CHAPTER 10

Differential Equations

IFFERENTIAL EQUATIONS OF FIRST ORDER ND FIRST DEGREE

1. An equation that involves independent and dependent variables and the derivatives of the dependent variable w.r.t. independent variable is called a differential equation.

   e.g. $\frac{dy}{dx} = x^2 \log x$, $dy = \sin x\, dx$, $y = x\frac{dy}{dx} + a$

2. A differential equation is said to be ordinary, if the differential coefficients have reference to only a single independent variable and it is said to be partial if there are two or more independent variables. We are concerned with ordinary differential equations only.

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

The above equation is an ordinary differential equation:

$$\frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial y}{\partial z} = 0;\quad \frac{\partial z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2 + y$$

The above equations are partial differential equations.

rder and Degree of Differential Equation

ıe order of a differential equation is the order of the highest fferential coefficient occurring in it.

The degree of a differential equation which is expressed or ın be expressed as a polynomial in the derivatives is the degree ʾ the highest order derivative occurring in it after it has been ɩpressed in a form free from radicals and fractions as far as ɜrivatives are concerned. Thus the differential equation:

$$f(x,y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x,y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q + \ldots = 0$$

is of order m and degree p.

ːxample 10.1 Find the order and degree of the following differential equations:

(i) $\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^6\right]^{1/4}$

(ii) $\frac{dy}{dx} + y = \frac{1}{\frac{dy}{dx}}$

(iii) $e^{\frac{d^3y}{dx^3}} - x\frac{d^2y}{dx^2} + y = 0$

(iv) $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

(v) $\ln\left(\frac{dy}{dx}\right) = ax + by$

Sol. (i) $\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^6\right]^{1/4}$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^4 = \left[y + \left(\frac{dy}{dx}\right)^6\right]$$

Hence order is 2 and degree is 4.

(ii) $\frac{dy}{dx} + y = \frac{1}{dy/dx} \Rightarrow \left(\frac{dy}{dx}\right)^2 + y\left(\frac{dy}{dx}\right) = 1$

Hence order is 1 and degree is 2.

(iii) $e^{\frac{d^3y}{dx^3}} - x\frac{d^2y}{dx^2} + y = 0$

Clearly order is 3, but degree is not defined as it cannot be written as a polynomial equation in derivatives, and hence it cannot be expressed as polynomial of derivatives.

(iv) $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y \Rightarrow \frac{dy}{dx} = \sin(x+y)$

Hence order is 1 and degree is 1.

(v) $\ln\left(\frac{dy}{dx}\right) = ax + by \Rightarrow \frac{dy}{dx} = e^{ax+by}$

Hence order is one and degree is also 1.

Concept Application Exercise 10.1

Find the order and degree (if defined) of the following differential equations:

1. $\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^4\right\}^{5/3}$
2. $\frac{d^3y}{dx^3} = x\ln\left(\frac{dy}{dx}\right)$
3. $\left(\frac{d^4y}{dx^4}\right)^3 + 3\left(\frac{d^2y}{dx^2}\right)^6 + \sin x = 2\cos x$
4. $\left(\frac{d^3y}{dx^3}\right)^{2/3} + 4 - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$
5. $a = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$, where a is constant.

FORMATION OF DIFFERENTIAL EQUATIONS

Consider a family of curves

$$f(x, y, \alpha_1, \alpha_2, \ldots, \alpha_n) = 0 \qquad (1)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are n independent parameters.

Equation (1) is known as an n parameter family of curves ..., $y = mx$ is a one-parameter family of straight lines and $x^2 + y^2 +$ ... $+ by = 0$ is a two-parameters family of circles.

If we differentiate equation (1) n times w.r.t. x, we will get n ...re relations between $x, y, \alpha_1, \alpha_2, \ldots, \alpha_n$ and derivates of y with ...spect to x. By eliminating $\alpha_1, \alpha_2, \ldots, \alpha_n$ from these n relations ...d equation (1), we get a differential equation.

Clearly order of this differential equation will be n, i.e., equal to ...e number of independent parameters in the family of curves. ...onsider the family of parabolas with vertex at the origin and axis the x-axis.

$y^2 = 4ax$. (1)

...ifferentiating w.r.t. x, we get $2y\dfrac{dy}{dx} = 4a = \dfrac{y^2}{x}$ [from equation (1)]

or, $2x\dfrac{dy}{dx} - y = 0$, which is the differential equation of (1) and is ...early of order 1.

Example 10.2 Form the differential equation of family of lines concurrent at the origin.

Sol. Such lines are given by $y = mx$ (1)

$$\Rightarrow \quad \frac{dy}{dx} = m$$

Putting the value of m in equation (1)

$$\Rightarrow \quad y = \frac{dy}{dx}x$$

$$\Rightarrow \quad xdy - ydx = 0$$

Note that the order is 1, same as the number of constants.

Example 10.3 Form the differential equation of all concentric circle at the origin.

Sol.

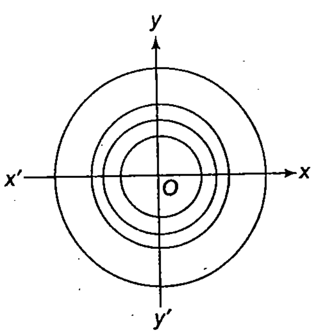


Fig. 10.1

Such circles are given by $x^2 + y^2 = r^2$.

Differentiating w.r.t. x, $2x + 2y\dfrac{dy}{dx} = 0$

$$\Rightarrow \quad x + y\frac{dy}{dx} = 0$$

Example 10.4 Form the differential equation of all circles touching the x axis at the origin and centre on the y-axis.

Sol. Such family of circles is given by

$$x^2 + (y - a)^2 = a^2$$

$$\Rightarrow \quad x^2 + y^2 - 2ay = 0 \qquad (1)$$

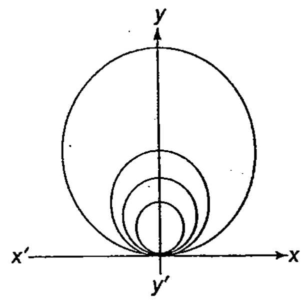


Fig. 10.2

Differentiating, $2x + 2y\dfrac{dy}{dx} = 2a\dfrac{dy}{dx}$

or $\quad x + y\dfrac{dy}{dx} = a\dfrac{dy}{dx}$

substituting the value of a in equation (1)

$\Rightarrow \quad (x^2 - y^2)\dfrac{dy}{dx} = 2xy$ (order is one again and degree 1)

Example 10.5 Form the differential equation of the family of parabolas with focus at the origin and the axis of symmetry along the x-axis.

Sol.

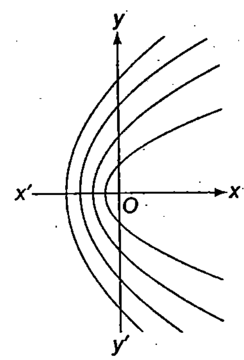


Fig. 10.3

Equation of such parabolas is $y^2 = 4A(A + x)$ (1)

Differentiating w.r.t. x, we get

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4A$$

$$\Rightarrow \quad y\frac{dy}{dx} = 2A \qquad (2)$$

Eliminating A from equations (2) and (1)

$$y^2 = \left(y\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx}x$$

or, $$y^2 = y^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}$$

which has order 1 and degree 2.

Example 10.6 Form the differential equation of family of lines situated at a constant distance p from the origin.

Sol. All such lines are tangent to the circle of radius p.

$$y = mx + p\sqrt{1 + m^2}$$

$$\Rightarrow \quad m = \frac{dy}{dx}$$

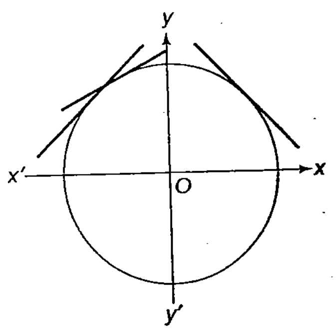


Fig. 10.4

By eliminating m, we get $y = \frac{dy}{dx}x + p\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$$\Rightarrow \quad \left(y - \frac{dy}{dx}x\right)^2 = p^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)$$

which has order 1 and degree 2.

Example 10.7 Find the differential equation of all parabolas whose axis are parallel to the x-axis and have latus rectum a.

Sol. Equation of parabola whose axis is parallel to the x-axis and have latus rectum 'a' is $(y - \beta)^2 = a(x - \alpha)$.
Here we have two effective constants α and β.
So it is required to differentiate twice.
Differentiating both sides, we get

$$2(y - \beta)\frac{dy}{dx} = a \qquad (1)$$

Differentiating equation (1) w.r.t. x, we get

$$\Rightarrow \quad 2(y - \beta)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0 \qquad (2)$$

Eliminating β from equations (1) and (2),

$\Rightarrow a\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 = 0$, which is the required differential equation.

Example 10.8 Form the differential equation having $y = (\sin^{-1}x)^2 + A\cos^{-1}x + B$, where A and B are arbitrary constants, as its general solution.

Sol. $y = (\sin^{-1}x)^2 + A\cos^{-1}x + B$

$$= (\sin^{-1}x)^2 - A\sin^{-1}x + \frac{\pi A}{2} + B$$

Differentiating w.r.t. x, we have

$$\frac{dy}{dx} = \frac{2\sin^{-1}x}{\sqrt{1 - x^2}} - \frac{A}{\sqrt{1 - x^2}}$$

$$\Rightarrow \quad (1 - x^2)\left(\frac{dy}{dx}\right)^2 = 4(\sin^{-1}x)^2 - 4A\sin^{-1}x + A^2$$

$$= 4y - 4B + A^2 - 2\pi A$$

Differentiating again w.r.t. x, we have

$$2(1 - x^2)\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) - 2x\left(\frac{dy}{dx}\right)^2 = 4\frac{dy}{dx}$$

$\Rightarrow (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$, which is the required differential equation.

Concept Application Exercise 10.2

1. Find the differential equation of all the parabolas having axis parallel to the x-axis.
2. Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-2x}$, where A and B are arbitrary constants.
3. Find the differential equation of all non-vertical lines in a plane.
4. Find the differential equation of all the ellipses whose center is at origin and axis are co-ordinate axis.
5. Consider the equation $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$ where a and b are specified constants and λ is an arbitrary parameter. Find a differential equation satisfied by it.
6. Find the degree of the differential equation satisfying the relation
$$\sqrt{1 + x^2} + \sqrt{1 + y^2} = \lambda\left(x\sqrt{1 + y^2} - y\sqrt{1 + x^2}\right).$$

SOLUTION OF A DIFFERENTIAL EQUATION

A solution of a differential equation is an equation which contains arbitrary constants as many as the order of the differential equation and is called the general solution. Other solutions, obtained by giving particular values to the arbitrary constants in the general solution, are called particular solutions.

Also, we know that the general integral of a function contains an arbitrary constant. Therefore, the solution of a differential equation, resulting as it does from the operations of integration, must contain arbitrary constants, equal in number to the number of times the integration is involved in obtaining the solution, and this latter is equal to the order of the differential equation.

Thus we see that the general solution of a differential equation of the nth order must contain n and only n independent arbitrary constants.

METHOD OF VARIABLE SEPARATION

If the coefficient of dx is only a function of x and dy is only a function of y in the given differential equation, then the equation can be solved using variable separation method.

…hus the general form of such an equation is

$$f(x)\,dx+g(y)\,dy=0 \qquad (1)$$

…ntegrating we get, $\int f(x)\,dx+\int g(y)\,dy=c$; where c is the …trary constant.

…his is a general solution of equation (1).

…f given differential equation is of type $\frac{dy}{dx}=f(ax+by+c)$, …0. If $b=0$ (this is directly variable separable), substitute …$ax+by+c$. Then the equation reduces to separable type in the …able t and x which can be easily solved.

…ample 10.9 Solve $\log\frac{dy}{dx}=4x-2y-2$, given that $y=1$ when $x=1$.

…ol. Given $\log\frac{dy}{dx}=4x-2y-2$

$$\Rightarrow \frac{dy}{dx}=e^{4x-2y-2}$$

$$\Rightarrow \int e^{2y+2}dy=\int e^{4x}dx$$

$$\Rightarrow \frac{e^{2y+2}}{2}=\frac{e^{4x}}{4}+c$$

$$x=1, y=1 \Rightarrow \frac{e^4}{2}=\frac{e^4}{4}+c \text{ or } c=e^4/4$$

…ample 10.10 Solve $e^{\frac{dy}{dx}}=x+1$, given that when $x=0, y=3$.

…ol. $e^{\frac{dy}{dx}}=x+1 \Rightarrow \frac{dy}{dx}=\log(x+1)$

$$\Rightarrow \int dy=\int\log(x+1)dx$$

$$\Rightarrow y=(x+1)\log(x+1)-x+c$$

when $x=0, y=3$ gives $c=3$

Hence the solution is $y=(x+1)\log(x+1)-x+3$.

…ample 10.11 Solve the differential equation

$$xy\frac{dy}{dx}=\frac{1+y^2}{1+x^2}\left(1+x+x^2\right).$$

…ol. Differential equation can be rewritten as

$$xy\frac{dy}{dx}=\left(1+y^2\right)\left(1+\frac{x}{1+x^2}\right)$$

$$\Rightarrow \frac{y}{1+y^2}\frac{dy}{dx}=\frac{1}{x}+\frac{1}{1+x^2}$$

Integrating, we get

$$\frac{1}{2}\ln\left(1+y^2\right)=\ln x+\tan^{-1}x+\ln c$$

$$\Rightarrow \sqrt{1+y^2}=cxe^{\tan^{-1}x}$$

Differential Equations Reducible to the Variable Separation Type

Sometimes differential equation of the first order cannot be solved directly by variable separation. By some substitution we can reduce it to a differential equation of variable separable type.

A differential equation of the form $\frac{dy}{dx}=f(ax+by+c)$ is solved by putting $ax+by+c=t$.

Example 10.12 Solve $\frac{dy}{dx}=(x+y)^2$.

Sol. $\frac{dy}{dx}=(x+y)^2 \qquad (1)$

Here the variables are not separable but by putting $x+y=v$, we have

$$1+\frac{dy}{dx}=\frac{dv}{dx}$$

$\Rightarrow$ Equation (1) reduces to

$$\frac{dv}{dx}=v^2+1 \text{ or } \int\frac{dv}{v^2+1}=\int dx \qquad (2)$$

in which variables are separated

Hence from equation (2),

$\tan^{-1}v=x+c$ or $x+y=\tan(x+c)$, which is a required solution.

Example 10.13 Solve $\frac{dy}{dx}\sqrt{1+x+y}=x+y-1$.

Sol. Putting $\sqrt{1+x+y}=v$, we have

$$\Rightarrow x+y-1=v^2-2$$

$$\Rightarrow 1+\frac{dy}{dx}=2v\frac{dv}{dx}$$

Then the given equation transforms to

$$\left(2v\frac{dv}{dx}-1\right)v=v^2-2$$

$$\Rightarrow \frac{dv}{dx}=\frac{v^2+v-2}{2v^2}$$

$$\Rightarrow \int\frac{2v^2}{v^2+v-2}dv=\int dx$$

$$\Rightarrow 2\int\left[1+\frac{1}{3(v-1)}-\frac{4}{3(v+2)}\right]dv=\int dx$$

$$\Rightarrow 2[v+\frac{1}{3}\log|v-1|-\frac{4}{3}\log|v+2|]=x+c$$

where $v=\sqrt{1+x+y}$

Concept Application Exercise 10.3

Solve the following equations:

1. $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$
2. $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$
3. $\dfrac{dy}{dx} = \dfrac{x+y+1}{x+y-1}$
4. $\dfrac{dy}{dx} + y f'(x) = f(x) f'(x)$, where $f(x)$ is a given integrable function of x.
5. $\dfrac{dy}{dx} = \cos(x+y) - \sin(x+y)$

HOMOGENEOUS EQUATIONS

The function $f(x, y)$ is said to be a homogeneous function of degree n if for any real number $t\ (\neq 0)$, we have $f(tx, ty) = t^n (x, y)$. For example, $f(x, y) = ax^{2/3} + hx^{1/3} \times y^{1/3} + by^{2/3}$ is a homogeneous function of degree 2/3.

A differential equation of the form $\dfrac{dy}{dx} = \dfrac{f(x,y)}{\phi(x,y)}$, where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of x and y, and of the same degree, is called *homogeneous*. This equation may also be reduced to the form $\dfrac{dy}{dx} = g\left(\dfrac{x}{y}\right)$ and is solved by putting $y = vx$ so that the dependent variable y is changed to another variable v, where v is some unknown function, the differential equation is transformed to an equation with variables separable.

Consider $\dfrac{dy}{dx} + \dfrac{y(x+y)}{x^2} = 0$.

Example 10.14 Solve $x^2\, dy + y(x+y)\, dx = 0$.

Sol. The given differential equation can be re-written as

$$\frac{dy}{dx} = -\frac{y(x+y)}{x^2} \text{ or } \frac{dy}{dx} = -\frac{y}{x} - \frac{y^2}{x^2}$$

Putting $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Given equation transforms to

$$v + x\frac{dv}{dx} = -v - v^2$$

$$\Rightarrow \int \frac{dv}{v^2+2v} = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\int\left[\frac{1}{v} - \frac{1}{v+2}\right]dv = -\int\frac{dx}{x}$$

$$\Rightarrow \log|v| - \log|v+2| = -2\log|x| + \log c\ (c>0)$$

$$\Rightarrow \left|\frac{vx^2}{v+2}\right| = c$$

$$\Rightarrow \left|\frac{x^2 y}{2x+y}\right| = c\ (c>0)$$

Example 10.15 Solve $\left(x \sin\dfrac{y}{x}\right)dy = \left(y \sin\dfrac{y}{x} - x\right)dx$.

Sol. $\left(\sin\dfrac{y}{x}\right)\dfrac{dy}{dx} = \left(\dfrac{y}{x}\sin\dfrac{y}{x} - 1\right)dx$

Put $y = vx$

$$\Rightarrow \sin v\left(v + x\frac{dv}{dx}\right) = (v\sin v - 1)$$

$$\Rightarrow \sin v\, \frac{xdv}{dx} = -1$$

$$\Rightarrow \int \sin v\, dv = -\int\frac{dx}{x}$$

$$\Rightarrow \cos v = \log_e x + c$$

$$\Rightarrow \cos\frac{y}{x} = \log_e x + c$$

Example 10.16 Solve $xdy = \left(y + x\dfrac{f(y/x)}{f'(y/x)}\right)dx$.

Sol. $xdy = \left(y + x\dfrac{f(y/x)}{f'(y/x)}\right)dx$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{f(y/x)}{f'(y/x)}$$

Putting $y/x = v$, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

The given equation transforms to

$$v + x\frac{dv}{dx} = v + \frac{f(v)}{f'(v)}$$

$$\Rightarrow \int\frac{f'(v)}{f(v)}dv = \int\frac{dx}{x}$$

$$\Rightarrow \log|f(v)| = \log|x| + \log c$$

$$\Rightarrow |f(v)| = c|x|\ (c>0),$$

$$\Rightarrow |f(y/x)| = c|x|, c>0$$

Equations Reducible to the Homogenous Form

Equations of the form $\dfrac{dy}{dx} = \dfrac{ax+by+c}{Ax+By+C}$ ($aB \neq Ab$ and $A + b \neq 0$) can be reduced to a homogenous form by changing the variable x, y, to X, Y by writing $x = X + h$ and $y = Y + k$, where h, k are constants to be chosen so as to make the given equation homogeneous. We have $\dfrac{dy}{dx} = \dfrac{d(Y+k)}{d(X+h)} = \dfrac{dY}{dX}$.

Hence the given equation becomes,

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{AX + BY + (Ah + Bk + C)}$$

Let h and k be chosen to satisfy the relation $ah + bk + c = 0$ and $Ah + Bk + C = 0$.

$\therefore\ h=\dfrac{bC-Bc}{aB-Ab}$ and $k=\dfrac{Ac-aC}{aB-Ab}$

which are meaningful when $aB \neq Ab$.

$\dfrac{dY}{dX}=\dfrac{aX+bY}{AX+BY}$ can now be solved by substiuting $Y=VX$.

n case $aB=Ab$, we write $ax+by=t$. This reduces the erential equation to the separable variable type.

f $A+b=0$, then a simple cross multiplication and substitution) for $xdy+ydx$ and integration term by term yields the result ly.

ample 10.17 Solve $\dfrac{dy}{dx}=\dfrac{x+2y+3}{2x+3y+4}$.

ol. Put $x=X+h, y=Y+k$

We have $\dfrac{dY}{dX}=\dfrac{X+2Y+(h+2k+3)}{2X+3Y+(2h+3k+4)}$

To determine h and k, we write

$h+2k+3=0, 2h+3k+4=0 \quad \Rightarrow h=1, k=-2$

so that $\dfrac{dY}{dX}=\dfrac{X+2Y}{2X+3Y}$

Putting $Y=VX$, we get

$$V+X\frac{dV}{dX}=\frac{1+2V}{2+3V} \Rightarrow \frac{2+3V}{3V^2-1}dV=-\frac{dX}{X}$$

$$\Rightarrow \left[\frac{2+\sqrt{3}}{2(\sqrt{3}V-1)}-\frac{2-\sqrt{3}}{2(\sqrt{3}V+1)}\right]dV=-\frac{dX}{X}$$

$$\Rightarrow \frac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}V-1)-\frac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}V+1)$$

$$=(-\log X+c)$$

$$\frac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}Y-X)-\frac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}Y+X)=A,$$

where A is another constant and $X=x-1, Y=y+2$

ample 10.18 Solve $\dfrac{dy}{dx}=\dfrac{x-2y+5}{2x+y-1}$.

ol. Here $A(2)+b(-2)=0$

Then cross multiplying, we get

$2xdy+ydy-dy=xdx-2ydx+5dx$

$\Rightarrow 2(xdy+ydx)+ydy-dy=xdx+5\,dx$

$\Rightarrow 2\,d(xy)+y\,dy-dy=xdx+5dx$

On integrating, we get $2(xy)+\dfrac{y^2}{2}-y=\dfrac{x^2}{2}+5x+c$

Concept Application Exercise 10.4

1. Solve $x\dfrac{dy}{dx}=y+2\sqrt{y^2-x^2}$.
2. Solve $x(dy/dx)=y(\log y-\log x+1)$.
3. Solve $(x+y\sin(y/x))dx=x\sin(y/x)\,dy$.
4. Show that the differential equation $y^3dy+(x+y^2)\,dx=0$ can be reduced to a homogeneous equation.
5. Solve $\dfrac{dy}{dx}=\dfrac{2x-y+1}{x+2y-3}$.

LINEAR DIFFERENTIAL EQUATIONS

Equation of the form $\dfrac{dy}{dx}+Py=Q$, where P and Q are functions of x alone, is called linear differential equation.

For solving such equation we multiply both sides by integrating factor = I.F. = $e^{\int Pdx}$

Multiplying given equations by I.F., we get

$$e^{\int Pdx}\left(\frac{dy}{dx}+Py\right)=Qe^{\int Pdx}$$

$$\Rightarrow \frac{dy}{dx}e^{\int Pdx}+yPe^{\int Pdx}=Qe^{\int Pdx}$$

$$\Rightarrow \frac{d}{dx}\left(ye^{\int Pdx}\right)=Qe^{\int Pdx} \quad \left[\text{since } \frac{d}{dx}\left(e^{\int Pdx}\right)=Pe^{\int Pdx}\right]$$

$$\Rightarrow \int\frac{d}{dx}\left(ye^{\int Pdx}\right)dx=\int Qe^{\int Pdx}dx$$

$$\Rightarrow ye^{\int Pdx}=\int Qe^{\int Pdx}dx+C$$

which is the required solution of the given differential equation.

In some cases a linear differential equation may be of the form $\dfrac{dx}{dy}+P_1x=Q_1$, where P_1 and Q_1 are functions of y alone or constants. In such a case the integrating factor is $e^{\int P_1 dy}$, and solutions is given by

$$x\,e^{\int Pdy}=\int Qe^{\int Pdy}dy+C$$

Example 10.19 Solve $x^2(dy/dx)+y=1$.

Sol. The given differential equation can be written as

$\dfrac{dy}{dx}+\dfrac{1}{x^2}y=\dfrac{1}{x^2}$, which is linear

Here $P=1/x^2$ and $Q=1/x^2$

I.F. $=e^{\int(1/x^2)dx}=e^{-1/x}$

Therefore the solution is

$$ye^{-1/x} = \int e^{-1/x}\left(1/x^2\right)dx + c$$

$$= e^{-1/x} + c$$

$$\Rightarrow \quad y = 1 + ce^{1/x}$$

Example 10.20 Solve $(x+2y^3)(dy/dx) = y$.

Sol. This equation can be re-written in the form

$$\frac{dx}{dy} = \frac{1}{y}x + 2y^2$$

$$\Rightarrow \quad \frac{dx}{dy} - \frac{1}{y}x = 2y^2$$

This is linear regarding y as independent variable.

Here, $\text{I.F.} = e^{-\int \frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$

$\therefore$ solution is $x\frac{1}{y} = \int \frac{1}{y}2y^2 dy + C$

$$\Rightarrow \quad \frac{x}{y} = y^2 + C$$

$$\Rightarrow \quad x = y^3 + cy$$

Example 10.21 Solve $y\,dx - x\,dy + \log x\,dx = 0$.

Sol. The given equation can be written as

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{x}\log x$$

$$\text{I.F.} = e^{\int -\frac{1}{x}} = e^{-\log x} = \frac{1}{x}$$

Therefore, the solution is

$$\frac{y}{x} = \int \frac{1}{x^2}\log x\,dx + c \qquad (1)$$

Putting $\log x = t$, so that $x = e^t$ and $(1/x)\,dx = dt$, we get

$$\frac{y}{x} = \int te^{-t}dt$$

$$= -te^{-t} - e^{-t} + c$$

$$= -(1/x)(1 + \log x) + c$$

Hence the required solution is $y + 1 + \log x = cx$.

Example 10.22 Solve $\left(1+y^2\right) + \left(x - e^{\tan^{-1} y}\right)\frac{dy}{dx} = 0$.

Sol. Differential equation can be rewritten as

$$\left(1+y^2\right)\frac{dx}{dy} + x = e^{\tan^{-1} y}$$

$$\text{or,} \quad \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{e^{\tan^{-1} y}}{1+y^2} \qquad (1)$$

$$\text{I.F.} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$

Hence, solution is

$$x \cdot e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y} \cdot e^{\tan^{-1} y}}{1+y^2}dy + c$$

$$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = e^{2\tan^{-1} y} + c$$

Concept Application Exercise 10.5

Solve the following equations:

1. $\frac{dy}{dx} + y\cot x = \sin x$
2. $(x+y+1)(dy/dx) = 1$
3. $(1-x^2)(dy/dx) + 2xy = x\sqrt{1-x^2}$
4. $\frac{dy}{dx} = \frac{y}{2y\ln y + y - x}$

Bernoulli's Equation

$$\frac{dy}{dx} + Py = Qy^n \qquad (1)$$

where P and Q are functions of x alone or are constants. Dividing each term of equation (1) by y^n, we get

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q \qquad (2)$$

Let $\frac{1}{y^{n-1}} = v$ so that $\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx}$

Substituting in equation (2), we get

$$\frac{dv}{dx} + (1-n)v.P = Q(1-n) \qquad (3)$$

Equation (3) is a linear differential equation.

Example 10.23 Solve $\left(xy^2 - e^{1/x^3}\right)dx - x^2 y\,dy = 0$.

Sol. Differential equation can be rewritten as

$$y\frac{dy}{dx} - \frac{y^2}{x} = -\frac{e^{1/x^3}}{x^2}$$

Putting $y^2 = t$, we get $y\frac{dy}{dx} = \frac{1}{2}\frac{dt}{dx}$

Therefore, $\frac{dt}{dx} - \frac{2}{x}t = -\frac{2}{x^2}e^{1/x^3}$ (1)

$$\text{I.F.} = e^{-2\int\frac{dx}{x}} = \frac{1}{x^2}$$

Hence, solution is

$$t\cdot\frac{1}{x^2} = -2\int\frac{e^{\frac{1}{x^3}}}{x^4}dx + c$$

$$= \frac{2}{3}e^{\frac{1}{x^3}} + c$$

$$\Rightarrow \quad \frac{y^2}{x^2} = \frac{2}{3}e^{\frac{1}{x^3}} + c$$

$$\Rightarrow \quad 3y^2 = 2x^2 e^{\frac{1}{x^3}} + cx^2$$

ferential Equation Reducible to the Linear m

ation of the form : $f'(y)\frac{dy}{dx} + f(y)P(x) = Q(x)$ (1)

'ut $f(y) = u \Rightarrow f'(y)\frac{dy}{dx} = \frac{du}{dx}$

n equation (1) reduces to $\frac{du}{dx} + uP(x) = Q(x)$

ch is of the linear differential equation form.

ample 10.24 Solve $(dy/dx) + (y/x) = y^3$.

ol. Dividing the given equation by y^3, we get

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}\frac{1}{x} = 1 \quad (1)$$

Putting $1/y^2 = v$, we have

$(-2/y^3)\, dy/dx = dv/dx$

∴ equation (1) becomes

$$-\frac{1}{2}\frac{dv}{dx} + \frac{1}{x}v = 1 \text{ or } \frac{dv}{dx} - \frac{2}{x}v = -2$$

This is a linear equation with v as the dependent variable.

$$\text{I.F.} = e^{-\int(2/x)dx} = e^{-2\log x} = 1/x^2$$

Therefore, the solution is $v(1/x^2) = -2\int(1/x^2)\,dx + c = 2/x + c$

or $2xy^2 + cx^2y^2 = 1$

Example 10.25 Solve $(dy/dx) = e^{x-y}(e^x - e^y)$.

Sol. Multiplying the given equation by e^y, we get

$$e^y\frac{dy}{dx} + e^x e^y = e^{2x} \quad (1)$$

Putting $e^y = v$, so that $e^y\frac{dy}{dx} = \frac{dv}{dx}$,

and equation (1) transform to $\frac{dv}{dx} + e^x v = e^{2x}$

$$\text{I.F.} = e^{\int e^x dx} = e^{e^x}$$

Hence solution is $v e^{e^x} = \int e^{2x}e^{e^x}dx + c$

Let $e^x = t \Rightarrow e^x dx = dt$

Hence solution is $v e^{e^x} = \int te^t dt + c$

$$\Rightarrow e^y e^{e^x} = te^t - e^t + c$$

$$\Rightarrow e^y e^{e^x} = e^x e^{e^x} - e^{e^x} + c$$

Example 10.26 Solve $(x-1)dy + y\,dx = x(x-1)y^{1/3}dx$.

Sol. Dividing by $dx\, y^{1/3}(x-1)$, the given equation reduces to

$$y^{-1/3}\frac{dy}{dx} + \frac{1}{x-1}y^{2/3} = x$$

put $y^{2/3} = z$, so that $\frac{2}{3}y^{-1/3}\frac{dy}{dx} = \frac{dz}{dx}$

Then given equation reduces to

$$\frac{dz}{dx} + \frac{2}{3(x-1)}z = \frac{2}{3}x \quad \text{(linear form)}$$

$$\text{I.F.} = e^{\frac{2}{3}\int\frac{1}{(x-1)}dx} = e^{\frac{2}{3}\log(x-1)} = (x-1)^{2/3}$$

∴ solution is given by

$$z(x-1)^{\frac{2}{3}} = \frac{2}{3}\int x(x-1)^{2/3}dx + c$$

Putting $(x-1) = t^3$ in the R.H.S., we get

$$\int x(x-1)^{2/3}\,dx$$

$$= \int(t^3+1)\,t^2 3t^2 dt$$

$$= 3\int(t^7 + t^4)\,dt$$

$$= 3\left[(1/8)t^8 + (1/5)t^5\right]$$

$$= (3/8)(x-1)^{8/3} + (3/5)(x-1)^{5/3}$$

Hence, the solution is $y^{2/3} = \frac{1}{4}(x-1)^2 + \frac{2}{5}(x-1) + c(x-1)^{-2/3}$.

Concept Application Exercise 10.6

Solve the following equations

1. $\dfrac{dy}{dx}+\dfrac{1}{x}=\dfrac{e^y}{x^2}$
2. $\dfrac{dy}{dx}+x\sin 2y=x^3\cos^2 y$
3. $\dfrac{dy}{dx}+\dfrac{xy}{(1-x^2)}=x\sqrt{y}$

GENERAL FORM OF VARIABLE SEPARATION

If we can write the differential equation in the form $f(f_1(x, y))\, d(f_1(x,y)) + \phi(f_2(x,y))\, d(f_2(x,y)) + \ldots = 0$, then each term can be easily integrated separately. For this the following results must be memorized.

(i) $xdy + ydx = d(xy)$

(ii) $\dfrac{xdy-ydx}{x^2}=d\left(\dfrac{y}{x}\right)$

(iii) $\dfrac{ydx-xdy}{y^2}=d\left(\dfrac{x}{y}\right)$

(iv) $\dfrac{xdy+ydx}{xy}=d(\ln xy)$

(v) $\dfrac{xdy-ydx}{xy}=d\left(\ln\dfrac{y}{x}\right)$

(vi) $\dfrac{xdy-ydx}{x^2+y^2}=d\left(\tan^{-1}\dfrac{y}{x}\right)$

(vii) $\dfrac{xdx+ydy}{\sqrt{x^2+y^2}}=d\left[\sqrt{x^2+y^2}\right]$

Example 10.27 Solve $xdx+ydy=\dfrac{xdy-ydx}{x^2+y^2}$.

Sol. The D.E. can be written as

$$\frac{1}{2}d(x^2+y^2)=d\{\tan^{-1}(y/x)\}$$

Integrating, we get

$$\frac{1}{2}\left(x^2+y^2\right)=\tan^{-1}(y/x)+c$$

Example 10.28 Solve $\{(x+1)(y/x)+\sin y\}\,dx+(x+\log x+x\cos y)\,dy=0.$

Sol. We can re-write the differential equation as

$$(y\,dx+x\,dy)+\left(\frac{y}{x}dx+\log x\,dy\right)+(\sin y\,dx+x\cos y\,dy)=0$$

$$\Rightarrow\ d(xy)+d(y\log x)+d(x\sin y)=0$$

Integrating both the sides we have

$xy+y\log x+x\sin y=c$

Example 10.29 Solve $y^4\,dx+2xy^3\,dy=\dfrac{y\,dx-x\,dy}{x^3\,y^3}$.

Sol. The given differential equation can be written as

$$y^4\,dx+2xy^3\,dy+\frac{1}{xy^3}\,\frac{(x\,dy-y\,dx)}{x^2}=0$$

$$\Rightarrow\ xy^7\,dx+2x^2y^6\,dy+d(y/x)=0$$

$$\Rightarrow\ \frac{x}{y}xy^7\,dx+\frac{x}{y}\times 2x^2\,y^6\,dy+\frac{x}{y}d\left(\frac{y}{x}\right)=0$$

$$\Rightarrow\ \frac{1}{3}(3x^2y^6\,dx+6x^3y^5\,dy)+\frac{d(y/x)}{y/x}=0$$

$$\Rightarrow\ \frac{1}{3}(y^6\,d(x^3)+x^3d(y^6))+\frac{d(y/x)}{y/x}=0$$

$$\Rightarrow\ \frac{1}{3}\int d\left(x^3y^6\right)+\int d(\log(y/x))=c$$

$$\Rightarrow\ x^3y^6+3\log y/x=\text{constant}$$

Example 10.30 Solve $\dfrac{dy}{dx}=\dfrac{yf'(x)-y^2}{f(x)}$.

Sol. $\dfrac{dy}{dx}=\dfrac{yf'(x)-y^2}{f(x)}$

$$\Rightarrow\ yf'(x)dx-f(x)\,dy=y^2\,dx$$

$$\Rightarrow\ \frac{yf'(x)dx-f(x)\,dy}{y^2}=dx$$

$$\Rightarrow\ d\left[\frac{f(x)}{y}\right]=d(x)$$

Integrating, we get

$$\frac{f(x)}{y}=x+c\ \text{or}\ f(x)=y\,(x+c)$$

Concept Application Exercise 10.7

Solve the following equations:

1. $y\,dx-x\,dy+3x^2y^2e^{x^3}dx=0$
2. $\dfrac{dy}{dx}=\dfrac{2xy}{x^2-1-2y}$
3. $y\,dx+(x+x^2y)dy=0$
4. $(xy^4+y)\,dx-x\,dy=0$

GEOMETRICAL APPLICATIONS OF DIFFERENTIAL EQUATION

We also use differential equations for finding the family of curves for which some conditions involving the derivatives are given. For this we proceed in the following way:

Equation of the tangent at a point (x, y) to the curve $y = f(x)$ is given by $Y - y = \frac{dy}{dx}(X - x)$.

At the X axis, $Y = 0$, and $X = x - \frac{y}{dy/dx}$ (intercept on X-axis).

At the Y axis, $X = 0$ and $Y = y - x\frac{dy}{dx}$ (intercept on Y-axis).

Similar information can be obtained for normals by writing its equation as $(Y - y)\frac{dy}{dx} + (X - x) = 0$.

Example 10.31 The slope of a curve, passing through (3, 4) at any point is the reciprocal of twice the ordinate of that point. Show that it is a parabola.

Sol. It is given that $\frac{dy}{dx} = \frac{1}{2y}$.

$\Rightarrow\ 2y\,dy = dx$

Integrating, we get $y^2 = x + c$.

Now when $x = 3, y = 4$, which gives $c = 13$

Hence the equation of the required curve is $y^2 = x + 13$, which is a parabola.

Example 10.32 Find the equation of the curve passing through (2, 1) which has constant sub-tangent.

Sol. We are given that

$$\text{sub-tangent} = \frac{y}{\frac{dy}{dx}} = k \text{ (constant)}$$

$$\Rightarrow\ k\frac{dy}{y} = dx$$

Integrating we get, $k \log y = x + c$

Given that curve passes through (2, 1) $\Rightarrow c = -2$

Hence the equation of such curve is $k \log y = x - 2$.

Example 10.33 Find the equation of the curve such that the square of the intercept cut off by any tangent from the y-axis is equal to the product of the coordinates of the point of tangency.

LO

Sol. Equation of tangent at any point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

On Y-axis, intercept is given by putting $X = 0$.

$$\therefore\ Y\text{-intercept} = y - x\frac{dy}{dx}$$

According to the question,

$$\left(y - x\frac{dy}{dx}\right)^2 = xy$$

$$\Rightarrow\ y - x\frac{dy}{dx} = \pm\sqrt{xy}$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{y \pm \sqrt{xy}}{x} \qquad \text{(Homogeneous)}$$

Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Hence $v + x\frac{dv}{dx} = v \pm \sqrt{v}$

$$\Rightarrow\ \pm\int\frac{dv}{\sqrt{v}} = \int\frac{dx}{x}$$

$$\Rightarrow\ \pm 2\sqrt{v} = \log x + \log c$$

$$\Rightarrow\ cx = e^{\pm 2\sqrt{v}}$$

$$\Rightarrow\ cx = e^{\pm 2\sqrt{y/x}}$$

Example 10.34 Find the curve such that the intercept on the x-axis cut off between the origin, and the tangent at a point is twice the abscissa and passes through the point (1, 2).

Sol. The equation of the tangent at any point $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x) \qquad (1)$$

Given that intercept on X-axis (putting $Y = 0$) = 2 (x-coordinates of P)

$$\Rightarrow\ x - y\frac{dx}{dy} = 2x$$

$$\Rightarrow\ -\frac{dy}{y} = \frac{dx}{x}$$

Integrating we get $xy = c$

Since the curve passes through (1, 2), $c = 2$.

Hence, the equation of the required curve is $xy = 2$.

Example 10.35 Find the equation of the curve which is such that the area of the rectangle constructed on the abscissa of any point and the intercept of the tangent at this point on the y-axis is equal to 4.

Sol.

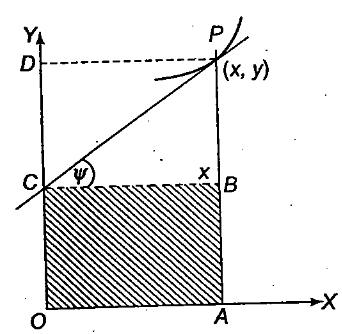


Fig. 10.5

Equation of tangent at $P(x, y)$ is $Y - y = \frac{dy}{dx}(X - x)$

$$\therefore\ Y\text{-intercept} = y - x\frac{dy}{dx}$$

$\therefore$ area of $OABC = \left|x\left(y - x\frac{dy}{dx}\right)\right| = 4$

$\Rightarrow xy - x^2\frac{dy}{dx} = \pm 4$

$\Rightarrow \frac{dy}{dx} - \frac{1}{x}y = \pm\frac{4}{x^2}$ (linear)

$\therefore$ I.F. $= e^{-\int\frac{1}{x}dx} = e^{-\log x} = 1/x$

$\therefore$ the solution is $y(1/x) = \pm 4\int\frac{1}{x^3}dx + c$

$\Rightarrow \frac{y}{x} = \pm\frac{2}{x^2} + c$

Example 10.36 Find the equation of the curve passing through the origin if the middle point of the segment of its normal from any point of the curve to the x-axis lies on the parabola $2y^2 = x$.

Sol. Equation of normal at any point $P(x, y)$ is

$\frac{dy}{dx}(Y - y) + (X - x) = 0$

This meets the x-axis at $A\left(x + y\frac{dy}{dx}, 0\right)$.

Mid point of AP is $\left(x + \frac{1}{2}y\frac{dy}{dx}, \frac{y}{2}\right)$ which lies on the parabola $2y^2 = x$.

$\therefore 2 \times \frac{y^2}{4} = x + \frac{1}{2}y\frac{dy}{dx}$ or $y^2 = 2x + y\frac{dy}{dx}$

Putting $y^2 = t$, so that $2y\frac{dy}{dx} = \frac{dt}{dx}$,

we get $\frac{dt}{dx} - 2t = -4x$ (linear)

I.F. $= e^{-2\int dx} = e^{-2x}$

Therefore, solution is given by

$t\,e^{-2x} = -4\int xe^{-2x}dx + c$

$= -4\left[-\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,1dx\right] + c$

$\Rightarrow y^2e^{-2x} = 2xe^{-2x} + e^{-2x} + c$

Since curve passes through (0, 0), $c = -1$

$\therefore y^2e^{-2x} = 2xe^{-2x} + e^{-2x} - 1$

or $y^2 = 2x + 1 - e^{2x}$ is the equation of the required curve.

Trajectories

Suppose we are given the family of plane curves $F(x, y, a) = 0$ depending on a single parameter a.

A curve making at each of its points a fixed angle α with the curve of the family passing through that point is called as isogonal trajectory of that family; if, in particular, $\alpha = \pi/2$, then it is called an *orthogonal trajectory*.

Finding Orthogonal Trajectories

We set up the differential equation of the given family of curves. Let it be of the form $F(x, y, y') = 0$

The differential equation of the orthogonal trajectories is of the form $F\left(x, y, -\frac{1}{y'}\right) = 0$ and its solution $\phi_1(x, y, C) = 0$ gives the family of orthogonal trajectories.

Example 10.37 Find the orthogonal trajectory of $y^2 = 4ax$ (a being the parameter).

Sol. $y^2 = 4ax$ (1)

$2y\frac{dy}{dx} = 4a$ (2)

Eliminating a from equation (1) and (2)

$y^2 = 2y\frac{dy}{dx}x$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get

$y = 2\left(-\frac{dx}{dy}\right)x$

$2x\,dx + y\,dy = 0$

Integrating each term,

$x^2 + \frac{y^2}{2} = c$

$2x^2 + y^2 = 2c$

which is the required orthogonal trajectory.

Example 10.38 Find the orthogonal trajectories of $xy = c$.

