$$

$$f''(x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$$
$$= 2e^x \cos x$$

Substitute for $x = -\frac{1}{4}\pi$ in $f''(x)$.

$$f''\left(-\frac{1}{4}\pi\right) = 2e^{-\frac{1}{4}\pi} \cos\left(-\frac{1}{4}\pi\right)$$
$$= 2e^{-\frac{1}{4}\pi} \cos\left(\frac{\pi}{4}\right)$$
$$> 0$$

Therefore, $x = -\frac{1}{4}\pi$ is a local minimum and $f\left(-\frac{\pi}{4}\right) =$ -0.3223

(g)

Concavity and inflection point:

$$f''(x) = 2e^x \cos x$$
$$= 2e^x \cos x < 0$$
$$= \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$$

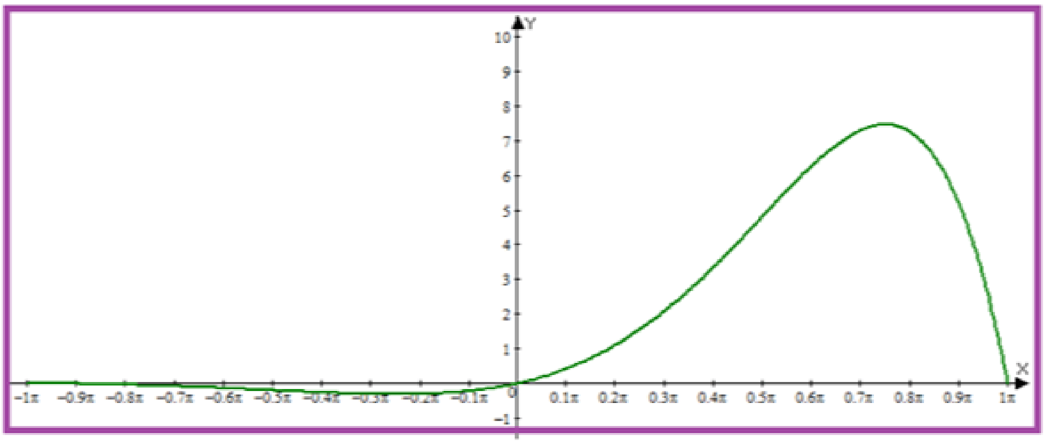
Concave down on $\left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$.

$$f''(x) = 2e^x \cos x$$
$$= 2e^x \cos x > 0$$
$$= \left(\frac{1}{2}\pi, \frac{3}{2}\pi\right)$$

Concave up on $\left(\frac{1}{2}\pi, \frac{3}{2}\pi\right)$.

(h)

Sketch for the given function is



Answer 80E.

(1) We have $y = f(x) = \sin^{-1}\left(\frac{1}{x}\right)$

Since $\sin^{-1}\left(\frac{1}{x}\right)$ is defined when $-1 < \frac{1}{x} < 1 \Rightarrow x \le -1, 1 \le x$

So domain is $= (-\infty, -1] \cup [1, \infty)$

(2) There is no intercept

(3) $f(-x) = \sin^{-1}\left(-\frac{1}{x}\right) = -\sin^{-1}\left(\frac{1}{x}\right) = -f(x)$

So the graph is symmetric about the origin

(4) $\lim_{x \to \pm\infty} \sin^{-1}\left(\frac{1}{x}\right) = \sin^{-1}(0) = 0$

So $y = 0$ is a horizontal asymptote

Since $\lim_{x \to \pm 1} \sin^{-1}\left(\frac{1}{x}\right) = \pm \pi/2$

So there is no vertical asymptote

(5) $f(x) = \sin^{-1}\left(\frac{1}{x}\right)$

Then

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\left(-\frac{1}{x^2}\right)$$

$$= -\frac{1}{x^2\sqrt{\left(1-\left(\frac{1}{x^2}\right)\right)}}$$

$f'(x) < 0$ for $x < -1$ and $x > 1$

So f is decreasing on $(-\infty, -1)$ and $(1, \infty)$

(6) There is no maximum or minimum

(7) From (5) we have

$$f'(x) = \frac{-1}{x^2\sqrt{1-\left(\frac{1}{x}\right)^2}}$$

$$\Rightarrow f'(x) = \frac{-1}{\sqrt{x^4 - x^2}}$$

$$\Rightarrow f'(x) = -\left(x^4 - x^2\right)^{-1/2}$$

Then $f''(x) = -\left(-\frac{1}{2}\right)\left(x^4 - x^2\right)^{-3/2}.\left(4x^3 - 2x\right)$

$$\Rightarrow f''(x) = \frac{\left(4x^3 - 2x\right)}{2\left(x^4 - x^2\right)\sqrt{x^4 - x^2}}$$

$$\Rightarrow f''(x) = \frac{2x^3 - x}{\left(x^4 - x^2\right)\sqrt{x^4 - x^2}}$$

$f''(x) < 0$ for $x < -1$ and $f''(x) > 0$ for $x > 1$

So $f(x)$ is concave downward on $(-\infty, -1)$

And $f(x)$ is concave upward on $(1, \infty)$

There is no inflection point

(8)

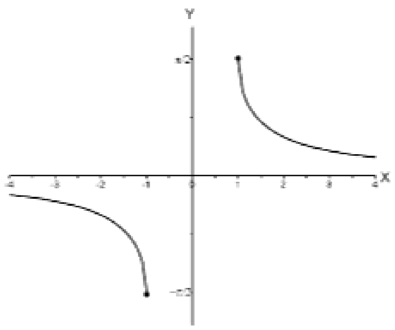


Fig.-1

Answer 81E.

We have $y = f(x) = x \ln x$

(1) Domain is $= (0, \infty)$

(2) For y – intercept putting x = 0, but $\ln x$ is not defined for x = 0
So there is no y-intercept.
For x – intercept, putting

$$y = 0$$
$$\Rightarrow x\ln x = 0$$
$$\Rightarrow \ln x = 0$$
$$\Rightarrow \boxed{x = 1}$$

So x-intercept is 1

(3) This is not symmetric

(4) $\lim_{x\to\infty} x\ln x = \infty$

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x}$$
$$\overset{H}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$
$$= \lim_{x\to 0^+} (-x) = 0$$

So there is no asymptote

(5) $f(x) = x\ln x$

Then $f'(x) = x\frac{1}{x} + \ln x$ [By product rule]

$$= 1 + \ln x$$

$f(x) = 0$ when $1 + \ln x = 0$

$$\Rightarrow \ln x = -1$$
$$\Rightarrow x = e^{-1}$$
$$\Rightarrow x = \frac{1}{e}$$

Since $f'(x) > 0$ for $x > \frac{1}{e}$, so f is decreasing on $\left(0, \frac{1}{e}\right)$ and increasing on $\left(\frac{1}{e}, \infty\right)$

(6) $f(x)$ has local minimum

$$f\left(\frac{1}{e}\right) = \frac{1}{e}\ln\frac{1}{e} = -\frac{1}{e}$$
$$\Rightarrow \boxed{f\left(\frac{1}{e}\right) = -\frac{1}{e}}$$

(7) Since from (5)

$$f'(x) = 1 + \ln x$$
$$\Rightarrow f''(x) = \frac{1}{x}$$

Since $f''(x) > 0$ for x > 0 so $f(x)$ is concave upward on $(0, \infty)$. There is no inflection point

(8)

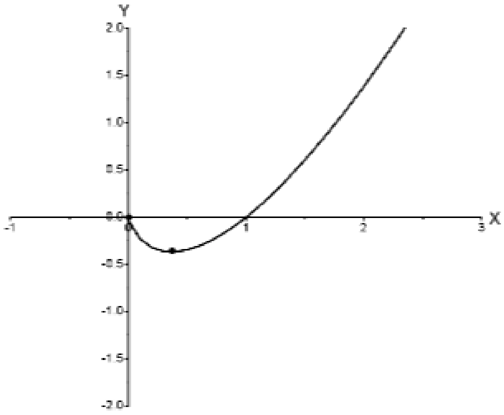


Fig.-1

Answer 82E.

We have $y = f(x) = e^{2x-x^2}$

(1) Domain is = R

(2) For y-intercept $y = e^0 = 1$
So y-intercept is 1
For x-intercept is $0 = e^{2x-x^2}$, but this is not possible
So there is no x-intercept

(3) This is not symmetric

(4) We calculate $\lim_{x\to-\infty} e^{2x-x^2}$
Let $2x - x^2 = y$ so $y \to -\infty$ as $x \to -\infty$
So $\lim_{x\to-\infty} e^{2x-x^2} = \lim_{y\to-\infty} e^y = 0$
$y \to -\infty$ as $x \to \infty$
And $\lim_{x\to\infty} e^{2x-x^2} = \lim_{y\to-\infty} e^y = 0$
So y = 0 is a horizontal asymptote

(5) $f(x) = e^{2x-x^2}$
Then $f'(x) = e^{2x-x^2}.(2-2x)$ By chain rule
$= 2(1-x)e^{2x-x^2}$
Sine $f'(x) > 0$ for x < 1 and $f'(x) < 0$ for x > 1
So $f(x)$ is increasing on $(-\infty,1)$ and $f(x)$ is decreasing on $(1,\infty)$

(6) $f(x)$ has maximum $f(1) = e^{2-1} = e$
$f(1) = e$

(7) From (5) we have
$f'(x) = 2(1-x)e^{2x-x^2}$
Then $f''(x) = 2\left[(1-x)e^{2x-x^2}(2-2x) + (-1)e^{2x-x^2}\right]$ [By product rule]
$\Rightarrow f''(x) = 2\left[2(1-x)^2 e^{2x-x^2} - e^{2x-x^2}\right]$
$\Rightarrow f''(x) = 2e^{2x-x^2}\left[2(1-x)^2 - 1\right]$
$\Rightarrow f''(x) = 2e^{2x-x^2}\left(2x^2 - 4x + 1\right)$