Sol. $xy = c$

Differentiating w.r.t. x, we get $x\frac{dy}{dx} + y = 0$.

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to get $x\frac{dx}{dy} - y = 0$

Integrating $x\,dx - y\,dy = 0$

$\Rightarrow x^2 - y^2 = c$

This is the family of the required orthogonal trajectories.

Concept Application Exercise 10.8

1. Find the equation of the curve in which the subnormal varies as the square of the ordinate.
2. Find the curve for which the length of normal is equal to the radius vector.

3. Find the curve for which the perpendicular from the foot of the ordinate to the tangent is of constant length.
4. A curve $y = f(x)$ passes through the origin. Through any point (x, y) on the curve, lines are drawn parallel to the co-ordinate axes. If the curve divides the area formed by these lines and co-ordinates axes in the ratio $m:n$, find the curve.
5. Find the orthogonal trajectories of family of curves $x^2 + y^2 = cx$.
6. Find the orthogonal trajectory of $y^2 = 4ax$ (a being the parameter).

TATISTICAL APPLICATIONS OF IFFERENTIAL EQUATION

xample 10.39 The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after two years the population has doubled, and after three years the population is 20,000, estimate the number of people initially living in the country.

Sol. Let N denote the number of people living in the country at any time t, and let N_0 denote the number of people initially living in the country.

Then, from $\frac{dN}{dt} \propto N$, $\frac{dN}{dt} - kN = 0$

which has the solution $N = ce^{kt}$ (1)

At $t = 0$, $N = N_0$; hence, equation (1) states that $N_0 = ce^{k(0)}$, or that $c = N_0$.

Thus, $N = N_0e^{kt}$ (2)

At $t = 2$, $N = 2N_0$.

Substituting these values into equation (2), we have

$2N_0 = N_0e^{2k}$ from which $k = \frac{1}{2}\ln 2$

Substituting this value into equation (1) gives

$$N = N_0 e^{\left(\frac{1}{2}\ln 2\right)t} \quad (3)$$

At $t = 3$, $N = 20{,}000$.

Substituting these values into equation (3), we obtain

$20{,}000 = N_0e^{(3/2)\ln 2} \Rightarrow N_0 = 20{,}000/2\sqrt{2} \approx 7071.$

Example 10.40 What constant interest rate is required if an initial deposit placed into an account accrues interest compounded continuously is to double its value in six years?

($\ln|2| = 0.6930$)

Sol. The balance $N(t)$ in the account at any time t,

$\frac{dN}{dt} - kN = 0$, its solution is $N(t) = ce^{kt}$ (1)

Let initial deposit be N_0.

At $t = 0$, $N(0) = N_0$, which when substituted into equation (1) yields

$$N_0 = ce^{k(0)} = c$$

and equation (1) becomes $N(t) = N_0e^{kt}$ (2)

We seek the value of k for which $N = 2N_0$ when $t = 6$, Substituting these values into (2) and solving for k we

find $2N_0 = N_0e^{k(6)} \Rightarrow e^{6k} = 2 \Rightarrow k = \frac{1}{6}\ln|2| = 0.1155$

An interest rate of 11.55 percent is required.

Concept Application Exercise 10.9

1. A person places ₹500 in an account that interest compounded continuously. Assuming no additional deposits or withdrawals, how much will be in the account after seven years if the interest rate is a constant 8.5 percent for the first four years and a constant 9.25 percent for the last thee years ($e^{0.340} = 1.404948$, $e^{0.37} = 1.447735$, $e^{0.6457} = 1.910758$).

PHYSICAL APPLICATIONS OF DIFFERENTIAL EQUATION

Example 10.41 Find the time required for a cylindrical tank of radius r and height H to empty through a round hole of area a at the bottom. The flow through the hole is according to the law $v(t) = k\sqrt{2g\,h(t)}$, where $v(t)$ and $h(t)$, are respectively, the velocity of flow through the hole and the height of the water level above the hole at time t and g is the acceleration due to gravity.

Sol.

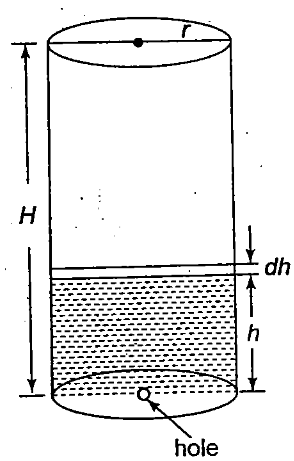


Fig. 10.6

Let at time t the depth of water is h and radius of water surface is r.
If in time dt the decrease of water level is dh, then

$$-\pi r^2\, dh = ak\sqrt{(2gh)}\, dt$$

$$\Rightarrow \quad \frac{-\pi r^2}{ak\sqrt{2g}\sqrt{h}}\, dh = dt$$

$$\Rightarrow \quad -\frac{\pi r^2}{ak\sqrt{2g}}\frac{dh}{\sqrt{h}} = dt$$

Now when $t = 0$, $h = H$ and when $t = t$, $h = 0$

$$\text{then} -\frac{\pi r^2}{ak\sqrt{2g}}\int_H^0 \frac{dh}{\sqrt{h}} = \int_0^t dt$$

$$\Rightarrow \quad -\frac{\pi r^2}{ak\sqrt{2g}}\left\{2\sqrt{h}\right\}_H^0 = t$$

$$\Rightarrow \quad t = \frac{\pi r^2\, 2\sqrt{H}}{ak\sqrt{2g}} = \frac{\pi r^2}{ak}\sqrt{\left(\frac{2H}{g}\right)}$$

Example 10.42 Suppose that a mothball loses volume by evaporation at a rate proportional to its instantaneous area. If the diameter of the ball decreases from 2 cm to 1 cm in 3 months, how long will it take until the ball has practically gone?

Sol. Let at any instance (t), radius of moth ball be r and v be its volume

$$\Rightarrow \quad v = \frac{4}{3}\pi r^3$$

$$\Rightarrow \quad \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Thus, as per the information

$$4\pi r^2 \frac{dr}{dt} = -k4\pi r^2, \text{ where } k \in R^+$$

$$\Rightarrow \quad \frac{dr}{dt} = -k$$

or $\quad r = -kt + c$

at $\quad t = 0, r = 2\text{cm}; t = 3 \text{ month}, r = 1 \text{ cm}$

$$\Rightarrow \quad c = 2, k = \frac{1}{3}$$

$$\Rightarrow \quad r = -\frac{1}{3}t + 2$$

now for $r \to 0, t \to 6$

Hence, it will take six months until the ball is practically gone.

Example 10.43 A body at a temperature of 50 °F is placed outdoors where the temperature is 100 °F. If the rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium. If after 5 min the temperature of the body is 60 °F, find (a) how long it will take the body to reach a temperature of 75 °F and (b) the temperature of the body after 20 min.

Sol. Let T be the temperature of the body at time t and $T_m = 100$ (the temperature of the surrounding medium). We have $\frac{dT}{dt} = -k(T - T_m)$ or $\frac{dT}{dt} + kT = kT_m$, where k is constant of proportionality.

$$\Rightarrow \quad \frac{dT}{dt} + kT = 100k$$

This differential equation whose solution is

$$T = ce^{-kt} + 100 \qquad (1)$$

Since $T = 50$ when $t = 0$,
then from equation (1), $50 = ce^{-k(0)} + 100$, or $c = -50$.
Substituting this value in equation (1), we obtain

$$T = -50e^{-kt} + 100 \qquad (2)$$

At $t = 5$, we are given that $T = 60$; hence, from equation (2), $60 = -50e^{-5k} + 100$.

Solving for k, we obtain $-40 = -50e^{-5k}$ or $k = -\frac{1}{5}\ln\frac{40}{50}$

Substituting this value in equation (2), we obtain the temperature of the body at any time t as

$$T = -50e^{(1/5)\ln(4/5)t} + 100 \qquad (3)$$

(a) We require t when $T = 75$. Substituting $T = 75$ in equation (3), we have

$$75 = -50e^{(1/5)\ln(4/5)t} + 100, \text{ from which we get } t$$

(b) We require T when $t = 20$. Substituting $t = 20$ in equation (3) and then solving for T, we find

$$T = -50e^{(1/5)(\ln 4/5)(20)} + 100$$

Concept Application Exercise 10.10

1. Find the time required for a cylindrical tank of radius 2.5 m and height 3 m to empty through a round hole of 2.5 cm with a velocity 2.5 $\sqrt{h}$ ms^{-1}, h being the depth of the water in the tank.
2. If the population of country doubles in 50 years, in how many years will it triple under the assumption that the rate of increase is proportional to the number of inhabitants.
3. The rate at which a substance cools in moving air is proportional to the difference between the temperatures of the substance and that of the air. If the temperature of the air is 290 K and the substance cools from 370 K to 330 K in 10 min., when will the temperature be 295 K.

EXERCISES

Subjective Type

Solutions on page 10.27

1. Solve $\dfrac{x + y\dfrac{dy}{dx}}{y - x\dfrac{dy}{dx}} = x^2 + 2y^2 + \dfrac{y^4}{x^2}$.

2. Solve $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right)dy = 0$.

3. Solve $\dfrac{dy}{dx} = \dfrac{(x+y)^2}{(x+2)(y-2)}$.

4. Solve $y\left(\dfrac{dy}{dx}\right)^2 + 2x\dfrac{dy}{dx} - y = 0$ given that $y(0) = \sqrt{5}$.

5. If $y + \dfrac{d}{dx}(xy) = x(\sin x + \log x)$, find $y(x)$.

6. If $\int_a^x t\,y(t)dt = x^2 + y(x)$, then find $y(x)$.

7. Given a function g which has a derivative $g'(x)$ for every real x and which satisfies $g'(0) = 2$ and $g(x+y) = e^y g(x) + e^x g(y)$ for all x and y. Find $g(x)$ and determine the range of the function.

8. Let the function $\ln(f(x))$ is defined where $f(x)$ exists for $x \ge 2$ and k is fixed positive real number. Prove that if $\dfrac{d}{dx}(x\,f(x)) \le -k\,f(x)$, then $f(x) \le Ax^{k-1}$ where A is independent of x.

9. If y_1 and y_2 are the solutions of the differential equation $\dfrac{dy}{dx} + Py = Q$, where P and Q are functions of x alone and $y_2 = y_1 z$, then prove that $z = 1 + c.\,e^{-\int \frac{Q}{y_1}dx}$, where c is an arbitrary constant.

10. If y_1 and y_2 are two solutions to the differential equation $\dfrac{dy}{dx} + P(x)y = Q(x)$. Then prove that $y = y_1 + c(y_1 - y_2)$ is the general solution to the equation where c is any constant.

11. Find a pair of curves such that
   a. the tangents drawn at points with equal abscissas intersect on the y-axis.
   b. the normal drawn at points with equal abscissas intersect on the x-axis.
   c. one curve passes through (1, 1) and other passes through (2, 3).

12. Given two curves: $y = f(x)$ passing through the point (0, 1) and $g(x) = \int_{-\infty}^{x} f(t)dt$ passing through the point $\left(0, \dfrac{1}{n}\right)$. The tangents drawn to both the curves at the points with equal abscissas intersect on the x-axis. Find the curve $y = f(x)$.

13. A cyclist moving on a level road at 4 m/s stops pedalling and lets the wheels come to rest. The retardation of the cycle has two components: a constant 0.08 m/s^2 due to friction in the working parts and a resistance of 0.02 v^2/m where v is speed in meters per second. What distance is traversed by the cycle before it comes to rest? (consider $\ln 5 = 1.61$).

14. The force of resistance encountered by water on a motor boat of mass m going in still water with velocity v is proportional to the velocity v. At $t = 0$ when its velocity is v_0, the engine is shut off. Find an expression for the position of motor boat at time t and also the distance travelled by the boat before it comes to rest. Take the proportionality constant as $k > 0$.

Objective Type

Solutions on page 10.31

Each question has four choices a, b, c, and d, out of which *only one* is correct.

1. The degree of the differential equation satisfying $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ is
   a. 1 **b.** 2
   c. 3 **d.** None of these

2. The differential equation whose solution is $Ax^2 + By^2 = 1$, where A and B are arbitrary constants, is of
   a. second order and second degree
   b. first order and second degree
   c. first order and first degree
   d. second order and first degree

3. The differential equation of the family of curves $y = e^x(A\cos x + B\sin x)$, where A and B are arbitrary constants, is
   a. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$
   b. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 2y = 0$
   c. $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 + y = 0$
   d. $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 2y = 0$

4. Differential equation of the family of circles touching the line $y = 2$ at (0, 2) is
   a. $x^2 + (y-2)^2 + \dfrac{dy}{dx}(y-2) = 0$
   b. $x^2 + (y-2)\left(2 - 2x\dfrac{dx}{dy} - y\right) = 0$

c. $x^2+(y-2)^2+\left(\frac{dx}{dy}+y-2\right)(y-2)=0$

d. None of these

5. The differential equation of all parabolas whose axis are parallel to the y-axis is

a. $\frac{d^3y}{dx^3}=0$ b. $\frac{d^2x}{dy^2}=C$

c. $\frac{d^3y}{dx^3}+\frac{d^2x}{dy^2}=0$ d. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=C$

6. A differential equation associated to the primitive $y=a+be^{5x}+ce^{-7x}$ is (where y_n is nth derivative w.r.t. x)

a. $y_3+2y_2-y_1=0$ b. $4y_3+5y_2-20y_1=0$

c. $y_3+2y_2-35y_1=0$ d. None of these

where y_n represents nth order derivative.

7. The differential equation of all circles which pass through the origin and whose centres lie on the y-axis is

a. $(x^2-y^2)\frac{dy}{dx}-2xy=0$ b. $(x^2-y^2)\frac{dy}{dx}+2xy=0$

c. $(x^2-y^2)\frac{dy}{dx}-xy=0$ d. $(x^2-y^2)\frac{dy}{dx}+xy=0$

8. The form of the differential equation of the central conics $ax^2+by^2=1$ is

a. $x=y\frac{dy}{dx}$ b. $x+y\frac{dy}{dx}=0$

c. $x\left(\frac{dy}{dx}\right)^2+xy\frac{d^2y}{dx^2}=y\frac{dy}{dx}$ d. None of these

9. The differential equation for the family of curve $x^2+y^2-2ay=0$, where a is an arbitrary constant, is

a. $2(x^2-y^2)y'=xy$ b. $2(x^2+y^2)y'=xy$

c. $(x^2-y^2)y'=2xy$ d. $(x^2+y^2)y'=2xy$

10. If $=(e^y-x)^{-1}$, where $y(0)=0$, then y is expressed explicitly as

a. $\frac{1}{2}\ln(1+x^2)$ b. $\ln(1+x^2)$

c. $\ln\left(x+\sqrt{1+x^2}\right)$ d. $\ln\left(x+\sqrt{1-x^2}\right)$

11. If $y=\frac{x}{\log|cx|}$ (where c is an arbitrary constant) is the general solution of the differential equation $dy/dx=y/x+\phi(x/y)$ then the function $\phi(x/y)$ is

a. x^2/y^2 b. $-x^2/y^2$

c. y^2/x^2 d. $-y^2/x^2$

12. The differential equation whose general solution is given by, $y=(c_1\cos(x+c_2)-(c_3e^{(-x+c_4)})+(c_5\sin x)$, where c_1, c_2, c_3, c_4, c_5 are arbitrary constants, is

a. $\frac{d^4y}{dx^4}-\frac{d^2y}{dx^2}+y=0$

b. $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$

c. $\frac{d^5y}{dx^5}+y=0$

d. $\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}+\frac{dy}{dx}-y=0$

13. The solution to the differential equation $y\log y+xy'=0$, where $y(1)=e$, is

a. $x(\log y)=1$ b. $xy(\log y)=1$

c. $(\log y)^2=2$ d. $\log y+\left(\frac{x^2}{2}\right)y=1$

14. If $y=y(x)$ and $\frac{2+\sin x}{y+1}\left(\frac{dy}{dx}\right)=-\cos x$, $y(0)=1$, then $y(\pi/2)$ equals

a. 1/3 b. 2/3

c. – 1/3 d. 1

15. The equation of the curves through the point (1, 0) and whose slope is $\frac{y-1}{x^2+x}$ is

a. $(y-1)(x+1)+2x=0$

b. $2x(y-1)+x+1=0$

c. $x(y-1)(x+1)+2=0$

d. None of these

16. The solution of the equation $\frac{dy}{dx}=\frac{x(2\log x+1)}{\sin y+y\cos y}$ is

a. $y\sin y=x^2\log x+\frac{x^2}{2}+c$

b. $y\cos y=x^2(\log x+1)+c$

c. $y\cos y=x^2\log x+\frac{x^2}{2}+c$

d. $y\sin y=x^2\log x+c$

17. The solution of the equation $\log(dy/dx)=ax+by$ is

a. $\frac{e^{by}}{b}=\frac{e^{ax}}{a}+c$ b. $\frac{e^{-by}}{-b}=\frac{e^{ax}}{a}+c$

c. $\frac{e^{-by}}{a}=\frac{e^{ax}}{b}+c$ d. None of these

18. The solution of the equation $(x^2y+x^2)dx+y^2(x-1)dy=0$ is given by

a. $x^2+y^2+2(x-y)+2\ln\frac{(x-1)(y+1)}{c}=0$

b. $x^2+y^2+2(x-y)+\ln\frac{(x-1)(y+1)}{c}=0$

c. $x^2+y^2+2(x-y)-2\ln\frac{(x-1)(y+1)}{c}=0$

d. None of these

19. Solution of differential equation $dy - \sin x \sin y dx = 0$ is

a. $e^{\cos x}\tan\frac{y}{2}=c$ b. $e^{\cos x}\tan y=c$

c. $\cos x\tan y=c$ d. $\cos x\sin y=c$

20. The solution of $\frac{dv}{dt}+\frac{k}{m}v=-g$ is

a. $v=ce^{-\frac{k}{m}t}-\frac{mg}{k}$ b. $v=c-\frac{mg}{k}e^{-\frac{k}{m}t}$

c. $ve^{-\frac{k}{m}t}=c-\frac{mg}{k}$ d. $ve^{\frac{k}{m}t}=c-\frac{mg}{k}$

21. The solution of the equation $dy/dx = \cos(x-y)$ is

a. $y+\cot\left(\frac{x-y}{2}\right)=C$ b. $x+\cot\left(\frac{x-y}{2}\right)=C$

c. $x+\tan\left(\frac{x-y}{2}\right)=C$ d. None of these

22. Solution of $\frac{dy}{dx}+2xy=y$ is

a. $y=ce^{x-x^2}$ b. $y=ce^{x^2-x}$

c. $y=ce^x$ d. $y=ce^{-x^2}$

23. The general solution of the differential equation $\frac{dy}{dx}+\sin\frac{x+y}{2}=\sin\frac{x-y}{2}$ is

a. $\log\tan\left(\frac{y}{2}\right)=c-2\sin x$

b. $\log\tan\left(\frac{y}{4}\right)=c-2\sin\left(\frac{x}{2}\right)$

c. $\log\tan\left(\frac{y}{2}+\frac{\pi}{4}\right)=c-2\sin x$

d. $\log\tan\left(\frac{y}{4}+\frac{\pi}{4}\right)=c-2\sin\left(\frac{x}{2}\right)$

24. The solutions of $(x+y+1)\,dy=dx$ is

a. $x+y+2=Ce^y$ b. $x+y+4=C\log y$

c. $\log(x+y+2)=Cy$ d. $\log(x+y+2)=C-y$

25. The solution of $x^2\frac{dy}{dx}-xy=1+\cos\frac{y}{x}$ is

a. $\tan\left(\frac{y}{2x}\right)=c-\frac{1}{2x^2}$ b. $\tan\frac{y}{x}=c+\frac{1}{x}$

c. $\cos\left(\frac{y}{x}\right)=1+\frac{c}{x}$ d. $x^2=(c+x^2)\tan\frac{y}{x}$

26. The slope of the tangent at (x, y) to a curve passing through $\left(1,\frac{\pi}{4}\right)$ is given by $\frac{y}{x}-\cos^2\left(\frac{y}{x}\right)$, then the equation of the curve is

a. $y=\tan^{-1}\left(\log\left(\frac{e}{x}\right)\right)$ b. $y=x\tan^{-1}\left(\log\left(\frac{x}{e}\right)\right)$

c. $y=x\tan^{-1}\left(\log\left(\frac{e}{x}\right)\right)$ d. None of these

27. If $x\frac{dy}{dx}=y(\log y-\log x+1)$, then the solution of the equation is

a. $\log\frac{x}{y}=cy$ b. $\log\frac{y}{x}=cy$

c. $\log\frac{x}{y}=cx$ d. None of these

28. The solution of differential equation

$$yy'=x\left(\frac{y^2}{x^2}+\frac{f(y^2/x^2)}{f'(y^2/x^2)}\right)\text{ is}$$

a. $f(y^2/x^2)=cx^2$ b. $x^2f(y^2/x^2)=c^2y^2$

c. $x^2f(y^2/x^2)=c$ d. $f(y^2/x^2)=cy/x$

29. The solution of $(x^2+xy)\,dy=(x^2+y^2)\,dx$ is

a. $\log x=\log(x-y)+\frac{y}{x}+c$

b. $\log x=2\log(x-y)+\frac{y}{x}+c$

c. $\log x=\log(x-y)+\frac{x}{y}+c$

d. None of these

30. The solution of $(y+x+5)\,dy=(y-x+1)\,dx$ is

a. $\log((y+3)^2+(x+2)^2)+\tan^{-1}\frac{y+3}{y+2}+C$

b. $\log((y+3)^2+(x-2)^2)+\tan^{-1}\frac{y-3}{x-2}=C$

c. $\log((y+3)^2+(x+2)^2)+2\tan^{-1}\frac{y+3}{x+2}=C$

d. $\log((y+3)^2+(x+2)^2)-2\tan^{-1}\frac{y+3}{x+2}=C$

31. The slope of the tangent at (x, y) to a curve passing through a point (2, 1) is $\frac{x^2+y^2}{2xy}$, then the equation of the curve is

a. $2(x^2-y^2)=3x$ b. $2(x^2-y^2)=6y$

c. $x(x^2-y^2)=6$ d. $x(x^2+y^2)=10$

32. Solution of the differential equation $(y + x\sqrt{xy}(x+y))\,dx + (y\sqrt{xy}(x+y)-x)dy = 0$ is

a. $\frac{x^2+y^2}{2}+\tan^{-1}\sqrt{\frac{y}{x}}=c$ b. $\frac{x^2+y^2}{2}+2\tan^{-1}\sqrt{\frac{x}{y}}=c$

c. $\frac{x^2+y^2}{2}+2\cot^{-1}\sqrt{\frac{x}{y}}=c$ d. None of these

33. The general solution of the differential equation, $y' + y\phi'(x) - \phi(x)\,.\,\phi'(x) = 0$, where $\phi(x)$ is a known function, is

a. $y = ce^{-\phi(x)} + \phi(x) - 1$ b. $y = ce^{+\phi(x)} + \phi(x) - 1$

c. $y = ce^{-\phi(x)} - \phi(x) + 1$ d. $y = ce^{-\phi(x)} + \phi(x) + 1$

where c is an arbitrary constant.

34. The solution of $\frac{dy}{dx} = \frac{x^2+y^2+1}{2xy}$ satisfying $y(1) = 1$ is given by

a. a system of parabolas b. a system of circles

c. $y^2 = x(1+x) - 1$ d. $(x-2)^2 + (y-3)^2 = 5$

35. The integrating factor of the differential equation $\frac{dy}{dx}(x\log_e x) + y = 2\log_e x$ is given by

a. x b. e^x

c. $\log_e x$ d. $\log_e(\log_e x)$

36. The solution of the differential equation $x(x^2+1)(dy/dx) = y(1-x^2) + x^3\log x$ is

a. $y(x^2+1)/x = \frac{1}{4}x^2\log x + \frac{1}{2}x^2 + c$

b. $y^2(x^2-1)/x = \frac{1}{2}x^2\log x - \frac{1}{4}x^2 + c$

c. $y(x^2+1)/x = \frac{1}{2}x^2\log x - \frac{1}{4}x^2 + c$

d. None of these

37. Integrating factor of differential equation

$\cos x\frac{dy}{dx} + y\sin x = 1$ is

a. $\cos x$ b. $\tan x$

c. $\sec x$ d. $\sin x$

38. Solution of the equation $\cos^2 x\frac{dy}{dx} - (\tan 2x)\,y = \cos^4 x$, $|x| < \frac{\pi}{4}$, when $y\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{8}$ is

a. $y = \tan 2x\cos^2 x$ b. $y = \cot 2x\cos^2 x$

c. $y = \frac{1}{2}\tan 2x\cos^2 x$ d. $y = \frac{1}{2}\cot 2x\cos^2 x$

39. If integrating factor of

$x(1-x^2)\,dy + (2x^2y - y - ax^3)\,dx = 0$ is $e^{\int p\,dx}$, then P is equal to

a. $\frac{2x^2-ax^3}{x(1-x^2)}$ b. $2x^3-1$

c. $\frac{2x^2-a}{ax^3}$ d. $\frac{2x^2-1}{x(1-x^2)}$

40. A function $y = f(x)$ satisfies $(x+1)f'(x) - 2(x^2+x)f(x) = \frac{e^{x^2}}{(x+1)}$, $\forall\ x > -1$.

If $f(0) = 5$, then $f(x)$ is

a. $\left(\frac{3x+5}{x+1}\right)e^{x^2}$ b. $\left(\frac{6x+5}{x+1}\right)e^{x^2}$

c. $\left(\frac{6x+5}{(x+1)^2}\right)e^{x^2}$ d. $\left(\frac{5-6x}{x+1}\right)e^{x^2}$

41. The solution of the differential equation $\frac{dy}{dx} = \frac{1}{xy[x^2\sin y^2+1]}$ is

a. $x^2(\cos y^2 - \sin y^2 - 2\,Ce^{-y^2}) = 2$

b. $y^2(\cos x^2 - \sin y^2 - 2Ce^{-y^2}) = 2$

c. $x^2(\cos y^2 - \sin y^2 - e^{-y^2}) = 4C$

d. None of these

42. The general solution of the equation $\frac{dy}{dx} = 1 + xy$ is

a. $y = ce^{-x^2/2}$ b. $y = ce^{x^2/2}$

c. $y = (x+c), e^{-x^2/2}$ d. None of these

43. The solution of the differential equation $(x+2y^3)\frac{dy}{dx} = y$ is

a. $\frac{x}{y^2} = y + c$ b. $\frac{x}{y} = y^2 + c$

c. $\frac{x^2}{y} = y^2 + c$ d. $\frac{y}{x} = x^2 + c$

44. The solution of the differential equation $x^2\frac{dy}{dx}\cos\frac{1}{x} - y\sin\frac{1}{x} = -1$, where $y \to -1$ as $x \to \infty$ is

a. $y = \sin\frac{1}{x} - \cos\frac{1}{x}$ b. $y = \frac{x+1}{x\sin\frac{1}{x}}$

c. $y = \cos\frac{1}{x} + \sin\frac{1}{x}$ d. $y = \frac{x+1}{x\cos 1/x}$

45. The solution of the differential equation $2x^2y\frac{dy}{dx} = \tan(x^2y^2) - 2xy^2$, given $y(1) = \sqrt{\frac{\pi}{2}}$, is

a. $\sin x^2y^2 = e^{x-1}$ b. $\sin(x^2y^2) = x$

c. $\cos x^2y^2 + x = 0$ d. $\sin(x^2y^2) = e\,e^x$

46. Solution of the differential equation

$\left\{\frac{1}{x} - \frac{y^2}{(x-y)^2}\right\}dx + \left\{\frac{x^2}{(x-y)^2} - \frac{1}{y}\right\}dy = 0$ is

a. $\ln\left|\frac{x}{y}\right| + \frac{xy}{x-y} = c$ b. $\frac{xy}{x-y} = ce^{x/y}$

c. $\ln|xy| = c + \frac{xy}{x-y}$ d. None of these

47. If $y+x\dfrac{dy}{dx}=x\dfrac{\phi(xy)}{\phi'(xy)}$, then $\phi(xy)$ is equal to

a. $ke^{x^2/2}$ b. $ke^{y^2/2}$

c. $ke^{xy/2}$ d. ke^{xy}

48. The solution of differential equation $(2y+xy^3)dx + (x+x^2y^2)\,dy=0$ is

a. $x^2y+\dfrac{x^3y^3}{3}=c$ b. $xy^2+\dfrac{x^3y^3}{3}=c$

c. $x^2y+\dfrac{x^4y^4}{4}=c$ d. None of these

49. The solution of $ye^{-x/y}dx-(xe^{(-x/y)}+y^3)dy=0$ is

a. $e^{-x/y}+y^2=C$ b. $xe^{-x/y}+y=C$

c. $2e^{-x/y}+y^2=C$ d. $e^{-x/y}+2y^2=C$

50. The curve satisfying the equation $\dfrac{dy}{dx}=\dfrac{y\,(x+y^3)}{x\,(y^3-x)}$ and passing through the point (4, – 2) is

a. $y^2=-2x$ b. $y=-2x$

c. $y^3=-2x$ d. None of these

51. The solution of differential equation

$$\frac{x+y\dfrac{dy}{dx}}{y-x\dfrac{dy}{dx}}=\frac{x\cos^2(x^2+y^2)}{y^3}\text{ is}$$

a. $\tan(x^2+y^2)=\dfrac{x^2}{y^2}+c$ b. $\cot(x^2+y^2)=\dfrac{x^2}{y^2}+c$

c. $\tan(x^2+y^2)=\dfrac{y^2}{x^2}+c$ d. $\cot(x^2+y^2)=\dfrac{y^2}{x^2}+c$

52. The solution of the differential equation

$$\frac{dy}{dx}=\frac{3x^2y^4+2xy}{x^2-2x^3y^3}\text{ is}$$

a. $\dfrac{y^2}{x}-x^3y^2=c$ b. $\dfrac{x^2}{y^2}+x^3y^3=c$

c. $\dfrac{x^2}{y}+x^3y^2=c$ d. $\dfrac{x^2}{3y}-2x^3y^2=c$

53. The solution of the differential equation

$$\{1+x\sqrt{(x^2+y^2)}\}dx+\{\sqrt{(x^2+y^2)}-1\}\,y\,dy=0$$

is equal to

a. $x^2+\dfrac{y^2}{2}+\dfrac{1}{3}(x^2+y^2)^{3/2}=c$

b. $x-\dfrac{y^3}{3}+\dfrac{1}{2}(x^2+y^2)^{1/2}=c$

c. $x-\dfrac{y^2}{2}+\dfrac{1}{3}(x^2+y^2)^{3/2}=c$

d. None of these

54. Which of the following is not the differential equation of family of curves whose tangent form an angle of $\pi/4$ with the hyperbola $xy=c^2$?

a. $\dfrac{dy}{dx}=\dfrac{x-y}{x+y}$ b. $\dfrac{dy}{dx}=\dfrac{x}{x-y}$

c. $\dfrac{dy}{dx}=\dfrac{x+y}{y-x}$ d. None of these

55. Tangent to a curve intercepts the y-axis at a point P. A line perpendicular to this tangent through P passes through another point (1, 0). The differential equation of the curve is

a. $y\dfrac{dy}{dx}-x\left(\dfrac{dy}{dx}\right)^2=1$ b. $\dfrac{xd^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=0$

c. $y\dfrac{dx}{dy}+x=1$ d. None of these

56. The differential equation of the curve for which the initial ordinate of any tangent is equal to the corresponding subnormal

a. is linear

b. is homogeneous of second degree

c. has separable variables

d. is of second order

57. Orthogonal trajectories of family of the curve $x^{2/3}+y^{2/3}=a^{2/3}$, where a is any arbitrary constant, is

a. $x^{2/3}-y^{2/3}=c$ b. $x^{4/3}-y^{4/3}=c$

c. $x^{4/3}+y^{4/3}=c$ d. $x^{1/3}-y^{1/3}=c$

58. The differential equation of all non-horizontal lines in a plane is

a. $\dfrac{d^2y}{dx^2}$ b. $\dfrac{d^2x}{dy^2}=0$

c. $\dfrac{dy}{dx}=0$ d. $\dfrac{dx}{dy}=0$

59. The curve in first quadrant for which the normal at any point (x, y) and the line joining the origin to that point form an isosceles triangle with the x-axis as base is

a. an ellipse b. a rectangular hyperbola

c. a circle d. None of these

60. The equation of the curve which is such that the portion of the axis of x cut off between the origin and tangent at any point is proportional to the ordinate of that point is

a. $x=y(a-b\log x)$ b. $\log x=by^2+a$

c. $x^2=y(a-b\log y)$ d. None of these

(b is a constant of proportionality)

61. The family of curves represented by $\dfrac{dy}{dx}=\dfrac{x^2+x+1}{y^2+y+1}$ and

$$\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$$

a. Touch each other b. Are orthogonal
c. Are one and the same d. None of these

62. A normal at $P(x, y)$ on a curve meets the x-axis at Q and N is the foot of the ordinate at P. If $NQ = \frac{x(1+y^2)}{1+x^2}$, then the equation of curve given that it passes through the point (3, 1) is
a. $x^2 - y^2 = 8$ b. $x^2 + 2y^2 = 11$
c. $x^2 - 5y^2 = 4$ d. None of these

63. A curve is such that the mid point of the portion of the tangent intercepted between the point where the tangent is drawn and the point where the tangent meets the y-axis lies on the line $y = x$. If the curve passes through (1, 0), then the curve is
a. $2y = x^2 - x$ b. $y = x^2 - x$
c. $y = x - x^2$ d. $y = 2(x - x^2)$

64. The equation of a curve passing through (2, 7/2) and having gradient $1 - \frac{1}{x^2}$ at (x, y) is
a. $y = x^2 + x + 1$ b. $xy = x^2 + x + 1$
c. $xy = x + 1$ d. None of these

65. A normal at any point (x, y) to the curve $y = f(x)$ cuts a triangle of unit area with the axis, the differential equation of the curve is
a. $y^2 - x^2\left(\frac{dy}{dx}\right)^2 = 4\frac{dy}{dx}$ b. $x^2 - y^2\left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$
c. $x + y\frac{dy}{dx} = y$ d. None of these

66. The normal to a curve at $P(x, y)$ meets the x-axis at G. If the distance of G from the origin is twice the abscissa of P, then the curve is a
a. parabola b. circle
c. hyperbola d. ellipse

67. The x-intercept of the tangent to a curve is equal to the ordinate of the point of contact. The equation of the curve through the point (1, 1) is
a. $ye^{x/y} = e$ b. $xe^{x/y} = e$
c. $xe^{y/x} = e$ d. $ye^{y/x} = e$

68. The equation of a curve passing through (1, 0) for which the product of the abscissa of a point P and the intercept made by a normal at P on the x-axis equals twice the square of the radius vector of the point P, is
a. $x^2 + y^2 = x^4$ b. $x^2 + y^2 = 2x^4$
c. $x^2 + y^2 = 4x^4$ d. None of these

69. The differential equation of all parabolas each of which has a latus rectum $4a$ and whose axis are parallel to the x-axis is
a. of order 1 and degree 2 b. of order 2 and degree 3
c. of order 2 and degree 1 d. of order 2 and degree 2

70. The curve, with the property that the projection of the ordinate on the normal is constant and has a length equal to a is
a. $a\ln\left(\sqrt{y^2 - a^2} + y\right) = x + c$
b. $x + \sqrt{a^2 - y^2} = c$
c. $(y - a)^2 = cx$
d. $ay = \tan^{-1}(x + c)$

71. The solution of the differential equation $y(2x^4 + y)\frac{dy}{dx} = (1 - 4xy^2)x^2$ is given by
a. $3(x^2y)^2 + y^3 - x^3 = c$
b. $xy^2 + \frac{y^3}{3} - \frac{x^3}{3} + c = 0$
c. $\frac{2}{5}yx^5 + \frac{y^3}{3} = \frac{x^3}{3} - \frac{4xy^3}{3} + c$
d. None of these

72. The solution of the differential equation $(x \cot y + \log \cos x)\,dy + (\log \sin y - y \tan x)\,dx = 0$
a. $(\sin x)^y (\cos y)^x = c$ b. $(\sin y)^x (\cos x)^y = c$
c. $(\sin x)^x (\cos y)^y = c$ d. None of these

73. Spherical rain drop evaporates at a rate proportional to its surface area. The differential equation corresponding to the rate of change of the radius of the rain drop if the constant of proportionality is $K > 0$ is
a. $\frac{dr}{dt} + K = 0$ b. $\frac{dr}{dt} - K = 0$
c. $\frac{dr}{dt} = Kr$ d. None of these

74. Water is drained from a vertical cylindrical tank by opening a valve at the base of the tank. It is known that the rate at which the water level drops is proportional to the square root of water depth y, where the constant of proportionality $k > 0$ depends on the acceleration due to gravity and the geometry of the hole. If t is measured in minutes and $k = \frac{1}{15}$, then the time to drain the tank if the water is 4 m deep to start with is
a. 30 min b. 45 min
c. 60 min d. 80 min

75. The population of a country increases at a rate proportional to the number of inhabitants. f is the population which doubles in 30 years, then the population will triple in approximately
a. 30 years b. 45 years
c. 48 years d. 54 years

76. An object falling from rest in air is subject not only to the gravitational force but also to air resistnace. Assume that the air resistance is proportional to the velocity with constant of proportionality as $k > 0$, and acts in a direction opposite to motion ($g = 9.8$ m/s^2). Then velocity cannot exceed
a. $9.8/k$ m/s b. $98/k$ m/s
c. $\frac{k}{9.8}$ m/s d. None of these

77. The solution of differential equation $x^2 = 1 + \left(\frac{x}{y}\right)^{-1}\frac{dy}{dx} + \frac{\left(\frac{x}{y}\right)^{-2}\left(\frac{dy}{dx}\right)^2}{2!} + \frac{\left(\frac{x}{y}\right)^{-3}\left(\frac{dy}{dx}\right)^3}{3!} + \dots$ is

a. $y^2 = x^2(\ln x^2 - 1) + c$
b. $y = x^2(\ln x - 1) + c$
c. $y^2 = x(\ln x - 1) + c$
d. $y = x^2 e^{x^2} + c$

78. The solution of the differential equation $y' y''' = 3(y'')^2$ is

a. $x = A_1 y^2 + A_2 y + A_3$
b. $x = A_1 y + A_2$
c. $x = A_1 y^2 + A_2 y$
d. None of these

79. Number of values of $m \in N$ for which $y = e^{mx}$ is a solution of the differential equation $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 12y = 0$

a. 0
b. 1
c. 2
d. More than 2

80. A curve passing through (2, 3) and satisfying the differential equation $\int_0^x t\,y(t)\,dt = x^2 y(x), (x > 0)$ is

a. $x^2 + y^2 = 13$
b. $y^2 = \frac{9}{2}x$
c. $\frac{x^2}{8} + \frac{y^2}{18} = 1$
d. $xy = 6$

81. The solution of the differential equation $\frac{d^2y}{dx^2} = \sin 3x + e^x + x^2$ when $y_1(0) = 1$ and $y(0) = 0$ is

a. $\frac{-\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x - 1$
b. $\frac{-\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x$
c. $\frac{-\cos 3x}{3} + e^x + \frac{x^4}{12} + \frac{1}{3}x + 1$
d. None of these

82. The solution of the differential equation

$$\frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots} = \frac{dx - dy}{dx + dy}$$ is

a. $2y e^{2x} = C e^{2x} + 1$
b. $2y e^{2x} = C e^{2x} - 1$
c. $y e^{2x} = C e^{2x} + 2$
d. None of these

83. The solution of the differential equation

$$x = 1 + xy\frac{dy}{dx} + \frac{x^2y^2}{2!}\left(\frac{dy}{dx}\right)^2 + \frac{x^3y^3}{3!}\left(\frac{dy}{dx}\right)^3 + \dots$$ is

a. $y = \ln(x) + c$
b. $y^2 = (\ln x)^2 + c$
c. $y = \log x + xy$
d. $xy = x^y + c$

84. The differential equation of the curve $\frac{x}{c-1} + \frac{y}{c+1} = 1$ is given by

a. $\left(\frac{dy}{dx} - 1\right)\left(y + x\frac{dy}{dx}\right) = 2\frac{dy}{dx}$
b. $\left(\frac{dy}{dx} + 1\right)\left(y - x\frac{dy}{dx}\right) = \frac{dy}{dx}$
c. $\left(\frac{dy}{dx} + 1\right)\left(y - x\frac{dy}{dx}\right) = 2\frac{dy}{dx}$
d. None of these

85. The function $f(\theta) = \frac{d}{d\theta}\int_0^{\theta}\frac{dx}{1 - \cos\theta\cos x}$ satisfies the differential equation

a. $\frac{df(\theta)}{d\theta} + 2f(\theta)\cot\theta = 0$
b. $\frac{df}{d\theta} - 2f(\theta)\cot\theta = 0$
c. $\frac{df}{d\theta} + 2f(\theta) = 0$
d. $\frac{df}{d\theta} - 2f(\theta) = 0$

86. Differential equation of the family of curves $v = A/r + B$, where A and B are arbitrary constants, is

a. $\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} = 0$
b. $\frac{d^2v}{dr^2} - \frac{2}{r}\frac{dv}{dr} = 0$
c. $\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0$
d. None of these

87. The solution of the differential equation $y''' - 8y'' = 0$ where $y(0) = \frac{1}{8}, y'(0) = 0, y''(0) = 1$ is

a. $y = \frac{1}{8}\left(\frac{e^{8x}}{8} + x - \frac{7}{9}\right)$
b. $y = \frac{1}{8}\left(\frac{e^{8x}}{8} + x + \frac{7}{8}\right)$
c. $y = \frac{1}{8}\left(\frac{e^{8x}}{8} - x + \frac{7}{8}\right)$
d. None of these

88. The solution of the differential equation

$$(e^{x^2} + e^{y^2})\,y\frac{dy}{dx} + e^{x^2}(xy^2 - x) = 0$$ is

a. $e^{x^2}(y^2 - 1) + e^{y^2} = C$
b. $e^{y^2}(x^2 - 1) + e^{x^2} = C$
c. $e^{y^2}(y^2 - 1) + e^{x^2} = C$
d. $e^{x^2}(y - 1) + e^{y^2} = C$

Multiple Correct Answers Type

Solutions on page 10.44

Each question has four choices a, b, c, and d, out of which *one or more* answers are correct.

1. Which one of the following function(s) is/are homogeneous?

a. $f(x, y) = \frac{x - y}{x^2 + y^2}$
b. $f(x, y) = x^{\frac{1}{3}} y^{-\frac{2}{3}} \tan^{-1}\frac{x}{y}$
c. $f(x, y) = x(\ln\sqrt{x^2 + y^2} - \ln y) + y e^{x/y}$
d. $f(x, y) = x\left[\ln\frac{2x^2 + y^2}{x} - \ln(x + y)\right] + y^2\tan\frac{x + 2y}{3x - y}$

2. For the differential equation whose solution is $(x-h)^2 + (y-k)^2 = a^2$ (a is a constant), its
   a. order is 2
   b. order is 3
   c. degree is 2
   d. degree is 3

3. The equation of the curve satisfying the differential equation $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$ can be a
   a. Circle
   b. Straight line
   c. Parabola
   d. Ellipse

4. Which of the following equation(s) is/are linear?
   a. $\frac{dy}{dx} + \frac{y}{x} = \log x$
   b. $y\left(\frac{dy}{dx}\right) + 4x = 0$
   c. $(2x + y^3)\left(\frac{dy}{dx}\right) = 3y$
   d. None of these

5. The solution of $\frac{dy}{dx} = \frac{ax+h}{by+k}$ represents a parabola when
   a. $a = 0, b \neq 0$
   b. $a \neq 0, b \neq 0$
   c. $b = 0, a \neq 0$
   d. $a = 0, b \in R$

6. The equation of the curve satisfying the differential equation $y_2(x^2+1) = 2xy_1$ passing through the point (0, 1) and having slope of tangent at $x = 0$ as 3 (where y_2 and y_1 represents 2nd and 1st order derivative), then
   a. $y = f(x)$ is a strictly increasing function
   b. $y = f(x)$ is a non-monotonic function
   c. $y = f(x)$ has three distinct real roots
   d. $y = f(x)$ has only one negative root

7. Identify the statement(s) which is/are true.
   a. $f(x, y) = e^{y/x} + \tan\frac{y}{x}$ is a homogeneous of degree zero.
   b. $x\ln\frac{y}{x}\,dx + \frac{y^2}{x}\sin^{-1}\frac{y}{x}\,dy = 0$ is a homogeneous differential equation.
   c. $f(x, y) = x^2 + \sin x \cos y$ is a not homogeneous.
   d. $(x^2 + y^2)dx - (xy^2 - y^3)dy = 0$ is a homegeneous differential equation.

8. The graph of the function $y = f(x)$ passing through the point (0, 1) and satisfying the differential equation $\frac{dy}{dx} + y\cos x = \cos x$ is such that
   a. It is a constant function
   b. It is periodic
   c. It is neither an even nor an odd function
   d. It is continuous and differentiable for all x

9. If $f(x)$, $g(x)$ be twice differential functions on [0, 2] satisfying $f''(x) = g''(x)$, $f'(1) = 2g'(1) = 4$ and $f(2) = 3g(2) = 9$, then
   a. $f(4) - g(4) = 10$
   b. $|f(x) - g(x)| < 2 \Rightarrow -2 < x < 0$
   c. $f(2) = g(2) \Rightarrow x = -1$
   d. $f(x) - g(x) = 2x$ has real root

10. The solution of the differential equation $(x^2y^2 - 1)\,dy + 2xy^3\,dx = 0$ is
   a. $1 + x^2y^2 = cx$
   b. $1 + x^2y^2 = cy$
   c. $y = 0$
   d. $y = -\frac{1}{x^2}$

11. $y = ae^{-1/x} + b$ is a solution of $\frac{dy}{dx} = \frac{y}{x^2}$, then
   a. $a \in R$
   b. $b = 0$
   c. $b = 1$
   d. a takes finite number of values

12. For equation of the curve whose subnormal is constant, then
   a. Its eccentricity is 1
   b. Its eccentricity is $\sqrt{2}$
   c. Its axis is the x-axis
   d. Its axis is the y-axis.

13. The solution of $\frac{xdx + ydy}{xdy - ydx} = \sqrt{\frac{1 - x^2 - y^2}{x^2 + y^2}}$ is
   a. $\sqrt{x^2 + y^2} = \sin\{\tan^{-1}(y/x) + C\}$
   b. $\sqrt{x^2 + y^2} = \cos\{(\tan^{-1} y/x) + C\}$
   c. $\sqrt{x^2 + y^2} = (\tan(\sin^{-1} y/x) + C)$
   d. $y = x\tan\left(c + \sin^{-1}\sqrt{x^2 + y^2}\right)$

14. The curves for which the length of the normal is equal to the length of the radius vector is/are
   a. circles
   b. rectangular hyperbola
   c. ellipses
   d. straight lines

15. In which of the following differential equation degree is not defined?
   a. $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x\log\frac{d^2y}{dx^2}$
   b. $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{d^2y}{dx^2}\right)$
   c. $x = \sin\left(\frac{dy}{dx} - 2y\right), |x| < 1$
   d. $x - 2y = \log\left(\frac{dy}{dx}\right)$

Reasoning Type

Solutions on page 10.46

Each question has four choices a, b, c, and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. if both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1
b. if both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1
c. if STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.
d. if STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

. **Statement 1:** The differential equation of all circles in a plane must be of order 3.
Statement 2: There is only one circle passing through three non-collinear points.

2. **Statement 1:** The differential equation of the family of curves represented by $y = Ae^x$ is given by $\frac{dy}{dx} = y$.
Statement 2: $\frac{dy}{dx} = y$ is valid for every member of the given family.

3. **Statement 1:** Degree of the differential equation $2x - 3y + 2 = \log\left(\frac{dy}{dx}\right)$ is not defined.
Statement 2: In the given differential equation, the power of highest order derivative when expressed as the polynomials of derivatives is called degree.

4. **Statement 1:** Order of a differential equation represents number of arbitrary constants in the general solution.
Statement 2: Degree of a differential equation represents number of family of curves.

5. **Statement 1:** The order of the differential equation whose general solution is $y = c_1\cos 2x + c_2\sin^2 x + c_3\cos^2 x + c_4e^{2x} + c_5\, e^{2x+c_6}$ is 3.
Statement 2: Total number of arbitrary parameters in the given general solution in the statement (1) is 3.

nked Comprehension ype

Solutions on page 10.46

ꞏed upon each paragraph, three multiple choice questions ꞏe to be answered. Each question has four choices a, b, c, l d, out of which *only one* is correct.

Problems 1–3

$f(x)$ be a non-positive continuous function and $F(x) = \int_0^x f(t)dt$ ≥ 0 and $f(x) \geq cF(x)$ where $c > 0$ and let $g : [0, \infty) \to R$ be a ction such that $\frac{dg(x)}{dx} < g(x)\ \forall\, x > 0$ and $g(0) = 0$.

1. The total number of root(s) of the equation $f(x) = g(x)$ is/are
   a. ∞ b. 1
   c. 2 d. 0
2. The number of solution(s) of the equation $|x^2 + x - 6| = f(x) + g(x)$ is/are
   a. 2 b. 1
   c. 0 d. 3
3. The solution set of inequation $g(x)(\cos^{-1}x - \sin^{-1}x) \leq 0$
   a. $\left[-1, \frac{1}{\sqrt{2}}\right]$ b. $\left[\frac{1}{\sqrt{2}}, 1\right]$
   c. $\left[0, \frac{1}{\sqrt{2}}\right]$ d. $\left(0, \frac{1}{\sqrt{2}}\right]$

r Problems 4–6

differential equation $y = px + f(p)$, (1)

where $p = \frac{dy}{dx}$, is known as Clairout's Equation. To solve equation (1), differentiate it with respect to x, which gives either

$\frac{dp}{dx} = 0 \Rightarrow p = c$ (2)

or $x + f'(p) = 0$ (3)

Note:

a. *If p is eliminated between equations (1) and (2), the solution obtained is a general solution of equation (1).*
b. *If p is eliminated between equation (1) and (3), then solution obtained does not contain any arbitrary constant and is not particular solution of equation (1). This solution is called singular solution of equation (1).*

4. Which of the following is true about solutions of differential equation $y = xy' + \sqrt{1 + y'^2}$?
   a. The general solution of equation is family of parabolas
   b. The general solution of equation is family of circles
   c. The singular solution of equation is circle
   d. The singular solution of equation is ellipse
5. The number of solution of the equation $f(x) = -1$ and the singular solution of the equation $y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$ is
   a. 1 b. 2
   c. 4 d. 0
6. The singular solution of the differential equation $y = mx + m - m^3$ where, $m = \frac{dy}{dx}$ passes through the point
   a. (0,0) b. (0,1)
   c. (1,0) d. (−1,0)

For Problems 7–9

For certain curves $y = f(x)$ satisfying $\frac{d^2y}{dx^2} = 6x - 4$, $f(x)$ has local minimum value 5 when $x = 1$.

7. Number of critical point for $y = f(x)$ for $x \in [0, 2]$
   a. 0 b. 1
   c. 2 d. 3
8. Global minimum value of $y = f(x)$ for $x \in [0, 2]$ is
   a. 5 b. 7
   c. 8 d. 9
9. Global maximum value of $y = f(x)$ for $x \in [0, 2]$ is
   a. 5 b. 7 c. 8 d. 9

For Problems 10–12

A certain radioactive material is known to decay at a rate proportional to the amount present. Initially there is 50 kg of the material present and after two hours it is observed that the material has lost 10 percent of its original mass. Based on these data answer the following questions.

10. The expression for the mass of the material remaining at any time t
    a. $N = 50e^{-(1/2)(\ln 0.9)t}$ b. $50e^{-(1/4)(\ln 9)t}$
    c. $N = 50e^{-(\ln 0.9)t}$ d. None of these

11. The mass of the material after four hours
    a. $50^{-0.5\ln 9}$ b. $50e^{-2\ln 9}$
    c. $50e^{-2\ln 0.9}$ d. None of these
12. The time at which the material has decayed to one half of its initial mass
    a. (ln 1/2) / (1/2 ln 9) hr
    b. (ln 2) / (–1/2 ln 0.9) hr
    c. (ln 1/2) / (–1/2 ln 0.9) hr
    d. None of these

For Problems 13 – 15

Consider a tank which initially holds V_0 ltr. of brine that contains a lb of salt. Another brine solution, containing b lb of salt/ltr., is poured into the tank at the rate of e ltr./min while, simultaneously, the well-stirred solution leaves the tank at the rate of f ltr./min. The problem is to find the amount of salt in the tank at any time t.

Let Q denote the amount of salt in the tank at any time. The time rate of change of Q, dQ/dt, equals the rate at which salt enters the tank minus the rate at which salt leaves the tank. Salt enters the tank at the rate of be lb/min. To determine the rate at which salt leaves the tank, we first calculate the volume of brine in the tank at any time t, which is the initial volume V_0 plus the volume of brine added et minus the volume of brine removed ft. Thus, the volume of brine at any time is

$$V_0 + et - ft \qquad \text{(a)}$$

The concentration of salt in the tank at any time is $Q/(V_0 + et - ft)$, from which it follows that salt leaves the tank at the rate of

$$f\left(\frac{Q}{V_0 + et - ft}\right) \text{ lb/min.}$$

Thus, $$\frac{dQ}{dt} = be - f\left(\frac{Q}{V_0 + et - ft}\right) \qquad \text{(b)}$$

or $$\frac{dQ}{dt} + \frac{f}{V_0 + et - ft}Q = be$$

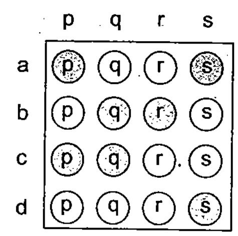
13. A tank initially holds 100 ltr. of a brine solution containing 20 lb of salt. At $t = 0$, fresh water is poured into the tank at the rate of 5 ltr./min, while the well-stirred mixture leaves the tank at the same rate. Then the amount of salt in the tank after 20 min.
    a. $20/e$ b. $10/e$
    c. $40/e^2$ d. $5/e$
14. A 50 ltr. tank initially contains 10 ltr. of fresh water. At $t = 0$, a brine solution containing 1 lb of salt per gallon is poured into the tank at the rate of 4 ltr./min, while the well-stirred mixture leaves the tank at the rate of 2 ltr./min. Then the amount of time required for overflow to occur is
    a. 30 min b. 20 min
    c. 10 min d. 40 min
15. In the above question, the amount of salt in the tank at the moment of overflow is
    a. 20 lb b. 50 lb
    c. 30 lb d. None of these

Matrix-Match Type

Solutions on page 10.48

Each question contains statements given in two columns which have to be matched.

Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct matches are a-p, a-s, b-q, r, c-p, q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:



1.

Column I	Column II : Differential equation
a. order 1	p. of all parabolas whose axis is the x-axis
b. order 2	q. of family of curves $y = a(x + a)^2$, where a is an arbitrary constant
c. degree 1	r. $\left(1 + 3\frac{dy}{dx}\right)^{2/3} = \frac{4d^3y}{dx^3}$
d. degree 3	s. of family of curve $y^2 = 2c\left(x + \sqrt{c}\right)$, where $c > 0$

2.

Column I	Column II
a. If the function $y = e^{4x} + 2e^{-x}$ is a solution of the differential equation $\frac{\frac{d^3y}{dx^3} - 13\frac{dy}{dx}}{y} = K$, then the value of $K/3$ is	p. 3

b. Number of straight lines which satisfy the differential equation $\frac{dy}{dx}+x\left(\frac{dy}{dx}\right)^2-y=0$ is	**q.** 4
c. If real value of m for which the substitution, $y=u^m$ will transform the differential equation, $2x^4y\frac{dy}{dx}+y^4=4x^6$ into a homogeneous equation, then the value of $2m$ is	**r.** 2
d. If the solution of differential equation $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=12y$ is $y=Ax^m+Bx^{-n}$, then $\lvert m+n\rvert$ is	**s.** 1

Integer Type

Solutions on page 10.49

1. If $y=y(x)$ and it follows the relation $4xe^{xy}=y+5\sin^2 x$, then $y'(0)$ is equal to

2. If $x\frac{dy}{dx}=x^2+y-2$, $y(1)=1$, then $y(2)$ equals

3. If the dependent variable y is changed to 'z' by the substitution $y=\tan z$ and the differential equation $\frac{d^2y}{dx^2}=1+\frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2$ is changed to $\frac{d^2z}{dx^2}=\cos^2 z+k\left(\frac{dz}{dx}\right)^2$, then the value of k equals

4. Let $y=y(t)$ be a solution to the differential equation $y'+2ty=t^2$, then $16\lim\limits_{t\to\infty}\frac{y}{t}$ is

5. If the solution of the differential equation $\frac{dy}{dx}=\frac{1}{x\cos y+\sin 2y}$ is $x=ce^{\sin y}-k(1+\sin y)$, then the value of k is

6. If the independent variable x is changed to y, then the differential equation $x\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3-\frac{dy}{dx}=0$ is changed to $x\frac{d^2x}{dy^2}+\left(\frac{dx}{dy}\right)^2=k$ where k equals

7. The curve passing through the point (1, 1) satisfies the differential equation $\frac{dy}{dx}+\frac{\sqrt{(x^2-1)(y^2-1)}}{xy}=0$. If the curve passes through the point $(\sqrt{2}, k)$ then the value of $[k]$ is (where [·] represents greatest integer function)

8. Tangent is drawn at the point (x_i, y_i) on the curve $y=f(x)$, which intersects the x-axis at $(x_{i+1}, 0)$. Now, again a tangent is drawn at (x_{i+1}, y_{i+1}) on the curve which intersects the x-axis at $(x_{i+2}, 0)$ and the process is repeated n times, i.e., $i=1, 2, 3, \ldots, n$. If $x_1, x_2, x_3, \ldots, x_n$ form an arithmetic progression with common difference equal to $\log_2 e$ and curve passes through (0, 2). Now if curve passes through the point (–2, k), then the value of k is

9. The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Also curve passes through the point (1, 1). Then the length of intercept of the curve on the x-axis is

10. If the eccentricity of the curve for which tangent at point P intersects the y-axis at M such that the point of tangency is equidistant from M and the origin is e, then the value of $5e^2$ is

11. If the solution of the differential equation $\frac{dy}{dx}-y=1-e^{-x}$ and $y(0)=y_0$ has a finite value, when $x\to\infty$, then the value of $\lvert 2/y_0\rvert$ is

Archives

Solutions on page 10.52

Subjective

1. A normal is drawn at a point $P(x, y)$ of a curve. It meets the x-axis at Q. If PQ has constant length k, then show that the differential equation describing such curves is $y\frac{dy}{dx}=\pm\sqrt{k^2-y^2}$. Find the equation of such a curve passing through $(0, k)$. **(IIT-JEE, 1994)**

2. Let $y=f(x)$ be a curve passing through (1, 1) such that the triangle formed by the coordinate axes and the tangent at any point of the curve lies in the first quadrant and has area 2. Form the differential equation and determine all such possible curves. **(IIT-JEE, 1995)**

3. Determine the equation of the curve passing through the origin, in the form $y=f(x)$, which satisfies the differential equation $\frac{dy}{dx}=\sin(10x+6y)$. **(IIT-JEE, 1996)**

4. A and B are two separate reservoirs of water. Capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled completely with water, their inlets are closed and then the water is released simultaneously form both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at the time. One hour after the water is released, the quantity of water is reservoir A is $1\frac{1}{2}$ times the quantity of water in reservoir B. After how many hours do both the reservoirs have the same quantity of water ? **(IIT-JEE, 1997)**

5. Let $u(x)$ and $v(x)$ satisfy the differential equation $\frac{du}{dx} + p(x)\, u = f(x)$ and $\frac{dv}{dx} + p(x)v = g(x)$ are continuous functions. If $u(x_1)$ for some x_1 and $f(x) > g(x)$ for all $x > x_1$, prove that any point (x, y) where $x > x_1$, does not satisfy the equations $y = u(x)$ and $y = v(x)$. **(IIT-JEE, 1997)**
6. *A* curve *C* has the property that if the tangent drawn at any point *P* on *C* meets the co-ordinate axis at *A* and *B*, then *P* is the mid–point of *AB*. The curve passes through the point (1, 1). Determine the equation of the curve. **(IIT-JEE, 1998)**
7. A curve passing through the point (1, 1) has the property that the perpendicular distance of the origin form the normal at any point *P* of the curve is equal to the distance of *P* form the *x*-axis. Determine the equation of the curve. **(IIT-JEE, 1999)**
8. A country has a food deficit of 10%. Its population grows continuously at a rate of 3% per year. Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self-sufficient in food after *n* years, where *n* is the smallest integer bigger than or equal to $\frac{\ln 10 - \ln 9}{\ln (1.04) - 0.03}$. **(IIT-JEE, 2000)**
9. Let $f(x)$, $x \geq 0$, be a non-negative continuous function, and let $F(x) = \int_0^x f(t)\, dt$, $x \geq 0$, if for some $c > 0$, $f(x) \leq c F(x)$ for all $x \geq 0$, then show that $f(x) = 0$ for all $x \geq 0$. **(IIT-JEE, 2001)**
10. A hemi-spherical tank of radius 2 m is initially full of water and has an outlet of 12 cm^2 cross-sectional area at the bottom. The outlet is opened at some instant. The flow through the outlet is according to the law $v(t) = 0.6\sqrt{2gh(t)}$, where $v(t)$ and $h(t)$ are, respectively, the velocity of the flow through the outlet and the height of water level above the outlet at time *t*, and *g* is the acceleration due to gravity. Find the time it takes to empty the tank. (Hint : Form a differential equation by relating the decrease of water level to the outflow).
11. A right circular cone with radius *R* and height *H* contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant $= k > 0$). Find the time after which the cone is empty. **(IIT-JEE, 2003)**
12. A curve *C* passes through (2, 0) and the slope at (x, y) as $\frac{(x+1)^2 + (y-3)}{x+1}$. Find the equation of the curve. Find the area bounded by curve and *x*-axis in fourth quadrant. **(IIT-JEE, 2004)**
13. If length of tangent at any point on the curve $y = f(x)$ intercepted between the point and the *x*-axis is of length *l*. Find the equation of the curve. **(IIT-JEE, 2005)**

Objective

Fill in the blanks

1. A spherical rain drop evaporates at a rate proportional to its surface area at any instant *t*. The differential equation giving the rate of change of the radius of the rain drop is __________. **(IIT-JEE, 1999)**

Multiple choice questions with one correct answer

1. A curve $y = f(x)$ passes through the point *P* (1, 1). The normal to the curve at *P* is a $(y-1) + (x-1) = 0$. If the slope of the tangent at any point on the curve is proportional to the ordinate of the point, then the equation of the curve is
   a. $y = e^{K(x-1)}$ b. $y = e^{Ke}$
   c. $y = e^{K(x+2)}$ d. None of these **(IIT-JEE, 1996)**
2. The solution of the differential equation $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$ is
   a. $y = 2$ b. $y = 2x$
   c. $y = 2x - 4$ d. $y = 2x^2 - 4$ **(IIT-JEE, 1999)**
3. If $y(t)$ is a solution of $(1+t)\frac{dy}{dt} - ty = 1$ and $y(0) = -1$, then $y(1)$ is
   a. $-1/2$ b. $e + 1/2$
   c. $e - 1/2$ d. $1/2$ **(IIT-JEE, 2003)**
4. If $y = y(x)$ and $\frac{2 + \sin x}{y+1}\left(\frac{dy}{dx}\right) = -\cos x$, $y(0) = 1$, then $y(\pi/2) =$
   a. $1/3$ b. $2/3$
   c. $-1/3$ d. 1 **(IIT-JEE, 2004)**
5. The solution of the primitive integral equation $(x^2 + y^2)\,dy = xy\, dx$ is $y = y(x)$. If $y(1) = 1$ and $y(x_0) = e$, then x_0 is
   a. $\sqrt{2(e^2 - 1)}$ b. $\sqrt{2(e^2 + 1)}$
   c. $\sqrt{3}\, e$ d. $\sqrt{\frac{e^2 + 1}{2}}$ **(IIT-JEE, 2005)**
6. For the primitive integral equation $y dx + y^2 dy = x\, dy$; $x \in R$, $y > 0$, $y(1) = 1$, then $y(-3)$ is
   a. 3 b. 2
   c. 1 d. 5 **(IIT-JEE, 2005)**
7. The differential equation $\frac{dy}{dx} = \frac{\sqrt{1 - y^2}}{y}$ determines a family of circles with
   a. Variable radii and a fixed centre at (0, 1).
   b. Variable radii and a fixed centre at (0, – 1).
   c. Fixed radius 1 and variable centres along the *x*-axis.
   d. Fixed radius 1 and variable centres along the *y*-axis.

Multiple choice questions with one or more than one correct answers

1. The order of the differential equation whose general solution is given by $y = (C_1 + C_2)\cos(x + C_3) - C_4 e^{x + C_5}$,

where C_1, C_2, C_3, C_4, C_5, are arbitrary constants, is

a. 5 b. 4
c. 3 d. 2 **(IIT-JEE, 1998)**

2. The differential equation representing the family of curves $y^2 = 2c\left(x + \sqrt{c}\right)$, where c is a positive parameter, is of
   a. order 1
   b. order 2
   c. degree 3
   d. degree 4 **(IIT-JEE, 1999)**

3. A curve $y = f(x)$ passes through (1, 1) and tangent at $P(x, y)$ cuts the x-axis and y-axis at A and B respectively such that $BP : AP = 3 : 1$, then
   a. equation of curve is $xy' - 3y = 0$
   b. normal at (1, 1) is $x + 3y = 4$
   c. curve passes through (2, 1/8)
   d. equation of curve is $xy' + 3y = 0$ **(IIT-JEE, 2006)**

Integer type

1. Let f be a real-valued differentiable function on R (the set of all real numbers) such that $f(1) = 1$. If the y-intercept of the tangent at any point $P(x, y)$ on the curve $y = f(x)$ is equal to the cube of the abscissa of P, then the value of f (– 3) is equal to **(IIT-JEE, 2010)**

2. Let $y'(x) + y(x)g'(x) = g(x)g'(x), y(0) = 0, x \in R$, where $f'(x)$ denotes $\frac{df(x)}{dx}$, and $g(x)$ is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. Then the value of $y(2)$ is **(IIT-JEE, 2011)**

3. Let $f: [1, \infty)$ be a differentiable function such that $f(1) = 2$. If $6\int_1^x f(t)\,dt = 3xf(x) - x^3$ for all $x \geq 1$, then the value of $f(2)$ is **(IIT-JEE, 2011)**

ANSWERS AND SOLUTIONS

Subjective Type

1. $$\frac{x + y\frac{dy}{dx}}{y - x\frac{dy}{dx}} = x^2 + 2y^2 + \frac{y^4}{x^2}$$

$$\Rightarrow \frac{xdx + ydy}{(x^2 + y^2)^2} = \frac{ydx - xdy}{y^2}\frac{y^2}{x^2}$$

$$\Rightarrow \int\frac{d(x^2 + y^2)}{(x^2 + y^2)^2} = 2\int\frac{1}{x^2/y^2}d\left(\frac{x}{y}\right)$$

Integrating both sides, we get

$$-\frac{1}{(x^2 + y^2)} = \frac{-2}{x/y} + c$$

$$\Rightarrow \frac{2y}{x} - \frac{1}{(x^2 + y^2)} = c$$

2. Given $\left(1 + e^{x/y}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

$$\Rightarrow \left(1 + e^{x/y}\right)\frac{dx}{dy} + e^{x/y}\left(1 - \frac{x}{y}\right) = 0$$

Put $x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$

$$\left(1 + e^v\right)\left(v + y\frac{dv}{dy}\right) + e^v(1 - v) = 0$$

$$\Rightarrow \frac{dy}{y} = \frac{-(1 + e^v)}{(v + e^v)}dv$$

Integrating, we get

$\ln y = -\ln(v + e^v) + \ln c$

$$\Rightarrow \ln\left[y\left(\frac{x}{y} + e^{\frac{x}{y}}\right)\right] = \ln c$$

$$\Rightarrow x + y\,e^{x/y} = c$$

3. $$\frac{dy}{dx} = \frac{(x + y)^2}{(x + 2)(y - 2)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x + 2 + y - 2)^2}{(x + 2)(y - 2)}$$

On putting $X = x + 2$ and $Y = y - 2$, the given differential equation reduces to

$$\frac{dY}{dX} = \frac{(X + Y)^2}{XY}$$

put $Y = VX \Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$

$$\Rightarrow V + X\frac{dV}{dX} = \frac{(1 + V)^2}{V}$$

$$\Rightarrow \frac{V}{2V + 1}dV = \frac{dX}{X}$$

$$\Rightarrow \int\left(1 - \frac{1}{1 + 2V}\right)dV = 2\int\frac{dX}{X}$$

$$\Rightarrow V - \frac{1}{2}\ln(1 + 2V) = 2\ln X + C$$

$$\Rightarrow \quad X^4\left(1+\frac{2Y}{X}\right)=Ce^{2Y/X}$$

where $X=x+2$, and $Y=y-2$

4. $y\left(\frac{dy}{dx}\right)^2+2x\frac{dy}{dx}-y=0$

Solving quadratic in $\frac{dy}{dx}$, we get

$$\frac{dy}{dx}=\frac{-2x\pm\sqrt{4x^2+4y^2}}{2y}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-x\pm\sqrt{x^2+y^2}}{y}$$ which is homogeneous.

Put $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

Then given equation transforms to

$$v+x\frac{dv}{dx}=\frac{-1\pm\sqrt{1+v^2}}{v}$$

$$\Rightarrow \quad v^2+xv\frac{dv}{dx}=-1\pm\sqrt{1+v^2}$$

$$\Rightarrow \quad (v^2+1)\pm\sqrt{1+v^2}=xv\frac{dv}{dx}$$

$$\Rightarrow \quad \int\frac{vdv}{(1+v^2)\pm\sqrt{v^2+1}}=\int\frac{dx}{x}$$

$$\Rightarrow \quad \int\frac{vdv}{\sqrt{v^2+1}\left(\sqrt{1+v^2}\pm 1\right)}=-\int\frac{dx}{x} \qquad (1)$$

Put $\sqrt{1+v^2}\pm 1=t \quad \Rightarrow \quad \frac{v}{\sqrt{1+v^2}}dv=dt$

Then equation (1) transforms to $\int\frac{dt}{t}=-\int\frac{dx}{x}$

$\Rightarrow \quad \ln t=-\ln x+\ln c$

$\Rightarrow \quad tx=c$

$$\Rightarrow \quad \left(\sqrt{1+v^2}\pm 1\right)x=c$$

$$\Rightarrow \quad \sqrt{x^2+y^2}\pm x=c$$

Given when $x=0; y=\sqrt{5}$

$\Rightarrow \quad [\sqrt{5}-0]=c \Rightarrow c=\sqrt{5}$

$\therefore \quad \sqrt{x^2+y^2}=\sqrt{5}\pm x$

$\Rightarrow \quad x^2+y^2=5+x^2\pm 2\sqrt{5}\,x$

$\Rightarrow \quad y^2=5\pm 2\sqrt{5}\,x$

5. The given differential equation is

$$y+\frac{d}{dx}(xy)=x(\sin x+\log x)$$

i.e., $x\frac{dy}{dx}+2y=x(\sin x+\log x)$

or $\quad \frac{dy}{dx}+\frac{2}{x}y=\sin x+\log x \qquad (1)$

This is a linear different equation

I.F. $=e^{2\int\frac{1}{x}dx}=e^{2\log x}=x^2 \qquad (2)$

$\therefore$ solution is given by

$$yx^2=\int x^2(\sin x+\log x)\,dx+c$$

$$=-x^2\cos x+2x\sin x+2\cos x+\frac{x^3}{3}\log x-\frac{x^3}{9}+c$$

or $y=-\cos x+\frac{2}{x}\sin x+\frac{2}{x^2}\cos x+\frac{x}{3}\log x-\frac{x}{3}+\frac{c}{x^2}$.

6. $\int_a^x t\,y(t)dt=x^2+y(x)$

Differentiating both sides w.r.t. x, we get

$xy(x)=2x+y'(x)$

hence $\frac{dy}{dx}-xy=-2x$ (linear)

I.F. $=e^{\int -x\,dx}=e^{-x^2/2}$

$\Rightarrow \quad$ solution is $ye^{-x^2/2}=\int -2xe^{-x^2/2}dx$

$\Rightarrow \quad y=2+ce^{-x^2/2}$

if $x=a \Rightarrow a^2+y=0 \quad \Rightarrow y=-a^2$

hence $-a^2=2+ce^{-a^2/2}$

$\Rightarrow \quad ce^{-a^2/2}=-(2+a^2)$

$\Rightarrow \quad c=-(2+a^2)\,e^{a^2/2}$

$\Rightarrow \quad y=2-(2+a^2)\,e^{(a^2-x^2)/2}$

7. Put $x=0; y=0 \Rightarrow g(0)=0$

and $g'(0)=\lim_{h\to 0}\frac{g(h)}{h}=2$

Now $g'(x) = \lim_{h \to 0} \dfrac{g(x+h) - g(x)}{h}$

$$= \lim_{h \to 0} \frac{e^h g(x) + e^x g(h) - g(x)}{h}$$

$$= g(x) \lim_{h \to 0}\left(\frac{e^h - 1}{h}\right) + e^x \lim_{h \to 0} \frac{g(h)}{h}$$

$\Rightarrow \quad g'(x) = g(x) + 2e^x$

Let $g(x) = y$ then $\dfrac{dy}{dx} = y + 2e^x$

$\Rightarrow \quad e^{-x}\dfrac{dy}{dx} - y\,e^{-x} = 2$

$\Rightarrow \quad \dfrac{d}{dx}(y\,e^{-x}) = 2$

$\Rightarrow \quad ye^{-x} = 2x + c$

Given if $x = 0, y = 0 \Rightarrow c = 0$

Then $ye^{-x} = 2x \Rightarrow y = 2xe^x$

Now $\dfrac{dy}{dx} = 2\,[e^x + x\,e^x] = 0 \Rightarrow x = -1$

$\Rightarrow$ minima at $x = -1$

$\Rightarrow$ range is $\left[-\dfrac{2}{e}, \infty\right)$

8. $\dfrac{d}{dx}(xf(x)) \le -kf(x)$

$\Rightarrow \quad xf'(x) + f(x) \le -kf(x)$

$\Rightarrow \quad xf'(x) + (k+1)f(x) \le 0$

$\Rightarrow \quad x^{k+1}f'(x) + (k+1)x^k f(x) \le 0$

$\Rightarrow \quad \dfrac{d\,[x^{k+1}f(x)]}{dx} \le 0$

Let $F(x) = x^{k+1}f(x)$

$F(x)$ is decreasing for $x \ge 2$

$\Rightarrow \quad F(x) \le F(2)$ for all $x \ge 2$

$\Rightarrow \quad F(x) \le (A)$

$\Rightarrow \quad x^{k+1}f(x) \le A$

$\Rightarrow \quad f(x) \le A\,x^{k-1}$

9. Given, $\dfrac{dy_1}{dx} + Py_1 = Q$ (1)

$\dfrac{dy_2}{dx} + Py_2 = Q$ (2)

Clearly, we have to eliminate P and y_2.

In equation (2), put $y_2 = zy_1$

$\Rightarrow \quad \dfrac{d(zy_1)}{dx} + Pzy_1 = Q$

$\Rightarrow \quad y_1\dfrac{dz}{dx} + z\dfrac{dy_1}{dx} + Pzy_1 = Q$

From equation (1) put the value of Py_1

We have $y_1\dfrac{dz}{dx} + z\dfrac{dy_1}{dx} + z\left(Q - \dfrac{dy_1}{dx}\right) = Q$

$\Rightarrow \quad y_1\dfrac{dz}{dx} = Q(1-z)$

$\Rightarrow \quad \displaystyle\int \frac{dz}{z-1} = -\int \frac{Q}{y_1}dx$

$\Rightarrow \quad \log(z-1) = -\displaystyle\int \frac{Q}{y_1}dx + \log c$

$\Rightarrow \quad \log\dfrac{z-1}{c} = -\displaystyle\int \frac{Q}{y_1}dx$

$\Rightarrow \quad z = 1 + c\,e^{\int -\frac{Q}{y_1}dx}$

10. y_1, y_2 are the solutions of the differential equation

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad (1)$$

then $\dfrac{dy_1}{dx} + P(x)y_1 = Q(x)$ (2)

and $\dfrac{dy_2}{dx} + P(x)y_2 = Q(x)$ (3)

From equations (1) and (2), we get

$$\frac{d\,(y - y_1)}{dx} + P(x)\,(y - y_1) = 0 \qquad (4)$$

And from equations (2) and (3), we get

$$\frac{d\,(y_1 - y_2)}{dx} + P(x)\,(y_1 - y_2) = 0 \qquad (5)$$

Also from equations (4) and (5), we get

$$\frac{\frac{d}{dx}(y - y_1)}{\frac{d}{dx}(y_1 - y_2)} = \frac{(y - y_1)}{(y_1 - y_2)}$$

$\Rightarrow \quad \displaystyle\int \frac{d(y - y_1)}{(y - y_1)} = \int \frac{d(y_1 - y_2)}{(y_1 - y_2)}$

$\Rightarrow \quad \ln(y - y_1) = \ln(y_1 - y_2) + \ln c$

$\Rightarrow \quad \ln(y - y_1) = \ln(c(y_1 - y_2))$

$\Rightarrow \quad y - y_1 = c\,(y_1 - y_2)$

$\Rightarrow \quad y = y_1 + c\,(y_1 - y_2)$

11. Let the curve be $y = f_1(x)$ and $y = f_2(x)$ equation of tangents with equal abscissa, x are

$Y - f_1(x) = f'(x)\,(X - x)$ and $Y - f_2(x) = f'(x)(X - x)$

These tangent intersect at y-axis,

so their Y-intercept are same.

$\Rightarrow -xf'_1(x)+f_1(x)=-xf'_2(x)+f_2(x)$

$\Rightarrow f_1(x)-f_2(x)=x(f'_1(x)-f'_2(x))$

$$\Rightarrow \int \frac{f_1'(x)-f_2'(x)}{f_1(x)-f_2(x)} = \int \frac{dx}{x}$$

$\Rightarrow \ln|f_1(x)-f_2(x)| = \ln|x| + \ln C_1$

$\Rightarrow f_1(x)-f_2(x) = \pm C_1 x$ (1)

Now equation of normal with equal abscissa x are

$$(Y-f_1(x)) = -\frac{1}{f_1'(x)}(X-x), \text{ and}$$

$$(Y-f_2(x)) = -\frac{1}{f_2'(x)}(X-x)$$

As these normal intersect on the x-axis

$\Rightarrow x+f_1(x)f_1'(x) = x+f_2(x).f_2'(x)$

$\Rightarrow f_1(x)f_1'(x)-f_2(x)f_2'(x)=0$

Integrating, we get $f_1^2(x)-f_1^2(x)=C_2$

$$\Rightarrow f_1(x)+f_2(x) = \frac{C_2}{f_1(x)-f_2(x)}$$

$$= \pm\frac{C_2}{C_1 x} = \frac{\pm\lambda_2}{x} \quad (2)$$

From equations (1) and (2), we get $2f_1(x) = \pm\left(\frac{\lambda_2}{x}+C_1 x\right)$,

$$2f_2(x) = \pm\left(\frac{\lambda_2}{x}-C_1 x\right)$$

we have $f_1(1)=1$ and $f(2)=3$

$$\Rightarrow f_1(x) = \frac{2}{x}-x,\ f_2(x) = \frac{2}{x}+x$$

12. Equation of tangent to the curve $y=f(x)$ is
$Y-y=f'(x)(X-x)$

Equation of tangents to the curve $g(x) = y_1 = \int_{-\infty}^{x} f(t)dt$

is $Y-y_1 = f(x)(X-x)$ $\quad \left(\frac{dy_1}{dx} = g'(x) = f(x)\right)$

Since the tangent with equal abscissas intersect on the x-axis

$$\Rightarrow x - \frac{y}{f'(x)} = x - \frac{y_1}{f(x)}$$

$$\Rightarrow \frac{f(x)}{f'(x)} = \frac{y_1}{f(x)}$$

$$\Rightarrow \frac{g'(x)}{g(x)} = \frac{g''(x)}{g'(x)}$$

$\Rightarrow \ln g(x) = \ln c\, g'(x)$

$\Rightarrow g(x) = c\, g'(x)$

$$\Rightarrow \frac{g'(x)}{g(x)} = c$$

$\Rightarrow g(x) = ke^{cx}$

$\Rightarrow f(x) = g'(x) = kce^{cx}$

The curve $y=f(x)$ passes through $(0, 1) \Rightarrow kc=1$

The curve $y=g(x)$ passes through $\left(0, \frac{1}{n}\right)$

$$\Rightarrow k = \frac{1}{n} \Rightarrow c = n \Rightarrow f(x) = e^{nx}$$

13. Let the cyclist starting to move from the point O and moving along OX, attain a velocity v at point P in time t such that $OP=x$. Let the acceleration of the moving cycle at P be a. Then we know that

$$v = \frac{dx}{dt} \text{ and } a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx} \quad (1)$$

By hypothesis, retardation $= 0.08 + 0.02\, v^2 = 0.02\,(4+v^2)$

$$\Rightarrow v\frac{dv}{dx} = -0.02(4+v^2) \text{ or}$$

$$dx = -\frac{1}{0.02}\frac{v\,dv}{4+v^2} \quad (2)$$

Integrating equation (2) between the limits $x=0$; $v=4$ m/s and $x=x'$ meters, $v=0$, we get

$$\int_0^{x'} dx = -\frac{1}{0.04}\int_4^0 \frac{2v\,dv}{4+v^2}$$

$$\text{or } x' = -\frac{1}{0.04}[\ln(4+v^2)]_4^0$$

$$x' = -\frac{1}{0.04}[\ln 4 - \ln 20]$$

$$= \frac{\ln 5}{0.04} = \frac{1.61}{0.04} = \frac{161}{4} \text{ m}$$

14. The resistance force opposing the

motion $= m \times$ acceleration $= m\,\frac{dv}{dt}$

Hence differential equation is $m\,\frac{dv}{dt} = -kv$

$$\Rightarrow \frac{dv}{v} = -\frac{k}{m}\,dt$$

Integrating, we get $\ln v = -\frac{k}{m}t + c$

at $t=0$, $v=v_0$. Hence $c = \ln v_0$

$$\therefore \ln\frac{v}{v_0} = -\frac{k}{m}t \Rightarrow v = v_0\, e^{-\frac{k}{m}t} \quad (1)$$

where v is the velocity at time t

now $\frac{ds}{dt} = v_0\, e^{-\frac{k}{m}t}$

$\Rightarrow \quad ds = v_0 e^{-\frac{k}{m}t}\, dt$

Boat's position at time t is,

$$s(t) = -\frac{v_0 m}{k} e^{-\frac{kt}{m}} + c$$

if $\quad t = 0, s = 0 \quad \Rightarrow c = \frac{v_0 m}{k}$

$$\therefore \quad s(t) = \frac{v_0 m}{k}\left[1 - e^{-\frac{kt}{m}}\right] \quad (2)$$

To find how far the boat goes,

we have to find $\lim_{t \to \infty} s(t) = \frac{m v_0}{k}$.

Objective Type

1. a. Putting $x = \sin A$ and $y = \sin B$ in the given relation, we get $\cos A + \cos B = a(\sin A - \sin B)$

$\Rightarrow A - B = 2\cot^{-1} a$

$\Rightarrow \sin^{-1} x - \sin^{-1} y = 2\cot^{-1} a$

Differentiating w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

Clearly, it is a differential equation of degree one.