$f''(x)=0 \qquad 2x^2-4x+1=0$

$$\Rightarrow x=\frac{4\pm\sqrt{16-8}}{4}$$
$$=\frac{4\pm\sqrt{8}}{4}$$
$$=\frac{2\pm\sqrt{2}}{2}$$
$$=\left(1\pm\frac{1}{\sqrt{2}}\right)$$

Since $f''(x)>0$ for $x<1-\frac{1}{\sqrt{2}}$ and $x>1+\frac{1}{\sqrt{2}}$

and $f''(x)<0$ for $1-\frac{1}{\sqrt{2}}<x<1+\frac{1}{\sqrt{2}}$

So $f(x)$ is concave upward on $\left(-\infty,1-\frac{1}{\sqrt{2}}\right)$ and $\left(1+\frac{1}{\sqrt{2}},\infty\right)$

And concave downward on $\left(1-\frac{1}{\sqrt{2}},1+\frac{1}{\sqrt{2}}\right)$

For $\quad x=1-\frac{1}{\sqrt{2}}\Rightarrow 2x-x^2=2\left(1-\frac{1}{\sqrt{2}}\right)-\left(1-\frac{1}{\sqrt{2}}\right)^2$

$$=2-\sqrt{2}-1-\frac{1}{2}+\sqrt{2}=\frac{1}{2}$$

So $\quad f\left(1-\frac{1}{\sqrt{2}}\right)=e^{1/2}=\sqrt{e}$

And for $x=1+\frac{1}{\sqrt{2}}\Rightarrow 2x-x^2=2\left(1+\frac{1}{\sqrt{2}}\right)-\left(1+\frac{1}{\sqrt{2}}\right)^2$

$$=2+\sqrt{2}-1-\frac{1}{2}-\sqrt{2}=\frac{1}{2}$$

So $\quad f\left(1+\frac{1}{\sqrt{2}}\right)=e^{1/2}=\sqrt{e}$

So inflection point $\left(1-\frac{1}{\sqrt{2}},\sqrt{e}\right)$ and $\left(1+\frac{1}{\sqrt{2}},\sqrt{e}\right)$

(8)

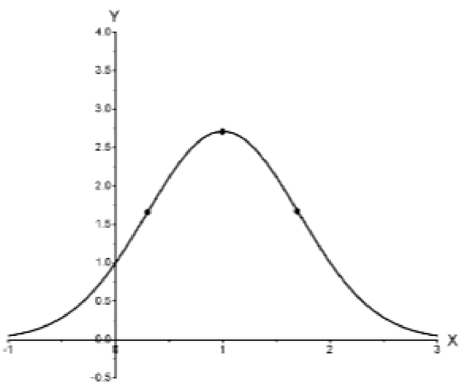


Fig.-1

Answer 83E.

Consider the following curve:

$y=(x-2)e^{-x}$

(a)

Domain:

The function $f(x)$ is a polynomial, so it is defined at every point in Real numbers system

So domain is $D=\mathbb{R}$

(b)

Intercepts:

The *x*- intercept is a point in the equation where the *y*-value is zero.

$y=0$ for the *x*-intercept,

$(x-2)e^{-x}=0$
$x=2$

$x=0$ for the *y*-intercept,

$y=(0-2)e^{0}$
$y=-2$

Therefore, the *x*- and *y*- intercepts are 2 and -2.

(c)

Symmetry

When the function $f(-x)=(-x-2)e^{-(-x)}$
$=-(x+2)e^{x}$
$\neq f(x)$

So, the function is neither even nor odd.

Therefore, the symmetry of the function is none.

(d)

Asymptote:

Since *f* is a polynomial

Therefore, there is no asymptote.

(e)

Interval of increase or decrease:

$f'(x)=e^{-x}+(x-2)(-e^{-x})$
$=e^{-x}(3-x)>0$
$(-\infty,\infty)$

$f(x)$ is increasing on $(-\infty,\infty)$

$f'(x)=e^{-x}+(x-2)(-e^{-x})$
$f'(x)=e^{-x}(3-x)<0$
$(-3,3)$

$f(x)$ is decreasing on $(-3,3)$.

(f)

Local maximum and minimum values:

$f'(x)=0$

$e^{-x}(3-x)=0$
$x=3$

$f''(x)=-e^{-x}+xe^{-x}-3e^{-x}$
$=e^{-x}(x-4)$

$f''(3)=e^{-3}(3-4)$
$=-0.0497$

$f''(3)=-0.0497<0$

$x=3$ is a local maximum and $f(3)=0.0497.$

(g)

Concavity and inflection point:

$$f''(x) = e^{-x}(x-4)$$
$$= e^{-x}(x-4) < 0$$
$$= (-\infty, 0)$$

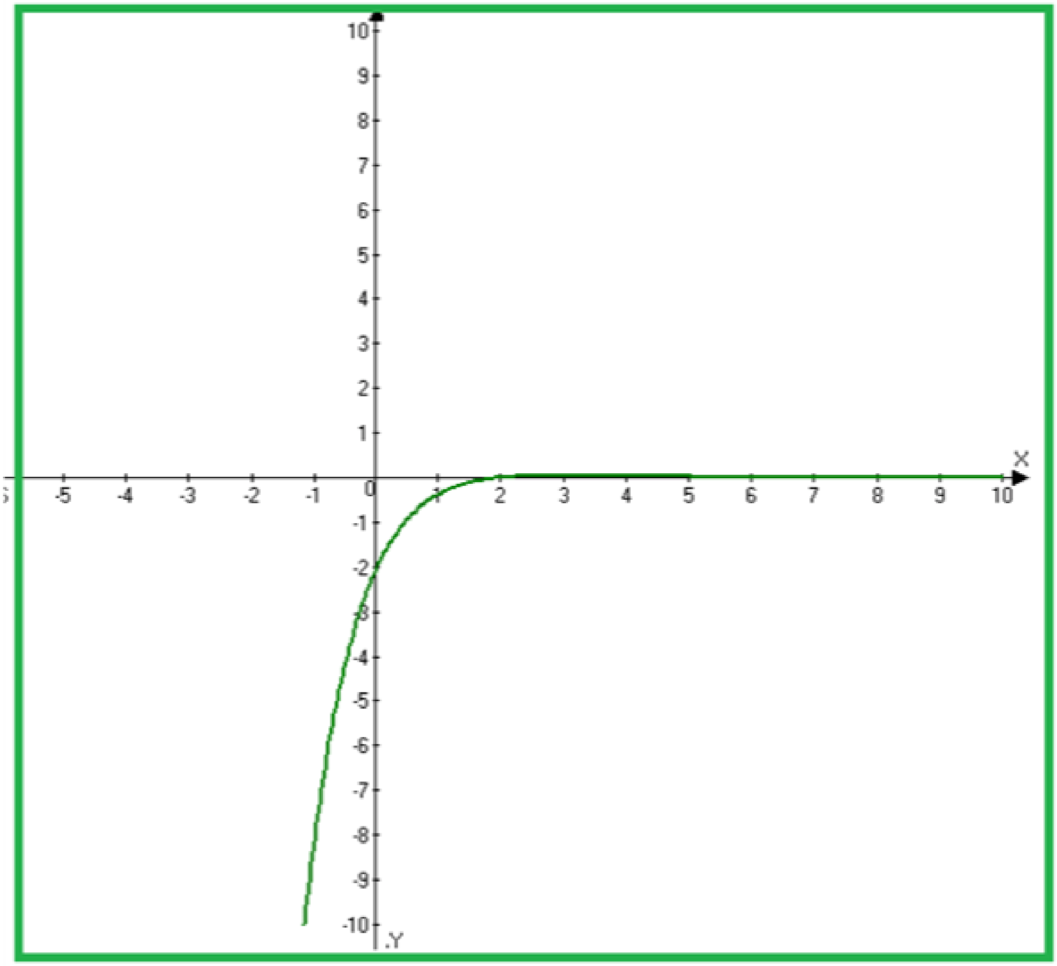
Concave down on $(-\infty, 0)$

$$f''(x) = e^{-x}(x-4)$$
$$= e^{-x}(x-4) > 0$$
$$= (0, \infty)$$

Concave up on $(0, \infty)$

Hence the inflections point is (0, 0)

(h) Sketch for the given function is



Answer 84E.

Consider the function,

$$y = x + \ln(x^2 + 1)$$

Sketch the curve for the above function.

The set of all real values of x satisfied for which the function y is defined.

So, the domain $D = \{\mathbb{R}\}$

Find the Intercepts:

To find the x -intercept, set $y=0$ and solve for x

$$\begin{aligned} x+\ln\left(x^2+1\right)&=0 \\ \ln\left(x^2+1\right)&=-x \\ x^2+1&=e^{-x} \\ x^2&=e^{-x}-1 \\ x&=\sqrt{e^{-x}-1} \end{aligned}$$

This equation is true for $x=0$ only.

So, the x -intercept is $x=0$.

To find the y -intercept, set $x=0$ and solve for y

$$\begin{aligned} y&=x+\ln\left(x^2+1\right) \\ y&=0+\ln\left(0^2+1\right) \\ y&=\ln(1) \\ y&=0 \end{aligned}$$

So, the y -intercept is $y=0$.