2. d. $Ax^2 + By^2 = 1$ (1)

Differentiating w.r.t. x, we get

$$2Ax + 2By\frac{dy}{dx} = 0 \Rightarrow Ax + By\frac{dy}{dx} = 0 \quad (2)$$

Again diff. $A + By\frac{d^2y}{dx^2} + B\left(\frac{dy}{dx}\right)^2 = 0$ (3)

From equations (2) and (3), we get

$$x\left[-By\frac{d^2y}{dx^2} - B\left(\frac{dy}{dx}\right)^2\right] + By\frac{dy}{dx} = 0$$

$$\Rightarrow xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

$\therefore$ order = 2 and degree = 1

3. a. $y = e^x(A\cos x + B\sin x)$

$$\frac{dy}{dx} = e^x[-A\sin x + B\cos x] + e^x[A\cos x + B\sin x]$$

$$\frac{dy}{dx} = e^x[-A\sin x + B\cos x] + y \quad (1)$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = e^x[-A\sin x + B\cos x] + e^x[-A\cos x - B\sin x] + \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx} - y\right) - y + \frac{dy}{dx} \quad \text{[using (1)]}$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

4. d. Equation of circle will be $x^2 + (y-2)^2 + \lambda(y-2) = 0$

Differentiating, we get $2x + 2(y-2)\frac{dy}{dx} + \lambda\frac{dy}{dx} = 0$

$\therefore$ the equation is $x^2 + (y-2)^2 - (y-2)\left(2x\frac{dx}{dy} + 2y - 4\right) = 0$

5. a. The equation of a member of the family of parabolas having axis parallel to y-axis is

$y = Ax^2 + Bx + C$ (1)

where A, B, and C are arbitrary constants

Differentiating equation (1) w.r.t. x, we get $\frac{dy}{dx} = 2Ax + B$ (2)

which on again differentiating w.r.t. x gives $\frac{d^2y}{dx^2} = 2A$ (3)

Differentiating (3) w.r.t. x, we get $\frac{d^3y}{dx^3} = 0$

6. c. Differentiating the given equation successively, we get

$y_1 = 5be^{5x} - 7ce^{-7x}$ (1)

$y_2 = 25be^{5x} + 49ce^{-7x}$ (2)

$y_3 = 125be^{5x} - 343ce^{-7x}$ (3)

Multiplying equation (1) by 7 and then adding to equation (2),

we get $y_2 + 7y_1 = 60be^{5x}$ (4)

Multiplying equation (1) by 5 and then subtracting it from equation (2),

we get $y_2 - 5y_1 = 84ce^{-7x}$ (5)

Putting the values of b and c, obtained from equation (4) and (5), respectively, in equation (1), we get

$y_3 + 2y_2 - 35y_1 = 0$

7. a. If $(0, k)$ be the centre on y-axis then its radius will be k as it passes through origin. Hence its equation is

$x^2 + (y-k)^2 = k^2$

or $\quad x^2 + y^2 = 2ky$ (1)

$$\therefore \quad 2x + 2y\frac{dy}{dx} = 2k\frac{dy}{dx}$$

$$= \frac{x^2+y^2}{y}\frac{dy}{dx} \quad \text{[by (1)]}$$

$$\therefore\ 2xy = (x^2+y^2-2y^2)\frac{dy}{dx}$$

$$\text{or } (x^2-y^2)\frac{dy}{dx} - 2xy = 0$$

8. c. $ax^2+by^2=1$

Differentiating w.r.t. x, we get

$$2ax + 2byy_1 = 0$$

$$\Rightarrow ax + byy_1 = 0 \Rightarrow \frac{-a}{b} = \frac{yy_1}{x} \qquad (1)$$

Again differentiating w.r.t. x, we get

$$\Rightarrow a + by_1^2 + byy_2 = 0 \Rightarrow \frac{-a}{b} = y_1^2 + yy_2 \qquad (2)$$

From equations (1) and (2), we get

$$\frac{yy_1}{x} = y_1^2 + yy_2$$

$$\Rightarrow yy_1 = xy_1^2 + xyy_2$$

9. c. The given family of curve is $x^2+y^2-2ay=0$ (1)

Differentiating w.r.t. x, we get $2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0$

$$\Rightarrow 2x + 2y\frac{dy}{dx} - \frac{x^2+y^2}{y}\frac{dy}{dx} = 0 \quad \text{[Using equation (1)]}$$

$$\Rightarrow 2xy + (2y^2-x^2-y^2)y' = 0$$

$$\Rightarrow (y^2-x^2)y' + 2xy = 0$$

$$\Rightarrow (x^2-y^2)y' = 2xy$$

10. c. We have $\frac{dy}{dx} = (e^y - x)^{-1} \Rightarrow \frac{dx}{dy} = e^y - x$

$$\Rightarrow \frac{dx}{dy} + x = e^y;$$

So I.F. $= e^{\int dy} = e^y$

$\therefore$ General solution is given by $xe^y = \frac{1}{2}e^{2y} + C$

$$\Rightarrow x = \frac{e^y}{2} + Ce^{-y}$$

As $y(0)=0$, so $C = \frac{-1}{2}$

$$\therefore\ x = \frac{e^y}{2} - \frac{1}{2}e^{-y}$$

$$\Rightarrow e^y - e^{-y} = 2x$$

$$\Rightarrow e^2y - 2xe^y - 1 = 0$$

$$\Rightarrow 2e^y = 2x \pm \sqrt{4x^2+4}$$

But $e^y = x - \sqrt{x^2+1}$. (Rejected)

Hence $y = \ln\left(x + \sqrt{x^2+1}\right)$

11. d. $\log c + \log|x| = \frac{x}{y}$

differentiating w.r.t. x, $\frac{1}{x} = \frac{y - x\frac{dy}{dx}}{y^2}$

$$\Rightarrow \frac{y^2}{x} = y - x\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \frac{y^2}{x^2}$$

$$\Rightarrow \phi\left(\frac{x}{y}\right) = -\frac{y^2}{x^2}$$

12. b. $y = c_1\cos(x+c_2) - (c_3e^{-x+c_4}) + (c_5\sin x)$

$$\Rightarrow y = c_1(\cos x\ \cos c_2 - \sin x\ \sin c_2) - (c_3e^{c_4}e^{-x}) + (c_5\sin x)$$

$$\Rightarrow y = (c_1\cos c_2)\cos x - (c_1\sin c_2 - c_5)\sin x - (c_3e^{c_4})e^{-x}$$

$$\Rightarrow y = l\cos x + m\sin x - ne^{-x} \qquad (1)$$

where l, m, n are arbitrary constant

$$\Rightarrow \frac{dy}{dx} = -l\sin x + m\cos x + ne^{-x} \qquad (2)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -l\cos x - m\sin x - ne^{-x} \qquad (3)$$

$$\Rightarrow \frac{d^3y}{dx^3} = l\sin x - m\cos x + ne^{-x} \qquad (4)$$

From equations (1) + (3), $\frac{d^2y}{dx^2} + y = -2ne^{-x}$ (5)

From equations (2) + (4), $\frac{d^3y}{dx^3} + \frac{dy}{dx} = 2ne^{-x}$ (6)

From equations (5) + (6), we get $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

13. a. $x\frac{dy}{dx} + y(\log y) = 0$

$$\Rightarrow \int \frac{dx}{x} + \int \frac{dy}{y(\log y)} = c$$

$\Rightarrow \log x + \log(\log y) = \log c$

$\Rightarrow x \log y = c$

$y(1) = e \Rightarrow c = 1$

Hence, the equation of the curve is $x \log y = 1$

4. a. $\frac{1}{y+1} dy = -\frac{\cos x}{2+\sin x} dx$

Integrating, we get

$\log(y+1) + \log k + \log(2+\sin x) = 0$

$\therefore \; k(y+1)(2+\sin x) = 1$ when $x = 0, y = 1$ where k is constant.

$\therefore \; 4k = 1$ or $k = 1/4$

$\therefore \; (y+1)(2+\sin x) = 4$

Now put $x = \pi/2 \quad \therefore (y+1)3 = 4$

$\therefore \; y = \frac{1}{3}$

15. a. Slope $= \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y-1}{x^2+x}$

$$\Rightarrow \frac{dy}{y-1} = \frac{dx}{x^2+x}$$

$$\Rightarrow \int \frac{1}{y-1} dy = \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx + C$$

$$\Rightarrow \frac{(y-1)(x+1)}{x} = k$$

Putting $x = 1, y = 0$, we get $k = -2$

The equation is $(y-1)(x+1) + 2x = 0$

16. d. $(y\cos y + \sin y)\, dy = (2x\log x + x)\, dx$

$y \sin y - \int \sin y\, dy + \int \sin y\, dx$

$= x^2 \log x - \int x^2 \frac{1}{x} dx + \int x\, dx + c$

$\therefore \; y\sin y = x^2 \log x + c$

17. b. $\frac{dy}{dx} = e^{ax+by} = e^{ax} e^{by}$

or $e^{-by} dy = e^{ax} dx$

$\therefore \; -\frac{1}{b} e^{-by} = \frac{1}{a} e^{ax} + c$

18. a. $x^2(y+1)\, dx + y^2(x-1) dy = 0$

$$\Rightarrow \frac{x^2 dx}{x-1} = -\frac{y^2 dy}{y+1}$$

$$\Rightarrow \int \left[x+1+\frac{1}{x-1}\right] dx = -\int \left[y-1+\frac{1}{y+1}\right] dy$$

$$\Rightarrow \frac{x^2}{2} + x + \ln(x-1) = -\left[\frac{y^2}{2} - y + \ln(y+1)\right] + \ln c$$

$$\Rightarrow \frac{x^2+y^2}{2} + (x-y) + \ln\left(\frac{(x-1)(y+1)}{c}\right) = 0$$

19. a. $dy - \sin x \sin y\, dx = 0$

$\Rightarrow dy = \sin x \sin y\, dx$

$$\Rightarrow \int \operatorname{cosec} y dy = \int \sin x dx$$

$$\Rightarrow \log \tan \frac{y}{2} = -\cos x + \log c$$

$$\Rightarrow \log \frac{\tan \frac{y}{2}}{c} = -\cos x$$

$$\Rightarrow \frac{\tan \frac{y}{2}}{c} = e^{-\cos x}$$

$$\Rightarrow e^{\cos x} \tan \frac{y}{2} = c$$

20. a. $\frac{dv}{dt} + \frac{k}{m} v = -g$

$$\Rightarrow \frac{dv}{dt} = -\frac{k}{m}\left(v + \frac{mg}{k}\right)$$

$$\Rightarrow \frac{dv}{v + mg/k} = -\frac{k}{m} dt$$

$$\Rightarrow \log\left(v + \frac{mg}{k}\right) = -\frac{k}{m} t + \log c$$

$$\Rightarrow v + \frac{mg}{k} = ce^{-k/mt}$$

$$\Rightarrow v = ce^{-\frac{k}{m}t} - \frac{mg}{k}$$

21. b. Putting $u = x - y$, we get $du/dx = 1 - dy/dx$. The given equation can be written as $1 - du/dx = \cos u$

$\Rightarrow (1 - \cos u) = du/dx$

$$\Rightarrow \int \frac{du}{1-\cos u} = \int dx + C$$

$$\Rightarrow \frac{1}{2} \int \operatorname{cosec}^2 (u/2)\, du = \int dx + C$$

$\Rightarrow x + \cot(u/2) = c$

$\Rightarrow x+\cot\dfrac{x-y}{2}=C$

22. a. $\dfrac{dy}{dx}+2xy=y$

$\Rightarrow \dfrac{dy}{dx}=y(1-2x)$

$\Rightarrow \dfrac{dy}{y}=(1-2x)\,dx$

$\Rightarrow \log y=x-x^2+c_1$

$\Rightarrow y=e^{x-x^2}\,e^{c_1}=c\,e^{x-x^2}$ where $c=e^{c_1}$

$\Rightarrow y=ce^{x-x^2}$ is the required solution.

23. b. We have $\dfrac{dy}{dx}=\sin\dfrac{x-y}{2}-\sin\dfrac{x+y}{2}$

$$=-2\cos\frac{x}{2}\sin\frac{y}{2}$$

$\Rightarrow \log\tan\dfrac{y}{4}=-\dfrac{\sin\frac{x}{2}}{\frac{1}{2}}+c$

$\Rightarrow \log\tan\left(\dfrac{y}{4}\right)=c-2\sin\dfrac{x}{2}$

24. a. Putting $x+y+1=u$, we have $du=dx+dy$ and the given equations reduces to

$u(du-dx)=dx$

$\Rightarrow \dfrac{u\,du}{u+1}=dx$

$\Rightarrow u-\log(u+1)=x+C$

$\Rightarrow \log(x+y+2)=y+C$

$\Rightarrow x+y+2=Ce^{y}$

25. a. $x^2\dfrac{dy}{dx}-xy=1+\cos\dfrac{y}{x}$

$\Rightarrow \dfrac{x(xdy-ydx)}{dx}=1+\cos\dfrac{y}{x}$

$\Rightarrow \dfrac{\frac{xdy-ydx}{x^2}}{1+\cos\frac{y}{x}}=\dfrac{dx}{x^3}$

$\Rightarrow \displaystyle\int\frac{d\left(\frac{y}{x}\right)}{1+\cos\frac{y}{x}}=\int\frac{dx}{x^3}$

$\Rightarrow \dfrac{1}{2}\displaystyle\int\frac{d\left(\frac{y}{x}\right)}{\cos^2\frac{y}{2x}}=\int\frac{dx}{x^3}$

$\Rightarrow \dfrac{1}{2}\displaystyle\int\sec^2\frac{y}{2x}.d\left(\frac{y}{x}\right)=\int\frac{dx}{x^3}$

$\Rightarrow \dfrac{1}{2}\cdot\dfrac{\tan\frac{y}{2x}}{\frac{1}{2}}=\dfrac{x^{-2}}{-2}+c$

$\Rightarrow \tan\dfrac{y}{2x}+\dfrac{1}{2x^2}=c$

26. c. We have, $\dfrac{dy}{dx}=\dfrac{y}{x}-\cos^2\left(\dfrac{y}{x}\right)$

Putting $y=vx$, so that $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$, we get

$v+x\dfrac{dv}{dx}=v-\cos^2 v$

$\Rightarrow \dfrac{dv}{\cos^2 v}=-\dfrac{dx}{x}$

$\Rightarrow \sec^2 u\,du=-\dfrac{1}{x}dx$

On integration, we get

$\tan u=-\log x+\log C$

$\Rightarrow \tan\left(\dfrac{y}{x}\right)=-\log x+\log C$

This passes through $(1, \pi/4)$, therefore $1=\log C$.

So, $\tan\left(\dfrac{y}{x}\right)=-\log x+1$

$\Rightarrow \tan\left(\dfrac{y}{x}\right)=-\log x+\log e$

$\Rightarrow y=x\tan^{-1}\left(\log\left(\dfrac{e}{x}\right)\right)$

27. d. $\dfrac{dy}{dx}=\dfrac{y}{x}\left[\log\dfrac{y}{x}+1\right]$

Put $y=vx$

$v+x\dfrac{dy}{dx}=v\log v+v$

$\therefore \dfrac{dv}{v\log v}=\dfrac{dx}{x}$

$\therefore \log(\log v)=\log x+\log c=\log cx$

$\therefore \log\dfrac{y}{x}=cx$

28. a. The given equation can be written as

$\dfrac{y}{x}\dfrac{dy}{dx}=\left\{\dfrac{y^2}{x^2}+\dfrac{f(y^2/x^2)}{f'(y^2/x^2)}\right\}$

Above equation is a homogeneous equation.

Putting $y = vx$, we get

$$v\left[v + x\frac{dv}{dx}\right] = v^2 + \frac{f(v^2)}{f'(v^2)}$$

$$\Rightarrow vx\frac{dv}{dx} = \frac{f(v^2)}{f'(v^2)} \text{ variable separable}$$

$$\Rightarrow \frac{2vf'(v^2)}{f(v^2)}dv = 2\frac{dv}{x}$$

Now integrating both sides, we get

$\log f(v^2) = \log x^2 + \log c$ [$\log c$ = constant]

or $\log f(v^2) = \log cx^2$

or $f(v^2) = cx^2$

or $f(y^2/x^2) = cx^2$

29. b. $(x^2 + xy)\,dy = (x^2 + y^2)dx \Rightarrow \dfrac{dy}{dx} = \dfrac{x^2+y^2}{x^2+xy}$

Let $\dfrac{y}{x} = v \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ equation reduces to

$$x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v$$

$$= \frac{1+v^2-v-v^2}{1+v}$$

$$= \frac{1-v}{1+v}$$

$$\Rightarrow \int \frac{1+v}{1-v}dv = \int \frac{dx}{x}$$

$$\Rightarrow -\int\left(1 - \frac{2}{1-v}\right)dv = \int \frac{dx}{x}$$

$$\Rightarrow -v - 2\log(1-v) = \log x + \log c$$

$$\Rightarrow -\frac{y}{x} - 2\log\left(\frac{x-y}{x}\right) = \log x + \log c$$

$$\Rightarrow \frac{-y}{x} - 2\log(x-y) + 2\log x = \log x + \log c$$

$$\Rightarrow \log x = 2\log(x-y) + \frac{y}{x} + k \text{ where } k = \log c$$

30. c. The intersection of $y - x + 1 = 0$ and $y + x + 5 = 0$ is $(-2, -3)$. Put $x = X - 2, y = Y - 3$.

The given equation reduces to $\dfrac{dY}{dX} = \dfrac{Y-X}{Y+X}$.

putting $Y = vX$, we get

$$X\frac{dv}{dX} = -\frac{v^2+1}{v+1}$$

$$\Rightarrow \left(-\frac{v}{v^2+1} - \frac{1}{v^2+1}\right)dv = \frac{dX}{X}$$

$$\Rightarrow -\frac{1}{2}\log\left(v^2+1\right) - \tan^{-1}v = \log|X| + \text{constant}$$

$$\Rightarrow \log(Y^2 + X^2) + 2\tan^{-1}\frac{Y}{X} = \text{constant}$$

$$\Rightarrow \log((y+3)^2 + (x+2)^2) + 2\tan^{-1}\frac{y+3}{x+2} = C$$

31. a. $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{2xy}$ (1)

Put $y = vx$ $\therefore \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ equation (1) transforms to

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2x\,vx} = \frac{1+v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1-v^2}{2v}$$

$$\Rightarrow \frac{2v\,dv}{1-v^2} = \frac{dx}{x}$$

$$\Rightarrow \log x + \log(1-v^2) = \log C$$

$$\Rightarrow x(1-v^2) = C$$

$$\Rightarrow x\left(1 - \frac{y^2}{x^2}\right) = C$$

$$\Rightarrow x^2 - y^2 = Cx$$

It passes through (2, 1).

$$\therefore 4 - 1 = 2C \Rightarrow C = \frac{3}{2}$$

$$\therefore x^2 - y^2 = \frac{3}{2}x \Rightarrow 2(x^2 - y^2) = 3x$$

32. b. The given equation is written as $y\,dx - x\,dy + x\sqrt{xy}$

$(x+y)dx + y\sqrt{xy}\,(x+y)\,dy = 0$

$\Rightarrow \; ydx - xdy + (x+y)\sqrt{xy}\,(xdx + ydy) = 0$

$$\Rightarrow \frac{ydx - xdy}{y^2} + \left(\frac{x}{y}+1\right)\sqrt{\frac{x}{y}}\left(d\left(\frac{x^2+y^2}{2}\right)\right) = 0$$

$$\Rightarrow d\left(\frac{x^2+y^2}{2}\right) + \frac{d\left(\frac{x}{y}\right)}{\left(\frac{x}{y}+1\right)\sqrt{\frac{x}{y}}} = 0$$

$$\Rightarrow \frac{x^2+y^2}{2} + 2\tan^{-1}\sqrt{\frac{x}{y}} = c$$

33. a. $\frac{dy}{dx} + y\,\phi'(x) = \phi(x)\,\phi'(x)$

I.F. $= e^{\int \phi'(x)dx} = e^{\phi(x)}$

Hence, the solution is

$ye^{\phi(x)} = \int e^{\phi(x)}\,\phi(x)\,\phi'(x)\,dx$

$= \int e^t t\,dt$, where $\phi(x) = t$

$= te^t - e^t + c$

$= \phi(x)\,e^{\phi(x)} - e^{\phi(x)} + c$

$\therefore \; y = ce^{-\phi(x)} + \phi(x) - 1$

34. c. Rewriting the given equation as

$2xy\frac{dy}{dx} - y^2 = 1 + x^2$

$\Rightarrow 2y\frac{dy}{dx} - \frac{1}{x}y^2 = \frac{1}{x} + x$

Putting $y^2 = u$, we have

$\frac{du}{dx} - \frac{1}{x}u = \frac{1}{x} + x$

I.F. $= e^{-\int \frac{1}{x}dx} = \frac{1}{x}$

$\therefore$ solution is $u\,\frac{1}{x} = \int\left(\frac{1}{x^2}+1\right)dx = -\frac{1}{x} + x + C$

$\Rightarrow y^2 = (x^2 - 1) + Cx$

Since $y(1) = 1$ so $C = 1$.

Hence $y^2 = x(1+x) - 1$ which represents a system of hyperbola.

35. c. $\because \; \frac{dy}{dx} + \frac{y}{x\log_e x} = \frac{2}{x}$

$\therefore$ I.F. $= e^{\int \frac{1}{x\log_e x}dx}$

$= e^{\log_e \log_e x}$

$= \log_e x$

36. c. The given equation can be rewritten as

$$\frac{dy}{dx} + \frac{x^2-1}{x(x^2+1)}y = \frac{x^2\log x}{(x^2+1)} \qquad (1)$$

which is linear. Also

$P = \frac{x^2-1}{x(x^2+1)}$ and $Q = \frac{x^2\log x}{(x^2+1)}$

$\int P\,dx = \int\left[\frac{2x}{x^2+1} - \frac{1}{x}\right]dx$

[resolving into partial fractions]

$= \log(x^2+1) - \log x$

$\therefore$ I.F. $= e^{\log[(x^2+1)/x]} = \frac{x^2+1}{x}$

Hence the required solution of equation (1) is

$$\frac{y(x^2+1)}{x} = \int \frac{(x^2+1)}{x}\,\frac{x^2\log x}{(x^2+1)}\,dx + c$$

$= \int x\log x\,dx + c$

$= \frac{1}{2}x^2\log x - \int \frac{1}{x}\frac{x^2}{2}\,dx + c$

$\therefore \; y(x^2+1)/x = \frac{1}{2}x^2\log x - \frac{1}{4}x^2 + c$

37. c. $\cos x\frac{dy}{dx} + y\sin x = 1$

$\Rightarrow \frac{dy}{dx} + y\frac{\sin x}{\cos x} = \sec x$

$\therefore \int P\,dx = \int \frac{\sin x}{\cos x}\,dx$

$= -\log\cos x$

$= \log\sec x$

$\therefore$ I.F. $= e^{\int P dx} = e^{\log\sec x} = \sec x$

38. c. The given differential equation can be written as

$\frac{dy}{dx} - \frac{\tan 2x}{\cos^2 x}y = \cos^2 x$ which is linear differential equation of first order.

$$\int P\,dx = \int \frac{-\sin 2x}{\cos 2x \cos^2 x} dx$$

$$= -\int \frac{2\sin 2x\,dx}{\cos 2x(1+\cos 2x)}$$

$$= \int \frac{dt}{t(1+t)}$$

$$= \int \left(\frac{1}{t} - \frac{1}{1+t}\right) dt$$

$$= \log \frac{t}{1+t} \text{ where } t = \cos 2x$$

$$= \log \frac{\cos 2x}{1+\cos 2x} \qquad \left[\because -\frac{\pi}{2} < 2x < \frac{\pi}{2}\right]$$

$$\therefore e^{\int P\,dx} = e^{\log \frac{\cos 2x}{1+\cos 2x}}$$

$$= \frac{\cos 2x}{1+\cos 2x} = \frac{\cos 2x}{2\cos^2 x}$$

$\therefore$ the solution is,

$$y \frac{\cos 2x}{2\cos^2 x} = \int \frac{\cos^2 x \cos 2x}{2\cos^2 x} dx + C$$

$$= \frac{1}{4}\sin 2x + C$$

When $x = \frac{\pi}{6}, y = \frac{3\sqrt{3}}{8}$

$$\therefore \frac{3\sqrt{3}}{8} \frac{4}{2\times 2\times 3} = \frac{1}{4}\frac{\sqrt{3}}{2} + C \Rightarrow C = 0$$

$$\therefore y = \frac{1}{2}\tan 2x \cos^2 x$$

39. d. $x(1-x^2)dy + (2x^2y - y - ax^3)dx = 0$

$$\Rightarrow x(1-x^2)\frac{dy}{dx} + 2x^2y - y - ax^3 = 0$$

$$\Rightarrow x(1-x^2)\frac{dy}{dx} + y(2x^2-1) = ax^3$$

$$\Rightarrow \frac{dy}{dx} + \frac{2x^2-1}{x(1-x^2)}y = \frac{ax^3}{x(1-x^2)}$$

which is of the form $\frac{dy}{dx} + Py = Q$.

Its integrating factor is $e^{\int P dx}$.

Here $P = \frac{2x^2-1}{x(1-x^2)}$

40. b. $f'(x) - \frac{2x(x+1)}{x+1} f(x) = \frac{e^{x^2}}{(x+1)^2}$

$$\text{I. F.} = e^{\int -2x\,dx} = e^{-x^2}$$

$\therefore$ solution is $f(x)\, e^{-x^2} = \int \frac{dx}{(x+1)^2} + C$

$$\Rightarrow f(x)e^{-x^2} = -\frac{1}{x+1} + C$$

Given $f(0) = 5 \Rightarrow C = 6$

$$\therefore f(x) = \left(\frac{6x+5}{x+1}\right)e^{x^2}$$

41. a. $\frac{dy}{dx} = \frac{1}{xy[x^2 \sin y^2 + 1]}$

$$\Rightarrow \frac{dx}{dy} = xy[x^2 \sin y^2 + 1]$$

$$\Rightarrow \frac{1}{x^3}\frac{dx}{dy} - \frac{1}{x^2}y = y\sin y^2$$

Putting $-1/x^2 = u$, the last equation can be written as

$$\frac{du}{dy} + 2uy = 2y\sin y^2.$$

$$\text{I.F.} = e^{y^2}$$

$\therefore$ solution is $ue^{y^2} = \int 2y \sin y^2 e^{y^2} dy + C$

$$= \int (\sin t)e^t dt + C$$

$$= \frac{1}{2}e^{y^2}(\sin y^2 - \cos y^2) + c'$$

$$\Rightarrow 2u = (\sin y^2 - \cos y^2) + 2Ce^{-y^2}$$

$$\Rightarrow 2 = x^2[\cos y^2 - \sin y^2 - 2Ce^{-y^2}]$$

42. d. $\frac{dy}{dx} = 1 + xy$

$$\Rightarrow \frac{dy}{dx} - xy = 1$$

$$\text{I.F.} = e^{\int -x\,dx} = e^{-x^2/2}$$

Hence solution is $y.e^{-x^2/2} = \int e^{-x^2/2}dx + c.$

$\int e^{-x^2/2}dx$ is not further integrable.

43. b. $\frac{dx}{dy} = \frac{x+2y^3}{y}$

$\Rightarrow \frac{dx}{dy} - \frac{1}{y}x = 2y^2$ which is linear

I.F. $= e^{\int -\frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$

$\therefore$ solution is $\frac{1}{y}x = \int \frac{1}{y}2y^2\,dy = y^2 + c$

$\Rightarrow \frac{x}{y} = y^2 + c$

44. a. $x^2\frac{dy}{dx}\cos\frac{1}{x} - y\sin\frac{1}{x} = -1$

$\Rightarrow \frac{dy}{dx} - \frac{y}{x^2}\tan\frac{1}{x} = -\sec\frac{1}{x}\frac{1}{x^2}$ (linear)

I.F. $= e^{-\int \frac{1}{x^2}\tan\frac{1}{x}dx} = \sec\frac{1}{x}$

$\Rightarrow$ solution is $y\sec\frac{1}{x} = -\int \sec^2\left(\frac{1}{x}\right)\frac{1}{x^2}dx = \tan\frac{1}{x} + c$

Given $y \to -1, x \to \infty \Rightarrow c = -1$

Hence equation of curve is $y = \sin\frac{1}{x} - \cos\frac{1}{x}$.

45. d. $2x^2y\frac{dy}{dx} = \tan(x^2y^2) - 2xy^2$

$\Rightarrow x^2\,2y\frac{dy}{dx} + y^2\,2x = \tan(x^2y^2)$

$\Rightarrow \frac{d}{dx}(x^2y^2) = \tan(x^2y^2)$

$\Rightarrow \int \cot(x^2y^2)\,d(x^2y^2) = \int dx$

$\Rightarrow \log(\sin(x^2y^2)) = x + c$

when $x = 1$, $y = \sqrt{\frac{\pi}{2}} \Rightarrow c = -1$

$\Rightarrow$ Equation of curve is $x = \log\sin(x^2y^2) + 1$

$\Rightarrow \log\sin(x^2y^2) = x + 1$

$\Rightarrow \sin(x^2y^2) = e^{x+1}$

46. a. $\left\{\frac{1}{x} - \frac{y^2}{(x-y)^2}\right\}dx + \left\{\frac{x^2}{(x-y)^2} - \frac{1}{y}\right\}dy = 0$

$\Rightarrow \left(\frac{dx}{x} - \frac{dy}{y}\right) + \left(\frac{x^2dy - y^2dx}{(x-y)^2}\right) = 0$

$\Rightarrow \left(\frac{dx}{x} - \frac{dy}{y}\right) + \left(\frac{dy/y^2 - dx/x^2}{(1/y - 1/x)^2}\right) = 0$

Integrating, we get $\ln|x| - \ln|y| - \frac{1}{(1/x - 1/y)} = c$

$\Rightarrow \ln\left|\frac{x}{y}\right| + \frac{xy}{x-y} = c$

47. a. Put $xy = v$ $\therefore y + x\frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow \frac{dv}{dx} = x\frac{\phi(v)}{\phi'(v)}$

$\therefore \frac{\phi'(v)}{\phi(v)}dv = x\,dx$. Integrating, we get

$\log\phi(v) = \frac{x^2}{2} + \log k$

$\Rightarrow \log\frac{\phi(v)}{k} = \frac{x^2}{2}$

or $\phi(v) = ke^{x^2/2} \Rightarrow \phi(xy) = ke^{x^2/2}$

48. a. $(2y + xy^3)dx + (x + x^2y^2)dy = 0$

$\Rightarrow (2y\,dx + xdy) + (xy^3dx + x^2y^2dy) = 0$

Multiplying by x, we get

$(2xy\,dx + x^2\,dy) + (x^2y^3\,dx + x^3y^2\,dy) = 0$

$\Rightarrow d(x^2y) + \frac{1}{3}d(x^3y^3) = 0$

Integrating, we get $x^2y + \frac{x^3y^3}{3} = c$

49. c. $ye^{-x/y}dx - (xe^{(-x/y)} + y^3)dy = 0$

$\Rightarrow (ydx - xdy)\,e^{-x/y} - y^3dy = 0$

$\Rightarrow \frac{ydx - xdy}{y^2}e^{-x/y} = ydy$

$\Rightarrow d(x/y)e^{-x/y} = ydy$

$\Rightarrow -e^{-x/y} = \frac{y^2}{2} + C$

$\Rightarrow 2e^{-x/y} + y^2 = C$

50. c. $(xy^3 - x^2)\,dy - (xy + y^4)\,dx = 0$

$\Rightarrow y^3(x\,dy - y\,dx) - x(x\,dy + y\,dx) = 0$

$\Rightarrow x^2y^3\frac{(x\,dy - y\,dx)}{x^2} - x(x\,dy + y\,dx) = 0$

$\Rightarrow x^2y^3\,d\left(\frac{y}{x}\right) - xd(xy) = 0$

$\Rightarrow$ Dividing by x^3y^2, we get

$\Rightarrow \frac{y}{x}d\left(\frac{y}{x}\right) - \frac{d(xy)}{x^2y^2} = 0$

Now integrating $\frac{1}{2}\left(\frac{y}{x}\right)^2 + \frac{1}{xy} = c$

It passes through the point (4, – 2).

$\Rightarrow \frac{1}{8} - \frac{1}{8} = c \Rightarrow c = 0$

$\therefore \; y^3 = -2x$

51. a. The given equation can be written as

$$\frac{x\,dx + y\,dy}{(y\,dx - x\,dy)/y^2} = y^2 \frac{x}{y^3} \cos^2(x^2 + y^2)$$

$$\Rightarrow \frac{xdx + ydy}{\cos^2(x^2 + y^2)} = \frac{x}{y}\left(\frac{ydx - xdy}{y^2}\right)$$

$$\Rightarrow \frac{1}{2}\sec^2(x^2 + y^2)\; d(x^2 + y^2) = \frac{x}{y} d\left(\frac{x}{y}\right)$$

On integrating, we get

$$\frac{1}{2}\tan(x^2 + y^2) = \frac{1}{2}\left(\frac{x}{y}\right)^2 + \frac{c}{2}$$

$$\text{or } \tan(x^2 + y^2) = \frac{x^2}{y^2} + c$$

52. c. Re-write the D.E. as

$(2xy\,dx - x^2\,dy) + y^2(3x^2y^2\,dx + 2x^3y\,dy) = 0$

Dividing by y^2, we get

$$\frac{y\,2x\,dx - x^2\,dy}{y^2} + y^2\,3x^2\,dx + x^3\,2y\,dy = 0$$

$$\text{or } d\left(\frac{x^2}{y}\right) + d\left(x^3y^2\right) = 0$$

Integrating, we get the solution

$$\frac{x^2}{y} + x^3y^2 = c$$

53. c. $\left\{1 + x\sqrt{(x^2 + y^2)}\right\}dx + \left\{\sqrt{(x^2 + y^2)} - 1\right\}y\,dy = 0$

$$\Rightarrow dx - y\,dy + \sqrt{(x^2 + y^2)}\,(x\,dx + y\,dy) = 0$$

$$\Rightarrow dx - y\,dy + \frac{1}{2}\sqrt{(x^2 + y^2)}\; d(x^2 + y^2) = 0$$

Integrating, we have

$$x - \frac{y^2}{2} + \frac{1}{2}\int \sqrt{t}\,dt = c, \left\{t = \sqrt{(x^2 + y^2)}\right\}$$

$$\text{or } x - \frac{y^2}{2} + \frac{1}{3}\left(x^2 + y^2\right)^{3/2} = c$$

54. b. $xy = C$

$$\Rightarrow x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x} = m_1$$

By condition,

$$\tan\frac{\pi}{4} = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

$$1 = \left|\frac{-\frac{y}{x} - m_2}{1 - \frac{y}{x}m_2}\right|$$

$$\Rightarrow \frac{y}{x} + m_2 = 1 - \frac{y}{x}m_2 \text{ or } \frac{y}{x}m_2 - 1$$

$$\Rightarrow m_2 = \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}} \text{ or } m_2 = \frac{\frac{y}{x} + 1}{\frac{y}{x} - 1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x - y}{x + y} \text{ or } \frac{dy}{dx} = \frac{x + y}{y - x}$$

55. a.

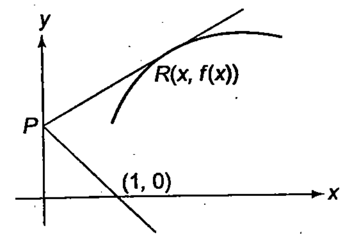


Fig. 10.8

The equation of the tangent at the point $R(x, f(x))$ is $Y - f(x) = f'(x)(X - x)$

The coordinates of the point P are $(0, f(x) - xf'(x))$

The slope of the perpendicular line through P is $\frac{f(x) - xf'(x)}{-1} = -\frac{1}{f'(x)}$

$$\Rightarrow f(x)f'(x) - x(f'(x))^2 = 1$$

$$\Rightarrow \frac{ydy}{dx} - x\left(\frac{dy}{dx}\right)^2 = 1$$ which is the required differential equation to the curve at $y = f(x)$.

56. a. If $y = f(x)$ is the curve,

$Y - y = \frac{dy}{dx}(X - x)$ is the equation of the tangent at (x, y)

Putting $X = 0$, the initial ordinate of the tangent is therefore $y - xf'(x)$.

The subnormal at this point is given by $y\frac{dy}{dx}$, so we have

$$y\frac{dy}{dx} = y - x\frac{dy}{dx} \Rightarrow \frac{y}{x + y} = \frac{dy}{dx}.$$

This is a homogeneous equation and, by rewriting it as $\frac{dx}{dy} = \frac{x + y}{y} = \frac{x}{y} + 1 \Rightarrow \frac{dx}{dy} - \frac{x}{y} = 1$ we see that it is also a linear equation.

57. b. $x^{2/3} + y^{2/3} = a^{2/3}$

$$\Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} \quad (1)$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$\Rightarrow \frac{dx}{dy} = \frac{x^{-1/3}}{y^{-1/3}}$$

$$\Rightarrow \int x^{1/3}dx = \int y^{1/3}dy$$

$$\Rightarrow x^{4/3} - y^{4/3} = c$$

58. b. The general equation of all non-horizontal lines in xy-plane is $ax + by = 1$, where $a \neq 0$.
Now, $ax + by = 1$

$$\Rightarrow a\frac{dx}{dy} + b = 0 \quad \text{[Diff. w.r.t. } y]$$

$$\Rightarrow a\frac{d^2x}{dy^2} = 0 \quad \text{[Diff. w.r.t. } y]$$

$$\Rightarrow \frac{d^2x}{dy^2} = 0 \quad [\because a \neq 0]$$

Hence, the required differential equation is $\frac{d^2x}{dy^2} = 0$

59. b. It is given that the triangle OPG is an isosceles triangle.

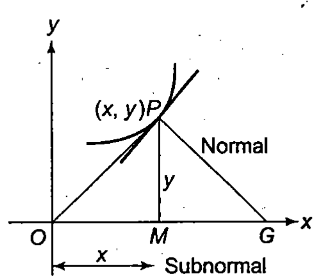


Fig. 10.9

Therefore, $OM = MG =$ sub-normal

$$\Rightarrow x = y\frac{dy}{dx} \Rightarrow x\,dx = y\,dx$$

On integration, we get $x^2 - y^2 = C$, which is a rectangular hyperbola.

60. a. Let the equation of the curve be $y = f(x)$.

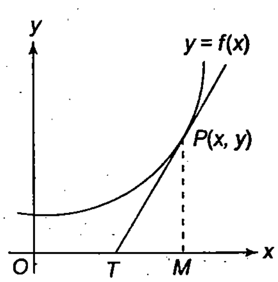


Fig. 10.10

It is given that $OT \propto y$

$$\Rightarrow OT = by$$

$$\Rightarrow OM - TM = by$$

$$\Rightarrow x - \frac{y}{dy/dx} = by \quad [\because TM = \text{Length of the subtangent}]$$

$$\Rightarrow x - y\frac{dx}{dy} = by$$

$$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = -b$$

It is linear differential equation.

Its solution is $\frac{x}{y} = -b\log y + a$.

$$\Rightarrow x = y(a - b\log y)$$

61. b. For the family of curves represented by the first differential equation the slope of the tangent at any point (x, y) is given by

$$\left(\frac{dy}{dx}\right)_{c_1} = \frac{x^2 + x + 1}{y^2 + y + 1}$$

For the family of curves represented by the second differential the slope of the tangent at any point is given by

$$\left(\frac{dy}{dx}\right)_{c_2} = -\frac{y^2 + y + 1}{x^2 + x + 1}$$

Clearly, $\left(\frac{dy}{dx}\right)_{c_1} \times \left(\frac{dy}{dx}\right)_{c_2} = -1$

Hence, the two curves are orthogonal.

62. c. Equation of normal at point $P(x, y)$, $Y - y = -\frac{dx}{dy}(X - x)$

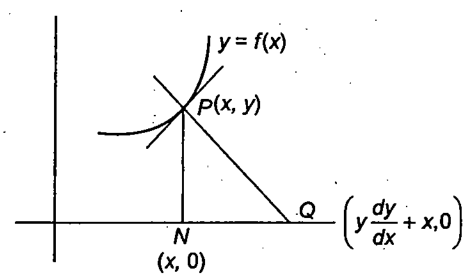


Fig. 10.11

$$NQ = y\frac{dy}{dx} = \frac{x(1+y^2)}{1+x^2}$$

$$\Rightarrow \frac{xdx}{1+x^2} = \frac{ydy}{1+y^2}$$

$$\Rightarrow \ln(1+x^2) = \ln(1+y^2) + \ln c$$

$$\Rightarrow 1 + y^2 = \frac{1+x^2}{c}$$

It passes through (3, 1) $\Rightarrow 1+1=\frac{1+(3)^2}{c} \Rightarrow c=5$

$\Rightarrow$ curve is $5+5y^2=1+x^2$ or $x^2-5y^2=4$

63. c. The point on y-axis is $\left(0,\ y-x\frac{dy}{dx}\right)$.

According to given condition,

$$\frac{x}{2}=y-\frac{x}{2}\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=2\frac{y}{x}-1$$

putting $\frac{y}{x}=v$, we get $x\frac{dv}{dx}=v-1$

$$\Rightarrow \ln\left|\frac{y}{x}-1\right|=\ln|x|+c$$

$$\Rightarrow 1-\frac{y}{x}=x \qquad [\text{as } y(1)=0]$$

64. b. We have, $\frac{dy}{dx}=1-\frac{1}{x^2} \Rightarrow y=x+\frac{1}{x}+C$

This passes through (2, 7/2),

Therefore, $\frac{7}{2}=2+\frac{1}{2}+C \Rightarrow C=1$

Thus the equation of the curve is

$$y=x+\frac{1}{x}+1 \Rightarrow xy=x^2+x+1$$

65. d. Equation of normal at point p is $Y-y=-\frac{dx}{dy}(X-x)$

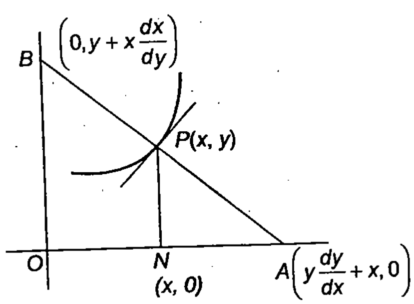


Fig. 10.12

Area of ΔOAB is 1 $\Rightarrow \frac{1}{2}\left(y\frac{dy}{dx}+x\right)\left(x\frac{dx}{dy}+y\right)=1$

$$\Rightarrow \left(y\frac{dy}{dx}+x\right)\left(y\frac{dy}{dx}+x\right)=2\frac{dy}{dx}$$

$$\Rightarrow y^2\left(\frac{dy}{dx}\right)^2+2(xy-1)\frac{dy}{dx}+x^2=0$$

66. c. Slope of tangent $=\frac{dy}{dx}$

$\therefore$ slope of normal $=-\frac{dx}{dy}$

$\therefore$ the equation of normal is:

$$Y-y=-\frac{dx}{dy}(X-x)$$

This meets x-axis ($y=0$), where

$$-y=-\frac{dx}{dy}(X-x) \Rightarrow X=x+y\frac{dy}{dx}$$

$\therefore$ G is $\left(x+y\frac{dy}{dx}, 0\right)$

$\therefore$ $OG=2x \Rightarrow x+y\frac{dy}{dx}=2x$

$$\Rightarrow y\frac{dy}{dx}=x \Rightarrow y\,dy=x\,dx$$

Integrating, we get $\frac{y^2}{2}=\frac{x^2}{2}+\frac{C}{2}$

$\Rightarrow y^2-x^2=c$, which is a hyperbola.

67. a. Equation of tangent is $Y-y=\frac{dy}{dx}(X-x)$

for X-intercept $Y=0 \Rightarrow X=x-y\frac{dx}{dy}$

According to question $x-y\frac{dx}{dy}=y$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x-y}$$

putting $y=vx$, we get

$$\Rightarrow v+x\frac{dv}{dx}=\frac{v}{1-v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v}{1-v}-v=\frac{v-v+v^2}{1-v}$$

$$\Rightarrow \int\frac{1-v}{v^2}dv=\int\frac{dx}{x}$$

$$\Rightarrow -\frac{1}{v}-\log v=\log x+c$$

$$\Rightarrow -\frac{x}{y}-\log\frac{y}{x}=\log x+c$$

$$\Rightarrow -\frac{x}{y}=\log y+c$$

Given when $x=1, y=1 \Rightarrow c=-1$

Hence equation of curve is $1-\frac{x}{y}=\log y$

$$\Rightarrow y=e\,e^{-x/y} \Rightarrow e^{x/y}=\frac{e}{y}$$

$$\Rightarrow ye^{x/y}=e$$

68. a. Tangent at point P is $Y-y=-\frac{1}{m}(X-x)$ where $m=\frac{dy}{dx}$

Let $Y=0 \Rightarrow X=my+x$

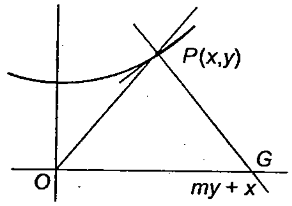


Fig. 10.13

According to question, $x(my+x)=2(x^2+y^2)$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{x^2+2y^2}{xy} \text{ (homogeneous)}$$

Putting $y=vx$, we get

$$\Rightarrow \quad v+x\frac{dv}{dx}=\frac{1+2v^2}{v}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{1+2v^2}{v}-v=\frac{1+v^2}{v}$$

$$\Rightarrow \quad \int\frac{v\,dv}{1+v^2}=\int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\log(1+v^2)=\log x+\log c, c>0$$

$$\Rightarrow \quad x^2+y^2=cx^4$$

Also it passes through (1, 0) then $c=1$.

69. c. Equation to the family of parabolas is $(y-k)^2=4a(x-h)$.

$$2(y-k)\frac{dy}{dx}=4a \text{ (diff. w.r.t. } x)$$

$$\Rightarrow (y-k)\frac{dy}{dx}=2a \quad (1)$$

$$\Rightarrow (y-k)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0 \text{ (diff. w.r.t. } x)$$

$$\Rightarrow 2a\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0 \quad \text{(substituting } y-k \text{ from equation (1))}$$

Hence the order is 2 and the degree is 1.

70. a.

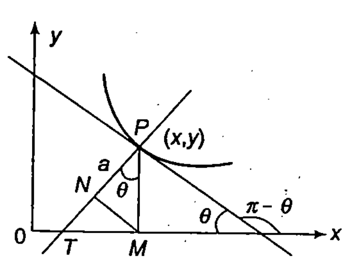


Fig. 10.14

Ordinate $=PM$. Let $P\equiv(x,y)$

Projection of ordinate on normal $=PN$.