Find symmetry:

Let $f(x)=x+\ln\left(x^2+1\right)$, then

Check for symmetry about $y-$ axis.

$$\begin{aligned} f(-x)&=-x+\ln\left((-x)^2+1\right) \\ &=\ln\left(x^2+1\right)-x \\ &\neq f(x) \end{aligned}$$

The function is neither even, odd, nor periodic. So, the symmetry for this function is none.

Find asymptotes:

The line $y=L$ is the horizontal asymptote if one of the following condition is must be satisfied.

$$\lim_{x\to\infty} f(x)=L$$

$$\lim_{x\to-\infty} f(x)=L$$

Find the limits of $f(x)$ at infinity.

$$\begin{aligned} \lim_{x\to\infty} x+\ln\left(x^2+1\right)&=\lim_{x\to\infty} x+\lim_{x\to\infty}\ln\left(x^2+1\right) \\ &=\infty+\ln\lim_{x\to\infty}\left(x^2+1\right) \\ &=\infty+\infty \\ &=\infty \end{aligned}$$

Find the limits of $f(x)$ at negative infinity.

$$\lim_{x\to-\infty}\left[x+\ln\left(x^2+1\right)\right]=\lim_{x\to-\infty}x+\lim_{x\to-\infty}\ln\left(x^2+1\right)$$
$$=-\infty$$

Here the growth rate of linear function x is faster than the logarithm function, so the limit is negative infinity.

Therefore, there are no horizontal asymptotes for the graph and also no vertical asymptotes.

Compute the derivative of $f(x)$

$$f(x)=x+\ln\left(x^2+1\right)$$
$$f'(x)=1+\frac{2x}{x^2+1}$$

Since $f'(x)>0$ for all values of x. So, it is an increasing function.

To find the critical numbers of $f(x)$

Set $f'(x)=0$ and solve for x

$$1+\frac{2x}{x^2+1}=0$$
$$\frac{2x}{x^2+1}=-1$$
$$2x=-\left(x^2+1\right)$$
$$x^2+2x+1=0$$
$$(x+1)^2=0$$
$$x=-1,-1 \qquad \text{Apply the zero product property.}$$

Since $f'(x)>0$ for all values of x . so, there is no local minimum and local maximum.

Find concavity and points of Inflection.

$$f'(x)=1+\frac{2x}{x^2+1}$$
$$f''(x)=0+\frac{\left(x^2+1\right)2-2x(2x)}{\left(x^2+1\right)^2}$$
$$f''(x)=\frac{2x^2-4x+2}{\left(x^2+1\right)^2}$$

Since $f''(x)=0$ at $x=-1$ and the sign of $f''(x)$ is negative at $x=-1.5$ and positive at $x=0$.

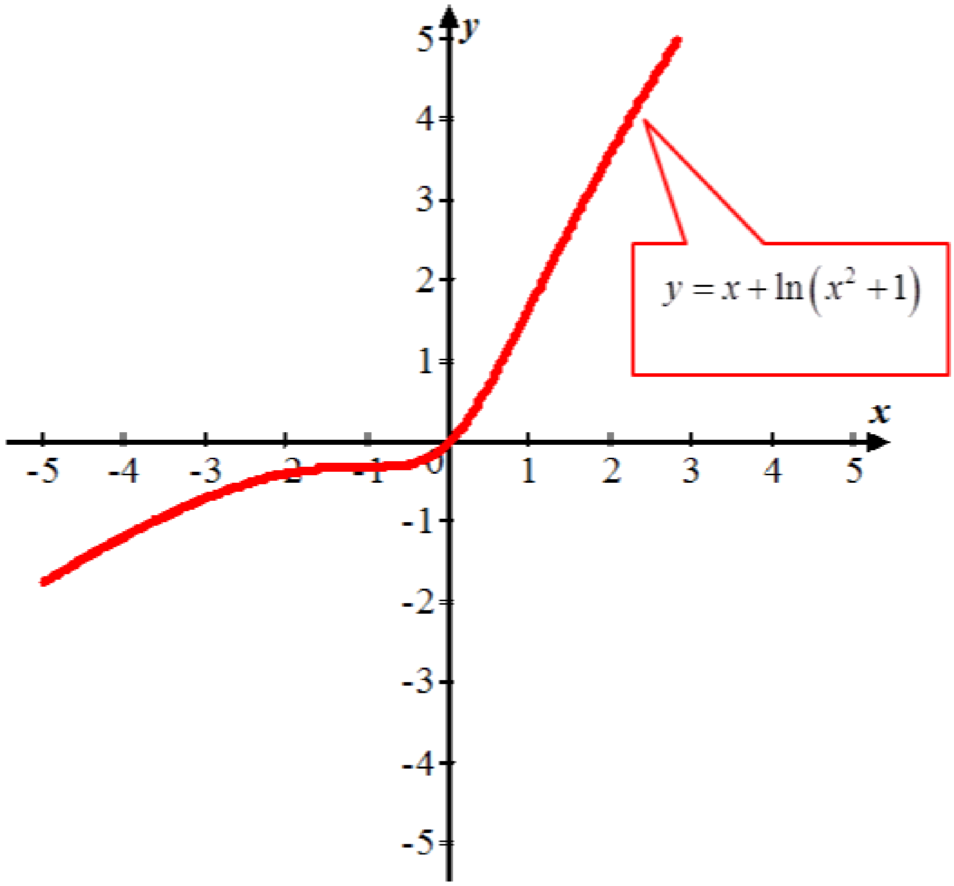
Since $f''(x)=0$ at $x=1$ and the sign of $f''(x)$ is positive at $x=0$ and negitive at $x=1.5$

.

So, the point of inflection are $x=-1$ and $x=1$

Thus, the curve concave upward in the interval $[-1,1]$ and concave downward in the interval $(-\infty,-1]\cup[1,\infty)$

Sketch the curve



Answer 86E.

Consider the family of functions $f(x) = xe^{-cx^2}$, where c is the real number.

The graph of a function has maximum or minimum values at its critical points.

Find the derivative of the function $f(x) = xe^{-cx^2}$.

$$\begin{aligned} f'(x) &= \left(xe^{-cx^2}\right)' \\ &= x\left(xe^{-cx^2}\right)' + x'e^{-cx^2} \\ &= x\left(e^{-cx^2} \cdot (-2xc)\right) + e^{-cx^2} \\ &= -2x^2ce^{-cx^2} + e^{-cx^2} \\ &= \left(-2x^2c+1\right)e^{-cx^2} \end{aligned}$$

Thus, $f'(x) = \left(-2x^2c+1\right)e^{-cx^2}$.

Set $f'(x) = 0$ and solve for x,

$$f'(x) = \left(-2x^2c+1\right)e^{-cx^2} = 0$$

$\left(-2x^2c+1\right) = 0$ Since the exponential function $e^{-cx^2} \neq 0$

$$-cx = -1$$

$$x = \pm\frac{1}{\sqrt{2c}}$$

So, the critical points are $x = \frac{1}{\sqrt{2c}}, -\frac{1}{\sqrt{2c}}$.

As $x = \frac{1}{\sqrt{2c}}, -\frac{1}{\sqrt{2c}}$.

So, c must be positive and $c > 0$ to avoid denominator 0 of the fraction .

So, the graphs of the family of the function $f(x) = xe^{-cx}$ has maximum or minimum value when $c > 0$.

And as c increases the max and minimum values of the graph of the function $f(x) = xe^{-cx}$ increases.

Inflection points:

Find the second derivative of the given function

$$f'(x) = (-2x^2c + 1)e^{-cx^2}$$

$$f''(x) = ((-2x^2c + 1)e^{-cx^2})'$$

$$= -6cxe^{-cx^2} + 4x^3c^2e^{-cx^2}$$

Set $f''(x) = 0$ and solve for x to find inflation points.

Using maple:

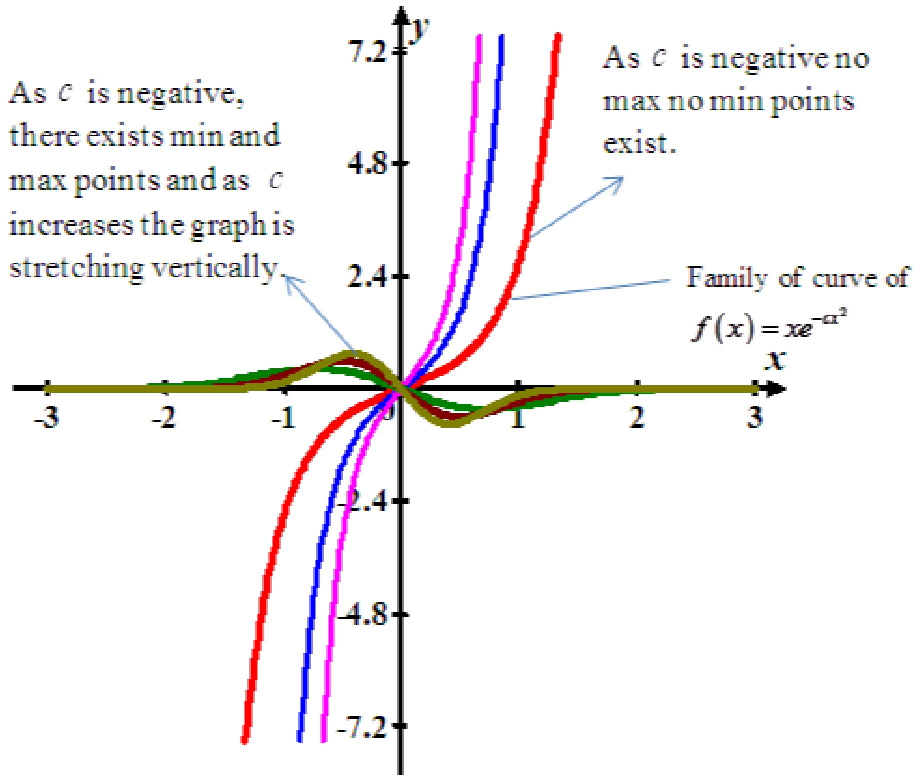
> $solve(-6cxe^{-cx^2} + 4x^3c^2e^{-cx^2}, x);$

$0, \frac{1}{2}\frac{\sqrt{6}}{\sqrt{c}}, -\frac{1}{2}\frac{\sqrt{6}}{\sqrt{c}}$

The, inflection points are 0, $\frac{\sqrt{6}}{2\sqrt{c}}, -\frac{\sqrt{6}}{2\sqrt{c}}$

As c increases, the inflection points are moving upward direction and for the negative values of c which are not existed because it is under square root. And square root of negative numbers not defined.