$\therefore\ PN=PM\cos\theta=a$ (given)

$$\therefore \quad \frac{y}{\sqrt{1+\tan^2\theta}}=a$$

$$\Rightarrow y=a\sqrt{1+(y_1)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{y^2-a^2}}{a}$$

$$\Rightarrow \int\frac{a\,dy}{\sqrt{y^2-a^2}}=\int dx$$

$$\Rightarrow a\ln|y+\sqrt{y^2-a^2}|=x+c$$

71. a. $y(2x^4+y)\frac{dy}{dx}=(1-4xy^2)x^2$

$\Rightarrow 2x^4y\,dy+y^2\,dy+4x^3y^2dx-x^2\,dx=0$

$\Rightarrow 2x^2y(x^2\,dy+2xy\,dx)+y^2\,dy-x^2\,dx=0$

$\Rightarrow 2x^2y\,d(x^2y)+y^2\,dy-x^2\,dx=0$

Integrating, we get $(x^2y)^2+\frac{y^3}{3}-\frac{x^3}{3}=c$

or $3(x^2y)^2+y^3-x^3=c$

72. b. $(x\cot y+\log\cos x)\,dy+(\log\sin y-y\tan x)dx=0$

$\Rightarrow (x\cot y\,dy+\log\sin y\,dx)+(\log\cos x\,dy-y\tan x\,dx)=0$

$\Rightarrow \int d(x\log\sin y)+\int d(y\log\cos x)=0$

$\Rightarrow x\log\sin y+y\log\cos x=\log c$

$\Rightarrow (\sin y)^x(\cos x)^y=c$

73. a. $\frac{dV}{dt}=-k4\pi r^2$ (1)

but $V=\frac{4}{3}\pi r^3$

$\Rightarrow \frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$ (2)

Hence, $\frac{dr}{dt}=-K$.

74. c. According to the question

$$\frac{dy}{dt}=-k\sqrt{y}$$

$$\Rightarrow \int_4^0\frac{dy}{\sqrt{y}}=-k\int_0^t dt$$

$$\Rightarrow 2\sqrt{y}\Big|_4^0=-kt=-\frac{t}{15}$$

$$\Rightarrow 0-4=-\frac{t}{15}$$

$$\Rightarrow t=60 \text{ min.}$$

75. c. Let population = x, at time t years. Given $\frac{dx}{dt} \propto x$

$\Rightarrow \frac{dx}{dt} = kx$ where k is a constant of proportionality

or $\frac{dx}{x} = kdt$. Integrating, we get $\ln x = kt + \ln c$

$\Rightarrow \frac{x}{c} = e^{kt}$ or $x = ce^{kt}$

If initially, i.e., when time $t = 0, x = x_0$ then $x_0 = ce^0 = c$.

$\Rightarrow x = x_0 e^{kt}$

Given $x = 2x_0$ when $t = 30$ then $2x_0 = x_0 e^{30k} \Rightarrow 2 = e^{30k}$

$\therefore \ln 2 = 30k$ (1)

To find t, when t triples, $x = 3x_0 \therefore 3x_0 = x_0 e^{kt} \Rightarrow 3 = e^{kt}$

$\therefore \ln 3 = kt$ (2)

Dividing equation (2) by (1) then $\frac{t}{30} = \frac{\ln 3}{\ln 2}$ or

$t = 30 \times \frac{\ln 3}{\ln 2} = 30 \times 1.5849 = 48$ years. (approx.)

76. a. Let $V(t)$ be the velocity of the object at time t.

Given $\frac{dV}{dt} = 9.8 - kV \Rightarrow \frac{dV}{9.8 - kV} = dt$.

Integrating, we get $\log(9.8 - kV) = -kt + \log C$

$\Rightarrow 9.8 - kV = Ce^{-kt}$

But $V(0) = 0 \Rightarrow C = 9.8$

Thus, $9.8 - kV = 9.8\, e^{-kt}$

$\Rightarrow kV = 9.8(1 - e^{-kt})$

$\Rightarrow V(t) = \frac{9.8}{k}(1 - e^{-kt}) < \frac{9.8}{k}$

for all t. Hence, $V(t)$ cannot exceed $\frac{9.8}{k}$ m/s.

77. a. $x^2 = e^{\left(\frac{x}{y}\right)^{-1}\left(\frac{dy}{dx}\right)}$

$\Rightarrow x^2 = e^{\left(\frac{y}{x}\right)\left(\frac{dy}{dx}\right)}$

$\Rightarrow \ln x^2 = \frac{y}{x}\frac{dy}{dx}$

$\Rightarrow \int x \ln x^2 dx = \int y\, dy$

Putting $x^2 = t$, we get $2xdx = dt$

$\Rightarrow \frac{1}{2}\int \ln t\, dt = \frac{y^2}{2}$

$\Rightarrow c + t\ln t - t = y^2$

$\Rightarrow y^2 = x^2 \ln x^2 - x^2 + c$

78. a. $y' y''' = 3(y'')^2$

$\Rightarrow \int \frac{y'''}{y''} dx = 3\int \frac{y''}{y_1'} dx$

$\Rightarrow \ln y'' = 3 \ln y' + \ln c$

$\Rightarrow y'' = c(y')^3$

$\Rightarrow \int \frac{y''}{(y')^2} dx = \int cy'\, dx$

$\Rightarrow -\frac{1}{y'} = cy + d$

$\Rightarrow -dx = (cy + d)\, dy$

$\Rightarrow -x = \frac{cy^2}{2} + dy + e$

79. c. $y = e^{mx}$ satisfies $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 12y = 0$

then $e^{mx}(m^3 - 3m^2 - 4m + 12) = 0$

$\Rightarrow m = \pm 2, 3$

$m \in N$ hence $m \in \{2, 3\}$

80. d. $\int_0^x t\, y(t)\, dt = x^2 y(x)$

Differentiating w.r.t. x, we get

$x\, y(x) = x^2 y'(x) + 2x\, y(x)$

$\Rightarrow x\, y(x) + x^2 y'(x) = 0$

$\Rightarrow x\frac{dy}{dx} + y = 0$

$\Rightarrow \log y + \log x = \log c$

$\Rightarrow xy = c$

81. a. Integrating the given differential equation, we have

$\frac{dy}{dx} = \frac{-\cos 3x}{3} + e^x + \frac{x^3}{3} + C_1$

but $y_1(0) = 1$

so $1 = \left(-\frac{1}{3}\right) + 1 + C_1 \Rightarrow C_1 = 1/3$

Again integrating, we get

$y = \frac{-\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x + C_2$

but $y(0) = 0$ so $0 = 0 + 1 + C_2 \Rightarrow C_2 = -1$

Thus $y = \frac{-\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x - 1$

82. b. Applying componendo and divedendo

we get $\frac{dy}{dx} = \frac{e^{-x}}{e^x} = e^{-2x}$

$\Rightarrow 2y = -e^{-2x} + C$

$\Rightarrow 2y\, e^{2x} = Ce^{2x} - 1$

83. b. The given equation is reduced to $x = e^{xy\,(dy/dx)}$

$\Rightarrow \log x = xy\frac{dy}{dx}$

$\Rightarrow \int y dy = \int \frac{1}{x}\log x dx$

$\Rightarrow \frac{y^2}{2} = \frac{(\log x)^2}{2} + C'$

84. c. $\frac{x}{c-1}+\frac{y}{c+1}=1$ (1)

$\Rightarrow \frac{1}{c-1}+\frac{y'}{c+1}=0$ (2)

$\Rightarrow \frac{y'}{1}=\frac{c+1}{1-c}$

$\Rightarrow \frac{y'-1}{y'+1}=c$

Put value of c in equation (1)

$$\Rightarrow \frac{x}{\frac{y'-1}{y'+1}-1}+\frac{y}{\frac{y'-1}{y'+1}+1}=1$$

$$\Rightarrow \frac{x(y'+1)}{-2}+\frac{y(y'+1)}{2y'}=1$$

$$\Rightarrow \frac{(y'+1)}{2}\left(\frac{y}{y'}-x\right)=1$$

$$\Rightarrow \left(1+\frac{dy}{dx}\right)\left(y-x\frac{dy}{dx}\right)=2\frac{dy}{dx}$$

85. a. We have

$$f(\theta)=\frac{d}{d\theta}\int_0^{\theta}\frac{dx}{1-\cos\theta\cos x}=\frac{1}{1-\cos^2\theta}=\operatorname{cosec}^2\theta$$

[using Leibnitz's Rule]

$$\Rightarrow \frac{df(\theta)}{d\theta}=-2\operatorname{cosec}^2\theta\cot\theta$$

$$\Rightarrow \frac{df(\theta)}{d\theta}+2f(\theta)\cot\theta=0 \qquad [\because f(\theta)=\operatorname{cosec}^2\theta]$$

86. c. $v=\frac{A}{r}+B$ (1)

$\frac{dv}{dr}=-\frac{A}{r^2}$ (2)

$\frac{d^2v}{dr^2}=\frac{2A}{r^3}$ (3)

Eliminating A between equations (2) and (3), we get

$$r\frac{d^2v}{dr^2}=\frac{2A}{r^2}=2\left(-\frac{dv}{dr}\right)$$

$$\therefore \frac{d^2v}{dr^2}+\frac{2}{r}\frac{dv}{dr}=0$$

87. c. $\frac{y'''}{y''}=8\Rightarrow \log y''=8x+c$

When $x=0, y''=1$ and $\log 1=0 \therefore c=0$

$\therefore y''=e^{8x}$. Integrating again

$y'=\frac{e^{8x}}{8}+\lambda$ when $x=0, y'(0)=0$

$\therefore \lambda=-1/8$

$\therefore y'=\frac{e^{8x}}{8}-\frac{1}{8}$. Integrate again

$$y=\frac{e^{8x}}{64}-\frac{x}{8}+k$$

Also when $x=0, y=\frac{1}{8}$ $\therefore k=\frac{7}{64}$

$$\therefore y=\frac{1}{8}\left(\frac{e^{8x}}{8}-x+\frac{7}{8}\right)$$

88. a. $y^2=t$; $2y\frac{dy}{dx}=\frac{dt}{dx}$; hence the differential equation becomes

$$\left(e^{x^2}+e^t\right)\frac{dt}{dx}+2e^{x^2}(xt-x)=0$$

$$e^{x^2}+e^t+2e^{x^2}x(t-1)\frac{dx}{dt}=0$$

put $e^{x^2}=z$; $e^{x^2}2x\frac{dx}{dt}=\frac{dz}{dt}$

$$\Rightarrow z+e^t+\frac{dz}{dt}(t-1)=0$$

$$\Rightarrow \frac{dz}{dt}+\frac{z}{(t-1)}=-\frac{e^t}{(t-1)};\ \text{I.F.}=e^{\int\frac{dt}{t-1}}=e^{\ln(t-1)}=t-1$$

$$\Rightarrow z(t-1)=-\int(e^t)dt$$

$$\Rightarrow z(t-1)=-e^t+C$$

$$\Rightarrow e^{x^2}(y^2-1)=-e^{y^2}+C$$

$$\Rightarrow e^{x^2}(y^2-1)+e^{y^2}=C$$

Multiple Correct Answers Type

1. a., b., c.

a. $f(\lambda x, \lambda y)=\frac{\lambda(x-y)}{\lambda^2(x^2+y^2)}=\lambda^{-1}f(x,y)$

$\Rightarrow$ homogeneous of degree (–1).

b. $f(\lambda x, \lambda y)=(\lambda x)^{1/3}(\lambda y)^{-2/3}\tan^{-1}\frac{x}{y}$

$$=\lambda^{-1/3}x^{1/3}y^{-2/3}\tan^{-1}\frac{x}{y}$$

$$=\lambda^{-\frac{1}{3}}f(x,y)$$

$\Rightarrow$ homogeneous

c. $f(\lambda x, \lambda y)=\lambda x\left(\ln\sqrt{\lambda^2(x^2+y^2)}-\ln\lambda y\right)+\lambda y e^{x/y}$

$$=\lambda x\left[\ln\left(\frac{\lambda\sqrt{(x^2+y^2)}}{\lambda y}\right)\right]+\lambda y e^{x/y}$$

$$=\lambda\left[x\left(\ln\sqrt{x^2+y^2}-\ln y\right)+ye^{x/y}\right]$$

$$=\lambda f(x,y)$$

$\Rightarrow$ homogeneous

d. $f(\lambda x, \lambda y) = \lambda x \left[\ln \dfrac{2\lambda^2 x^2 + \lambda^2 y^2}{\lambda x \lambda(x+y)}\right] + \lambda^2 x^2 \tan \dfrac{x+2y}{3x-y}$

$= \lambda x \left[\ln \dfrac{2x^2 + y^2}{x(x+y)}\right] + \lambda^2 x^2 \tan \dfrac{x+2y}{3x-y}$

$\Rightarrow$ non homogeneous

a, c. We have $(x-h)^2 + (y-k)^2 = a^2$ (1)

Differentiating w.r.t. x, we get

$2(x-h) + 2(y-k)\dfrac{dy}{dx} = 0$

$\Rightarrow (x-h) + (y-k)\dfrac{dy}{dx} = 0$ (2)

Differentiating w.r.t. x, we get

$1 + \left(\dfrac{dy}{dx}\right)^2 + (y-k)\dfrac{d^2y}{dx^2} = 0$ (3)

From equation (3), $y - k = -\left(\dfrac{1+p^2}{q}\right)$, where $p = \dfrac{dy}{dx}$,

$q = \dfrac{d^2y}{dx^2}$

Putting the value of $y-k$ in equation (2), we get

$x - h = \dfrac{(1+p^2)p}{q}$

Substituting the values of $x-h$ and $y-k$ in equation (1), we get

$\left(\dfrac{1+p^2}{q}\right)^2 (1+p^2) = a^2 \Rightarrow \left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^3 = a^2\left(\dfrac{d^2y}{dx^2}\right)^2$

which is the required differential equation.

3. a, b. $y\left(\dfrac{dy}{dx}\right)^2 + (x-y)\dfrac{dy}{dx} - x = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{(y-x) \pm \sqrt{(x-y)^2 + 4xy}}{2y}$

$\Rightarrow \dfrac{dy}{dx} = 1$ which gives straight line

or $\dfrac{dy}{dx} = -\dfrac{x}{y}$ which gives circle.

4. a, c.

Obviously (a) is linear D.E. with $P = \dfrac{1}{x}$ and $Q = \log x$

$y\left(\dfrac{dy}{dx}\right) + 4x = 0 \Rightarrow \dfrac{dy}{dx} + \dfrac{4x}{y} = 0$. Hence not linear.

$(2x + y^3)\left(\dfrac{dy}{dx}\right) = 3y$

$\Rightarrow \dfrac{dx}{dy} = \dfrac{2x}{3y} + \dfrac{y^2}{3}$

$\Rightarrow \dfrac{dx}{dy} - \dfrac{2x}{3y} = \dfrac{y^2}{3}$ which is linear with $P = \dfrac{2}{3y}$ and

$Q = \dfrac{y^2}{3}$

5. a, c. $\dfrac{dy}{dx} = \dfrac{ax+h}{by+k} \Rightarrow (by+k)\,dy = (ax+h)\,dx$

$\Rightarrow b\dfrac{y^2}{2} + ky = \dfrac{a}{2}x^2 + hx + C$

For this to represent a parabola, one of the two terms x^2 or y^2 is zero.

Therefore, either $a = 0, b \neq 0$ or $a \neq 0, b = 0$

6. a, d. The given differential equation is

$y_2(x^2+1) = 2xy_1 \Rightarrow \dfrac{y_2}{y_1} = \dfrac{2x}{x^2+1}$

Integrating both sides, we get

$\log y_1 = \log(x^2+1) + \log C$

$\Rightarrow y_1 = C(x^2+1)$ (1)

It is given that $y_1 = 3$ at $x = 0$

Putting $x = 0, y_1 = 3$ in equation (1), we get $3 = C$

Substituting the value of C in (1), we obtain

$y_1 = 3(x^2+1)$ (2)

Integrating both sides w.r.t. to x, we get

$y = x^3 + 3x + C_2$

This passes through the point (0, 1). Therefore, $1 = C_2$

Hence, the required equation of the curve is $y = x^3 + 3x + 1$

Obviously it is strictly increasing from equation (2)

Also $f(0) = 1 > 0$, then the only root is negative.

7. a, b, c.

8. a, b, d.

$\dfrac{dy}{dx} + y\cos x = \cos x$ (linear)

I.F. $= e^{\int \cos x\,dx} = e^{\sin x}$

$\therefore$ solution is $y\,e^{\sin x} = \int e^{\sin x}\cos x\,dx = e^{\sin x} + c$

when $x = 0, y = 1$ then $c = 0$

$\Rightarrow y = 1$. Hence options (a), (b), (d) are true.

9. a, b, c.

We have, $f''(x) = g''(x)$. On integration, We get

$f'(x) = g'(x) + C$ (1)

Putting $x = 1$, we get

$f'(1) = g'(1) + C \Rightarrow 4 = 2 + C \Rightarrow C = 2$

$\therefore f'(x) = g'(x) + 2$

Integrating w.r.t. x, we get $f(x) = g(x) + 2x + c_1$ (2)

Putting $x = 2$, we get

$f(2) = g(2) + 4 + c_1 \Rightarrow 9 = 3 + 4 + c_1 \Rightarrow c_1 = 2$

$\therefore f(x) = g(x) + 2x + 2$. Putting $x = 4$, we get $f(4) - g(4) = 10$

$|f(x) - g(x)| < 2 \Rightarrow |2x+2| < 2 \Rightarrow |x+1| < 1 \Rightarrow -2 < x < 0$

Also $f(2) = g(2) \Rightarrow x = -1$

$f(x) - g(x) = 2x$ has no solution.

10. b.

$(x^2y^2 - 1)\,dy + 2xy^3\,dx = 0$

$\Rightarrow x^2y^2dy + 2xy^3\,dx = dy$

$$\Rightarrow x^2\,dy + 2xy\,dx = \frac{dy}{y^2}$$

$$\Rightarrow \int d\left(x^2 y\right) = \int \frac{dy}{y^2} + c$$

$$\Rightarrow x^2 y = \frac{y^{-1}}{-1} + c$$

$$\Rightarrow x^2y^2 = -1 + cy$$

i.e., $1 + x^2y^2 = cy$

11. a, b.

$$\frac{dy}{dx} = \frac{y}{x^2} \Rightarrow \frac{dy}{y} = \frac{dx}{x^2} \Rightarrow \ln y = -\frac{1}{x} + \ln c \Rightarrow \frac{y}{c} = e^{-\frac{1}{x}}$$

$$\Rightarrow y = ce^{-\frac{1}{x}}$$

Comparing with $y = a\,e^{-1/x} + b, a \in R, b = 0$

12. b. We have $y\frac{dy}{dx} = k$ (constant)

$$\Rightarrow y\,dy = k\,dx \Rightarrow \frac{y^2}{2} = kx + C \Rightarrow y^2 = 2kx + 2C$$

$$\Rightarrow y^2 = 2ax + b, \text{ where } a = k, b = 2C$$

13. a, d.

The D.E. can be re-written as

$$\frac{x\,dx + y\,dy}{\sqrt{1-\left(x^2+y^2\right)}} = \frac{x\,dy - y\,dx}{\sqrt{x^2+y^2}}$$

Since $d\tan^{-1}(y/x) = \dfrac{x\,dy - y\,dx}{x^2+y^2}$, and $d(x^2+y^2)$

$= 2(x\,dx + y\,dy)$,

$$\therefore \text{ we have } \frac{\frac{1}{2}d\left(x^2+y^2\right)}{\sqrt{x^2+y^2}\sqrt{1-\left(x^2+y^2\right)}} = \frac{x\,dy - y\,dx}{x^2+y^2}$$

$$= d\{\tan^{-1}(y/x)\}$$

Put $x^2 + y^2 = t^2$ in the L.H.S and get

$$\frac{t\,dt}{t\sqrt{1-t^2}} = d\{\tan^{-1}(y/x)\}$$

Integrating both sides, we get

$\sin^{-1} t = \tan^{-1}(y/x) + c$

i.e., $\sin^{-1}\sqrt{\left(x^2+y^2\right)} = \tan^{-1}(y/x) + c$

14. a, b.

We have length of the normal = radius vector

$$\Rightarrow y\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{x^2+y^2}$$

$$\Rightarrow y^2\left(1+\left(\frac{dy}{dx}\right)^2\right) = x^2+y^2$$

$$\Rightarrow y^2\left(\frac{dy}{dx}\right)^2 = x^2$$

$$\Rightarrow x = \pm y\frac{dy}{dx}$$

$$\Rightarrow x = y\frac{dy}{dx} \text{ or } x = -y\frac{dy}{dx}$$

$$\Rightarrow x\,dx - y\,dy = 0 \text{ or } x\,dx + y\,dy = 0$$

$$\Rightarrow x^2 - y^2 = c_1 \text{ or } x^2 + y^2 = c_2$$

Clearly, $x^2 - y^2 = c_1$ represents a rectangular hyperbola and $x^2 + y^2 = c_2$ represents circles.

15. a, b.

$$x = \sin\left(\frac{dy}{dx} - 2y\right) \Rightarrow \frac{dy}{dx} - 2y = \sin^{-1} x$$

$$x - 2y = \log\left(\frac{dy}{dx}\right) \Rightarrow \frac{dy}{dx} = e^{x-2y}$$

Reasoning Type

1. a. The equation of circle contains. Three independent constants if it passes through three non-collinear points, therefore statement 1 is true and follows from statement 2.

2. a. $y = Ae^x$

On differentiation, we get $\dfrac{dy}{dx} = Ae^x$

3. d. Statement 2 is obviously true. But statement 1 is false as

$$2x - 3y + 2 = \log\left(\frac{dy}{dx}\right)$$

$\Rightarrow \left(\dfrac{dy}{dx}\right) = e^{2x-3y+2}$ which has degree 1.

4. b. Statement 1 is obviously true.

Even statement 2 is also obviously true but it does not explain statement 1.

5. a. $y = c_1\cos 2x + c_2\sin^2 x + c_3\cos^2 x + c_4e^{2x} + c_5e^{2x+c_6}$

$$= c_1\cos 2x + c_2\left(\frac{1-\cos 2x}{2}\right) + c_3\left(\frac{\cos 2x+1}{2}\right) + c_4e^{2x}$$

$$+ c_5e^{c_6}e^{2x}$$

$$= \left(c_1 - \frac{c_2}{2} + \frac{c_3}{2}\right)\cos 2x + \left(\frac{c_2}{2} + \frac{c_3}{2}\right) + (c_4 + c_5e^{c_6})e^{2x}$$

$$= \lambda_1\cos 2x + \lambda_2 e^{2x} + \lambda_3$$

$\Rightarrow$ Total number of independent parameters in the given general solution is 3.

Hence statement 1 is true, also statement 2 is true which explains statement 1.

Linked Comprehension Type

For Problems 1–3

1. b., 2. c., 3. a.

Sol.

1. b. $f(x) \le 0$ and $F'(x) = f(x)$

$\Rightarrow\ f(x)\geq cF(x)$
$\Rightarrow\ F'(x)-cF(x)\geq 0$
$\Rightarrow\ e^{-cx}F'(x)-c\,e^{-cx}F(x)\geq 0$

$$\Rightarrow\ \frac{d}{dx}\left(e^{-cx}F(x)\right)\geq 0$$

$\Rightarrow\ e^{-cx}F(x)$ is an increasing function
$\Rightarrow\ e^{-cx}F(x)\geq e^{-c(0)}F(0)$
$\Rightarrow\ e^{-cx}F(x)\geq 0$
$\Rightarrow\ F(x)\geq 0$
$\Rightarrow\ f(x)\geq 0$ (as $f(x)\geq cF(x)$ and c is positive)
$\Rightarrow\ f(x)=0$

Also $\left(\frac{d\,g(x)}{dx}\right)<g(x)\ \forall\ x>0$

$$\Rightarrow\ e^{-x}\frac{d(g(x))}{dx}-e^{-x}g(x)<0$$

$$\Rightarrow\ \frac{d}{dx}(e^{-x}g(x))<0$$

$\Rightarrow\ e^{-x}g(x)$ is a decreasing function
$\Rightarrow\ e^{-x}g(x)<e^{-(0)}g(0)$
$\Rightarrow\ g(x)<0$ (as $g(0)=0$)
Thus $f(x)=g(x)$ has one solution $x=0$.

2. c. $|x^2+x-6|=f(x)+g(x)\Rightarrow|x^2+x-6|=g(x)$
$\Rightarrow$ no solution

3. a. $g(x)(\cos^{-1}x-\sin^{-1}x)\leq 0$

$$\Rightarrow\ (\cos^{-1}x-\sin^{-1}x)\geq 0\Rightarrow x\in\left[-1,\frac{1}{\sqrt{2}}\right]$$

For Problems 4 – 6

4. c., 5. b., 6. d.

Sol.

4. c. Given equation can be rewritten as

$$y=xp+\sqrt{(1+p^2)},\ p=\frac{dy}{dx} \qquad (1)$$

differentiating w.r.t. x, we get

$$p=p+x\frac{dp}{dx}+\frac{1}{2\sqrt{1+p^2}}2p\frac{dp}{dx}$$

$$\Rightarrow\ \frac{dp}{dx}=0 \text{ or } \frac{p}{\sqrt{1+p^2}}=-x$$

$$\Rightarrow\ p=c \text{ or } p=\frac{x}{\sqrt{1-x^2}}$$

$\Rightarrow\ y=cx+\sqrt{(1+c^2)}$ gives the general solution and $x^2+y^2=1$ as singular solution.

5. b. $y=xp+p^2\left(p=\frac{dy}{dx}\right)$ (1)

differentiating equation (1) w.r.t. x, we get

$$p=p+x\frac{dp}{dx}+2p\frac{dp}{dx}$$

$$\Rightarrow\ \frac{dp}{dx}(x+2p)=0$$

$$\Rightarrow\ \frac{dp}{dx}=0 \text{ or } p=-\frac{x}{2}$$

Eliminating p from equation (1), we get

$$y=-\frac{x^2}{2}+\frac{x^2}{4}=-\frac{x^2}{4} \qquad (4)$$

Clearly $f(x)=-\frac{x^2}{4}=-1$ has two solutions.

6. d. $y=mx+m-m^3$ (1)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=m+x\frac{dm}{dx}+\frac{dm}{dx}-3m^2\frac{dm}{dx}$$

$$m=m+x\frac{dm}{dx}+\frac{dm}{dx}-3m^2\frac{dm}{dx}$$

$$\frac{dm}{dx}(x+1-3m^2)=0$$

$$\frac{dm}{dx}=0\ \Rightarrow m=c \qquad (2)$$

$$\text{or } x+1-3m^2=0\ \Rightarrow m^2=\frac{x+1}{3} \qquad (3)$$

Eliminating m between equations (1) and (3), we get
$y=m(x+1-m^2)$

$$\Rightarrow\ y=\left(\frac{x+1}{3}\right)^{1/2}\left(x+1-\frac{x+1}{3}\right)$$

$$\Rightarrow\ y=\left(\frac{x+1}{3}\right)^{1/2}\frac{2}{3}(x+1)$$

$$\Rightarrow\ y=2\left(\frac{x+1}{3}\right)^{3/2}$$

$$\Rightarrow\ y^2=\frac{4}{27}(x+1)^3$$

$$\Rightarrow 27y^2=4(x+1)^3$$

For Problems 7–9

7. c, 8. a, 9. b.

Sol. Integrating $\frac{d^2y}{dx^2}=6x-4$, we get $\frac{dy}{dx}=3x^2-4x+A$

When $x=1$, $\frac{dy}{dx}=0$ so that $A=1$. Hence

$$\frac{dy}{dx}=3x^2-4x+1 \qquad (1)$$

Integrating, we get $y=x^3-2x^2+x+B$.
When $x=1, y=5$, so that $B=5$.
Thus, we have $y=x^3-2x^2+x+5$
From equation (1), we get the critical points $x=1/3, x=1$

At the critical point $x=\frac{1}{3}$, $\frac{d^2y}{dx^2}$ is – ve

Therefore, at $x=1/3$, y has a local maximum.

At $x=1$, $\frac{d^2y}{dx^2}$ is +ve

Therefore, at $x=1$ y has a local minimum.

Also $f(1)=5, f\left(\frac{1}{3}\right)=\frac{139}{27}, f(0)=5, f(2)=7$

Hence the global maximum value = 7
and the global minimum value = 5

For Problems 10 – 12

10. a., 11. c., 12. c.

10. a. Let N denote the amount of material present at time t. Then,

$$\frac{dN}{dt}-kN=0$$

This differential equation is separable and linear, its solution is $N=ce^{kt}$ (1)

At $t=0$, we are given that $N=50$. Therefore, from equation (1), $50=ce^{k(0)}$, or $c=50$.

Thus, $N=50\,e^{kt}$ (2)

At $t=2$, 10 percent of the original mass of 50 mg or 5 mg, has decayed.

Hence, at $t=2$, $N=50-5=45$.

Substituting these values into equation (2) and solving for k, we have $45=50e^{2k}$ or $k=\frac{1}{2}\log\frac{45}{50}$

Substituting this value into (2), we obtain the amount of mass present at any time t as

$$N=50e^{-(1/2)(\ln 0.9)\,t} \quad (3)$$

where t is measured in hours.

11. c. We require N at $t=4$. Substituting $t=4$ into (3) and then solving for N, we find

$N=50e^{-2\ln 0.9}$

12. c. We require t when $N=50/2=25$. Substituting $N=25$ into equation (3) and solving for t, we find

$25=50\,e^{-(1/2)(\ln 0.9)\,t}$ or $t=(\ln 1/2)/(-1/2\ln 0.9)$ hr.

For Problems 13 – 15

13. a., 14. b., 15. d.

13. a. Here, $V_0=100$, $a=20$, $b=0$, and $e=f=5$. Hence

$$\frac{dQ}{dt}+\frac{1}{20}Q=0$$

The solution of this linear equation is $Q=ce^{-t/20}$ (1)

At $t=0$, we are given that $Q=a=20$.

Substituting these values into equation (1), we find that $c=20$, so that equation (1) can be rewritten as $Q=20\,e^{-t/20}$.

For $t=20$, $Q=20/e$

14. b. Here $a=0$, $b=1$, $e=4$, $f=2$, and $V_0=10$.

The volume of brine in the tank at any time t is given as $V_0+et-ft=10+2t$.

We require t when $10+2t=50$, hence, $t=20$ min.

15. d. For the equation $\frac{dQ}{dt}+\frac{2}{10+2t}Q=4$

This is a linear equation; its solution is $Q=\frac{40t+4t^2+c}{10+2t}$ (1)

At $t=0$, $Q=a=0$. Substituting these values into equation (1), we find that $c=0$. We require Q at the moment of overflow, which from part (a) is $t=20$. Thus

$$Q=\frac{40(20)+4(20)^2}{10+2(20)}=40\text{ lb}$$

Matrix-Match Type

1. a → q, s; b → p; c → p; d → q, r, s

a. Equation of the required parabola is of the form $y^2=4a(x-h)$. Differentiating, we have

$$2y\frac{dy}{dx}=4a\Rightarrow y\frac{dy}{dx}=2a\Rightarrow\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}=0$$

The degree of this differential equation is 1 and the order is 2.

b. We have $y=a(x+a)^2$ (1)

$$\Rightarrow \frac{dy}{dx}=2a(x+a) \quad (2)$$

Dividing equations (1) by (2), we get $\frac{y}{\frac{dy}{dx}}=\frac{x+a}{2}$

$$\Rightarrow x+a=\frac{2y}{y_1}, \text{ where } y_1=\frac{dy}{dx}$$

Substituting $a=\frac{2y}{y_1}-x$ in equation (1),

we get $y=\left(\frac{2y}{y_1}-x\right)\left(\frac{2y}{y_1}\right)^2 \Rightarrow y_1^3y=4(2y-xy_1)y^2$

Clearly, it is a differential equation of degree 3.

c. The given equation is $\left(1+3\frac{dy}{dx}\right)^{2/3}=4\frac{d^3y}{dx^3}$

Cubing, we get $\left(1+3\frac{dy}{dx}\right)^2=64\left(\frac{d^3y}{dx^2}\right)^3$

Hence order = degree = 3

d. We have $y^2=2c\left(x+\sqrt{c}\right)$ (1)

Diff. w.r.t. x, we get $2y\frac{dy}{dx}=2c$

$$\Rightarrow c=y\frac{dy}{dx}$$

Putting in equation (1), we get $y^2=2\left(y\frac{dy}{dx}\right)x+2\left(y\frac{dy}{dx}\right)^{3/2}$

$$\Rightarrow \left(y^2-2xy\frac{dy}{dx}\right)^2=4y^3\left(\frac{dy}{dx}\right)^3$$

Its order is 1 and degree is 3

2. a → q; b → r; c → p; d → s

a. $y=e^{4x}+2e^{-x}$; $y_1=4e^{4x}-2e^{-x}$; $y_2=16e^{4x}+2e^{-x}$;

$y_3=64e^{4x}-2e^{-x}$

Now, $y_3-13y_1=(64e^{4x}-2e^{-x})-13(4e^{4x}-2e^{-x})$

$=12e^{4x}+24e^{-x}$

$y_3 - 13y = 12(e^{4x} + 2e^{-x}) = 12y$

$\therefore K = 12$ and $K/3 = 4$

b. Since equation is 2 degree, two linew are possible

c. $y = u^m \Rightarrow \dfrac{dy}{dx} = m\,u^{m-1}\dfrac{du}{dx}$

Substituting the value of y and $\dfrac{dy}{dx}$ in $2x^4 y\dfrac{dy}{dx} + y^4 = 4x^6$

we have $2x^4 u^m\, m\, u^{m-1}\dfrac{du}{dx} + u^{4m} = 4x^6$.

$$\Rightarrow \frac{du}{dx} = \frac{4x^6 - u^{4m}}{2m\; x^4\; u^{2m-1}}$$

For homogeneous $4m = 6 \Rightarrow m = \dfrac{3}{2}$

and $2m - 1 = 2 \Rightarrow m = \dfrac{3}{2}$

d. $y = Ax^m + Bx^{-n}$

$$\Rightarrow \frac{dy}{dx} = Amx^{m-1} - nBx^{-n-1}$$

$$\Rightarrow \frac{d^2y}{dx^2} = Am(m-1)x^{m-2} + n(n+1)Bx^{-n-2}$$

Putting these values in $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 12y$

We have $= m(m+1)Ax^m + n(n-1)Bx^{-n} = 12(Ax^m + Bx^{-n})$

$\Rightarrow m(m+1) = 12$ or $n(n-1) = 12$

$\Rightarrow m = 3, -4$ or $n = 4, -3$

Integer Type

1.(4) We have $4xe^{xy} = y + 5\sin^2 x$ (1)

Put $x = 0$, in equation (1), we get $y = 0$

Therefore, (0, 0) lies on the curve.

Now on differentiating equation (1) w.r.t. x, we get

$$4e^{xy} + 4xe^{xy}\left(x\frac{dy}{dx} + y\right) = \frac{dy}{dx} + 10\sin x\cos x$$

$\Rightarrow y'(0) = 4$

2.(2) Given $\dfrac{dy}{dx} - \dfrac{1}{x}y = \left(x - \dfrac{2}{x}\right)$

$$\text{I.F.} = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

Now general solution is given by $\dfrac{y}{x} = \int\left(x - \dfrac{2}{x}\right)\dfrac{1}{x}dx$

$$\Rightarrow \frac{y}{x} = x + \frac{2}{x} + C$$

As $y(1) = 1 \Rightarrow C = -2$

$\therefore \dfrac{y}{x} = x + \dfrac{2}{x} - 2 \Rightarrow y = x^2 - 2x + 2$

Hence $y(2) = (2)^2 - 2(2) + 2 = 2$

3.(2) Given $y = \tan z$

$$\frac{dy}{dx} = \sec^2 z\cdot\frac{dz}{dx} \quad (1)$$

Now $\dfrac{d^2y}{dx^2} = \sec^2 z\cdot\dfrac{d^2z}{dx^2} + \dfrac{dz}{dx}\cdot\dfrac{d}{dx}(\sec^2 z)$ [using product rule]

$$= \sec^2 z\cdot\frac{d^2z}{dx^2} + \frac{dz}{dx}\cdot\frac{d}{dz}(\sec^2 z)\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = \sec^2 z\cdot\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\cdot 2\sec^2 z.\tan z \quad (2)$$

Now $1 + \dfrac{2(1+y)}{1+y^2}\left(\dfrac{dy}{dx}\right)^2$

$$= 1 + \frac{2(1+\tan z)}{\sec^2 z}\cdot\sec^4 z.\left(\frac{dz}{dx}\right)^2$$

$$= 1 + 2(1+\tan z)\cdot\sec^2 z\cdot\left(\frac{dz}{dx}\right)^2$$

$$= 1 + 2\sec^2 z\left(\frac{dz}{dx}\right)^2 + 2\tan z\cdot\sec^2 z\left(\frac{dz}{dx}\right)^2 \quad (3)$$

from (2) and (3), we have RHS of (2) = RHS of (3)

$$\sec^2 z\cdot\frac{d^2z}{dx^2} = 1 + 2\sec^2 z\left(\frac{dz}{dx}\right)^2$$

$$\Rightarrow \frac{d^2z}{dx^2} = \cos^2 z + 2\left(\frac{dz}{dx}\right)^2$$

$\Rightarrow k = 2$

4.(8) $\dfrac{dy}{dt} + 2ty = t^2$

I.F. $= e^{t^2}$

$\therefore$ Solution is $y\cdot e^{t^2} = \int t^2 e^{t^2}dt = \dfrac{1}{2}\int t\cdot(2t\cdot e^{t^2})\,dt$

$$\therefore \quad y\cdot e^{t^2} = t\cdot\frac{e^{t^2}}{2} - \frac{1}{2}\int e^{t^2}dt + C$$

$$\therefore \quad y = \frac{t}{2} - e^{-t^2}\int\frac{e^{t^2}}{2}dt + Ce^{-t^2}$$

$$\therefore \quad \lim_{t\to\infty}\frac{y}{t} = \frac{1}{2} - \lim_{t\to\infty}\frac{\int\frac{e^{t^2}}{2}}{te^{t^2}} + \frac{c}{t.e^{t^2}} = \frac{1}{2}$$

5.(2) $\dfrac{dy}{dx} = \dfrac{1}{x\cos y + 2\sin y\cos y}$

$$\therefore \quad \frac{dx}{dy} = x\cos y + 2\sin y\cos y$$

$\therefore$ I.F. $= e^{-\int \cos y\, dx} e = e^{-\sin y}$

$\therefore$ The solution is

$x \cdot e^{-\sin y} = 2\int e^{-\sin y} \cdot \sin y \cos y\, dy$

$= -2 \sin y\, e^{-\sin y} - 2 \int (-e^{-\sin y}) \cos y\, dx$

$= -2 \sin y\, e^{-\sin y} + 2 \int -e^{-\sin y} \cos y\, dy$

$= -2 \sin y\, e^{-\sin y} - 2\, e^{-\sin y} + c$

i.e. $x = -2 \sin y - 2 + c\, e^{\sin y} = ce^{\sin y} - 2\,(1 + \sin y)$

$\therefore\ k = 2$

6.(1) $\dfrac{dy}{dx} = \dfrac{1}{dx/dy};\ \dfrac{d^2y}{dx^2} = \dfrac{d}{dy}\left(\dfrac{1}{dx/dy}\right)\cdot\dfrac{dy}{dx} = -\dfrac{1}{(dx/dy)^3}\dfrac{d^2x}{dy^2}$

hence $x\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 - \dfrac{dy}{dx} = 0$

becomes $-x\cdot\dfrac{1}{(dx/dy)^3}\dfrac{d^2x}{dy^2} + \dfrac{1}{(dx/dy)^3} - \dfrac{1}{(dx/dy)} = 0$

or $x\dfrac{d^2x}{dy^2} - 1 + \left(\dfrac{dx}{dy}\right)^2 = 0 \ \Rightarrow\ x\dfrac{d^2x}{dy^2} + \left(\dfrac{dx}{dy}\right)^2 = 1;$

$\therefore\ k = 1$

7.(3) $\dfrac{dy}{dx} = -\dfrac{\sqrt{(x^2-1)(y^2-1)}}{xy}$

$\int \dfrac{y}{\sqrt{y^2-1}}\,dy = -\int \dfrac{\sqrt{x^2-1}}{x}\,dx$

let $y^2 - 1 = t^2 \ \Rightarrow\ 2y\,dy = 2t\,dt$

$\therefore\ \int \dfrac{t}{t}\,dt = -\int \dfrac{x^2-1}{x\sqrt{x^2-1}}\,dx$

$\therefore\ t = -\int \dfrac{x}{\sqrt{x^2-1}}\,dx + \int \dfrac{1}{x\sqrt{x^2-1}}\,dx$

$\therefore\ \sqrt{y^2-1} = -\sqrt{x^2-1} + \sec^{-1}x + c$

Curve passes through the point (1, 1), then the value of $c = 0$.

Hence the curve is $\sqrt{y^2-1} = -\sqrt{x^2-1} + \sec^{-1}x$

8.(8) Equation of tangent at $P(x_1, y_1)$ of $y = f(x)$

$y - y_1 = \dfrac{dy}{dx}(x - x_1)$ (1)

this tangent cuts the x-axis so

$x_2 = x_1 - \dfrac{y_1}{\left(\dfrac{dy}{dx}\right)}$

$\because\ x_1, x_2, x_3 \ldots x_n$ are in AP

$x_2 - x_1 = -\dfrac{y_1}{\dfrac{dy}{dx}} = \log_z e$ given

$-y = \log_z e\dfrac{dy}{dx}$

$\dfrac{dy}{y}\log_z e = -dx$ Integrating both sides

$\log_e y = -x\log_z e + c$

$y = ke^{-x\log_e 2}$

$\because\ y = f(x)$ passes through (0, 2)

$\Rightarrow\ k = 2$

$\therefore\ y = 2\cdot e^{-x\log_e 2}$

$\therefore\ y = 2^{1-x}$

9.(2) Equation of tangent is $X\dfrac{dy}{dx} - y - Y\dfrac{dy}{dx} + y = 0$ perpendicular distance from origin is

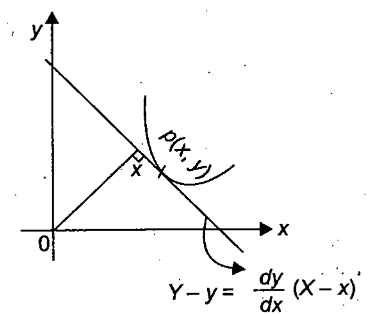


Fig. 10.15

$\therefore\ \perp$ from $(0, 0) = x$

$\left|\dfrac{0 - 0 - x\dfrac{dy}{dx} + y}{\sqrt{\left(\dfrac{dy}{dx}\right)^2 + 1}}\right| = x$

$\therefore\ \left|\dfrac{x\dfrac{dy}{dx} - y}{\sqrt{\left(\dfrac{dy}{dx}\right)^2 + 1}}\right| = x \ \Rightarrow\ \left(x\dfrac{dy}{dx} - y\right)^2 = x^2\left(1 + \left(\dfrac{dy}{dx}\right)^2\right)$

$\Rightarrow\ x^2\left(\dfrac{dy}{dx}\right)^2 + y^2 - 2xy\dfrac{dy}{dx} = x^2 + x^2\left(\dfrac{dy}{dx}\right)^2$

$\Rightarrow\ \dfrac{y^2 - x^2}{2xy} = \dfrac{dy}{dx}$ (1) (Homogeneous)

put $y = vx$ in (1)

$v + x\dfrac{dv}{dx} = \dfrac{v^2 - 1}{2v}$

$$\int \frac{2v}{v^2+1} dv = -\int \frac{dx}{x}$$

$$\ell n(v^2+1) = -\ell nx + \ell nc$$

$$v^2 + 1 = \frac{c}{x}$$

$$\frac{y^2+x^2}{x^2} = \frac{c}{x} \Rightarrow y^2 + x^2 = cx$$

passes through (1, 1), then $c = 2$

$x^2 + y^2 - 2x = 0.$

For intercept of curve on x-axis, put $y = 0$

We have $x^2 - 2x = 0$ or $x = 0, 2.$

Hence length of intercept is 2.

0.(5)

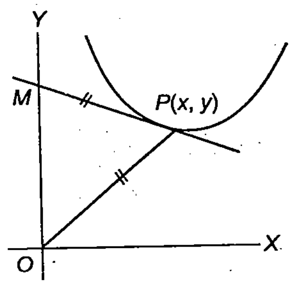


Fig. 10.16

$\because OP = OM$

$$y - x\frac{dy}{dx} = \sqrt{x^2+y^2}$$

$$\frac{dy}{dx} = \frac{y - \sqrt{x^2+y^2}}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x} - \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

put $\frac{y}{x} = v \Rightarrow y = vx \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore v + x\frac{dv}{dx} = v - \sqrt{1+v^2}$$

$$\therefore \log(v + \sqrt{1+v^2}) = \log\frac{c}{x}$$

$$\therefore v + \sqrt{1+v^2} = \frac{c}{x}$$

$$\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = \frac{c}{x}$$

$$y + \sqrt{x^2+y^2} = c$$

Hence curve is parabola, ~~which~~ has eccentricity 1.

11.(4) $\frac{dy}{dx} - y = 1 - e^{-x}$

$P = -1 \qquad Q = 1 - e^{-x}$

$$\text{I.F.} = e^{\int P dx} = e^{\int -1 dx} = e^{-x}$$

$$\therefore y \cdot e^{-x} = \int e^{-x}(1 - e^{-x})\, dx + C$$

$$ye^{-x} = -e^{-x} + \frac{1}{2}e^{-2x} + C$$

$$y = -1 + \frac{1}{2}e^{-x} + Ce^{x}$$

$\because x = 0 \qquad y = y_0$

So $C = y_0 + \frac{1}{2}$

$$y = -1 + \frac{1}{2}e^{-x} + (y_0 + 1/2)e^{x}$$

$x \to \infty\ y \to$ finite value so $y_0 + 1/2 = 0$

$y_0 = -1/2$

Archives

Subjective

1. The equation of normal to required curve at $P(x, y)$ is given by,

$$Y - y = -\frac{1}{\left(\frac{dY}{dX}\right)_{(x,y)}}(X - x)$$

or $(X - x) + \frac{dy}{dx}(Y - y) = 0$

For point Q, where this normal meets X-axis, put $Y = 0$, we get,

$$X = x + y\frac{dy}{dx}$$

$$\therefore Q\left(x + y\frac{dy}{dx}, 0\right)$$

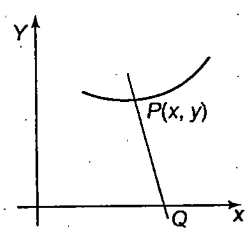


Fig. 10.17

According to question length of $PQ = k$.

$$\Rightarrow \left(y\frac{dy}{dx}\right)^2 + y^2 = k^2$$

$$\Rightarrow y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$$

which is the required differential equation of given curve. Solving this, we get

$$\int \frac{y\,dy}{\sqrt{k^2 - y^2}} = \int \pm dx$$

$$\Rightarrow \quad -\frac{1}{2}\, 2\sqrt{k^2 - y^2} = \pm x + C$$

$$\Rightarrow \quad -\sqrt{k^2 - y^2} = \pm x + C$$

As it passes through $(0, k)$ we get $c = 0$.

$\therefore$ equation of curve is $-\sqrt{k^2 - y^2} = \pm x$

or $x^2 + y^2 = k^2$

2. Equation of the tangent to the curve $y = f(x)$ at point (x, y) is $Y - y = f'(x)(X - x)$.

The line (1) meets X-axis at $P\left(x - \dfrac{y}{f'(x)}, 0\right)$

and Y-axis in $Q\,[0, y - xf'(x)]$.

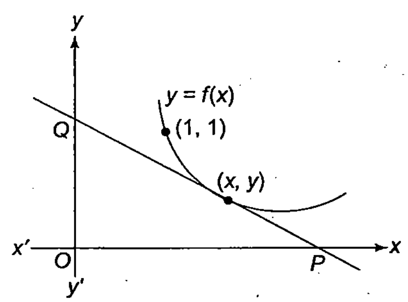


Fig. 10.18

Area of triangle OPQ is

$$= \frac{1}{2}(OP)(OQ)$$

$$= \frac{1}{2}\left(x - \frac{y}{f'(x)}\right)(y - xf'(x))$$

$$= -\frac{(y - xf'(x))^2}{2f'(x)}$$

Given that area of $\Delta OPQ = 2$

$$\Rightarrow \quad \frac{-(y - xf'(x))^2}{2f'(x)} = 2$$

$$\Rightarrow \quad (y - xf'(x))^2 + 4f'(x) = 0$$

$$\Rightarrow \quad (y - px)^2 + 4p = 0 \qquad (2)$$

where $p = f'(x) = dy/dx$

From the diagram $y - xf'(x) > 0$ and $p = f'(x) < 0$

So we can write equation (2) as $y - px = 2\sqrt{-p}$

$$\Rightarrow \quad y = px + 2\sqrt{-p} \qquad (3)$$

Differentiating equation (3) with respect to x, we get

$$p = \frac{dy}{dx} = p + \frac{dp}{dx}x + 2\left(\frac{1}{2}\right)(-p)^{-1/2}(-1)\frac{dp}{dx}$$

$$\Rightarrow \quad \frac{dp}{dx}x - (-p)^{-1/2}\frac{dp}{dx} = 0$$

$$\Rightarrow \quad \frac{dp}{dx}[x - (-p)^{-1/2}] = 0$$

$$\Rightarrow \quad \frac{dp}{dx} = 0 \text{ or } x = (-p)^{-1/2}$$

If $\dfrac{dp}{dx} = 0$, then $p = c$ where $c < 0$ $\quad [\because p < 0]$

Putting the value in equation (3), we get

$$y = cx + 2\sqrt{-c} \qquad (4)$$

this curve will pass through (1, 1) if

$$1 = c + 2\sqrt{-c}$$

$$\Rightarrow \quad -c - 2\sqrt{-c} + 1 = 0$$

$$\Rightarrow \quad (\sqrt{-c} - 1)^2 = 0$$

$$\Rightarrow \sqrt{-c} = 1 \Rightarrow -c = 1 \text{ or } c = -1$$

Putting the value of c in equation (4), we get $y = -x + 2$

Next, Putting $x = (-p)^{-1/2}$ or $-p = x^{-2}$ in equation (3), we get

$$y = \frac{-x}{x^2} + 2\left(\frac{1}{x}\right) = \frac{1}{x}$$

$$\Rightarrow \quad xy = 1\ (x > 0, y > 0)$$

Thus, the two required curves are $x + y = 2$ and $xy = 1$, $(x > 0, y > 0)$.

3. $\dfrac{dy}{dx} = \sin(10x + 6y) \qquad (1)$

Put $10x + 6y = v \Rightarrow 10 + 6\dfrac{dy}{dx} = \dfrac{dv}{dx}$

Then equation (1) transforms to $\dfrac{dv}{dx} - 10 = 6\sin v$

$$\Rightarrow \quad \int \frac{dv}{6\sin v + 10} = \int dx$$

$$\Rightarrow \quad \int \frac{dv}{12\sin\frac{v}{2}\cos\frac{v}{2} + 10} = \int dx$$

Divide above and below by $\cos^2(v/2)$ and put $\tan(v/2) = t$

$$\Rightarrow \quad \int \frac{2dt}{12t + 10(1 + t^2)} = \int dx$$

$$\Rightarrow \quad \int \frac{dt}{5t^2 + 6t + 5} = \int dx$$

$$\Rightarrow \quad \int \frac{dt}{\left(t + \frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 5\int dx$$

$$\Rightarrow \quad \frac{5}{4}\tan^{-1}\frac{5t + 3}{4} = 5x + 5c$$

$$\Rightarrow \quad \tan^{-1}\frac{5t + 3}{4} = 4x + c \qquad (2)$$

At origin $x = 0, y = 0 \therefore v = 0 \therefore t = \tan\dfrac{v}{2} = 0$

$$\Rightarrow \quad \tan^{-1}\frac{3}{4} = c$$

Then from equation (2) we get $\tan^{-1}\frac{5t+3}{4} - \tan^{-1}\frac{3}{4} = 4x$

$$\Rightarrow \frac{\frac{5t+3}{4}-\frac{3}{4}}{1+\frac{5t+3}{4}\frac{3}{4}} = \tan 4x$$

$$\Rightarrow \frac{20t}{25+15t} = \tan 4x$$

$\Rightarrow$ $4t = (5+3t)\tan 4x$

$\Rightarrow$ $t(4-3\tan 4x) = 5\tan 4x$

$$\Rightarrow \tan\frac{v}{2} = \frac{5\tan 4x}{4-3\tan 4x}$$

$$\Rightarrow \tan(5x+3y) = \frac{5\tan 4x}{4-3\tan 4x}$$

$$\Rightarrow 5x+3y = \tan^{-1}\left(\frac{5\tan 4x}{4-3\tan 4x}\right)$$

$$\Rightarrow y = \frac{1}{3}\left[\tan^{-1}\left(\frac{5\tan 4x}{4-3\tan 4x}\right) - 5x\right]$$

4. Let at any instant t, x be the volume of water in the reserver A and y of that in B.

Then $\frac{dx}{dt} \propto x$

$\Rightarrow$ $\frac{dx}{dt} = k_1 x$

$\Rightarrow$ $\frac{dx}{x} = k_1\,dt$

$\Rightarrow$ $\log x = k_1 t + C_1$

$\Rightarrow$ $x = e^{k_1 t}\,e^{C_1}$ (1)

Similarly for B, $\frac{dy}{dt} \propto y$

$\Rightarrow$ $\log y = k_2 t + C_2$

$\Rightarrow$ $y = e^{k_2 t}\,e^{C_2}$ (2)

Now at $t = 0$, $x = 2y$, i.e., $\frac{x}{y} = 2$

$\therefore$ form equations (1) and (2) we get $\frac{e^{C_1}}{e^{C_2}} = 2$ (3)

Also at $t = 1$, $x = \frac{3}{2}y$, i.e., $\frac{x}{y} = \frac{3}{2}$

$$\Rightarrow \frac{e^{k_1}}{e^{k_2}}\frac{e^{C_1}}{e^{C_2}} = \frac{3}{2}$$

$$\Rightarrow e^{k_1-k_2} = \frac{3}{4}$$

Let at $t = T$, $x = y$, i.e., $\frac{x}{y} = 1$

then $\frac{e^{k_1 T}\,e^{C_2}}{e^{k_2 T}\,e^{C_2}} = 1$

$\Rightarrow$ $(e^{k_1-k_2})^T\,2 = 1$

$$\Rightarrow \left(\frac{3}{4}\right)^T = \frac{1}{2}$$

Taking log on both sides, we get

$T\log(3/4) = \log(1/2)$

$$\Rightarrow T = \frac{-\log 2}{-\log 4/3}$$

$$\Rightarrow T = \left(\frac{\log 2}{\log 4/3}\right)$$

5. **a.** $y = u(x)$ and $y = v(x)$ are solutions of given differential equations.

b. $u(x_1) > v(x_1)$ for some x_1

c. $f(x) > g(x)$, $\forall x > x_1$

$$\frac{du}{dx} + p(x)u = f(x) \text{ and } \frac{dv}{dx} + p(x)v = g(x)$$

$$\Rightarrow \frac{d(u-v)}{dx} + p(x)(u-v) = f(x) - g(x)$$

$$\Rightarrow e^{\int pdx}\frac{d(u-v)}{dx} + e^{\int pdx}p(x)(u-v) = e^{\int pdx}(f(x)-g(x))$$

$$\Rightarrow \frac{d}{dx}\left[(u-v).e^{\int pdx}\right] = [f(x)-g(x)]e^{\int pdx}$$

Given $f(x) > g(x)$, $\forall x > x_1$ and exponential function is always +ive, then R.H.S. is + ive.

$$\therefore \frac{d}{dx}\left[(u-v).e^{\int pdx}\right] > 0$$

Hence the function $F(x) = (u-v)\,e^{\int pdx}$ is an increasing function.

Again $u(x_1) > v(x_1)$ for some x_1

$\therefore$ $F = (u-v)\,e^{\int pdx}$ is +ive at $x = x_1$

$\Rightarrow$ $F = (u-v)\,e^{\int pdx}$ is +ive $\forall x > x_1$ (F being increasing function)

$\therefore$ $u(x) > u(x_1)$, $\forall x > x_1$

$\therefore$ Hence there is no point (x, y) such that $x > x_1$ which can satisfy the equations.

$y = u(x)$ and $y = v(x)$

6. Equation of the tangent at point (x, y) on the curve is

$Y - y = \frac{dy}{dx}(X - x)$.

This meet axis in $A\left(x - y\frac{dx}{dy}, 0\right)$ and $B\left(0, y - x\frac{dy}{dx}\right)$

Mid-point of AB is $\left[\frac{1}{2}\left(x - y\frac{dx}{dy}\right), \frac{1}{2}\left(y - x\frac{dy}{dx}\right)\right]$

we are given $\frac{1}{2}\left(x - y\frac{dx}{dy}\right) = x$ and $\frac{1}{2}\left(y - x\frac{dy}{dx}\right) = y$

$\Rightarrow \; x\dfrac{dy}{dx} = -y \Rightarrow \dfrac{dy}{y} = -\dfrac{dx}{x}$

Integrating both sides $\int \dfrac{dy}{y} = -\int \dfrac{dy}{x}$, we get

$\Rightarrow \; \log y = -\log x + c$

put $x=1, y=1, \Rightarrow \log 1 - \log 1 = c \Rightarrow c = 0$

$\Rightarrow \; \log y + \log x = 0 \Rightarrow \log yx = 0$

$\Rightarrow \; yx = e^0 = 1$ which is a rectangular hyperbola.

7. Equation of normal is $\dfrac{dx}{dy}(X-x)+Y-y=0$

Given that perpendicular distance of the origin form the normal at P = distance of P form the x-axis

$$\Rightarrow \frac{\left|x\dfrac{dx}{dy}+y\right|}{\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}} = |y|$$

$$\Rightarrow x^2\left(\frac{dx}{dy}\right)^2 + y^2 + 2xy\frac{dx}{dy} = y^2 + y^2\left(\frac{dx}{dy}\right)^2$$

$$\Rightarrow \left(\frac{dx}{dy}\right) = 0 \quad \text{or} \quad \frac{dx}{dy} = \left(\frac{2xy}{y^2-x^2}\right)$$

If $\dfrac{dx}{dy} = 0$, then $x = c$, when $x = 1, y = 1 \Rightarrow c = 1$

$\therefore x = 1$ (1)

When $\dfrac{dx}{dy} = \dfrac{2xy}{y^2-x^2}$ (homogeneous) (2)

Putting, $x = vy \Rightarrow \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$

$\therefore$ equation (2) transforms to

$$v + y\frac{dv}{dy} = \frac{2v}{1-v^2}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{2v}{1-v^2} - v = \frac{2v-v+v^3}{1-v^2} = \frac{v+v^3}{1-v^2}$$

$$\Rightarrow \int\frac{(1-v^2)dv}{v(1+v^2)} = \int\frac{dv}{y}$$

$$\Rightarrow \int\left(\frac{1}{v} - \frac{2v}{1+v^2}\right)dy = \int\frac{dy}{y}$$

$\Rightarrow \; \log v - \log(1+v^2) = \log y + C$

$$\Rightarrow \frac{v}{1+v^2} = cy$$

$$\Rightarrow \frac{x}{x^2+y^2} = c$$

when $x = 1, y = 1$ gives $c = 1/2$

$\therefore$ solution is $x^2 + y^2 - 2x = 0$

Hence the solution are $x^2 + y^2 - 2x = 0, x - 1 = 0$.

8. Let P_0 be the initial population of country and P be the population of country in year t then

$\dfrac{dP}{dt}$ = rate of change of population $= \dfrac{3}{100}P = 0.03\,P$

$\therefore$ population of P at the end of n years is given by

$$\int_{P_0}^{P}\frac{dP}{P} = \int_0^n 0.03\,dt$$

$\Rightarrow \; \ln P - \ln P_0 = (0.03)\,n$

$\Rightarrow \; \ln P = \ln P_0 + (0.03)\,n$ (1)

If F_0 be its initial food production and F be the food production in year n.

Then $F_0 = 0.9\,P_0$

and $F = (1.04)^n F_0$

$\Rightarrow \; \ln F = n\ln(1.04) + \ln F_0$ (2)

$\because$ the country will be self sufficient if $F \geq P$

$\Rightarrow \; \ln F \geq \ln P$

$\Rightarrow \; n\ln(1.04) + \ln F_0 \geq \ln P_0 + (0.03)\,n$

$$\Rightarrow n \geq \frac{\ln P_0 - \ln F_0}{\ln(1.04) - (.03)} = \frac{\ln 10 - \ln 9}{\ln(1.04) - 0.03}$$

Hence $n \geq \dfrac{\ln 10 - \ln 9}{\ln(1.04) - 0.03}$

Thus, the least integral values of the year n, when the country becomes self-sufficient, is the smallest integer greater than or equal to $\dfrac{\ln 10 - \ln 9}{\ln(1.04) - 0.03}$.

9. Given that $F(x) = \int_0^x f(t)\,dt$

$\therefore \; F'(x) = f(x)$ (1) [Using Leibnitz, theorem]

Also given that $f(x) \leq c\,F(x)$, $\forall x \geq 0$

$\therefore$ we get $f(0) \leq cF(0) = 0$

$\therefore \; f(0) \leq 0$ (2)

But given that $f(x)$ is non-negative function on $[0, \infty)$

$\therefore \; f(x) \geq 0$

$\therefore \; f(0) \geq 0$ (3)

$\therefore$ from equations (2) and (3) $f(0) = 0$

Again $f(x) \leq cF(x)$, $\forall x \geq 0$, we get

$f(x) - cF(x) \leq 0$

$\Rightarrow \; F'(x) - cF(x) \leq 0, \forall x \geq 0$ [Using equation (1)]

$\Rightarrow \; e^{-cx}F'(x) - ce^{-cx}F(x) \leq 0$

[Multiplying both sides by e^{-cx} (I.F.) and keeping in mind that $e^{-cx} > 0, \forall x$]

$$\Rightarrow \frac{d}{dx}\left[e^{-cx}F(X)\right] \leq 0$$

$\Rightarrow \; g(x) = e^{-cx}F(x)$ is a decreasing function on $[0, \infty)$.

That is $g(x) \leq g(0)$ for all $x \geq 0$

But $g(0) = F(0) = 0$

$\therefore g(x) \leq 0, \forall x \geq 0$

$\Rightarrow \; e^{-cx}F(x) \leq 0, \forall x \geq 0$

$\Rightarrow \; F(x) \leq 0, \forall x \geq 0$

$\Rightarrow \; \therefore f(x) \leq cF(x) \leq 0; \forall x \geq 0$ [$\because c > 0$ and using $f(x) \leq cF(x)$]

$\Rightarrow \; f(x) \leq 0, \forall x \geq 0$

But given $f(x) \geq 0$

$\Rightarrow \quad f(x) = 0, \forall x \geq 0$

10. Let the water level be at a height h after time t, and water level falls by dh in time dt and the corresponding volume of water gone out be dV.

$\Rightarrow \quad dV = -\pi r^2\, dh$

$$\Rightarrow \quad \frac{dV}{dt} = -\pi r^2 \frac{dh}{dt} \qquad (\because \text{ as } t \text{ increases, } h \text{ decreases})$$

Now, velocity of water, $v = \frac{3}{5}\sqrt{2gh}$

Rate of flow of water = Av ($A = 12$ cm^2)

$$\Rightarrow \quad \frac{dV}{dt} = \left(\frac{3}{5}\sqrt{2gh}A\right) = -\pi r^2 \frac{dh}{dt}$$

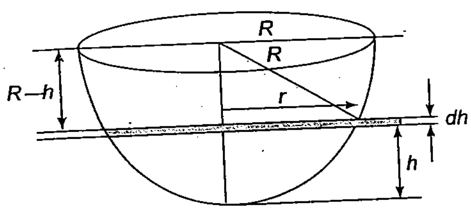


Fig. 10.19

Also from the figure,

$R^2 = (R-h)^2 + r^2 \Rightarrow r^2 = 2hR - h^2$

$$\text{So, } \frac{3}{5}\sqrt{2g}\ \sqrt{h}A = -\pi(2hR - h^2) \times \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{2hR - h^2}{\sqrt{h}}\, dh = -\frac{3}{5\pi}\sqrt{2g}\, A dt$$

Integrating, we get

$$\int_R^0 (2R\sqrt{h} - h^{3/2})dh = -\frac{3\sqrt{2g}}{5\pi} A \int_0^T dt$$

$$\Rightarrow \quad T = -\frac{5\pi}{3A\sqrt{2g}}\left(2R\frac{h^{3/2}}{3/2} - \frac{h^{5/2}}{5/2}\right)\Bigg|_R^0$$

$$= \frac{5\pi}{3A\sqrt{2g}}\left(\frac{4R}{3}R^{3/2} - \frac{2}{5}R^{5/2}\right)$$

$$= \frac{5\pi}{3A\sqrt{2g}}\frac{14}{15}R^{5/2}$$

$$= \frac{56\pi}{9A\sqrt{g}}(10)^5 \quad (R = 200 \text{ cm})$$

$$= \frac{56\pi}{9 \times 12\sqrt{g}}(10)^5$$

$$= \frac{14\pi}{27\sqrt{g}}(10)^5 \text{ units}$$

11. Let at time t, r, and h be the radius and height of cone of water.

$\therefore$ at time t, surface area of liquid in contact with air $= \pi r^2$

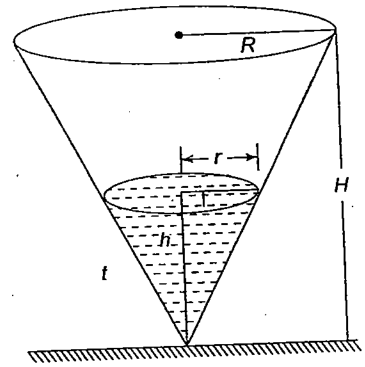


Fig. 10.20

Now according to questions $-\frac{dV}{dt} \propto \pi r^2$

[$\because$ '–' sign shows that V decreases with time]

$$\Rightarrow \quad \frac{1}{3}\pi \frac{dV}{dt} = -k\pi r^2$$

But from the figure, we get $\frac{r}{h} = \frac{R}{H}$ [Similar Δ's]

$$\Rightarrow \quad h = \frac{rH}{R}$$

$$\Rightarrow \quad \frac{1}{3}\frac{d}{dt}\left[r^2 \frac{rH}{R}\right] = -kr^2$$

$$\Rightarrow \quad \frac{r^2 H}{R}\frac{dr}{dt} = -kr^2$$

$$\Rightarrow \quad \frac{dr}{dt} = -\frac{kR}{H}$$

$$\Rightarrow \quad r = \frac{-kR}{H}t + C \qquad \text{(integrating)}$$

Now at $t = 0, r = R$

$\Rightarrow \quad R = 0 + C \Rightarrow C = R$

$$\Rightarrow \quad r = \frac{-kRt}{H} + R$$

Now let the time at which cone is empty be T then at T, $r = 0$ (no liquid is left)

$$\therefore \quad 0 = \frac{-kRT}{H} + R \Rightarrow T = H/k$$

12. According to the question, slope of curve C at (x, y)

$$= \frac{(x+1)^2 + (y-3)}{(x+1)}$$

$$\Rightarrow \quad \frac{dy}{dx} = (x+1) + \frac{y-3}{x+1}$$

$$\Rightarrow \quad \frac{dy}{dx} - \left(\frac{1}{x+1}\right)y = x + 1 - \frac{3}{x+1}$$

which is a linear differential equation.

$$\text{I.F.} = e^{-\int \frac{dx}{x+1}} = e^{-\log(x+1)} = \frac{1}{x+1}$$

$$\therefore \quad \text{solution is } y\frac{1}{x+1} = \int\left[1 - \frac{3}{(x+1)^2}\right]dx$$

$\Rightarrow \quad \frac{y}{x+1} = x + \frac{3}{x+1} + C$

$\Rightarrow \quad y = x(x+1) + 3 + C(x+1)$ (1)

As the curve passes through (2, 0),

$\therefore 0 = 2.3 + 3 + C.3$

$\Rightarrow \quad C = -3$

$\therefore$ equation (1) becomes

$y = x(x+1) + 3 - 3x - 3$

$y = x^2 - 2x$

which is the required equation of curve.

This can be written as $(x-1)^2 = (y+1)$

[Upward parabola with vertex at $(1, -1)$ meeting x-axis at (0, 0) and (2, 0)]

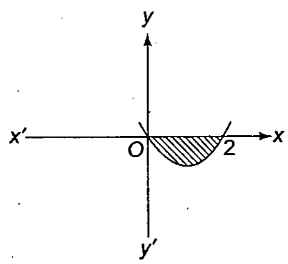


Fig. 10.21

Area bounded by curve and x-axis in fourth quadrant is as shaded region in the figure given by

$$A = \left|\int_0^2 y dx\right| = \left|\int_0^2 (x^2 - 2x)\right| dx = \left|\frac{x^3}{3} - x^2\right|_0^2$$

$$= \left|\frac{8}{3} - 4\right| = \frac{4}{3} \text{ sq. units}$$

13. Given length of tangent to curve $y = f(x) = 1$

$$\Rightarrow \left|\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\left(\frac{dy}{dx}\right)}\right| = 1$$

$$\Rightarrow y^2\left(1+\left(\frac{dy}{dx}\right)^2\right) = \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{y^2}{1-y^2}$$

$$\Rightarrow \frac{dy}{dx} = \pm\frac{y}{\sqrt{1-y^2}}$$

$$\Rightarrow \int\frac{\sqrt{1-y^2}}{y} dy = \int \pm dx$$

put $y = \sin\theta$ so that $dy = \cos\theta d\theta$

$$\Rightarrow \int\frac{\cos\theta}{\sin\theta}\cos\theta d\theta = \pm x + c$$

$$\Rightarrow \int(\operatorname{cosec}\theta - \sin\theta)\, d\theta = \pm x + c$$

$$\Rightarrow \log|\operatorname{cosec}\theta - \cot\theta| + \cos\theta = \pm x + c$$

$$\Rightarrow \log\left|\frac{1-\sqrt{1-y^2}}{y}\right| + \sqrt{1-y^2} = \pm x + c$$

Objective

Fill in the blanks

1. If S denotes the surface area and V the volume of the rain drop then according to the question

$$\frac{dV}{dt} \propto -S$$

$$\Rightarrow \frac{dV}{dt} = -kS \text{ where } k > 0$$

(– ve sign shows V decreases with time)

$$\Rightarrow \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] = -k(4\pi r^2)$$

$$\Rightarrow 4\pi r^2\frac{dr}{dt} = -k(4\pi r^2)$$

$$\Rightarrow \frac{dr}{dt} = -k$$

Multiple choice questions with one correct answer

1. a. Slope of the normal at $(1, 1) = -\frac{1}{a}$

Slope of tangent at $(1, 1) = a$

i.e., $\left(\frac{dy}{dx}\right)_{(1,1)} = a$

Since $\frac{dy}{dx}$ is proportional to y,

$\therefore \quad \frac{dy}{dx} = Ky$

$\Rightarrow \quad \frac{dy}{y} = K\,dx$

$\Rightarrow \quad \log y = Kx + C$

$\Rightarrow \quad y = e^{Kx+C} = Ae^{Kx}$ where $A = e^c$

It passes through (1, 1)

$\therefore \quad 1 = Ae^K \therefore A = e^{-K}$

$\therefore \quad y = e^{-K}e^{Kx} = e^{K(x-1)}$

2. c. $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$

By verification we find that the choice (c), i.e., $y = 2x - 4$ satisfies the given differential equations.

Alternate

$$\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0.$$

$$\Rightarrow \frac{dy}{dx} = \frac{x \pm\sqrt{x^2-4y}}{2} \quad (1)$$

Let $x^2 - 4y = t^2$

$$\Rightarrow 2x - 4\frac{dy}{dx} = 2t\frac{dt}{dx}$$

$$\Rightarrow x - 2\frac{dy}{dx} = t\frac{dt}{dx}$$

Then equation (1) changes to $x - t\frac{dt}{dx} = x \pm t$

$\Rightarrow \quad \frac{dt}{dx} = \pm 1$ or $t = 0$

$\Rightarrow \quad t = \pm x + c$ or $x^2 = 4y$

$\Rightarrow \quad x^2 - 4y = x^2 \pm 2cx + c^2$

$\Rightarrow \quad -4y = \pm 2cx + c^2$

For $c = 4$

$4y = \pm 8x - 16$ or $y = 2x - 4$

3. a. The given differential equation is

$$\frac{dy}{dt} - \frac{t}{1+t} y = \frac{1}{1+t}$$

$$\text{I.F.} = e^{-\int \frac{t}{1+t} dt}$$

$$= e^{-\int \left(1 - \frac{1}{1+t}\right) dt}$$

$$= e^{-(t - \log(1+t))}$$

$$= e^{-t} e^{\log(1+t)} = (1+t)e^{-t}$$

$\therefore$ solution is

$$ye^{-t}(1+t) = \int \frac{1}{(1+t)} e^{-t}(1+t) dt + C$$

$\Rightarrow \quad ye^{-t}(1+t) = -e^{-t} + C$

Given that $y(0) = -1$

$\Rightarrow \quad -1 = -1 + C$

$\Rightarrow \quad C = 0$

$\therefore \quad y = -\frac{1}{1+t}$

$\therefore \quad y(1) = -\frac{1}{1+1} = -1/2$

4. a. $\frac{dy}{dx}\left(\frac{2 + \sin x}{1+y}\right) = -\cos x, y(0) = 1$

$\Rightarrow \quad \frac{dy}{1+y} = \frac{-\cos x}{2 + \sin x} dx$

Integrating both sides, we get

$\Rightarrow \quad \ln(1+y) = -\ln(2 + \sin x) + C$

Put $x = 0$ and $y = 1$

$\Rightarrow \quad \ln 2 = -\ln 2 + C$

$\Rightarrow C = \ln 4$

Put $x = \pi/2$

$\ln(1+y) = -\ln 3 + \ln 4 = \ln 4/3$

$\Rightarrow \quad y = 1/3$

5. c. The given differential equation is $(x^2 + y^2)dy = xy\, dx$ such that $y(1) = 1$ and $y(x_0) = e$

The given equation can be written as

$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$ (homogeneous equation)

Put $y = vx$ to get $v + x\frac{dv}{dx} = \frac{v}{1+v^2}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v^3}{1+v^2}$

$\Rightarrow \quad \int \frac{1+v^2}{v^3} dv + \int \frac{dx}{x} = 0$

$\Rightarrow \quad -\frac{1}{2v^2} + \log|v| + \log|x| = C$

$\Rightarrow \quad \log y = C + \frac{x^2}{2y^2}$ (using $v = y/x$)

Also $y(1) = 1$

$\Rightarrow \quad \log 1 = C + \frac{1}{2} \Rightarrow C = -\frac{1}{2}$

$\therefore \quad \log y = \frac{x^2 - y^2}{2y^2}$

Given $y(x_0) = e$

$\Rightarrow \quad \log e = \frac{x_0^2 - e^2}{2e^2}$

$\Rightarrow \quad x_0^2 = 3e^2 \Rightarrow x_0 = \sqrt{3}e$

6. a. The given equation is

$ydx + y^2 dy = x\, dy; x \in R, y > 0, y(1) = 1$

$\Rightarrow \quad \frac{ydx - xdy}{y^2} + dy = 0$

$\Rightarrow \quad \frac{d}{dx}\left(\frac{x}{y}\right) + dy = 0$

On integrating, we get $\frac{x}{y} + y = C$

$y(1) = 1 \Rightarrow 1 + 1 = C \Rightarrow C = 2$

$\therefore \quad \frac{x}{y} + y = 2$

Now to find $y(-3)$, putting $x = -3$ in the above equation

we get, $-\frac{3}{y} + y = 2$

$\Rightarrow \quad y^2 - 2y - 3 = 0 \Rightarrow y = 3, -1$

but given that $y > 0 \therefore y = 3$

7. c. $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y}$

$\Rightarrow \quad \frac{-2y}{\sqrt{1-y^2}} dy + 2dx = 0$

$\Rightarrow \quad 2\sqrt{1-y^2} + 2x = 2c$

$\Rightarrow \quad \sqrt{1-y^2} + x = c$

$\Rightarrow \quad (x-c)^2 + y^2 = 1$

which is a circle of fixed radius 1 and variable centre $(c, 0)$ lying on x-axis.

Multiple choice questions with one or more than one correct answers

1. c. $y = (C_1 + C_2)\cos(x + C_3) - C_4 e^{x + C_5}$

$= (C_1 + C_2)\cos(x + C_3) - C_4 e^{C_5} e^x$

$= A\cos(x + C_3) - Be^x$ [Taking $C_1 + C_2 = A, C_4 e^{C_5} = B$]

Thus, there are actually three arbitrary constants and hence this differential equation should be of order 3.

2. a, c. $y^2 = 2c\left(x + \sqrt{c}\right)$

Differentiating w.r.t. x, we get

$2yy' = 2c \Rightarrow c = yy'$

Eliminating c, we get

$y^2 = 2yy_1(x + \sqrt{yy_1})$ or $(y^2 - 2xyy_1)^2 = 4y^3y_1^3$

It involves only first order derivative, its order is 1 but its degree is 3 as y_1^3 is there.

3. c, d. Tangent to the curve $y = f(x)$ at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

$$\therefore \quad A\left(\frac{x\frac{dy}{dx} - y}{\frac{dy}{dx}}, 0\right); B\left(0, -x\frac{dy}{dx} + y\right)$$

$\because \quad BP : PA = 3 : 1$

$$\Rightarrow \quad x = \frac{\frac{3\left(x\frac{dy}{dx} - y\right)}{\frac{dy}{dx}} + 1 \times 0}{4}$$

$$\Rightarrow \quad x\frac{dy}{dx} + 3y = 0$$

$$\Rightarrow \quad \int \frac{dy}{y} = \int -3\frac{dx}{x}$$

$$\Rightarrow \quad \log y = -3\log x + \log c$$

$$\Rightarrow \quad y = \frac{c}{x^3}$$

As curve passes through (1, 1), $c = 1$

$\therefore$ curve is $x^3 y = 1$ which also passes through (2, 1/18).

Integer type

1.(9) Equation of tangent to $y = f(x)$ at point (x_1, y_1) is

$y - y_1 = m(x - x_1)$

Put $x = 0$, to get y intercept

$y_1 - mx_1 = x_1^3$ (given)

$$y_1 - x_1\frac{dy}{dx} = x_1^3$$

$$\therefore \quad x\frac{dy}{dx} - y = -x^3$$

$$\therefore \quad \frac{dy}{dx} - \frac{y}{x} = -x^2$$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\therefore \quad \text{Solution is } y \times \frac{1}{x} = \int -x^2 \times \frac{1}{x}\,dx$$

$$\text{Or} \quad \frac{y}{x} = -\frac{x^2}{2} + c$$

$$\Rightarrow \quad f(x) = -\frac{x^3}{2} + \frac{3}{2}x \text{ (as } f(1) = 1)$$

$\therefore \quad f(-3) = 9.$

2.(0) $y'(x) + y(x)g'(x) = g(x)g'(x)$

$$\Rightarrow \quad e^{g(x)}y'(x) + e^{g(x)}g'(x)y(x) = e^{g(x)}g(x)g'(x)$$

$$\Rightarrow \quad \frac{d}{dx}(y(x)e^{g(x)}) = e^{g(x)}g(x)g'(x)$$

$$\therefore \quad y(x)e^{g(x)} = \int e^{g(x)}g(x)g'(x)dx$$

$$= \int e^t\, t\,dt, \text{ where } g(x) = t$$

$$= (t - 1)e^t + c$$

$$\therefore \quad y(x)e^{g(x)} = (g(x) - 1)e^{g(x)} + c$$

Put $x = 0 \Rightarrow 0 = (0 - 1)\cdot 1 + c \Rightarrow c = 1$

Put $x = 2 \Rightarrow y(2)\cdot 1 = (0 - 1)\cdot(1) + 1$

$y(2) = 0.$

3.(6) $6\int_1^x f(t)\,dt = 3xf(x) - x^3$

$$\Rightarrow \quad 6f(x) = 3f(x) + 3xf'(x) - 3x^2$$

$$\Rightarrow \quad xf'(x) - f(x) = x^2$$

$$\Rightarrow \quad x\frac{dy}{dx} - y = x^2$$

$$\Rightarrow \quad \frac{xdy - ydx}{x^2} = dx$$

$$\Rightarrow \quad \int \frac{xdy - ydx}{x^2} = \int dx$$

$$\Rightarrow \quad \frac{y}{x} = x + c$$

Given $f(1) = 2$

$\Rightarrow \quad c = 1$

$\Rightarrow \quad y = x^2 + x$

Note

If we put $x = 1$ in the given equation we get $f(1) = 1/3$.

Appendix

Solutions to Concept Application Exercises

Chapter 1

Exercise 1.1

1. $f(x)=\dfrac{x-3}{(x+3)\sqrt{x^2-4}}$

   we must have $x^2-4>0$ and $x\neq -3$
   $\Rightarrow$ Domain is $x\in(-\infty,-3)\cup(-3,-2)\cup(2,\infty)$.

2. $f(x)=\sqrt{2-x}-\dfrac{1}{\sqrt{9-x^2}}$

   We must have (i) $x\le 2$ and (ii) $9-x^2>0 \Rightarrow |x|<3$ or $-3<x<3$.
   Hence, domain is $(-3, 2]$

3. $f(x)=\sqrt{\dfrac{x-2}{x+2}}+\sqrt{\dfrac{1-x}{1+x}}$

   We must have $\dfrac{x-2}{x+2}\ge 0$ and $\dfrac{1-x}{1+x}\ge 0$.

   $\dfrac{x-2}{x+2}\ge 0 \Rightarrow x\ge 2$ or $x<-2$.

   $\dfrac{1-x}{1+x}\ge 0 \Rightarrow -1<x\le 1$

   Hence, the given function has empty domain.

4. $f(x)=\sqrt{\left(\dfrac{2}{x^2-x+1}-\dfrac{1}{x+1}-\dfrac{2x-1}{x^3+1}\right)}$

   We must have $\dfrac{2}{x^2-x+1}-\dfrac{1}{x+1}-\dfrac{2x-1}{x^3+1}\ge 0$

   $\Rightarrow \dfrac{2(x+1)-(x^2-x+1)-(2x-1)}{(x+1)(x^2-x+1)}\ge 0$

   $\Rightarrow \dfrac{-(x^2-x-2)}{(x+1)(x^2-x+1)}\ge 0$

   $\Rightarrow \dfrac{-(x-2)(x+1)}{(x+1)(x^2-x+1)}\ge 0$

   $\Rightarrow \dfrac{2-x}{x^2-x+1}\ge 0$, where $x\neq -1$

   $\Rightarrow 2-x\ge 0, x\neq -1$ (as $x^2-x+1>0$ for $\forall\ x\in R$)
   $\Rightarrow x\le 2, x\neq -1$
   Hence, domain of the function is $(-\infty,-1)\cup(-1,2]$.

5. $f(x)=\sqrt{x-\sqrt{1-x^2}}$ to get defined $x-\sqrt{1-x^2}\ge 0$

   $\Rightarrow x\ge\sqrt{1-x^2}$
   $\Rightarrow x$ is positive and $x^2\ge 1-x^2$
   $\Rightarrow x^2\ge 1/2$
   $\Rightarrow x\in\left[\dfrac{1}{\sqrt{2}},1\right]$ $(\because -1\le x\le 1)$

6. $f(x)=\dfrac{x^2+1}{x^2+2}=\dfrac{x^2+2-1}{x^2+2}=1-\dfrac{1}{x^2+2}$

   Now $x^2+2\ge 2, \forall x\in R$

   $\Rightarrow\ 0<\dfrac{1}{x^2+2}\le\dfrac{1}{2}\ \Rightarrow\ -\dfrac{1}{2}\le-\dfrac{1}{x^2+2}<0$

   $\Rightarrow\ \dfrac{1}{2}\le 1-\dfrac{1}{x^2+2}<1$

7. Using wavy curve method and the fact that $x=0$ and 3 are the repeated roots of $x(e^x-1)(x+2)(x-3)^2=0$, we get the sign scheme of the given expression as

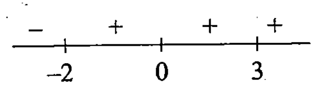


Fig. S-1.1

   Thus, the complete solution set is $x\in(-\infty,-2]\cup\{0,3\}$.

Exercise 1.2

1. Let $\dfrac{x^2+34x-71}{x^2+2x-7}=y$

   $\Rightarrow x^2(1-y)+2(17-y)x+(7y-71)=0$
   For the real value of x, $b^2-4ac\ge 0$
   $\Rightarrow y^2-14y+45\ge 0$
   $\Rightarrow y\le 5$ or $y\ge 9$
   Hence, range is $(-\infty,5]\cup[9,\infty)$.

2. Let $y=\sqrt{x-1}+\sqrt{5-x}$

   $\Rightarrow y^2=x-1+5-x+2\sqrt{(x-1)(5-x)}$

   $\Rightarrow y^2=4+2\sqrt{-x^2-5+6x}$

   $\Rightarrow y^2=4+2\sqrt{4-(x-3)^2}$
   Then y^2 has minimum value 4 [when $4-(x-3)^2=0$] and the maximum value 8 when $x=3$.

   $\Rightarrow y\in[2, 2\sqrt{2}]$

3. $f(x)=\sqrt{x^2+ax+4}$ is defined for all x.

   $\Rightarrow x^2+ax+4\ge 0$ for all x

$\Rightarrow D = a^2 - 16 \le 0$
$\Rightarrow a \in [-4, 4]$

4. $f(x) = \sqrt{3 - 2x - x^2}$ is defined if $3 - 2x - x^2 \ge 0$
$\Rightarrow x^2 + 2x - 3 \le 0$
$\Rightarrow (x-1)(x+3) \le 0$
$\Rightarrow x \in [-3, 1]$
Also $f(x) = \sqrt{4 - (x+1)^2}$ which has maximum value when $x + 1 = 0$.
Hence, the range is $[0, 2]$.

Exercise 1.3

1. a. $1 \le |x-2| \le 3$
We know that $a \le |x| \le b \Leftrightarrow x \in [-b, -a] \cup [a, b]$.
Given that $1 \le |x-2| \le 3$
$\Rightarrow (x-2) \in [-3, -1] \cup [1, 3]$
$\Rightarrow x \in [-1, 1] \cup [3, 5]$

b. $0 < |x-3| \le 5$
$\Rightarrow x - 3 \ne 0$ and $|x-3| \le 5$
$\Rightarrow x \ne 3$ and $-5 \le x - 3 \le 5$
$\Rightarrow x \ne 3$ and $-2 \le x \le 8$
$\Rightarrow x \in [-2, 3) \cup (3, 8]$

c. $|x-2| + |2x-3| = |x-1|$
$\Rightarrow |x-2| + |2x-3| = |(2x-3) + (2-x)|$
$\Rightarrow (x-2)(2x-3) \le 0$
$\Rightarrow 3/2 \le x \le 2$
$\Rightarrow x \in [3/2, 2]$

d. $\left|\dfrac{x-3}{x+1}\right| \le 1$

$$\Rightarrow -1 \le \frac{x-3}{x+1} \le 1$$

$$\Rightarrow \frac{x-3}{x+1} - 1 \le 0 \text{ and } 0 \le \frac{x-3}{x+1} + 1$$

$$\Rightarrow \frac{-4}{x+1} \le 0 \text{ and } 0 \le \frac{2x-2}{x+1}$$

$\Rightarrow x > -1$ and $\{x < -1$ or $x \ge 1\}$
$\Rightarrow x \ge 1$

2. a. $f(x) = \dfrac{1}{\sqrt{x - |x|}}$

$$x - |x| = \begin{cases} x - x = 0, & \text{if } x \ge 0 \\ x + x = 2x, & \text{if } x < 0 \end{cases}$$

$\Rightarrow x - |x| \le 0$ for all x

$\Rightarrow \dfrac{1}{\sqrt{x - |x|}}$ does not take real values for any $x \in R$

$\Rightarrow f(x)$ is not defined for any $x \in R$
Hence, the domain (f) is ϕ.

b. $f(x) = \dfrac{1}{\sqrt{x + |x|}}$

$$x + |x| = \begin{cases} x + x, & \text{if } x \ge 0 \\ x - x, & \text{if } x < 0 \end{cases}$$

$$\Rightarrow x + |x| = \begin{cases} 2x, & \text{if } x \ge 0 \\ 0, & \text{if } x < 0 \end{cases} \quad (1)$$

Now, $f(x) = \dfrac{1}{\sqrt{x + |x|}}$ assumes real values, if $x + |x| > 0$

$\Rightarrow x > 0$ [Using (1)] $\Rightarrow x \in (0, \infty)$
Hence, domain $(f) = (0, \infty)$.