Sketch the graph of the families of the given function:



Answer 87E.

We have $S = Ae^{-ct}\cos(\omega t+\delta)$ --- (1)

Then velocity

$$V(t) = S'(t) = A\left[-ce^{-ct}.\cos(\omega t+\delta)+e^{-ct}\left(-\sin(\omega t+\delta)\right).\omega\right]$$

[Product rule]

$$= A\left[-ce^{-ct}.\cos(\omega t+\delta)-\omega e^{-ct}\sin(\omega t+\delta)\right]$$

Or $\boxed{V(t) = S'(t) = -Ae^{-ct}\left[c\cos(\omega t+\delta)+\omega\sin(\omega t+\delta)\right]}$ --- (2)

Acceleration $a(t) = V'(t)$

$$\Rightarrow a(t) = -A\left[-ce^{-ct}\left(c\cos(\omega t+\delta)+\omega\sin(\omega t+\delta)\right)+e^{-ct}\left(-\omega c\sin(\omega t+\delta)+\omega^2\cos(\omega t+\delta)\right)\right]$$

$$= -Ae^{-ct}\left[-c^2\cos(\omega t+\delta)-c\omega\sin(\omega t+\delta)-c\omega\sin(\omega t+\delta)+\omega^2\cos(\omega t+\delta)\right]$$

$$= -Ae^{-ct}\left[-\left(c^2-\omega^2\right)\cos(\omega t+\delta)-2c\omega\sin(\omega t+\delta)\right]$$

$$\boxed{a(t) = Ae^{-ct}\left[\left(c^2-\omega^2\right)\cos(\omega t+\delta)+2c\omega\sin(\omega t+\delta)\right]}$$

Answer 88E.

(A) Let $f(x) = \ln x - 3 + x$

$f(2) = \ln 2 - 3 + 2$

$= \ln 2 - 1 < 0$

And $f(e) = \ln e - 3 + e$

$= 1 - 3 + e$

$= -2 + e > 0$ because $e > 2$

So by intermediate value theorem

$f(x) = \ln x - 3 + x = 0$ has at least one root in (2, e)

Now suppose that $f(x) = \ln x - 3 + x$ has two roots a, b so $f(a) = f(b) = 0$

By the mean value theorem $f'(x) = \dfrac{f(b)-f(a)}{b-a}$

$= \dfrac{0}{b-a}$

$= 0$

But $f'(x) = \dfrac{1}{x}+1 \neq 0$ for $x > 0$

This is a contradiction

So $f(x) = \ln x - 3 + x = 0$ has exactly one root in (2, e)

(B) Taking first approximation $x_1 = 2$

Newton's formula for $(n+1)^{th}$ approximation becomes

$$x_{n+1} = x_n - \frac{\ln x_n - 3 + x_n}{1/x_n + 1}$$

For $x_1 = 2$

$$x_2 = 2 - \frac{\ln 2 - 3 + 2}{1/2 + 1} \approx 2.2046$$

Similarly $x_3 \approx 2.2079$,

$x_4 \approx 2.2079$

Since $x_3 \approx x_4$ (up to four decimal places)

So root of the equation $\ln x = 3 - x$ is about $\boxed{2.2079}$

Answer 89E.

Consider a bacteria culture contains 200 cells initially.

After half an hour, the population increased to 360 cells.

The initial population of the culture is 60 cells.

(a)

Now find an expression for the number of cells after *t* hours.

Find the relative growth rate.

Let, the population size is denoted by $P(t)=P(0)e^{kt}$.

Where $P(0)$, is the initial population , *k* is the relative growth rate, *t* is the time in hours.

Now substitute $P(0)=200$, $P(t)=360$, $t=\frac{1}{2}\text{hour}$ in $P(t)=P(0)e^{kt}$.

Then,

$$360=200e^{k\left(\frac{1}{2}\right)}$$
$$e^{k\left(\frac{1}{2}\right)}=\frac{360}{200}$$
$$e^{\frac{k}{2}}=1.8 \quad \text{Simplify}$$
$$\frac{k}{2}=\ln 1.8 \quad \text{Take natural logarithms on both sides}$$
$$k=2\ln 1.8$$

Substitute $P(0)=200$, $k=2\ln 1.8$ in $P(t)=P(0)e^{kt}$.

Then,

$$\begin{aligned}P(t)&=200e^{t2\ln 1.8}\\&=200e^{\ln 1.8^{2t}}\\&=200\left(1.8^{2t}\right)\\&=200\left(3.24^t\right) \quad \text{Simplify}\end{aligned}$$

Therefore, the expression for the number of cells after *t* hours is, $\boxed{P(t)=200\left(3.24^t\right)}$.

(b)

Now find the number of cells after 4 hours.

From part (a), the expression for the number of cells after *t* hours is, $P(t)=200\left(3.24^t\right)$.

Substitute $t=4$ in $P(t)=200\left(3.24^t\right)$.

Then,

$$\begin{aligned}P(4)&=200\left(3.24^4\right)\\&\approx 22040 \quad \text{Use CAS}\end{aligned}$$

Therefore, the number of cells after 8 hours is $\boxed{22040}$.

(c)

Now find the rate of growth after 4 hours.

The relative growth rate is $k=2\ln 1.8$.

From part(b), the number of cells after 8 hours is 22040.

Therefore, the rate of growth after 4 hours is,

$$\begin{aligned}k\times 22040&=2\ln 1.8\times 22040\\&\approx \boxed{25910 \text{ bacteria / hour}} \quad \text{Simplify}\end{aligned}$$

(d)

Now find the time to reach 10,000 cells.

From part (a), the expression for the number of cells after t hours is, $P(t)=200(3.24^t)$

Substitute $P(t)=10{,}000$ in $P(t)=200(3.24^t)$ and solve for t.

Then,

$$\begin{aligned}10{,}000 &= 200(3.24^t)\\ 3.24^t &= \frac{10000}{200}\\ 3.24^t &= 50\\ t\ln 3.24 &= \ln 50 \quad \text{Apply logarithms on both sides}\\ t &= \frac{\ln 50}{\ln 3.24}\\ &\approx 3.33\end{aligned}$$

Therefore, the time to reach 20,000 cells is about $\boxed{\text{3.33 hours}}$

Answer 90E.

Consider Cobalt-60 has a half-life of 5.24 years.

(a)

Now find mass that remains from a 100-mg sample after 20 years.

Let, the expression mass of Cobalt-60 after t years is denoted by

$m(t)=m(0)e^{kt}$.

Where $m(0)$, is the initial mass , k is the relative decay rate, t is the time in years.

Now substitute $m(0)=100$, $m(t)=50$, $t=5.24$ years in $m(t)=m(0)e^{kt}$.

Then,

$$\begin{aligned}50 &= 100e^{k(5.24)}\\ e^{k(5.24)} &= \frac{50}{100}\\ e^{5.24k} &= \frac{1}{2} \quad \text{Simplify}\\ 5.24k &= \ln\frac{1}{2} \quad \text{Take natural logarithms on both sides}\\ k &= \frac{\ln\frac{1}{2}}{5.24}\\ &\approx -0.00138\end{aligned}$$

Substitute $m(0)=100$, $k=0.00138, t=20$ in $m(t)=m(0)e^{-0.00138t}$.

Then,

$$\begin{aligned}m(t) &= 100e^{-0.00138(20)}\\ &= 100e^{-0.0276}\\ &\approx 97.28\end{aligned}$$

Therefore, mass that remains from a 100-mg sample of Cobalt-60 after 20 years is $\boxed{\text{97.28 mg}}$.

(b)

Now find the time to take the mass of 100-mg sample of Cobalt-60 to decay to 1 mg.

From part (a), the expression for the mass after t years is, $m(t)=m(0)e^{-0.00138t}$.

Substitute $m(t)=1, m(0)=100$ in $m(t)=m(0)e^{-0.00138t}$.

Then,

$$1=100\left(e^{-0.00138t}\right)$$

$$e^{-0.00138t}=\frac{1}{100}$$

$$-0.00138t=\ln\frac{1}{100} \qquad \text{Apply logarithms on both sides}$$

$$t=\frac{\ln\frac{1}{100}}{-0.00138}$$

$$t\approx 3337 \qquad \text{Use CAS}$$

Therefore, the time to take the mass of 100-mg sample of Cobalt-60 to decay to 1 mg is about 3337 years.

Answer 91E.

Population function is $P(t)=\dfrac{64}{1+31e^{-0.7944t}}$

Let a = 64, b = 31 and c = -0.7944

So we have $P(t)=\dfrac{a}{1+be^{ct}}=a(1+be^{ct})^{-1}$

The population is increasing most rapidly when its graph changes its concavity

(At which time $\dfrac{d^2P}{dt^2}=0$)

So we calculate $P''(t)$

$$P'(t)=-a\left(1+be^{ct}\right)^{-2}.\left(bce^{ct}\right)$$

$$P'(t)=-abc\ e^{ct}\left(1+be^{ct}\right)^{-2}$$

Then $P''(t)=-abc\left[ce^{ct}\left(1+be^{ct}\right)^{-2}+e^{ct}(-2)\left(1+be^{ct}\right)^{-3}\left(bce^{ct}\right)\right]$ [Product rule]

$$=-abc^2e^{ct}\left(1+be^{ct}\right)^{-3}\left[\left(1+be^{ct}\right)-2be^{ct}\right]$$

$$=-abc^2e^{ct}\frac{\left(1-be^{ct}\right)}{\left(1+be^{ct}\right)^3}$$

$P''(t)=0$ When $1-be^{ct}=0$

$$\Rightarrow be^{ct}=1$$

$$\Rightarrow e^{ct}=\frac{1}{b}$$

$$\Rightarrow ct=\ln\frac{1}{b}$$

$$\Rightarrow t=\frac{1}{c}\ln\frac{1}{b}$$

$$\Rightarrow t=\frac{1}{-0.7944}\ln\left(\frac{1}{31}\right)$$

$$\boxed{\approx 4.32\text{ days}}$$

Answer 92E.

We have to evaluate $\displaystyle\int_0^4\frac{1}{16+t^2}\,dt$

$$=\int_0^4\frac{1}{(4)^2+t^2}\,dt$$

We use the formula $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

So $\int_0^4 \frac{1}{(4)^2+t^2}\,dt = \left[\frac{1}{4}\tan^{-1}\left(\frac{t}{4}\right)\right]_0^4$

$$= \frac{1}{4}\left[\tan^{-1}(1)-\tan^{-1}(0)\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{4}-0\right] = \frac{\pi}{16}$$

So $\boxed{\int_0^4 \frac{1}{16+t^2}\,dt = \frac{\pi}{16}}$

Answer 93E.

We have to evaluate $\int_0^1 ye^{-2y^2}dy$

Let $-2y^2 = t \Rightarrow -4y\,dy = dt$

$$\Rightarrow ydy = -\frac{1}{4}dt$$

And when y = 0, t = 0 and when y = 1, t = -2

So $\int_0^1 ye^{-2y^2}dy = -\int_0^{-2}\frac{1}{4}e^t\,dt$

Since $\int_a^b f(x)dx = -\int_b^a f(x)dx$

So $\int_0^1 ye^{-2y^2}dy = \frac{1}{4}\int_{-2}^0 e^t\,dt$

$$= \frac{1}{4}\left[e^t\right]_{-2}^0 \qquad \text{Since } \left[\int e^t dt = e^t + c\right]$$

$$= \frac{1}{4}\left[e^0 - e^{-2}\right]$$

$$= \frac{1}{4}\left[1-\frac{1}{e^2}\right]$$

$$\boxed{\int_0^1 ye^{-2y^2}dy = \frac{1}{4}\left(1-\frac{1}{e^2}\right)}$$

Answer 94E.

We have to evaluate $\int_2^5 \frac{dr}{1+2r}$

Let $1+2r = t$ then $2dr = dt$

$$\Rightarrow dr = \frac{dt}{2}$$

When r = 2, t = 5

And when r = 5, t = 11

Then we have

$$\int_2^5 \frac{dr}{1+2r} = \int_5^{11}\frac{dt}{2t}$$

$$= \frac{1}{2}\int_5^{11}\frac{dt}{t}$$

$$= \frac{1}{2}\left[\ln|t|\right]_5^{11} \qquad [\text{F.T.C} - 2]$$

$$= \frac{1}{2}\left[\ln 11 - \ln 5\right]$$

$$= \frac{1}{2}\left[\ln\frac{11}{5}\right] \qquad \left[\ln m - \ln n = \ln\frac{m}{n}\right]$$

$$= \ln\sqrt{\frac{11}{5}} \qquad \left[r\ln x = \ln x^r\right]$$

$$\Rightarrow \boxed{\int_2^5 \frac{dr}{1+2r} = \ln\sqrt{\frac{11}{5}}}$$

Answer 95E.

Evaluate the following integral:

$$\int_0^1 \frac{e^x}{1+e^{2x}} dx$$

Rewrite the given integral as shown below:

$$\int_0^1 \frac{e^x}{1+e^{2x}} dx = \int_0^1 \frac{e^x dx}{1+(e^x)^2}.$$

The value of a definite integral can be evaluated using substitution method.

Substitute, $u = e^x$.

This follows that, $du = e^x dx.$

Find the new limit of integration.

Substitute 0 for *x* in e^x to obtain the lower limit of variable *u*.

$$u = e^0 \Rightarrow u = 1$$

Hence, the lower limit of the variable *u* is 1.

Substitute 1 for *x* in e^x to obtain the upper limit of variable *u*.

$$u = e^1 \Rightarrow u = e$$

Hence, the upper limit of the variable *u* is e.

Substitute, e^x for *u* and du for $e^x dx$ and apply the new limits in $\int_0^1 \frac{e^x dx}{1+(e^x)^2}$.

$$\begin{aligned} \int_0^1 \frac{e^x}{1+e^{2x}} dx &= \int_0^1 \frac{e^x dx}{1+(e^x)^2} \\ &= \int_1^e \frac{du}{1+u^2} \\ &= \int_1^e \left(\frac{1}{1+u^2}\right) du \\ &= \left[\tan^{-1} u\right]_1^e \\ &= \left[\tan^{-1} e - \tan^{-1} 1\right] \\ &= \tan^{-1} e - \frac{\pi}{4} \end{aligned}$$

Hence, $\int_0^1 \frac{e^x}{1+e^{2x}} dx = \boxed{\tan^{-1} e - \frac{\pi}{4}}$.

Answer 96E.

We have to evaluate $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x} dx$

Let $\sin x = t \Rightarrow \cos x dx = dt$

And when $x = 0, t = 0$

And when $x = \frac{\pi}{2}, t = 1$

Then we have

$$\int_0^{\pi/2}\frac{\cos x}{1+\sin^2 x}dx=\int_0^1\frac{dt}{1+t^2}$$

$$=\left[\tan^{-1}t\right]_0^1 \qquad \left[\int\frac{1}{1+x^2}dx=\tan^{-1}x+c\right]$$

$$=\left[\tan^{-1}(1)-\tan^{-1}(0)\right]$$

$$=\left[\frac{\pi}{4}-0\right] \qquad \left[\tan^{-1}1=\tan^{-1}\left(\tan\frac{\pi}{4}\right)=\frac{\pi}{4}\right]$$

$$=\frac{\pi}{4}$$

$$\Rightarrow \boxed{\int_0^{\pi/2}\frac{\cos x}{1+\sin^2 x}dx=\frac{\pi}{4}}$$

Answer 97E.

We have to evaluate $\int\frac{e^{\sqrt{x}}}{\sqrt{x}}dx$.

We make the substitution $u=\sqrt{x}$ because the differential $du=\frac{1}{2\sqrt{x}}dx$ occurs(except for the constant factor $\frac{1}{2}$). Thus $\frac{1}{\sqrt{x}}dx=2du$ and

$$\int\frac{e^{\sqrt{x}}}{\sqrt{x}}dx=\int e^u(2du)$$

$$=2\int e^u du$$

$$=2e^u+C$$

$$=2e^{\sqrt{x}}+C$$

Answer 98E.

We have to evaluate $\int\frac{\cos(\ln x)}{x}dx$

Let $\ln x=t\Rightarrow\frac{1}{x}dx=dt$

Then we have

$$\int\frac{\cos(\ln x)}{x}dx=\int\cos t\,dt$$

$$=\sin t+c \qquad \text{Where c is any constant}$$

$$\Rightarrow \boxed{\int\frac{\cos(\ln x)}{x}dx=\sin(\ln x)+c}$$

Answer 99E.

We have to evaluate $\int\frac{x+1}{x^2+2x}dx$

Let $x^2+2x=t\Rightarrow(2x+2)dx=dt$

$$\Rightarrow(x+1)dx=\frac{dt}{2}$$

Then we have

$$\int\frac{x+1}{x^2+2x}dx=\frac{1}{2}\int\frac{1}{t}dt$$

$$=\frac{1}{2}\ln|t|+C \qquad \text{Where C is any constant}$$

$$\Rightarrow \boxed{\int\frac{x+1}{x^2+2x}dx=\frac{1}{2}\ln|x^2+2x|+C}$$

Answer 100E.

Evaluate the following integral:

$$\int \frac{\csc^2 x}{1+\cot x}dx$$

Value of the indefinite integral can be evaluated by substitution method.

Substitute, $u = \cot x$.

This follows that, $du = \csc^2 x\,dx$.

Substitute $\cot x$ for u, du for $\csc^2 x\,dx$ in $\int \frac{\csc^2 x}{1+\cot x}dx$.

$$\int \frac{\csc^2 x}{1+\cot x}dx = \int \frac{du}{1+u}$$

Substitution $t = 1+u$ to solve the integral $\int \frac{du}{1+u}$.

This follows that, $dt = du$.

Substitute, t for $1+u$, dt for du in $\int \frac{du}{1+u}$.

$$\begin{aligned}\int \frac{du}{1+u} &= \int \frac{dt}{t} \\ &= \int \frac{1}{t}dt \\ &= \ln|t| + C\end{aligned}$$

Substitute $1+u$ for t in $\ln|t|+C$ then $\int \frac{du}{1+u} = \ln|1+u|+C$.

Substitute $\cot x$ for u in $\ln|1+u|+C$ then $\int \frac{\csc^2 x}{1+\cot x}dx = \ln|1+\cot x|+C$.