3. Given $|2x+3| + |2x-3| = \begin{cases} 4x & \text{if } x \ge \frac{3}{2} \\ 6 & \text{if } -\frac{3}{2} < x < \frac{3}{2} \\ -4x & \text{if } x \le -\frac{3}{2} \end{cases}$

and $y = ax + 6$

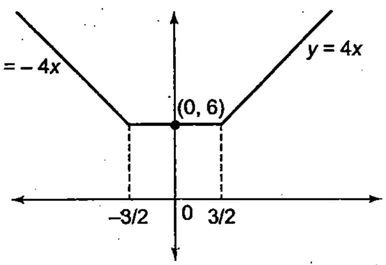


Fig. S-1.2

From the graph, it is obvious that if,
$a = 0$ we have infinite solutions in the range $\left[-\frac{3}{2}, \frac{3}{2}\right]$
if $0 < a < 4$ or $-4 < a < 0$, two solutions,
if $a = 4$ or -4 we have $x = 0$ is the only solution.

4. $f(x)$ can be rewritten as

$$f(x) = \begin{cases} a + b + c - 3x, & x < a \\ b + c - a - x, & a \le x < b \\ c - a - b + x, & b \le x < c \\ 3x - a - b - c, & x \ge c \end{cases}$$

Graph of $f(x)$ is shown in Fig. S-1.3
Clearly the minimum value of $f(x)$ will occur at $x = b$ which is $c - a$.

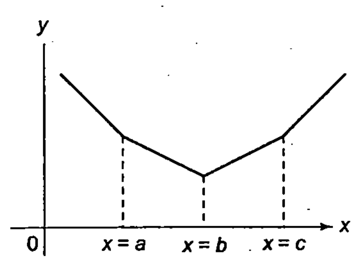


Fig. S-1.3

5. $f(x) = \sqrt{1 - \sqrt{x^2 - 6x + 9}} = \sqrt{1 - \sqrt{(x-3)^2}} = \sqrt{1 - |x-3|}$
$\Rightarrow$ Range of $f(x)$ is $[0, 1]$.

Exercise 1.4

1. $f(x)=\sqrt{\sin x}+\sqrt{16-x^2}$

$\Rightarrow \sin x \geq 0$ and $16-x^2 \geq 0$

$\Rightarrow 2n\pi \leq x \leq (2n+1)\pi$ and $-4 \leq x \leq 4$

$\therefore$ domain is $[-4,-\pi] \cup [0, \pi]$.

2. a. We know that $\tan x$ is periodic with period π. So, check the solution in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

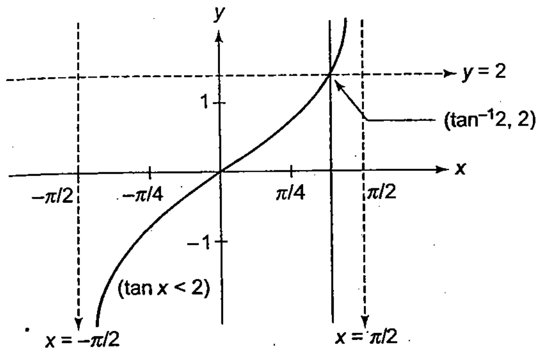


Fig. S-1.4

It is clear from Fig. S-1.4, $\tan x<2$ when $-\frac{\pi}{2}<x<\tan^{-1} 2$

$\Rightarrow$ General solution is $n\pi-\frac{\pi}{2}<x<n\pi+\tan^{-1} 2$

$\Rightarrow n \in \left(n\pi-\frac{\pi}{2}, n\pi+\tan^{-1} 2\right)$

b. $\cos x$ is periodic with period 2π.
So, check the solution in $[0, 2\pi]$.

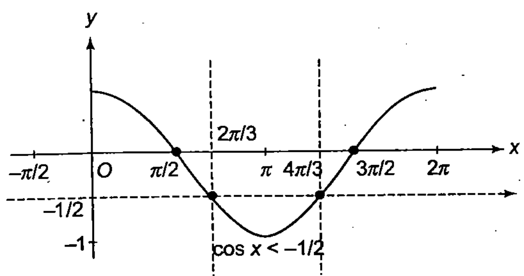


Fig. S-1.5

It is clear from Fig. S-1.5, $\cos x \leq -\frac{1}{2}$ when $\frac{2\pi}{3} \leq x \leq \frac{4\pi}{3}$.

On generalizing the above solution, we get

$2n\pi+\frac{2\pi}{3} \leq x \leq 2n\pi+\frac{4\pi}{3}$; $n \in Z$

$\therefore$ solution of $\cos x \leq -\frac{1}{2}=x \in \left[2n\pi+\frac{2\pi}{3}, 2n\pi+\frac{4\pi}{3}\right]$;

$n \in Z$

3. Let $f(x)=\tan x$ and $g(x)=x+1$; which could be shown as

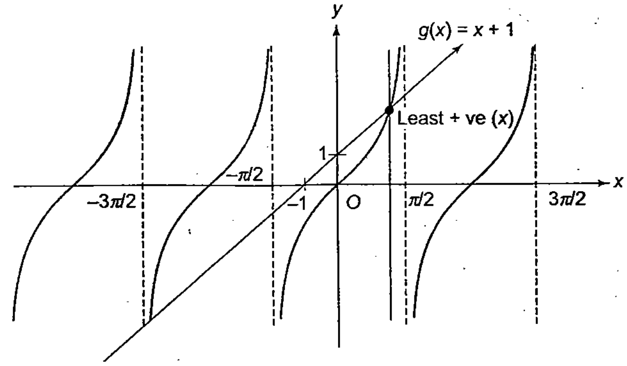


Fig. S-1.6

From Fig. S-1.6, $\tan x=x+1$ has infinitely many solutions but the least positive value of $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

4. $f(x)=\sec\left(\frac{\pi}{4}\cos^2 x\right)$

We know that, $0 \leq \cos^2 x \leq 1$.

$\Rightarrow \quad 0 \leq \frac{\pi}{4}\cos^2 x \leq \frac{\pi}{4}$

For the above value of $\theta=\frac{\pi}{4}\cos^2 x$, $\sec x$ is an increasing function.

at $\cos x=0$, $f(x)=1$ and at $\cos x=1$, $f(x)=\sqrt{2}$

$\therefore \quad 1 \leq x \leq \sqrt{2} \quad \Rightarrow x \in [1, \sqrt{2}]$

5. $f(x)=\tan x, x \in [1, 2]$ (see Fig. S-1.7)

Here the limited values of x are given

The best way to get the range of $\tan x$ for such values of x is graphical one.

Consider the graph of $f(x)=\tan x$ for $x \in [1, 2]$

Clearly from the graph, $\tan x \in (-\infty, \tan 2] \cup [\tan 1, \infty)$

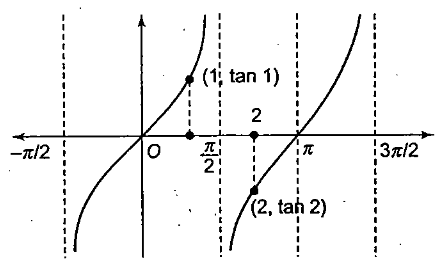


Fig. S-1.7

6. $f(x)=\dfrac{1}{1-3\sqrt{1-\sin^2 x}}$

$=\dfrac{1}{1-3\sqrt{\cos^2 x}}$

$=\dfrac{1}{1-3|\cos x|}$

Now $-3|\cos x| \in [-3, 0]$

$1-3|\cos x| \in [-2, 1]$

$\Rightarrow \quad \dfrac{1}{1-3|\cos x|} \in (-\infty, -1/2] \cup [1, \infty)$

Exercise 1.5

1.

a. $f(x)$ is defined if $x \in [-1, 1]$ and $x \neq 0$

$\Rightarrow \quad x \in [-1, 0) \cup = (0, 1]$

b. $f(x) = \sin^{-1}(|x-1|-2)$

Since the domain of $\sin^{-1} x$ is $[-1, 1]$. Therefore, $f(x)$ is defined, if $-1 \le |x-1|-2 \le 1$

$\Rightarrow 1 \le |x-1| \le 3$

$\Rightarrow -3 \le x-1 \le -1$, or $1 \le x-1 \le 3$

$\Rightarrow -2 \le x \le 0$, or $2 \le x \le 4$

$\Rightarrow$ domain $= [-2, 0] \cup [2, 4]$

c. $-1 \le 1 + 3x + 2x^2 \le 1$

$2x^2 + 3x + 1 \ge -1$

or $2x^2 + 3x + 2 \ge 0$ (1)

and $2x^2 + 3x \le 0$ (2)

From equation (2), $2x^2 + 3x \le 0 \Rightarrow 2x\left(x + \frac{3}{2}\right) \le 0$

$\Rightarrow \frac{-3}{2} \le x \le 0 \quad \Rightarrow x \in \left[-\frac{3}{2}, 0\right]$

In equation (1), we get imaginary root for $2x^2 + 3x + 2 = 0$ and $2x^2 + 3x + 2 \ge 0$ for all x.

$\therefore$ domain of function $= \left[-\frac{3}{2}, 0\right]$

d. To define $f(x)$, $9 - x^2 > 0 \Rightarrow -3 < x < 3$ (1)

$-1 \le (x-3) \le 1 \Rightarrow 2 \le x \le 4$ (2)

From equations (1) and (2), $2 \le x < 3$, i.e., $[2, 3)$

e. $f(x) = \cos^{-1}\left(\frac{6-3x}{4}\right) + \operatorname{cosec}^{-1}\left(\frac{x-1}{2}\right)$

For $\cos^{-1}\left(\frac{6-3x}{4}\right)$

$\Rightarrow -1 \le \frac{6-3x}{4} \le 1$

$\Rightarrow -4 \le 6 - 3x \le 4$

$\Rightarrow -10 \le -3x \le -2$

$\Rightarrow 2/3 \le x \le 10/3$ (1)

For $\operatorname{cosec}^{-1}\left(\frac{x-1}{2}\right)$

$\frac{x-1}{2} \le -1$ or $\frac{x-1}{2} \ge 1$

$\Rightarrow x \le -1$ or $x \ge 3$ (2)

From equations (1) and (2), $x \in \left[3, \frac{10}{3}\right]$

f. $f(x) = \sqrt{\sec^{-1}\left(\frac{2-|x|}{4}\right)}$

$\sec^{-1}$ function always takes positive values which are $[0, \pi] - \{\pi/2\}$

Hence, the given function is defined, if $\frac{2-|x|}{4} \le -1$ or $\frac{2-|x|}{4} \ge 1$.

$\Rightarrow |x| \ge 6$ or $|x| \le -2 \quad \Rightarrow x \in (-\infty, -6] \cup [6, \infty)$

2. $f(x) = \tan^{-1}\left(\sqrt{(x-1)^2 + 1}\right)$

Now $(x-1)^2 + 1 \in [1, \infty)$

$\Rightarrow \quad \tan^{-1}\left(\sqrt{(x-1)^2 + 1}\right) \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$

3. For $x \ge 0$, $\cos^{-1}\sqrt{1-x^2} = \sin^{-1} x$

$\Rightarrow f(x) = 0$

For $x < 0$, $\cos^{-1}\sqrt{1-x^2} = -\sin^{-1} x$

$\Rightarrow f(x) = \sqrt{-2\sin^{-1} x}$

$\Rightarrow$ range of (x) is $[0, \sqrt{\pi}\,]$.

4. $y = (x^2-1)^2 + 2 \ge 2$

$\Rightarrow \log_{0.5}(x^4 - 2x^2 + 3) \le -1$

$\Rightarrow \cot^{-1} \log_{0.5}\left(x^4 - 2x^2 + 3\right) \in \left[\frac{3\pi}{4}, \pi\right)$

Exercise 1.6

1. $4^x + 8^{\frac{2}{3}(x-2)} - 13 - 2^{2(x-1)} \ge 0$

$\Rightarrow \quad 4^x + \frac{4^x}{16} - \frac{4^x}{4} \ge 13$

$\Rightarrow \quad 4^x \ge 4^2 \quad \Rightarrow x \in [2, \infty)$

2. $f(x) = \sin^{-1}(\log_2 x)$

Since the domain of $\sin^{-1} x$ is $[-1, 1]$.

Therefore, $f(x) = \sin^{-1}(\log_2 x)$ is defined, if $-1 \le \log_2 x \le 1$

$\Rightarrow \quad 2^{-1} \le x \le 2^1$

$\Rightarrow \quad \frac{1}{2} \le x \le 2$

$\Rightarrow \quad$ domain is $\left[\frac{1}{2}, 2\right]$

3. $f(x) = \log_{(x-4)}(x^2 - 11x + 24)$.

$f(x)$ is defined if $x - 4 > 0$ and $\neq 1$ and $x^2 - 11x + 24 > 0$

$\Rightarrow \quad x > 4$ and $\neq 5$ and $(x-3)(x-8) > 0$

$\Rightarrow \quad x > 4$ and $\neq 5$ and $x < 3$ or $x > 8$

$\Rightarrow \quad x > 8$

$\Rightarrow \quad$ domain $(y) = (8, \infty)$

4. $f(x) = \frac{3}{4 - x^2} + \log_{10}(x^3 - x)$

f is defined when $x \neq \pm 2$ and $x^3 - x > 0$

$\Rightarrow \quad x \neq \pm 2$ and $x(x^2 - 1) > 0$

$\Rightarrow \quad x \neq \pm 2, x \in (-1, 0) \cup (1, \infty)$

$\Rightarrow \quad x \in (-1, 0) \cup (1, 2) \cup (2, \infty)$

5. $f(x) = \sqrt{\frac{\log_{0.3}|x-2|}{|x|}}$. Here $|x| > 0 \;\forall\, x \in R - \{0\}$ (1)

$\Rightarrow \quad$ For $f(x)$ to get defined $\log_{0.3}|x-2| \ge 0$

$\Rightarrow\ 0<|x-2|\le 1$
$\Rightarrow\ |x-2|\le 1$ and $x\ne 2$
$\Rightarrow\ -1\le x-2\le 1$ and $x\ne 2$
$\Rightarrow\ 1\le x\le 3$ and $x\ne 2$
$\Rightarrow\ x\in[1,2)\cup(2,3]$

6. $f(x)=\sqrt{\log_{10}\left\{\dfrac{\log_{10}x}{2(3-\log_{10}x)}\right\}}$. Clearly, $f(x)$ is defined, if

$\log_{10}\left\{\dfrac{\log_{10}x}{2(3-\log_{10}x)}\right\}\ge 0,\ \dfrac{\log_{10}x}{2(3-\log_{10}x)}>0$ and $x>0$

$\Rightarrow\ \dfrac{\log_{10}x}{2(3-\log_{10}x)}\ge 1,\ \dfrac{\log_{10}x}{\log_{10}x-3}<0$ and $x>0$

$\Rightarrow\ \dfrac{3(\log_{10}x-2)}{2(\log_{10}x-3)}\le 0,\ \dfrac{\log_{10}x}{\log_{10}x-3}<0$ and $x>0$

$\Rightarrow\ 2\le\log_{10}x<3,\ 0<\log_{10}x<3$ and $x>0$

$\Rightarrow\ 10^2\le x<10^3,\ 10^0<x<10^3$ and $x>0$

$\Rightarrow\ x\in[10^2,10^3)$

7. $f(x)=\dfrac{1}{\sqrt{\log_{1/2}(x^2-7x+13)}}$ exists

If $\log_{1/2}(x^2-7x+13)>0$ (1)
$\Rightarrow\ x^2-7x+13<1$ (2)
and $x^2-7x+13>0$

$\Rightarrow\ x^2-7x+12<0$ and $\left(x-\dfrac{7}{2}\right)^2+\dfrac{3}{4}>0$

$\Rightarrow\ 3<x<4$ and $x\in R$
$\Rightarrow\ 3<x<4$

8. $-\sqrt{2}\le\sin x-\cos x\le\sqrt{2}$

$\Rightarrow\ 2\sqrt{2}\le\sin x-\cos x+3\sqrt{2}\le 4\sqrt{2}$

$\Rightarrow\ 2\le\dfrac{\sin x-\cos x+3\sqrt{2}}{\sqrt{2}}\le 4$

$\Rightarrow\ \log_2 2\le\log_2\left(\dfrac{\sin x-\cos x+3\sqrt{2}}{\sqrt{2}}\right)\le\log_2 4$

$\Rightarrow\ 1\le\log_2\left(\dfrac{\sin x-\cos x+3\sqrt{2}}{\sqrt{2}}\right)\le 2$

Exercise 1.7

1. $[x]^2-5[x]+6=0$
$\Rightarrow\ [x]=2,3$
$\Rightarrow\ x\in[2,4)$

2. $y=3[x]+1=4[x-1]-10=4[x]-14$
$\Rightarrow\ [x]=15$ and $y=3.15+1=46$
$\Rightarrow\ [x+2y]=2y+[x]=2.46+15=107$

3. a. We have, $f(x)=\dfrac{1}{\sqrt{x-[x]}}$

We know that $0\le x-[x]<1$ for all $x\in R$. Also, $x-[x]=0$ for $x\in Z$.

Now, $f(x)=\dfrac{1}{\sqrt{x-[x]}}$ is defined, if $x-[x]>0$

$\Rightarrow\ x\in R-Z$ $\left[\because\ x-[x]=0 \text{ for } x\in Z \text{ and } 0<x-[x]<1 \text{ for } x\in R-Z\right]$

Hence, the domain $=R-Z$

b. $f(x)=\dfrac{1}{\log[x]}$

We must have $[x]>0$ and $[x]\ne 1$ (as for $[x]=1$, $\log[x]=0$)

$\Rightarrow\ [x]\ge 2\ \Rightarrow\ x\in[2,\infty)$

c. $f(x)=\log\{x\}$ is defined if $\{x\}>0$ which is true for all real numbers except integers.

Hence, the domain is $R-Z$.

4. $f(x)=\dfrac{1}{\sqrt{|[|x|-1]|-5}}$ is defined

$|[|x|-1]|-5>0$
$\Rightarrow\ |[|x|-1]|>5$
$\Rightarrow\ [|x|-1]<-5$ or $[|x|-1]>5$
$\Rightarrow\ |x|-1<-5$ or $|x|-1\ge 6\ \Rightarrow\ |x|\ge 7$
$\Rightarrow\ x\in(-\infty,-7]\cup[7,\infty)$

5. a. $1-\sin x\ge 0\Rightarrow\sin\le 1\Rightarrow x\in R$

b. $1-4x^2>0\Rightarrow x\in(-1/2,1/2)$

c. $\log_5(1-4x^2)\ne 0\Rightarrow 1-4x^2\ne 1\Rightarrow x\ne 0$

d. $-1\le 1-\{x\}\le 1\Rightarrow 0\le\{x\}\le 2\Rightarrow x\in R$

Hence, domain is common values of a, b, c, and d, i.e.,

$x\in\left(-\dfrac{1}{2},\dfrac{1}{2}\right)-\{0\}$.

6. $f(x)=\cos(\log_e\{x\})$.

For the given function to define $0<\{x\}<1$
$\Rightarrow\ -\infty<\log_e\{x\}<0$

For this values of $\theta=(\log_e\{x\})$, $\cos\theta$ takes all its possible values.

Hence, the range is $[-1,1]$.

7. $\log_{[x]}\dfrac{|x|}{x}$ is defined, if $\dfrac{|x|}{x}>0$, $[x]>0$ and $[x]\ne 1$

$\Rightarrow\ x>0, x\in[1,\infty)$ and $x\notin[1,2)$
$\Rightarrow\ x\in[2,\infty)$

For $x\in[2,\infty)$, we have $\log_{[x]}\dfrac{|x|}{x}=\log_{[x]}1=0$

$\therefore\ f(x)=\cos^{-1}0=\pi/2$ for all $x\in[2,\infty)$

Hence, domain $(f)=[2,\infty)$ and range $(f)=\{\pi/2\}$.

8. $f(x)=\log_{[x-1]}\sin x$, where [.] denotes the greatest integer.

To get the range of $f(x)$, let us examine the values of x for which the function is defined.

$f(x)$ is defined if $\sin x>0$ and $[x-1]>0$ and $[x-1]\ne 1$

$\Rightarrow\ 0<\sin x\le 1$ and $[x]\ge 2$

Now for base of the logarithm ≥ 2 and $\sin x\in(0,1]$, clearly $\log_{[x-1]}\sin x\in(-\infty,0]$.

9. For $x\ge 2$, LHS is always non-negative and RHS is always negative.

Hence for $x\ge 2$ no solution.

If $1\le x<2$, then $(x-2)=(x-1)-1=x-2$, which is an identity

For $0\le x<1$, LHS is '0' and RHS is (–)ve

$\Rightarrow$ No solution.

For $x<0$, LHS is (+) ve, RHS is (–)ve

$\Rightarrow$ No solution

Hence $x\in[1,2)$

Exercise 1.8

1. $f(3) = \max.\{1, |3-1|, \min.\{4, |9-1|\}$
$= \max.\{1, 2, 4\}$
$= 4.$

2. Here, for maximum, let us consider $f_1(x) = x^2$, $f_2(x) = (1-x)^2$ and $f_3(x) = 2x(1-x)$
Now taking graph for $f_1(x)$, $f_2(x)$ and $f_3(x)$;

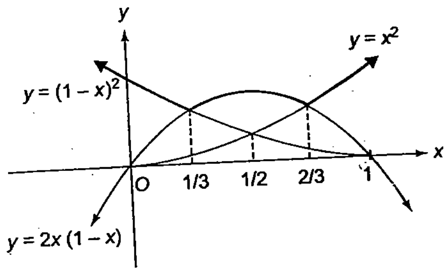


Fig. S-1.8

Here, neglecting the graph, i.e., below the point of intersection.
Since we want to find the maximum of three functions $f_1(x)$, $f_2(x)$ and $f_3(x)$:

$$\therefore \quad f(x) = \begin{cases} (1-x)^2, & 0 \le x < \frac{1}{3} \\ 2x(1-x), & \frac{1}{3} \le x < \frac{2}{3} \\ x^2, & \frac{2}{3} \le x \le 1 \end{cases}$$

3. a.

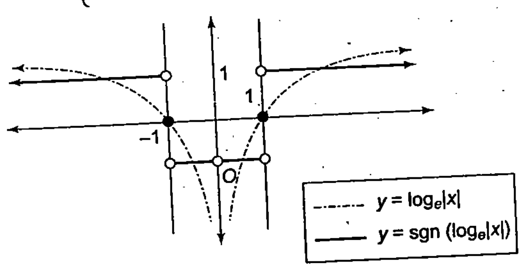


Fig. S-1.9

$$\text{From the graph } f(x) = \begin{cases} 1, & |x| > 1 \\ -1, & 0 < |x| < 1 \\ 0, & |x| = 1 \end{cases}$$

b.

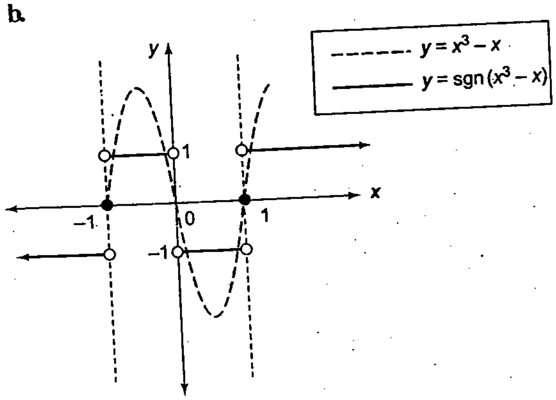


Fig. S-1.10

$$\text{From the graph } f(x) = \begin{cases} -1, & x < -1,\ 0 < x < 1 \\ 1, & -1 < x < 0,\ x > 1 \\ 0, & x = -1, 0\ 1 \end{cases}$$

Exercise 1.9

1. b. Clearly $f(x)$ must be $x+2$ as for this function each image has its preimage and each image has one and only one preimage.

2. When n is even
Let $f(2m_1) = f(2m_2)$

$$\Rightarrow \quad -\frac{2m_1}{2} = -\frac{2m_2}{2}$$

$$\Rightarrow \quad m_1 = m_2$$

When n is odd
Let $f(2m_1+1) = f(2m_2+1)$

$$\Rightarrow \quad \frac{2m_1+1-1}{2} = \frac{2m_2+1-1}{2} \quad \Rightarrow m_1 = m_2$$

$\therefore$ $f(x)$ is one-one.

Also when n is even, $-\frac{n}{2} = -\frac{2m}{2} = -m$

When n is odd, $\frac{n-1}{2} = \frac{2m+1-1}{2} = m$

Hence, the range of the function is Z.
$\Rightarrow$ function is onto.

3. $f(x) = f(-x)$. So, f is many-one.
Also, $f(x) = 1 - \frac{5}{x^2+1} > 1 - 5 = -4$. So, f is into.

4. $f(x) = \sin x - \sqrt{3}\cos x + 1$

$$= 2\left(\sin x \frac{1}{2} - \cos x \frac{\sqrt{3}}{2}\right) + 1$$

$$= 2\left(\sin x \cos\frac{\pi}{3} - \cos x \sin\frac{\pi}{3}\right) + 1$$

$$= 2\sin\left(x - \frac{\pi}{3}\right) + 1$$

Clearly, f is onto, when the interval of S is $[-1, 3]$.

5. c. For $-1 < x < 1$, $\tan^{-1}\frac{2x}{1-x^2} = 2\tan^{-1}x$

$\therefore$ range of $f(x) = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

6. $g(x)$ is surjective if $\frac{1}{2} \le \frac{x^2-k}{1+x^2} < 1, \forall x \in R,$

$$\Rightarrow \quad \frac{1}{2} \le 1 - \frac{(k+1)}{x^2+1} < 1 \quad \forall x \in R$$

$$\Rightarrow \quad -\frac{1}{2} \le -\frac{(k+1)}{x^2+1} < 0 \quad \forall x \in R$$

$\Rightarrow\ 0 < \dfrac{(k+1)}{x^2+1} \le \dfrac{1}{2},\ \forall x \in R$

$\Rightarrow\ k+1 > 0$; So $k > -1$ (1)

and $\dfrac{k+1}{x^2+1} \le \dfrac{1}{2}, \forall x \in R$

or $x^2-(2k+1) \ge 0\ \forall x \in R$

$\Rightarrow\ 4(2k+1) \le 0$

$\therefore\ k \le -\dfrac{1}{2}$ (2)

From (1) and (2), $k \in \left(-1, -\dfrac{1}{2}\right]$

Exercise 1.10

1. $f(-x) = (g(-x)-g(x))^3 = -(g(x)-g(-x))^3 = -f(x)$
   Hence, $f(x)$ is an odd function.

2. $\log\left(\dfrac{x^4+x^2+1}{x^2+x+1}\right) = \log(x^2-x+1)$, which is neither odd nor even.

3. $f(-x) = (-x)g(-x)\cdot g(x) + \tan(\sin(-x))$
   $= -(xg(x)\,g(-x) - \tan(\sin x)) = -f(x)$
   Hence, $f(x)$ is an odd function.

4. $0 \le \left|\dfrac{\sin x}{2}\right| \le \dfrac{1}{2}\ \Rightarrow\ \left[\left|\dfrac{\sin x}{2}\right|\right] = 0$
   $\Rightarrow\ f(x) = \cos x$ which is even

5. $f(x) = \log\left(x + \sqrt{x^2+1}\right)$
   $f(-x) = \log\left(-x + \sqrt{x^2+1}\right)$
   $\Rightarrow\ f(x) + f(-x)$
   $= \log\left(x+\sqrt{x^2+1}\right) + \log\left(-x+\sqrt{x^2+1}\right)$
   $= \log\left(\sqrt{x^2+1}+x\right) + \log\left(\sqrt{x^2+1}-x\right)$
   $= \log(x^2+1-x^2) = \log 1 = 0$
   $\Rightarrow\ f(-x) = -f(x)$
   Hence, $f(x)$ is an odd function.

6. $$f(-x) = \begin{cases} -x\,|-x|, & -x \le -1 \\ [-x+1]+[1+x], & -1 < -x < 1 \\ -(-x)\,|-x|, & -x \ge 1 \end{cases}$$
   $$= \begin{cases} -x\,|x|, & x \ge 1 \\ [1-x]+[1+x], & -1 < x < 1 \\ x\,|x|, & x \le 1 \end{cases}$$
   $= f(x)$
   Hence, the function is even.

Exercise 1.11

1. p. $f(x) = \sin^3 x + \cos^4 x$,
   $\sin^3 x$ has period 2π and $\cos^4 x$ has period π, and L.C.M. of π and 2π is 2π. Hence, period is 2π.

   q. $f(x) = \sin^4 x + \cos^4 x$
   Both $\sin^4 x$ and $\cos^4 x$ have the same period π, and L.C.M. of π and π is π.
   But $f(x+\pi/2) = f(x)$, then period is $\pi/2$.

   r. Both $\sin^3 x$ and $\cos^3 x$ have the same a period 2π, and L.C.M. of 2π and 2π is 2π,
   Hence, period is 2π, $[(f(x+\pi) \ne f(x)]$.

   s. $f(x) = \cos^4 x - \sin^4 x$
   Both $\sin^4 x$ and $\cos^4 x$ have the same period π, and L.C.M. of π and π is π,
   Hence, period is π $(f(x+\pi/2) \ne f(x))$.

2. b. Since $\cos\sqrt{x}$ is not periodic, therefore, $\cos\sqrt{x} + \cos^2 x$ is not periodic although $\cos^2 x$ is periodic.

3. Clearly, $f(x) = \tan(\sqrt{[n]}\,x)$ has period $\dfrac{\pi}{3}$,
   but, it is given that $\tan(\sqrt{[n]}\,x)$ has a period $\dfrac{\pi}{3}$.
   $\Rightarrow\ \dfrac{\pi}{\sqrt{[n]}} = \dfrac{\pi}{3}$
   $\Rightarrow\ [n] = 9\ \Rightarrow\ n \in [9, 10)$.

4. a. Period of $|\sin 4x| + |\cos 4x|$ is $\dfrac{\pi}{8}$
   Period of $|\sin 4x - \cos 4x| + |\sin 4x + \cos 4x| = \dfrac{\pi}{8}$
   Because period of $|\sin x - \cos x| + |\sin x + \cos x| = \dfrac{\pi}{8}$
   The period of given function is $\dfrac{\pi}{8}$.

   b. $f(x) = \sin\dfrac{\pi x}{n!} - \cos\dfrac{\pi x}{(n+1)!}$
   Period of $\sin\dfrac{\pi x}{n!}$ is $\dfrac{2\pi}{\frac{\pi}{n!}} = 2n!$ and period of $\cos\dfrac{\pi x}{(n+1)!}$ is
   $\dfrac{2\pi}{\frac{\pi}{(n+1)!}} = 2(n+1)!$
   Hence, period of $f(x)$ is = L.C.M. of $\{2n!, 2(n+1)!\}$ $= 2(n+1)!$

   c. $f(x) = \sin x + \tan\dfrac{x}{2} + \sin\dfrac{x}{2^2} + \tan\dfrac{x}{2^3} + \cdots + \sin\dfrac{x}{2^{n-1}} + \tan\dfrac{x}{2^n}$
   Period of $\sin x$ is 2π
   Period of $\tan\dfrac{x}{2}$ is 2π
   Period of $\sin\dfrac{x}{2^2}$ is 8π
   Period of $\tan\dfrac{x}{2^3}$ is 8π
   ⋮
   Period of $\tan\dfrac{x}{2^n}$ is $2^n\pi$
   Hence, period of $f(x)$ = L.C.M. of $(2\pi, 8\pi, \ldots, 2^n\pi) = 2^n\pi$

5. Since the period of $|\sin x| + |\cos x| = \pi/2$ it is possible when $\lambda = 1$.

6. Given $f(x) + f(x+4) = f(x+2) + f(x+6)$ (1)
Replace x by $x+2$
$\Rightarrow f(x+2) + f(x+6) = f(x+4) + f(x+8)$ (2)
From equations (1) and (2), we have $f(x) = f(x+8)$
Hence, $f(x)$ is periodic with period 8.

Exercise 1.12

1. Given $(gof)\left(\frac{-5}{3}\right) - (fog)\left(\frac{-5}{3}\right)$

$$= g\left\{f\left(\frac{-5}{3}\right)\right\} - f\left\{g\left(\frac{-5}{3}\right)\right\} = g(-2) - f\left(\frac{5}{3}\right) = 2 - 1 = 1$$

2. $f(x) = \begin{cases} 1+|x|, & x < -1 \\ [x], & x \geq -1 \end{cases}$

$f(-2.3) = 1 + |-2.3|$
$= 1 + 2.3 = 3.3$. Now $f(f(-2.3)) = f(3.3) = [3.3] = 3$

3. $$f(x) = \log\left[\frac{1+x}{1-x}\right]$$

$$\Rightarrow f\left(\frac{2x}{1+x^2}\right) = \log\left[\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right] = \log\left[\frac{x^2+1+2x}{x^2+1-2x}\right]$$

$$= \log\left[\frac{1+x}{1-x}\right]^2 = 2\log\left[\frac{1+x}{1-x}\right] = 2f(x)$$

4. Here, $f(x)$ is defined by $[-3, 2]$
$\Rightarrow x \in [-3, 2]$.
For $g(x) = f(|[x]|)$ to be defined, we must have
$-3 \leq |[x]| \leq 2$
$\Rightarrow 0 \leq |[x]| \leq 2$ [as $|x| \geq 0$ for all x]
$\Rightarrow -2 \leq [x] \leq 2$ [as $|x| \leq a \Rightarrow -a \leq x \leq a$]
$\Rightarrow -2 \leq x < 3$ [by the definition of greatest integral function]

Hence, domain $g(x) \in [-2, 3)$.

5. g is meaningful if $0 \leq 9x^2 - 1 \leq 2$
$\Leftrightarrow 1 \leq 9x^2 \leq 3$

i.e., $x \in \left[-\frac{1}{\sqrt{3}}, -\frac{1}{3}\right] \cup \left[\frac{1}{3}, \frac{1}{\sqrt{3}}\right]$.

6. $f(x) = \begin{cases} \log_e x, & 0 < x < 1 \\ x^2 - 1, & x \geq 1 \end{cases}$ and $g(x) = \begin{cases} x+1, & x < 2 \\ x^2 - 1, & x \geq 2 \end{cases}$

$$g(f(x)) = \begin{cases} f(x)+1, & f(x) < 2 \\ (f(x))^2 - 1, & f(x) \geq 2 \end{cases}$$

$$= \begin{cases} \log_e x + 1, & \log_e x < 2,\ 0 < x < 1 \\ x^2 - 1 + 1, & x^2 - 1 < 2,\ x \geq 1 \\ (\log_e x)^2 - 1, & \log_e x \geq 2,\ 0 < x < 1 \\ (x^2-1)^2 - 1, & x^2 - 1 \geq 2,\ x \geq 1 \end{cases}$$

$$= \begin{cases} \log_e x + 1, & x < e^2,\ 0 < x < 1 \\ x^2 - 1 + 1, & -\sqrt{3} < x < \sqrt{3},\ x \geq 1 \\ (\log_e x)^2 - 1, & x \geq e^2,\ 0 < x < 1 \\ (x^2-1)^2 - 1, & x \leq -\sqrt{3} \text{ or } x \geq \sqrt{3},\ x \geq 1 \end{cases}$$

$$= \begin{cases} \log_e x + 1, & 0 < x < 1 \\ x^2, & 1 \leq x < \sqrt{3} \\ (x^2-1)^2 - 1, & x \geq \sqrt{3} \end{cases}$$

7. $g(f(x)) = \tan\left(x - \frac{\pi}{4}\right) = \frac{\tan x - 1}{\tan x + 1}$

$$\Rightarrow g(x) = \frac{x-1}{x+1} \Rightarrow f(g(x)) = \tan\left(\frac{x-1}{x+1}\right)$$

8. $g(x)$ is defined if $f(x+1)$ is defined.
Hence the domain of g is all x such that $(x+1) \in [0, 2]$
$\Rightarrow -2 \leq x \leq 1$
Also $f(x+1) \in [0, 1]$
$\therefore -f(x+1) \in [-1, 0]$
$\therefore 1 - f(x+1) \in [0, 1]$
$\therefore$ range of $g(x)$ is $[0, 1]$

Exercise 1.13

1. $y = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 2$

$$\Rightarrow y = \frac{e^{2x} - 1}{e^{2x} + 1} + 2$$

$$\Rightarrow e^{2x} = \frac{1-y}{y-3} = \frac{y-1}{3-y}$$

$$\Rightarrow x = \frac{1}{2}\log_e\left(\frac{y-1}{3-y}\right)$$

$$\Rightarrow f^{-1}(y) = \log_e\left(\frac{y-1}{3-y}\right)^{1/2}$$

$$\Rightarrow f^{-1}(x) = \log_e\left(\frac{x-1}{3-x}\right)^{1/2}$$

2. Let $y = 1 - 2^{-x}$
$\Rightarrow 2^{-x} = 1 - y$
$\Rightarrow -x = \log_2(1-y)$
$\Rightarrow f^{-1}(x) = g(x) = -\log_2(1-x)$

3. Given $f: (2, 3) \to (0, 1)$ and $f(x) = x - [x]$
$\therefore f(x) = y = x - 2$
$\Rightarrow y + 2 = f^{-1}(y)$
$\Rightarrow f^{-1}(x) = x + 2$

4. Since the domain of the function is I, we have $f(x) = x + 1$
$\Rightarrow f^{-1}(x) = x - 1$

5. $f(x) = \begin{cases} x^3 - 1, & x < 2 \\ x^2 + 3, & x \geq 2 \end{cases}$

For $f(x) = x^3 - 1, x < 2, f^{-1}(x) = (x+1)^{1/3}, x < 7$
(as $x < 2 \Rightarrow x^3 < 8 \Rightarrow x^3 - 1 < 7$)

For $f(x) = x^2 + 3, x \geq 2, f^{-1}(x) = (x-3)^{1/2}, x \geq 7$
(as $x \geq 2 \Rightarrow x^2 \geq 4 \Rightarrow x^2 + 3 \geq 7$)

Hence, $f^{-1}(x)=\begin{cases}(x+1)^{1/3}, & x<7\\ (x-3)^{1/2}, & x\geq 7\end{cases}$

6. $f:[-1,1]\to[-1,1]$ defined by $f(x)=x|x|$

$$f(x)=\begin{cases}x^2, & x\geq 0\\ -x^2, & x<0\end{cases}$$

$$\Rightarrow\quad f^{-1}(x)=\begin{cases}\sqrt{x}, & x\geq 0\\ -\sqrt{-x}, & x<0\end{cases}$$

Also $y=\text{sgn}(x)\sqrt{|x|}=\begin{cases}\sqrt{x}, & x>0\\ 0, & x=0\\ -\sqrt{-x}, & x<0\end{cases}$

Hence, proved.

7. $y=2^{x(x-2)}$

$\Rightarrow x^2-2x=\log_2 y$

$\Rightarrow x^2-2x-\log_2 y=0$

$\Rightarrow x=1\pm\sqrt{1+\log_2 y}$

$\Rightarrow f^{-1}(x)=1-\sqrt{1+\log_2 x}$

as $f^{-1}:\left[\frac{1}{2},\infty\right)\to(-\infty,1]$

Exercise 1.14

1.

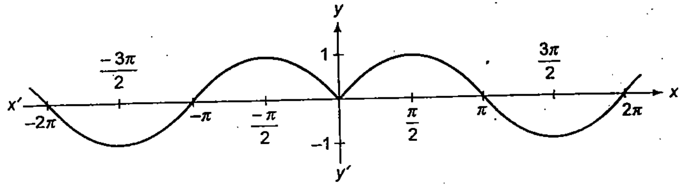


Fig. S-1.11

2.

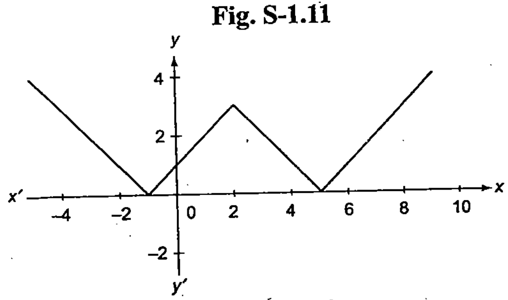


Fig. S-1.12

3.

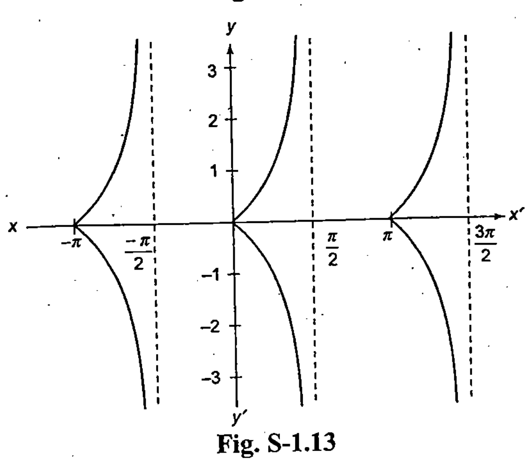


Fig. S-1.13

4.

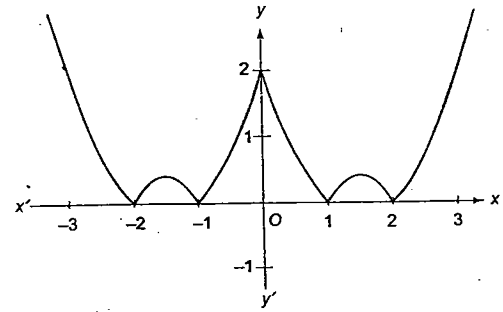


Fig. S-1.14

5.

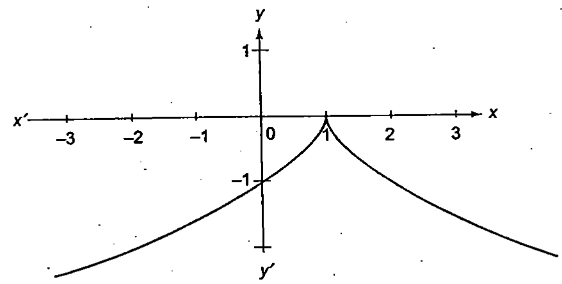


Fig. S-1.15

6. There are exactly six solutions.

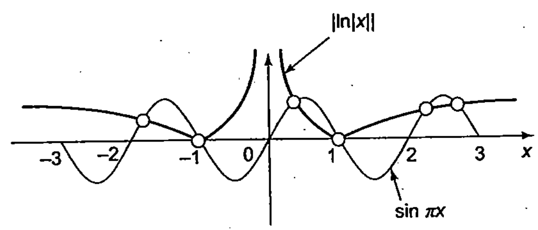


Fig. S-1.16

7. $\left|\frac{x^2}{x-1}\right|\leq 1 \Rightarrow x^2\leq|x-1|, x\neq 1$

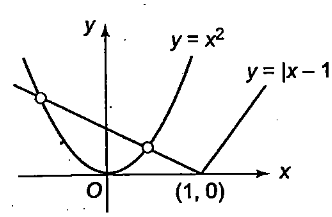


Fig. S-1.17

Figure S-1.17 represents the graph of $y=x^2$ and $y=|x-1|$

Solving $x^2=1-x$, we get $x=\frac{-1\pm\sqrt{5}}{2}$.