Therefore, $\int \frac{\csc^2 x}{1+\cot x}dx = \boxed{\ln|1+\cot x|+C}$.

Answer 101E.

We have to evaluate $\int \tan x \ln(\cos x)dx$.

We make the substitution $u = \ln(\cos x)$ because the differential

$$\begin{aligned}\frac{du}{dx} &= \frac{1}{\cos x}\frac{d}{dx}(\cos x) \qquad \text{[By Chain Rule]} \\ &= \frac{1}{\cos x}(-\sin x) \\ &= -\tan x\end{aligned}$$

$\Rightarrow du = -\tan x dx$

occurs (except for the constant factor -1). Thus $\tan x dx = du$ and

$$\begin{aligned}\int \tan x \ln(\cos x)dx &= \int \ln(\cos x)(\tan x dx) \\ &= -\int u\,du \\ &= -\frac{u^2}{2}+C \\ &= -\frac{(\ln\cos x)^2}{2}+C\end{aligned}$$

$$\therefore \boxed{\int \tan x \ln(\cos x)dx = -\frac{[\ln(\cos x)]^2}{2}+C}.$$

Answer 102E.

We have to evaluate $\int \frac{x}{\sqrt{1-x^4}}dx$. We make the substitution $u = x^2$ because the differential $du = 2x\,dx$ occurs(except for the factor 2). Thus $x\,dx = \frac{1}{2}du$ and

$$\int \frac{x}{\sqrt{1-x^4}}dx = \frac{1}{2}\int \frac{2x}{\sqrt{1-\left(x^2\right)^2}}dx \quad \text{[By multiplying and dividing with 2]}$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}du \qquad [\because du = 2x\,dx]$$

$$= \frac{1}{2}\sin^{-1}u + C \qquad \left[\because \int \frac{1}{\sqrt{1-u^2}} = \sin^{-1}u + C\right]$$

$$= \frac{1}{2}\sin^{-1}\left(x^2\right) + C$$

$$\therefore \boxed{\int \frac{x}{\sqrt{1-x^4}}dx = \frac{1}{2}\sin^{-1}\left(x^2\right) + C}$$, where C is any constant.

Answer 103E.

We have to evaluate $\int 2^{\tan\theta}.\sec^2\theta\,d\theta$. We make the substitution $u = \tan\theta$ because the differential $du = \sec^2\theta\,d\theta$ occurs.

Then $\int 2^{\tan\theta}.\sec^2\theta\,d\theta = \int 2^u\,du$

$$= \frac{2^u}{\ln 2} + C \qquad \left[\int a^x dx = \frac{a^x}{\ln a} + C \quad a \neq 1\right]$$

$$= \frac{2^{\tan\theta}}{\ln 2} + C$$

$$\therefore \boxed{\int 2^{\tan\theta}.\sec^2\theta\,d\theta = \frac{2^{\tan\theta}}{\ln 2} + C}$$, *where C is any* constant.

Answer 104E.

We have to evaluate $\int \sinh\,au\,du$

We can evaluate this integral by two methods

Method 1

Let $au = x \Rightarrow a\,du = dx$

$$\Rightarrow du = \frac{dx}{a}$$

Then $\int \sinh\,au\,du = \int \sinh x \frac{dx}{a}$

$$= \frac{1}{a}\int \sinh x\,dx$$

$$= \frac{1}{a}\cosh x + C \qquad \left[\frac{d}{dx}\left(\frac{1}{a}\cosh x + C\right) = \frac{1}{a}\sinh x\right]$$

So $\boxed{\int \sinh\,au\,du = \frac{1}{a}\cosh au + C}$ where C is any constant

Method 2

We have $\sinh au = \frac{e^{au} - e^{-au}}{2}$

Then $\int \sinh au\,du = \frac{1}{2}\int \left(e^{au} - e^{-au}\right)du$

Let $au = x \Rightarrow a\,du = dx \Rightarrow du = \frac{dx}{a}$

So $\int \sinh au\, du = \frac{1}{2a}\int\left(e^{x}-e^{-x}\right)dx$

$$= \frac{1}{2a}\left(e^{x}-\frac{e^{-x}}{(-1)}\right)+C$$

$$= \frac{1}{2a}\left(e^{x}+e^{-x}\right)+C$$

$$= \frac{1}{a}\left(\frac{e^{au}+e^{-au}}{2}\right)+C$$

$$\Rightarrow \boxed{\int \sin h au\, du = \frac{1}{a}\cosh au + C} \qquad \left[\cosh au = \frac{e^{au}+e^{-au}}{2}\right]$$

Answer 105E.

Consider the following integral:

$$\int\left(\frac{1-x}{x}\right)^2 dx$$

Solve the integral as shown below:

$$\int\left(\frac{1-x}{x}\right)^2 dx = \int\frac{(1-x)^2}{x^2}dx$$

$$= \int\left(\frac{1-2x+x^2}{x^2}\right)dx$$

$$= \int\left(\frac{1}{x^2}-\frac{2x}{x^2}+\frac{x^2}{x^2}\right)dx$$

$$= \int\frac{1}{x^2}dx - \int\frac{2x}{x^2}dx + \int\frac{x^2}{x^2}dx$$ Distribute the integral.

$$= \int x^{-2}dx - 2\int\frac{1}{x}dx + \int dx$$

$$= \frac{x^{-2+1}}{-2+1} - 2\ln x + x + C$$ Where C is the integration constant.

$$= \frac{x^{-1}}{-1} - 2\ln x + x + C$$

$$= \frac{-1}{x} - 2\ln x + x + C$$

Hence, $\int\left(\frac{1-x}{x}\right)^2 dx = \boxed{\frac{-1}{x} - 2\ln x + x + C}$.

Answer 106E.

Since $1+e^{2x} \ge e^{2x}$ for all x

$$\Rightarrow \sqrt{1+e^{2x}} \ge \sqrt{e^{2x}}$$

$$\Rightarrow \sqrt{1+e^{2x}} \ge e^{x}$$

We have

If $f(x) \ge g(x)$ and $f(x)\ \&\ g(x)$ are continuous on [a, b]

Then $\int_a^b f(x)dx \ge \int_a^b g(x)dx$

Then $\int_0^1 \sqrt{1+e^{2x}}dx \ge \int_0^1 e^{x}dx$

$$\Rightarrow \int_0^1 \sqrt{1+e^{2x}}dx \ge \left[e^{x}\right]_0^1$$

$$\Rightarrow \int_0^1 \sqrt{1+e^{2x}}dx \ge \left(e^{1}-e^{0}\right)$$

$$\Rightarrow \boxed{\int_0^1 \sqrt{1+e^{2x}}dx \ge e-1}$$

Answer 107E.

Since $\cos x \le 1$ for all x

$$\Rightarrow e^{x}\cos x \le e^{x}$$

We have by the property of integral

If $f(x) \le g(x)$ and $f(x)$ & $g(x)$ are continuous on (a, b)

Then $\int_a^b f(x)dx \le \int_a^b g(x)dx$

So $\int_0^1 e^x \cos x\,dx \le \int_0^1 e^x dx$

$\Rightarrow \int_0^1 e^x \cos x\,dx \le \left[e^x\right]_0^1$

$\Rightarrow \int_0^1 e^x \cos x\,dx \le \left[e - e^0\right]$

$\Rightarrow \boxed{\int_0^1 e^x \cos x\,dx \le e - 1}$

Answer 108E.

Since $\sin^{-1} x \le \frac{\pi}{2}$ Where $-1 \le x \le 1$

$\Rightarrow x\sin^{-1} x \le x\frac{\pi}{2}$

By the property of integral if $f(x)$ and $g(x)$ are continuous on (a, b) and $f(x) \le g(x)$ then $\int_a^b f(x)\,dx \le \int_a^b g(x)dx$

Since $\sin^{-1} x$ is continuous on [0, 1]]

Then $\int_0^1 x\sin^{-1} x \le \frac{\pi}{2}\int_0^1 x\,dx$

$\Rightarrow \int_0^1 x\sin^{-1} x \le \frac{\pi}{2}\left[\frac{x^2}{2}\right]_0^1$

$\Rightarrow \int_0^1 x\sin^{-1} x \le \frac{\pi}{2}\left[\frac{1}{2}\right]$

$\Rightarrow \boxed{\int_0^1 x\sin^{-1} x\,dx \le \frac{\pi}{4}}$

Answer 109E.

We have $f(x) = \int_1^{\sqrt{x}} \frac{e^S}{S}dS$

Then $f'(x) = \frac{d}{dx}\int_1^{\sqrt{x}} \frac{e^S}{S}dS$

Let $\sqrt{x} = t$ then $\frac{1}{2\sqrt{x}} = \frac{dt}{dx}$

Then $f'(x) = \frac{d}{dt}\int_1^t \frac{e^S}{S}dS.\frac{dt}{dx}$ [By chain rule]

By the fundamental theorem of calculus part 1 we have

$\frac{d}{dx}\int_a^x f(t)dt = f(x)$

Then $f'(x) = \frac{e^t}{t}.\frac{dt}{dx}$

$\Rightarrow f'(x) = \frac{e^{\sqrt{x}}}{\sqrt{x}} \times \frac{1}{2\sqrt{x}}$

$\Rightarrow \boxed{f'(x) = \frac{e^{\sqrt{x}}}{2x}}$

Answer 110E.

We have $\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$

Then $f(x) = \int_{\ln x}^{2x} e^{-t^2}dt = \int_{\ln x}^{1} e^{-t^2}dt + \int_1^{2x} e^{-t^2}dt$

Or $f(x) = -\int_1^{\ln x} e^{-t^2}dt + \int_1^{2x} e^{-t^2}dt$

Then $f'(x) = -\frac{d}{dx}\int_1^{\ln x} e^{-t^2}dt + \frac{d}{dx}\int_1^{2x} e^{-t^2}dt$

$$= -e^{-(\ln x)^2}\cdot\frac{d}{dx}(\ln x) + e^{-(2x)^2}\cdot\frac{d}{dx}(2x) \qquad \text{[By F.T.C – 1 and chain rule]}$$

$$= -\frac{e^{-(\ln x)^2}}{x} + 2.e^{-4x^2}$$

So $\boxed{f'(x) = 2e^{-4x^2} - \frac{e^{-(\ln x)^2}}{x}}$

Answer 111E.

We have to find the average value of $f(x) = \frac{1}{x}$ on $[1, 4]$

The average value of the function $f(x)$ on $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

Here $f(x) = \frac{1}{x}$ and $a = 1, b = 4$

So $f_{ave} = \frac{1}{4-1}\int_1^4 \frac{1}{x}dx$

$$\Rightarrow f_{ave} = \frac{1}{3}\left[\ln|x|\right]_1^4$$

$$\Rightarrow f_{ave} = \frac{1}{3}[\ln 4 - \ln 1]$$

$$\Rightarrow f_{ave} = \frac{1}{3}[\ln 4 - 0] \qquad [\ln 1 = 0]$$

$$\Rightarrow \boxed{f_{ave} = \frac{1}{3}\ln 4}$$

Answer 112E.

First we sketch the curves

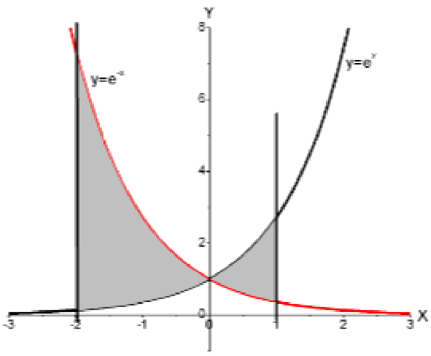


Fig.1

We have to find the area of shaded region

Here we see that $e^{-x} > e^x$ for $x < 0$ and $e^{-x} < e^x$ for $x > 0$

So the area is

$$A = \int_{-2}^{0}\left(e^{-x} - e^x\right)dx + \int_0^1\left(e^x - e^{-x}\right)dx$$

$$= \left[-e^{-x} - e^x\right]_{-2}^{0} + \left[e^x + e^{-x}\right]_0^1$$

$$= \left[-e^0 - e^0 + e^{+2} + e^{-2}\right] + \left[e + e^{-1} - e^0 - e^0\right]$$

$$= \left[-1 - 1 + e^2 + \frac{1}{e^2}\right] + \left[e + \frac{1}{e} - 2\right]$$

$$= e^2 + \frac{1}{e^2} + e + \frac{1}{e} - 4$$

Or $\boxed{A = e(1+e) + \frac{1}{e}\left(1 + \frac{1}{e}\right) - 4}$

Answer 113E.

First we sketch the curve

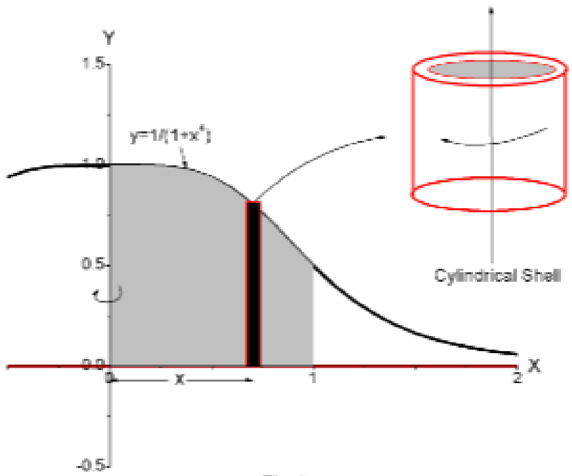


Fig.1

We have rotate this shaded region about y – axis, if we consider a vertical strip in this region then after rotation we get a cylindrical shell with the radius x and height $\frac{1}{(1+x^4)}$

Then volume of the solid is

$$V = \int_0^1 (\text{circumference}) \times (\text{height})\,dx$$

$$= \int_0^1 (2\pi x) \times \left(\frac{1}{1+x^4}\right) dx$$

$$= \pi \int_0^1 \frac{2x}{1+x^4}\,dx$$

Let $x^2 = t \Rightarrow 2x\,dx = dt$

And when x = 0, t = 0, and $x = 1$, $t = 1$

So the volume

$$V = \pi \int_0^1 \frac{dt}{1+t^2}$$

$$V = \pi \left[\tan^{-1} t\right]_0^1$$

$$\Rightarrow V = \pi\left[\tan^{-1} 1 - \tan^{-1} 0\right] \qquad \left[\tan^{-1} 1 = \frac{\pi}{4}\right]$$

$$\Rightarrow \boxed{V = \frac{\pi^2}{4}}$$

Answer 114E.

We have $f(x) = x + x^2 + e^x$ and $g(x) = f^{-1}(x)$

Put x = 0

$$f(0) = 0 + 0 + e^0$$

$$\Rightarrow f(0) = 1$$

$$\Rightarrow 0 = f^{-1}(1) = g(1)$$

And $f'(x) = 1 + 2x + e^x$

Now we have

$$g'(x) = \frac{1}{f'(g(x))}$$

Then $g'(1) = \frac{1}{f'(g(1))}$

$$\Rightarrow g'(1) = \frac{1}{f'(0)}$$

$$\Rightarrow g'(1) = \frac{1}{(1+0+e^0)}$$

$$\Rightarrow g'(1) = \frac{1}{1+1}$$

$$\Rightarrow \boxed{g'(1) = \frac{1}{2}}$$

Answer 115E.

Consider the following:

$f(x) = \ln x + \tan^{-1} x$.

Find $(f^{-1})'\left(\frac{\pi}{4}\right)$.

Let $(f^{-1})\left(\frac{\pi}{4}\right) = x$

Then,

$$f(x) = \frac{\pi}{4}$$

$$\ln x + \tan^{-1} x = \frac{\pi}{4}$$

Now, find the values of *x*, which satisfy $\ln x + \tan^{-1} x = \frac{\pi}{4}$.

$\ln(1) = 0, \tan^{-1}(1) = \frac{\pi}{4}$.

$x = 1$ satisfies $\ln x + \tan^{-1} x = \frac{\pi}{4}$.

So, $f(1) = \frac{\pi}{4}$.

That is, $(f^{-1})\left(\frac{\pi}{4}\right) = 1$.

Now,

$$f(x) = \ln x + \tan^{-1} x$$

$$f'(x) = \frac{1}{x} + \frac{1}{1+x^2}$$

$$f'(1) = \frac{1}{1} + \frac{1}{1+1^2}$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$

Thus,

$$\left(f^{-1}\right)'\left(\frac{\pi}{4}\right)=\frac{1}{f'\left(f^{-1}\left(\frac{\pi}{4}\right)\right)} \qquad \left(f^{-1}\right)'(a)=\frac{1}{f'\left(f^{-1}(a)\right)}$$

$$=\frac{1}{f'(1)} \qquad f^{-1}\left(\frac{\pi}{4}\right)=1$$

$$=\frac{1}{\frac{3}{2}} \qquad f'(1)=\frac{3}{2}$$

$$=\frac{2}{3}$$

Therefore, $\left(f^{-1}\right)'\left(\frac{\pi}{4}\right)=\boxed{\frac{2}{3}}$.

Answer 116E.

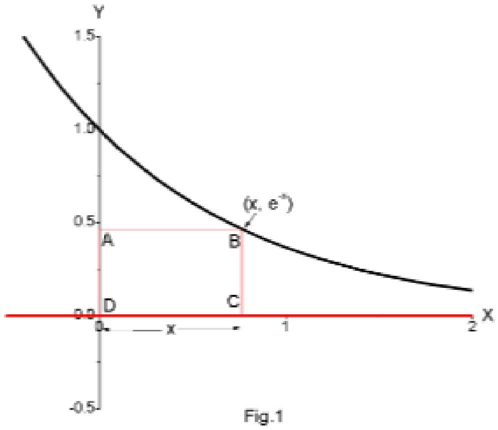


Fig.1

Let ABCD be a rectangle whose one side CD be on the x-axis, vertex D be at the origin and vertex B be on the curve $y=e^{-x}$
Then coordinates of the point B are (x, e^{-x})
So Length of the rectangle = x
and width of the rectangle = e^{-x}
So Area of the rectangle

$$A(x)=x.e^{-x} \qquad \text{--- (1)}$$

Now we have to maximize the area $A(x)$
Differentiating (1) with respect to x

$$A'(x)=x.\frac{d}{dx}e^{-x}+e^{-x}.\frac{d}{dx}x \qquad \text{(By product rule)}$$

$$=-xe^{-x}+e^{-x}$$

$$\Rightarrow A'(x)=e^{-x}(1-x)$$

Since $A'(x)=0$ when $(1-x)=0$ or $x=1$, because $e^{-x}\neq 0$

Since $A'(x)<0$ when $x>1$ and $A'(x)>0$ when $x<1$

So $A(x)$ has an absolute maximum at $x=1$

(Since 1 is only one critical number)
So maximum area at $x=1$

$$A(1)=1.e^{-1}$$

$$\Rightarrow \boxed{A(1)=\frac{1}{e}}$$

Answer 117E.

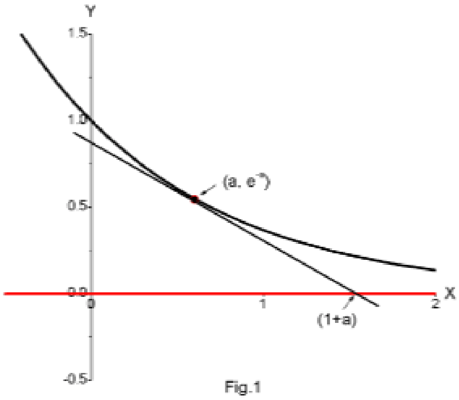


Fig.1

Equation of the curve is $y = e^{-x}$ --- (1)

Slope of the tangent line of the curve at any point is $= \dfrac{dy}{dx}$

So $\dfrac{dy}{dx} = -e^{-x}$

Slope of the tangent line of the curve at the point $\left(a, e^{-a}\right)$ is

$$\left.\frac{dy}{dx}\right|_{x=a} = -e^{-a}$$

Then equation of the tangent line passing though $\left(a, e^{-a}\right)$ is

$$y - e^{-a} = -e^{-a}(x - a)$$

$$\Rightarrow y = e^{-a}(a - x + 1) \qquad \text{--- (2)}$$

Now we find x and y-intercepts of the tangent line

For x intercepts putting y = 0

$$\Rightarrow e^{-a}(a - x + 1) = 0$$

$$\Rightarrow a - x + 1 = 0$$

$$\Rightarrow \boxed{x = 1 + a}$$

For y-intercept putting $x = 0$, $\boxed{y = e^{-a}(a+1)}$

So the height of the triangle is $= e^{-a}(a+1)$

And base of the triangle is $= (1+a)$

Then the area of the triangle

$$A(a) = \frac{1}{2} \times \text{base} \times \text{height}$$

$$\Rightarrow A(a) = \frac{1}{2} \times e^{-a}(1+a) \times (1+a)$$

$$\Rightarrow A(a) = \frac{1}{2} e^{-a}(1+a)^2 \qquad \text{--- (1)}$$

Now we have to maximize the area of the triangle

Differentiating (1) with respect to a

$$A'(a) = \frac{1}{2}\left[e^{-a} \frac{d}{da}(1+a)^2 + \left(1+a^2\right)^2 \frac{d}{da}.e^{-a}\right]$$

$$= \frac{1}{2}\left[2e^{-a}(1+a) - (1+a)^2 e^{-a}\right]$$

$$= \frac{1}{2}\left[e^{-a}(1+a)(2-1-a)\right]$$

$$= \frac{1}{2} e^{-a}(1+a)(1-a)$$

$$\Rightarrow A'(a) = \frac{1}{2} e^{-a}\left(1 - a^2\right)$$

Since $A'(a) = 0$ when $1 - a^2 = 0 \Rightarrow a = \pm 1$

Since $A(a)$ is defined only on $(0, 1+a)$ (from figure) and a = -1 does not exist in the domain, so we leave this root, now we consider only a = 1

Since $A'(a) > 0$ for a < 1 and $A'(a) < 0$ for $a > 1$

So by first derivative test $\Rightarrow A(a)$ has an absolute maximum at $a = 1$

So maximum area of the triangle

$$A(1) = \frac{1}{2}e^{-1}(1+1)^2$$

$$\Rightarrow \boxed{A(1) = \frac{2}{e}}$$

Answer 118E.

We have $f(x) = e^x$

And interval is [0, 1]

Now we divide the interval in to n sub intervals, so sub intervals are

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \left[\frac{2}{n}, \frac{3}{n}\right] \ldots\ldots \left[\frac{n-1}{n}, \frac{n}{n}\right]$$

And width of the sub interval is $\Delta x = \frac{1}{n}$

We have by definition of definite integral

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

Where x_i is the right end point of the interval $[x_{i-1}, x_i]$

So $\quad \int_0^1 e^x dx = \lim_{n\to\infty}\left[e^{1/n} + e^{2/n} + e^{3/n} + \ldots\ldots\ldots + e\right] \cdot \frac{1}{n}$

Here $\quad e^{1/n} + e^{2/n} + e^{3/n} + \ldots\ldots\ldots\ldots + e$ is a geometric series

First term of the series is $= e^{1/n}$

And number of terms = n

And common ratio $= e^{1/n}$

So the sum of geometric series $= \dfrac{e^{1/n} \cdot \left[1 - \left(e^{1/n}\right)^n\right]}{\left(1 - e^{1/n}\right)} = \dfrac{e^{1/n}(1-e)}{\left(1 - e^{1/n}\right)}$

$$\int_0^1 e^x dx = \lim_{n\to\infty} \frac{(1-e).e^{1/n}}{\left(1-e^{1/n}\right)} \cdot \frac{1}{n}$$

$$= (1-e)\lim_{n\to\infty} \frac{\frac{1}{n}.e^{1/n}}{\left(1-e^{1/n}\right)}$$

This is the form of $\frac{0}{0}$ so by L-hospital rule

$$\int_0^1 e^x dx = (1-e)\lim_{n\to\infty} \frac{\frac{1}{n}.e^{1/n}\left(-\frac{1}{n^2}\right) - \frac{1}{n^2}e^{1/n}}{-e^{1/n}\left(-\frac{1}{n^2}\right)}$$

$$= (1-e)\lim_{n\to\infty} \frac{\left(-\frac{1}{n} - 1\right)}{1} = (1-e)(0-1)$$

$$\Rightarrow \boxed{\int_0^1 e^x dx = e - 1}$$

Answer 119E.

If $F(x) = \int_a^b t^x\,dt$ where a, b > 0 then by fundamental theorem

We have

$$F(x) = \frac{b^{x+1} - a^{x+1}}{x+1} \qquad x \neq -1$$

And $\quad F(-1) = \ln b - \ln a$

Now we find $\lim_{x\to -1} F(x)$

$$\lim_{x\to -1} F(x) = \lim_{x\to -1} \frac{b^{x+1} - a^{x+1}}{x+1}$$

Since is the form of $\frac{0}{0}$ so we use L-hospitals rule

$$\lim_{x\to -1} F(x) = \lim_{x\to -1} \frac{\frac{d}{dx}\left(b^{x+1}\right) - \frac{d}{dx}\left(a^{x+1}\right)}{\frac{d}{dx}(x) + \frac{d}{dx}(1)}$$

$$= \lim_{x\to -1} \frac{b^{x+1}.\ln b - a^{x+1}.\ln a}{1+0} \qquad \left[\frac{d}{dx}a^x = a^x \ln a\right]$$

$$= \lim_{x\to -1} b^{x+1}.\ln b - a^{x+1}.\ln a$$

$$= b^0 \ln b - a^0 \ln a$$

$$= \ln b - \ln a$$

$$= F(-1) \qquad \text{[Limit exists]}$$

So F is continuous at -1

Answer 120E.

We have to evaluate $\cos\left\{\arctan\left[\sin\left(arc\cot x\right)\right]\right\} = \cos\left\{\tan^{-1}\left[\sin\left(\cot^{-1} x\right)\right]\right\}$

Let $\cot^{-1} x = y$

$$\Rightarrow x = \cot y$$

$$\Rightarrow \cot^2 y = x^2 \qquad \text{(Squaring both sides)}$$

$$\Rightarrow \csc^2 y - 1 = x^2 \qquad \left[\cot^2 y = \csc^2 y - 1\right]$$

$$\Rightarrow \csc y = \sqrt{1+x^2}$$

$$\Rightarrow \sin y = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \sin\left(\cot^{-1} x\right) = \frac{1}{\sqrt{1+x^2}}$$

So $\cos\left\{\tan^{-1}\left[\sin\left(\cot^{-1} x\right)\right]\right\} = \cos\left\{\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right\}$

Let $\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = z \Rightarrow \tan z = \frac{1}{\sqrt{1+x^2}}$

Squaring both sides

$$\Rightarrow \tan^2 z = \frac{1}{\left(1+x^2\right)}$$

$$\Rightarrow \sec^2 z - 1 = \frac{1}{1+x^2} \qquad \left[\tan^2 z = \sec^2 z - 1\right]$$

$$\Rightarrow \sec^2 z = \frac{1}{1+x^2} + 1$$

$$\Rightarrow \sec^2 z = \frac{2+x^2}{1+x^2}$$

$$\Rightarrow \sec z = \sqrt{\frac{2+x^2}{1+x^2}}$$

$$\Rightarrow \cos z = \sqrt{\frac{1+x^2}{2+x^2}}$$

$$\Rightarrow \boxed{\cos\left\{\arctan\left[\sin\left(\text{arc}\cot x\right)\right]\right\} = \sqrt{\frac{x^2+1}{x^2+2}}}$$

Answer 121E.

We have $\int_0^x f(t)dt = xe^{2x} + \int_0^x f(t)dt$

Differentiating with respect to x

$$\frac{d}{dx}\int_0^x f(t)dt = \frac{d}{dx}\left(xe^{2x}\right) + \frac{d}{dx}\int_0^x e^{-t} f(t)dt$$

$$\Rightarrow f(x) = \frac{d}{dx}\left(xe^{2x}\right) + e^{-x}.f(x) \qquad \text{[By F.T.C. - 1]}$$

$$\Rightarrow f(x) = e^{2x} + 2xe^{2x} + e^{-x} f(x) \qquad \text{(By product rule)}$$

$$\Rightarrow f(x) - e^{-x} f(x) = e^{2x}(1+2x)$$

$$\Rightarrow f(x)\left(1-e^{-x}\right) = e^{2x}(1+2x)$$

$$\Rightarrow \boxed{f(x) = \frac{e^{2x}(1+2x)}{\left(1-e^{-x}\right)}}$$