Thus, the solution is $\left\{\frac{-1-\sqrt{5}}{2},\frac{-1+\sqrt{5}}{2}\right\}$.

8.a. Domain of both $f(x)=\frac{\sec x}{\cos x}-\frac{\tan x}{\cot x}$, $g(x)=\frac{\cos x}{\sec x}+\frac{\sin x}{\text{cosec}\,x}$

is $x\in R-\left\{\frac{n\pi}{2}, n\in Z\right\}$

Also both functions simplify to 1
Hence both functions are identical.

b. As $x^2-6x+10=(x-3)^2+1>0$
Hence $f(x)=1\ \forall x\in R$.

Also $\cos^2 x+\sin^2\left(x+\frac{\pi}{6}\right)>0$

Hence $g(x)=1\ \forall x\in R$.
$\Rightarrow$ $f(x)$ and $g(x)$ are identical.

c. $f(x)=e^{\ln(x^2+3x+3)}$

As $x^2+3x+3=\left(x+\frac{3}{2}\right)^2+\frac{3}{4}>0\ \forall x\in R$.

Hence $f(x)=x^2+3x+3\ \forall x\in R$.
$\Rightarrow$ $f(x)$ is identical to $g(x)$.

d. $f(x)=\frac{\sin x}{\sec x}+\frac{\cos x}{\operatorname{cosec} x}$

$=2\sin x\cos x$

$=\frac{2\cos^2 x}{\cot x}$

$=g(x)$

Also domain of both the functions is $x\in R-\left\{\frac{n\pi}{2}, n\in Z\right\}$

Chapter 2

Exercise 2.1

1. L.H.L of $f(x)$ at $x=0$

$=\lim_{x\to0^-} f(x)$

$=\lim_{h\to0} f(0-h)=\lim_{h\to0}\frac{-h-|-h|}{(-h)}$

$=\lim_{h\to0}\frac{-h-h}{-h}=\lim_{h\to0}\frac{-2h}{-h}=\lim_{h\to0} 2=2$

R.H.L of $f(x)$ at $x=0$

$=\lim_{h\to0^+} f(x)$

$=\lim_{h\to0} f(0+h)=\lim_{h\to0}\frac{h-|h|}{h}$

$=\lim_{h\to0}\frac{h-h}{h}=\lim_{h\to0}\frac{0}{h}=\lim_{h\to0} 0=0.$

Clearly, $\lim_{x\to0^-} f(x)\neq\lim_{x\to0^+} f(x)$

So, $\lim_{x\to0} f(x)$ does not exist.

2. Let $f(x)=\frac{e^{1/x}-1}{e^{1/x}+1}$

L.H.L of $f(x)$ at $x=0$

$=\lim_{x\to0^-} f(x)=\lim_{h\to0}(0-h)=\lim_{h\to0}\frac{e^{-1/h}-1}{e^{-1/h}+1}$

$=\lim_{h\to0}\left(\frac{\frac{1}{e^{1/h}}-1}{\frac{1}{e^{1/h}}+1}\right)=-1$

$\left[\because h\to0\Rightarrow\frac{1}{h}\to\infty\Rightarrow e^{1/h}\to\infty\Rightarrow\frac{1}{e^{1/h}}\to0\right]$

R.H.L of $f(x)$ at $x=0$

$=\lim_{x\to0} f(x)=\lim_{h\to0} f(0+h)$

$=\lim_{h\to0}\frac{e^{1/h}-1}{e^{1/h}+1}$

$=\lim_{h\to0}\left(\frac{1-\frac{1}{e^{1/h}}}{1+\frac{1}{e^{h}}}\right)$ [Dividing N^r and D^r by $e^{1/h}$]

$=\frac{1-0}{1+0}=1$

Clearly, $\lim_{x\to0^-} f(x)\neq\lim_{x\to0^+} f(x)$

Hence, $\lim_{x\to0} f(x)$ does not exist.

3. $\lim_{x\to0^-}\frac{3x+|x|}{7x-5|x|}=\lim_{x\to0^-}\frac{3x-x}{7x+5x}=\frac{1}{6}$

and $\lim_{x\to0^+}\frac{3x+|x|}{7x-5|x|}=\lim_{x\to0}\frac{3x+x}{7x-5x}=2$

Hence the limit does not exist.

4. We have, $\lim_{x\to0^-} f(x)=\lim_{x\to0} x=0$

and $\lim_{x\to0^+} f(x)=\lim_{x\to0} x^2=0$.

Hence, $\lim_{x\to0} f(x)$ is equal to 0.

5.

a. $\lim_{x\to1^+} f(x)=3$ and $\lim_{x\to1^-} f(x)=2\Rightarrow\lim_{x\to1} f(x)$ does not exist

b. $\lim_{x\to2^+} f(x)=\lim_{x\to2^-} f(x)=3\Rightarrow\lim_{x\to2} f(x)$ exists

c. $\lim_{x\to3^+} f(x)=\lim_{x\to3^-} f(x)=3\Rightarrow\lim_{x\to3} f(x)$ exists

d. $\lim_{x\to1.99^+} f(x)=\lim_{x\to1.99^-} f(x)=3\Rightarrow\lim_{x\to1.99} f(x)$ exists

Exercise 2.2

1. We have $x-1<[x]\le x$

$\Rightarrow 1-\frac{1}{x}<\frac{[x]}{x}\le1$

Now $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)=1$.

Therefore, by Sandwich theorem, $\lim_{x\to\infty}\frac{[x]}{x}=1$.

2. As $0 \le \log_e x \le \sqrt{x}\ (x > 1)$

$$\Rightarrow 0 \le \frac{\log_e x}{x} \le \frac{1}{\sqrt{x}}\ (x > 1)$$

$$0 \le \lim_{x\to\infty} \frac{\log_e x}{x} \le \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

$$\Rightarrow \lim_{x\to\infty} \frac{\ln x}{x} = 0$$

Exercise 2.3

1. $$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$$

$$= \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots - x + \frac{x^3}{6}}{x^5}$$

$$= \lim_{x\to 0} \left(\frac{1}{5!} - \frac{x^2}{7!} + \cdots\right) = \frac{1}{120}$$

2. $$\lim_{x\to 0} \frac{\sin x + \log(1-x)}{x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) + \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right)}{x^2}$$

$$= \lim_{x\to 0} \frac{\frac{-x^2}{2} - x^3\left(\frac{1}{3!} + \frac{1}{3}\right) - \frac{x^4}{4} \cdots}{x^2} = -\frac{1}{2}$$

3. $$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$$

$$= \lim_{x\to 0} \frac{\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right) - 1 - x}{x^2} = \frac{1}{2}$$

Exercise 2.4

1. $$\lim_{x\to 1} \frac{(2x-3)\left(\sqrt{x}-1\right)\times\left(\sqrt{x}+1\right)}{(x-1)(2x+3)\times\left(\sqrt{x}+1\right)} = \frac{-1}{5\times 2} = \frac{-1}{10}$$

2. $$\lim_{x\to 1} \frac{\left[\sum_{k=1}^{100} x^k\right] - 100}{(x-1)}$$

$$= \lim_{x\to 1} \frac{(x + x^2 + x^3 + \cdots + x^{100}) - 100}{(x-1)}$$

$$= \lim_{x\to 1} \frac{(x-1) + (x^2-1) + (x^3-1) + \cdots + (x^{100}-1)}{(x-1)}$$

$$= \lim_{x\to 1} \left\{\left(\frac{x-1}{x-1}\right) + \left(\frac{x^2-1}{x-1}\right) + \left(\frac{x^3-1}{x-1}\right) + \cdots + \left(\frac{x^{100}-1}{x-1}\right)\right\}$$

$$= \lim_{x\to 1}\left(\frac{x-1}{x-1}\right) + \lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right) + \lim_{x\to 1}\left(\frac{x^3-1}{x-1}\right) + \cdots \lim_{x\to 1}\left(\frac{x^{100}-1}{x-1}\right)$$

$$= 1 + 2 + 3 + \cdots + 100$$

$$= \frac{100\times 101}{2} = 5050$$

3. $$\lim_{x\to\infty} \left[\sqrt{a^2x^2 + ax + 1} - \sqrt{a^2x^2 + 1}\right]$$

$$= \lim_{x\to\infty} \frac{ax}{\sqrt{a^2x^2 + ax + 1} + \sqrt{a^2x^2 + 1}} \qquad \text{(Rationalizing)}$$

$$= \lim_{x\to\infty} \frac{a}{\sqrt{a^2 + \frac{a}{x} + \frac{1}{x^2}} + \sqrt{a^2 + \frac{1}{x^2}}}$$

(Dividing numerator and denominator by x)

$$= \lim_{x\to\infty} \frac{a}{\sqrt{a^2 + \frac{a}{x} + \frac{1}{x^2}} + \sqrt{a^2 + \frac{1}{x^2}}} = \frac{a}{2a} = \frac{1}{2}$$

4. $$\lim_{x\to a} \frac{\sqrt{3x-a} - \sqrt{x+a}}{x-a}$$

$$\lim_{x\to a} \frac{\sqrt{3x-a} - \sqrt{x+a}}{(x-a)} \times \frac{\sqrt{3x-a} + \sqrt{x+a}}{\sqrt{3x-a} + \sqrt{x+a}}$$

$$= \frac{2}{2\sqrt{2a}} = \frac{1}{\sqrt{2a}}$$

5. When n is even:

Given series $1^2 - 2^2 + 3^2 - 4^2 + \cdots - n^2$

$= 1^2 - 2^2 + 3^2 - 4^2 + \cdots - n^2$

$= (1^2 - 2^2) + (3^2 - 4^2) + \cdots [(n-1)^2 - n^2]$

$= -(1 + 2 + 3 + 4 + \cdots + n)$

$$= -\frac{n(n+1)}{2}$$

$$\Rightarrow \text{Given } L = \lim_{n\to\infty} -\frac{n(n+1)}{2n^2} = -1/2$$

When n is odd:

Given series

$1^2 - 2^2 + 3^2 - 4^2 + \cdots + n^2$

$= -1(1 + 2 + 3 + \cdots + (n-1)) + n^2$

$$= -\frac{n(n-1)}{2} + n^2$$

$$= \frac{n(n+1)}{2}$$

$$\Rightarrow \text{Given } L = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = 1/2$$

6. $$\lim_{h\to 0} \left[\frac{2 - \sqrt[3]{8+h}}{2h\sqrt[3]{8+h}}\right]$$

$$= -\lim_{h\to 0}\left[\frac{\left(1+\frac{h}{8}\right)^{1/3}-1}{8\cdot\frac{h}{8}\sqrt[3]{8+h}}\right] = -\frac{1}{48}$$

Exercise 2.5

1. $\lim_{x\to 0}\frac{\sin x^\circ}{x} = \lim_{x\to 0}\frac{\sin\frac{\pi x}{180}}{x} = \frac{\pi}{180}$ $\left\{\because x^\circ = \frac{\pi x}{180}\text{ radian}\right\}$

2. $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx} = \lim_{x\to 0}\left\{\frac{2\sin^2\frac{mx}{2}}{2\sin^2\frac{n}{2}x}\right\}$

$$= \lim_{x\to 0}\left[\left\{\frac{\sin\frac{mx}{2}}{\frac{mx}{2}}\right\}^2\frac{m^2x^2}{4}\times\frac{1}{\left\{\frac{\sin\frac{nx}{2}}{\frac{nx}{2}}\right\}^2}\times\frac{4}{n^2x^2}\right] = \frac{m^2}{n^2}$$

3. $\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\cos x-1}{\cot x-1}$

$$= \lim_{x\to\frac{\pi}{4}}\frac{2\cos^2 x-1}{\cos x-\sin x}\cdot\frac{\sin x}{\sqrt{2}\cos x+1}$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{\cos^2 x-\sin^2 x}{\cos x-\sin x}\cdot\frac{1/\sqrt{2}}{\sqrt{2}\cdot\frac{1}{\sqrt{2}}+1}$$

$$= \frac{1}{2\sqrt{2}}\lim_{x\to\frac{\pi}{4}}(\cos x+\sin x) = \frac{1}{2}$$

4. We have $\lim_{x\to 0}\frac{\cot 2x-\operatorname{cosec} 2x}{x}$

$$= \lim_{x\to 0}\frac{\frac{\cos 2x}{\sin 2x}-\frac{1}{\sin 2x}}{x} = \lim_{x\to 0}\frac{\cos 2x-1}{x\sin 2x}$$

$$= \lim_{x\to 0}\frac{-(1-\cos 2x)}{x\sin 2x} = -\lim_{x\to 0}\frac{2\sin^2 x}{x(2\sin x\cos x)} = -\lim_{x\to 0}\frac{\tan x}{x} = -1$$

5. $\lim_{x\to 0}\frac{\tan 2x-x}{3x-\sin x} = \lim_{x\to 0}\left\{\frac{\frac{2\tan 2x}{2x}-1}{3-\frac{\sin x}{x}}\right\} = \frac{1}{2}$

6. $\lim_{h\to 0}\frac{2\left[\sqrt{3}\sin\left(\frac{\pi}{6}+h\right)-\cos\left(\frac{\pi}{6}+h\right)\right]}{\sqrt{3}h\left(\sqrt{3}\cosh-\sin h\right)}$

$$= \lim_{h\to 0}\frac{\frac{4}{\sqrt{3}}\left[\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{6}+h\right)-\frac{1}{2}\cos\left(\frac{\pi}{6}+h\right)\right]}{h\left(\sqrt{3}\cos h-\sin h\right)}$$

$$= \lim_{h\to 0}\frac{4}{\sqrt{3}}\times\frac{\sin h}{h}\,\frac{1}{\left(\sqrt{3}\cos h-\sin h\right)} = \frac{4}{3}$$

7. $L = \lim_{n\to\infty} n\cos\left(\frac{\pi}{4n}\right)\sin\left(\frac{\pi}{4n}\right)$

$$= \lim_{n\to\infty}\frac{\cos\left(\frac{\pi}{4n}\right)\sin\left(\frac{\pi}{4n}\right)}{\left(\frac{\pi}{4n}\right)\frac{4}{\pi}}$$

$$= \cos(0)\times 1\times\frac{\pi}{4} = \frac{\pi}{4}$$

8. $\lim_{\substack{x\to 0\\ y\to 0}}\frac{y^2+\sin x}{x^2+\sin y^2} = \lim_{x\to 0}\frac{x+\sin x}{x^2+\sin x} = \lim_{x\to 0}\frac{1+\frac{\sin x}{x}}{x+\frac{\sin x}{x}} = 2$

9. $\lim_{x\to 0}\frac{\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)}{\sin^{-1}x}$

$$= \lim_{x\to 0}\frac{2\tan^{-1}x}{\sin^{-1}x} = 2$$

Exercise 2.6

1. $\lim_{x\to\infty} x\left(a^{1/x}-1\right) = \lim_{x\to\infty}\left[\frac{a^{1/x}-1}{1/x}\right]$

$$= \log_e a = -\log_e\frac{1}{a}$$

2. $\lim_{x\to 0}\frac{x\left(2^x-1\right)}{1-\cos x} = \lim_{x\to 0}\frac{2^x-1}{x}\times\frac{x^2}{1-\cos x}$

$$= \log 2\lim_{x\to 0}\frac{x^2}{2\sin^2\frac{x}{2}} = 2\log 2 = \log 4$$

3. $\lim_{x\to 2}\frac{\sin(e^{x-2}-1)}{\log(x-1)}$

$$= \lim_{t\to 0}\frac{\sin(e^t-1)}{\log(1+t)}$$ {Putting $x = 2+t$}

$$= \lim_{t\to 0}\frac{\sin(e^t-1)}{e^t-1}\times\frac{e^t-1}{t}\times\frac{t}{\log(1+t)}$$

$$= 1\times 1\times 1 = 1$$

4. $\lim_{x\to 0}\frac{e^{x^2}-\cos x}{x^2}$

$$= \lim_{x\to 0}\left[\frac{e^{x^2}-1}{x^2}+\frac{1-\cos x}{x^2}\right]$$

$$= 1+\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2} = 1+\frac{1}{2} = \frac{3}{2}$$

5. $\lim_{x\to 0}\frac{e^x+e^{-x}-2}{x^2}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0} \frac{1}{e^x}\left(\frac{e^x-1}{x}\right)^2$$
$$=1$$

6. $\lim_{x\to a} \frac{\log(x-a)}{\log(e^x-a^a)}$ $\left(\frac{0}{0}\text{form}\right)$

$$= \lim_{x\to a} \frac{\frac{1}{x-a}}{\frac{e^x}{e^x-e^a}}$$ (Applying L' Hopital's rule)

$$= \lim_{x\to a} \frac{e^x-e^a}{e^x(x-a)}$$
$$= \lim_{x\to a} \frac{e^a(e^{x-a}-1)}{e^x(x-a)}$$
$$=1$$

7. $\lim_{x\to 0} a^{\sin x} \times \frac{(a^{\tan x-\sin x}-1)}{(\tan x-\sin x)}$
$$= a^0 \ln a = \ln a$$

8. $\lim_{x\to 0} \frac{(1-3^x-4^x+12^x)}{\sqrt{(2\cos x+7)}-3}$

$$= \lim_{x\to 0} \frac{(3^x-1)(4^x-1)}{\sqrt{(2\cos x+7)}-3}$$
$$= \lim_{x\to 0} \frac{(3^x-1)(4^x-1)(\sqrt{(2\cos x+7)}+3)}{(2\cos x+7-9)}$$
$$= \lim_{x\to 0} \frac{\frac{(3^x-1)}{x}\times\frac{(4^x-1)}{x}(\sqrt{(2\cos x+7)}+3)}{\frac{-2(1-\cos x)}{x^2}}$$
$$= \frac{(\ln 3)(\ln 4)6}{-2\times\frac{1}{2}} = -6\ln 3\times\ln 4$$
$$= -12\ln 2\times\ln 3$$

9. $L = \lim_{x\to 0} \frac{(729)^x-(243)^x-(81)^x+9^x+3^x-1}{x^3}$

$$= \lim_{x\to 0} \frac{(3^x-1)}{x}\times\frac{(9^x-1)}{x}\times\frac{(27^x-1)}{x}$$
$$= (\ln 3)(\ln 9)(\ln 27)$$
$$= 6(\ln 3)^3$$

Exercise 2.7

1. Let $A = \lim_{x\to\infty}\left(\frac{x+2}{x+1}\right)^{x+3}$

$$= \lim_{x\to\infty}\left(\frac{x+2}{x+1}\right)^{x+3}$$
$$= \lim_{x\to\infty}\left[\left(1+\frac{1}{x+1}\right)^{x+1}\right]^{\frac{(x+3)}{(x+1)}}$$
$$= \left[\lim_{x\to\infty}\left(1+\frac{1}{x+1}\right)^{x+1}\right]^{\lim_{x\to\infty}\frac{1+\frac{3}{x}}{1+\frac{1}{x}}}$$
$$= e^1$$

2. $\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx} = \lim_{x\to\infty}\left\{\left(1+\frac{1}{a+bx}\right)^{a+bx}\right\}^{\frac{c+dx}{a+bx}} = e^{d/b}$

$$\left\{\because \lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{a+bx} = e \text{ and } \lim_{x\to\infty}\frac{c+dx}{a+bx}=\frac{d}{b}\right\}$$

3. $\left[\lim_{x\to\infty}\left(1+\frac{1}{x/2}\right)^{x/2}\right]^2 = e^2$

4. Given limit takes 1^∞ form

$$\Rightarrow L = \lim_{x\to 7/2}(2x^2-9x+8)^{\cot(2x-7)}$$
$$= \lim_{x\to 7/2}((2x-7)(x-1)+1)^{\cot(2x-7)}$$
$$= e^{\lim_{x\to 7/2}((2x-7)(x-1))\cot(2x-7)}$$
$$= e^{\lim_{x\to 7/2}\frac{(2x-7)(x-1)}{\tan(2x-7)}}$$
$$= e^{5/2}$$

5. Given limit takes 1^∞ form

$$\Rightarrow L = \lim_{x\to 0}\left\{\sin^2\left(\frac{\pi}{2-px}\right)\right\}^{\sec^2\left(\frac{\pi}{2-qx}\right)}$$
$$= e^{\lim_{x\to 0}\left[\sin^2\left(\frac{\pi}{2-px}\right)-1\right]\sec^2\left(\frac{\pi}{2-qx}\right)}$$
$$= e^{-\lim_{x\to 0}\frac{\cos^2\left(\frac{\pi}{2-px}\right)}{\cos^2\left(\frac{\pi}{2-qx}\right)}}$$
$$= e^{-\lim_{x\to 0}\frac{\sin^2\left(\frac{\pi}{2}-\frac{\pi}{2-px}\right)}{\sin^2\left(\frac{\pi}{2}-\frac{\pi}{2-qx}\right)}}$$

Now, $\lim_{x\to 0}\frac{\sin^2\left(\frac{\pi}{2}-\frac{\pi}{2-px}\right)}{\sin^2\left(\frac{\pi}{2}-\frac{\pi}{2-qx}\right)}$

$$= \lim_{x\to 0} \frac{\dfrac{\sin^2\left(\dfrac{\pi}{2}-\dfrac{\pi}{2-px}\right)}{\left(\dfrac{\pi}{2}-\dfrac{\pi}{2-px}\right)^2}}{\dfrac{\sin^2\left(\dfrac{\pi}{2}-\dfrac{\pi}{2-qx}\right)}{\left(\dfrac{\pi}{2}-\dfrac{\pi}{2-qx}\right)^2}} \times \frac{\left(\dfrac{\pi}{2}-\dfrac{\pi}{2-px}\right)^2}{\left(\dfrac{\pi}{2}-\dfrac{\pi}{2-qx}\right)^2}$$

$$= \lim_{x\to 0} \frac{\left(\dfrac{\pi}{2}-\dfrac{\pi}{2-px}\right)^2}{\left(\dfrac{\pi}{2}-\dfrac{\pi}{2-qx}\right)^2}$$

$$= \lim_{x\to 0} \frac{\left(\dfrac{-\pi px}{2(2-px)}\right)^2}{\left(\dfrac{-\pi qx}{2(2-qx)}\right)^2}$$

$$= p^2/q^2 \Rightarrow L = e^{p^2/q^2}$$

Exercise 2.8

1. Let $y = \lim_{x\to 0} x^x$

$$\Rightarrow \log y = \log\left(\lim_{x\to 0} x^x\right)$$

$$= \lim_{x\to 0} (\log x^x)$$

$$= \lim_{x\to 0} (x \cdot \log x) = \lim_{x\to 0} \frac{\log x}{\frac{1}{x}}$$

$$= \lim_{x\to 0} \frac{1/x}{-1/x^2} \quad \text{(Applying L'Hopital rule)}$$

$$= \lim_{x\to 0} (-x) = 0$$

2. $\lim_{x\to \pi/2} \tan x \log \sin x = \lim_{x\to\pi/2} \dfrac{\log \sin x}{\cot x}$

$$= \lim_{x\to\pi/2} \frac{\dfrac{1}{\sin x}\cos x}{-\operatorname{cosec}^2 x} = 0 \quad \text{(Applying L' Hopital's Rule)}$$

3. $\lim_{x\to 0} \dfrac{\log \cos x}{x}$

$$= \lim_{x\to 0} \frac{-\tan x}{1}$$

$$= 0$$

4. $\lim_{x\to 0} \dfrac{2^x - 1}{(1+x)^{1/2} - 1}$

$$= \lim_{x\to 0} \frac{2^x \log 2}{\frac{1}{2}(1+x)^{-1/2}} \quad \text{(Applying L' Hopital's Rule)}$$

$$= 2\log 2$$

$$= \log 4$$

5. $\lim_{x\to\pi/4} (2-\tan x)^{1/\ln \tan x}$

$$= e^{\lim_{x\to\pi/4}(2-\tan x - 1)\times\frac{1}{\ln \tan x}} \quad (1^\infty \text{ form})$$

$$= e^{\lim_{x\to\pi/4}\left(\frac{1-\tan x}{\ln \tan x}\right)}$$

$$= e^{\lim_{x\to\pi/4}\frac{-\sec^2 x}{\frac{1}{\tan x}\sec^2 x}}$$

$$= e^{-\lim_{x\to\pi/4}\tan x} = e^{-1}$$

6. Since $\lim_{x\to a} \dfrac{a^x - x^a}{x^x - a^a} = -1$

$$\Rightarrow \lim_{x\to a} \frac{a^x \log a - a x^{a-1}}{x^x(1+\log x)} = -1$$

$$\Rightarrow \frac{a^a[\log a - 1]}{a^a[1+\log a]} = -1$$

$$\Rightarrow \log a - 1 = -1 - \log a$$

$$\Rightarrow 2\log a = 0$$

$$\Rightarrow \log a = 0$$

$$\Rightarrow a = 1$$

Exercise 2.9

1. $\lim_{x\to 0} \dfrac{ae^x - b}{x} = 2$

$$\Rightarrow \lim_{x\to 0} \frac{ae^x - a + a - b}{x} = 2$$

$$\Rightarrow a\lim_{x\to 0}\frac{e^x - 1}{x} + \lim_{x\to 0}\frac{a-b}{x} = 2$$

$$\Rightarrow a + \lim_{x\to 0}\frac{a-b}{x} = 2$$

$$\Rightarrow a - b = 0 \text{ and } a = 2$$

$$\Rightarrow a = 2, b = 2$$

2. We have $\lim_{x\to\infty}\left\{\dfrac{x^2+1}{x+1} - (ax+b)\right\} = 0$

$$\Rightarrow \lim_{x\to\infty}\left\{\frac{x^2(1-a)-x(a+b)+1-b}{x+1}\right\}=0$$

Since the limit of the given expression is zero. Therefore, the degree of numerator is less than that of denominator. Denominator on L.H.S. is a polynomial of degree one. So, numerator must be a constant. For this, we must have coeff. of $x^2=0$ and coeff. of $x=0 \Rightarrow 1-a=0$ and $-(a+b)=0$

$\Rightarrow a=1, b=-1$

3. Let $P=\lim_{x\to 0}(1+ax+bx^2)^{2/x}$ (1^∞ form)

$$= e^{\lim_{x\to 0}(1+ax+bx^2-1)\frac{2}{x}}$$

$$= e^{\lim_{x\to 0}(2a+2bx)}$$

$= e^{2a}$

$= e^3$ (given)

$\therefore\ a=3/2$ and $b\in R$

Chapter 3

Exercise 3.1

1. Given that $f(x+y)=f(x)+f(y)$ for all x and y

Put $x=y=0 \Rightarrow f(0+0)=f(0)+f(0) \Rightarrow f(0)=0$

Consider some arbitrary point, $x=a$

$$\text{L.H.L.}=\lim_{h\to 0}f(a-h)$$

$$=\lim_{h\to 0}[f(a)+f(-h)]$$

$$=f(a)+\lim_{h\to 0}f(-h)$$

$=f(a)+f(0)$ (as $f(x)$ is continuous at $x=0$)

$=f(a)$

Similarly, we can prove that R.H.L. $=f(a)$

Hence, $f(x)$ is continuous for all x.

2. Given relation $f(x.y)=f(x)f(y)$

Put $x=y=1 \Rightarrow f(1)=f(1)f(1) \Rightarrow f(1)=0$ or 1

If $f(1)=0$, then $f(x\times 1)=f(x)f(1)=0$ or $f(x)$ is identically zero which is continuous for all x.

For $f(1)=1$:

Consider some arbitrary point, $x=a$

$$\text{L.H.L.}=\lim_{h\to 0}f(a-h)$$

$$=\lim_{h\to 0}f\left(a\left(1-\frac{h}{a}\right)\right)$$

$$=\lim_{h\to 0}f(a)f\left(1-\frac{h}{a}\right)$$

$$=f(a)\lim_{h\to 0}f\left(1-\frac{h}{a}\right)$$

$=f(a)f(1)$ [as $f(x)$ is continuous at $x=1$]

$=f(a)$

Similarly, we can prove that R.H.L. $=\lim_{h\to 0}f(a+h)=f(a)$

Hence $f(x)$ is continuous for all x.

3. Consider some arbitrary point $x=a$

$$\text{L.H.L.}=\lim_{h\to 0}f(a-h)$$

$$=\lim_{h\to 0}f(a)f(-h)$$

$$=f(a)\lim_{h\to 0}f(-h)$$

$$=f(a)\lim_{h\to 0}[1+g(-h)G(-h)]$$

$$=f(a)\left[1+\lim_{h\to 0}g(-h)\lim_{h\to 0}G(-h)\right]$$

$=f(a)[1+(0)\times(\text{any finite value})]$

[as it is given that $\lim_{x\to 0}g(x)=0$ and $\lim_{x\to 0}G(x)$ exists]

$=f(a)$

Similarly, we can prove that R.H.L. $=\lim_{h\to 0}f(a+h)=f(a)$.

Hence, $f(x)$ is continuous for all x.

Exercise 3.2

1. We must have

$$f(0)=\lim_{x\to 0}f(x)=\underset{x\to 0}{\text{Lt}}\frac{(1+x)^{\frac{1}{2}}-(1+x)^{\frac{1}{3}}}{x}$$

$$=\lim_{x\to 0}\left[\frac{(1+x)^{\frac{1}{2}}-1}{x}-\frac{(1+x)^{\frac{1}{3}}-1}{x}\right]$$

$$=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

2. Since $f(x)$ is continuous at $x=2$

$$\therefore f(2)=\lim_{x\to 2}f(x)$$

$$=\lim_{x\to 2}\frac{x^2-(A+2)x+A}{x-2}$$

$$=\lim_{x\to 2}\frac{x(x-2)-A(x-1)}{x-2}$$

Now $f(2)$ is finite only when $A=0$

3. For continuity at $x=0$

$$f(0)=\lim_{x\to 0}\left(\frac{1}{x}-\frac{2}{e^{2x}-1}\right)=\lim_{x\to 0}\frac{e^{2x}-1-2x}{x(e^{2x}-1)}$$

$$=\lim_{x\to 0}\frac{\left(1+(2x)+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots\right)-1-2x}{x\left(1+(2x)+\frac{(2x)^2}{2!}+\cdots-1\right)}$$

$$=\lim_{x\to 0}\frac{\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots}{x\left((2x)+\frac{(2x)^2}{2!}+\cdots\right)}$$

$$= \lim_{x\to 0} \frac{\frac{2^2}{2!} + \frac{2^3}{3!}x + \cdots}{2 + \frac{(2)^2}{2!}x + \cdots}$$

$$= \frac{2 + 0 + \cdots}{2 + 0 + \cdots} = \frac{2}{2} = 1$$

4. f is continuous at $\frac{\pi}{4}$,

if $f\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{4x-\pi}$

$$= \lim_{x\to\frac{\pi}{4}} \frac{0-\sec^2 x}{4} \quad \text{[L' Hopital's Rule]}$$

$$= -\frac{1}{4}\sec^2\frac{\pi}{4} = -\frac{1}{4}(2) = -\frac{1}{2}$$

5. $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^2}{|x|} = \lim_{x\to 0} |x| = 0 = f(0)$

$\therefore$ $f(x)$ is continuous at $x = 0$

Also if $x \neq 0, f(x) = |x|$, which is continuous for non-zero x.

$\therefore$ $f(x)$ is continuous everywhere.

6. Clearly continuous at $x = 1$

To check continuity at $x = 0$, $f(0) = e^3$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}(1+3x)^{1/x} \quad (1^\infty \text{ form})$$

$$= e^{\lim_{x\to 0} 3x\left(\frac{1}{x}\right)}$$

$$= e^{\lim_{x\to 0}(3)} = e^3$$

$\therefore \lim_{x\to 0} f(x) = f(0)$

Thus, continuous at $x = 0$.

7. $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} \frac{h}{e^{\frac{1}{h}}+1} = 0$ and

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} \frac{-h}{e^{-\frac{1}{h}}+1} = 0$$

Also

$f(1) = 0$

Therefore, $f(x)$ is continuous at $x = 0$.

8. d. $f_1(x) = \sqrt{2\sin x + 3}$

$-1 \le \sin x \le 1$

$\Rightarrow -2 \le 2\sin x \le 2$

$\Rightarrow 1 \le 2\sin x + 3 \le 5$

$\Rightarrow \sqrt{2\sin x + 3}$ is defined and hence continuous $\forall x \in R$

$f_2(x) = \frac{e^x+1}{e^x+3}$. Here $e^x + 3 > 3, \forall x \in R$

$\Rightarrow f_2(x)$ is continuous, $\forall x \in R$

$$f_3(x) = \left(\frac{2^{2x}+1}{2^{3x}+5}\right)^{5/7}.$$

Here $2^{3x} + 5 = 8^x + 5 > 5, \forall x \in R$

$\Rightarrow f_3(x)$ is continuous, $\forall x \in R$

$f_4(x) = \sqrt{\text{sgn}(x)+1}$

$$= \begin{cases} \sqrt{\frac{|x|}{x}+1} & x \neq 0 \\ \sqrt{1}, & x = 0 \end{cases}$$

$$= \begin{cases} 0, & x < 0 \\ \sqrt{2}, & x > 0. \\ 1, & x = 0 \end{cases}$$

Clearly $f_4(x)$ is discontinuous at $x = 0$.

9. Since $f(x)$ is continuous at $x = 1$, therefore,

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$\Rightarrow A - B = 3 \Rightarrow A = 3 + B$ (1)

If $f(x)$ is continuous at $x = 2$, then

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

$\Rightarrow 6 = 4B - A$ (2)

Solving equations (1) and (2), we get $B = 3$

But $f(x)$ is not continuous at $x = 2$, therefore $B \neq 3$

Hence, $A = 3 + B$ and $B \neq 3$.

10. For any $x \neq 1, 2$, we find that $f(x)$ is the quotient of two polynomials and a polynomial is everywhere continuous. Therefore, $f(x)$ is continuous for all $x \neq 1, 2$.

$$f(x) = \begin{cases} \frac{x^4 - 5x^2 + 4}{|(x-1)(x-2)|}, & x \neq 1, 2 \\ 6, & x = 1 \\ 12, & x = 2 \end{cases}$$

$$= \begin{cases} \frac{(x-1)(x-2)}{|(x-1)(x-2)|}(x^2+3x+2), & x \neq 1, 2 \\ 6, & x = 1 \\ 12, & x = 2 \end{cases}$$

$$= \begin{cases} (x^2+3x+2), & x < 1 \text{ or } x > 2 \\ -(x^2+3x+2), & 1 < x < 2 \\ 6, & x = 1 \\ 12, & x = 2 \end{cases}$$

$f(1^+) = -6, f(1^-) = 6$ and $f(2^+) = 12$ and $f(2^-) = -12$

Hence $f(x)$ is discontinuous at $x = 1$ and $x = 2$

11. a → s, q; b → t, p; c → r, q; d → u, q

a. $f(x) = \frac{1}{x-1} \Rightarrow \lim_{x\to 1^+} \frac{1}{x-1} = \infty$ and $\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$

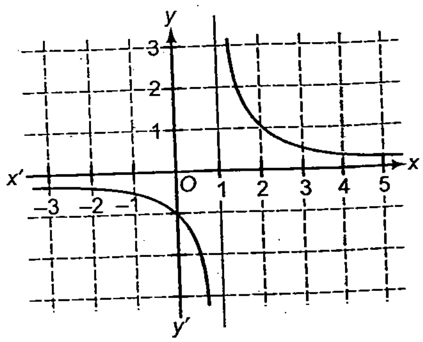


Fig. S-3.1

Thus, $f(x)$ has vertical asymptote at $x = 1$, Hence it has non-removable discontinuity at $x = 1$.

b. $f(x) = \dfrac{x^3 - x}{x^2 - 1} = x, x \neq \pm 1,$

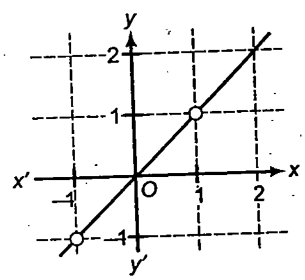


Fig. S-3.2

Hence $f(x)$ has a missing point discontinuity at $x = 1$ which is removable.

c. $f(x) = \dfrac{|x-1|}{x-1} = \begin{cases} 1, & x > 1 \\ -1, & x < 1 \end{cases}$

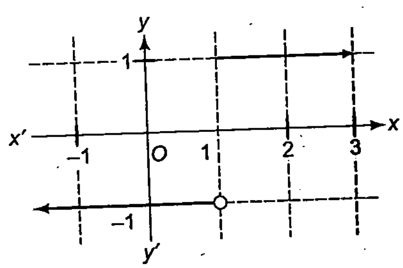


Fig. S-3.3

Hence $f(x)$ has jump discontinuity at $x = 1$ which is non-removable.

d. $f(x) = \sin\left(\dfrac{1}{x-1}\right)$

$\Rightarrow \lim\limits_{x \to 1+} \sin\left(\dfrac{1}{x-1}\right) = \sin(\infty) =$ any value between -1 and 1

Similarly $\lim\limits_{x \to 1-} \sin\left(\dfrac{1}{x-1}\right) = \sin(-\infty) =$ any value between -1 and 1.

Thus, $f(x)$ oscillates between -1 and 1. Hence, it has non-removable discontinuity.

Exercise 3.3

1. $f(x) = [x^2 + 1] = [x^2] + 1$
   Now x^2 is monotonic in the range of $[1, 3]$.
   Hence $[x^2]$ is discontinuous when x^2 is integer, or $x^2 = 2, 3, 4, \ldots, 9$
   $\Rightarrow x = \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{9}$.
   Note that it is right continuous at $x = 1$ but not left continuous at $x = 3$.
   or $\lim\limits_{x \to 1^+} [x^2 + 1] = 2 = f(1)$ and
   $\lim\limits_{x \to 3^-} [x^2 + 1] = 9 \neq 10 [= f(3))]$

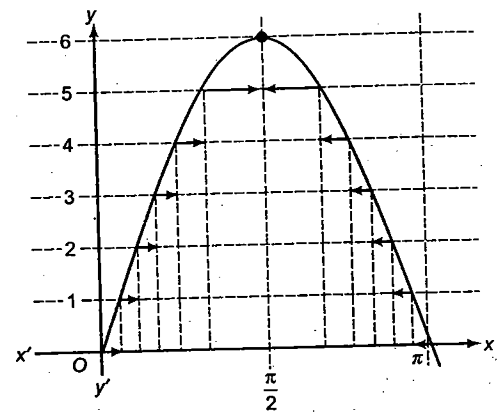
2. 

Fig. S-3.4

Clearly, from Fig. S-3.4, the number of points of discontinuity are 11.

3. Since $g(x) = \tan^{-1} x$ is a strictly increasing function, then $f(x) = [\tan^{-1} x]$ is discontinuous when $\tan^{-1} x$ is an integer.
   Now integral values of $\tan^{-1} x$ are -1, 0 and 1.
   Hence $f(x)$ is discontinuous when $\tan^{-1} x = -1, 0, 1$.
   $\Rightarrow x = \tan(-1), \tan 0, \tan 1$
   $\Rightarrow x = -\tan 1, 0, \tan 1$
   Graphically, also this can be analyzed.
   Clearly from the graph given in Fig. S-3.5 $f(x)$ is discontinuous when
   $\tan^{-1} x = 0, \pm 1$ or $x = 0, \pm \tan 1$

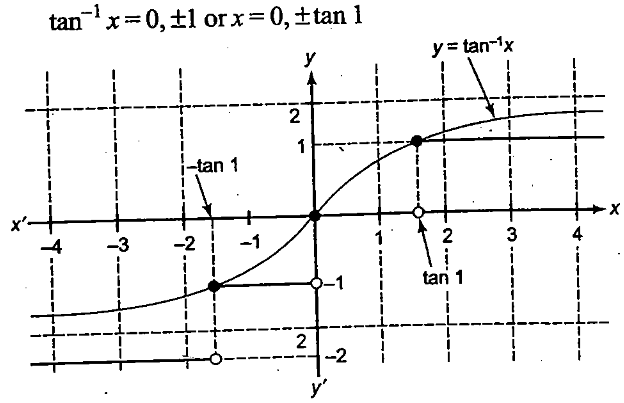


Fig. S-3.5

4. $f(x) = \{\cot^{-1} x\}$

$= \cot^{-1} x - [\cot^{-1} x]$

$f(x)$ is discontinuous where $\cot^{-1} x$ is an integer. Clearly from graph shown in Fig. S.3.6 $f(x)$ is discontinuous when $\cot^{-1} x = 1, 2, 3$ or

$x = \cot 1, \cot 2, \cot 3$

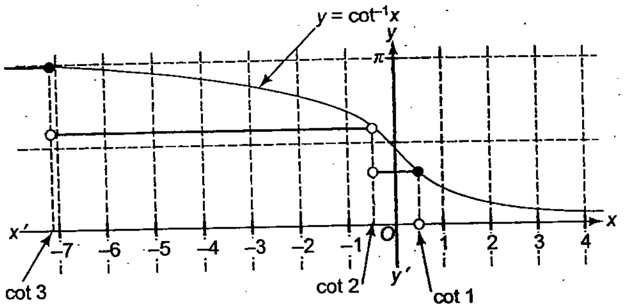


Fig. S-3.6

5. $f(x) = \lim_{n\to\infty} \dfrac{1 - \dfrac{\sin x^n}{x^n}}{1 + \dfrac{\sin x^n}{x^n}} : f(x) = \begin{bmatrix} 1 \text{ for } x>1 \\ 0 \text{ for } x=1 \\ 0 \text{ for } x<1 \end{bmatrix}$

Hence, $f(x)$ is discontinuous at $x = 1$.

6. Obviously if $g(x) = \left[\dfrac{f(x)}{c}\right]$ is continuous then c must exceed the greatest value of $f(x)$ to restrict the ratio $f(x)/c$ between 0 and 1, for which least positive integral value of c is 6.

($\because$ maximum value of $f(x)$ is $\sqrt{16}$ which lies between 5 and 6.)

7. Since, $\lim_{n\to\infty} x^{2n} = \begin{cases} 0, & |x|<1 \\ 1, & |x|=1 \end{cases}$

$\therefore f(x) = \lim_{n\to\infty}\left(\sin\left(\dfrac{\pi x}{2}\right)\right)^{2n} = \begin{cases} 0; & \left|\sin\dfrac{\pi x}{2}\right|<1 \\ 1; & \left|\sin\dfrac{\pi x}{2}\right|=1 \end{cases}$

Thus, $f(x)$ is continuous for all x, except for those values of x for which $\left|\sin\dfrac{\pi x}{2}\right| = 1$, i.e., x is an odd integer.

$\Rightarrow x = (2n+1)$ where $n \in I$

Check continuity at $x = (2n+1)$

L.H.L $= \lim_{x\to 2n+1} f(x) = 0$ and $f(2n+1) = 1$

Thus, L.H.L $\neq f(2n+1)$

$\Rightarrow f(x)$ is discontinuous at $x = (2n+1)$

(i.e., at odd integers)

8. $f(x) = \begin{cases} x^2, & x \text{ is rational} \\ -x^2, & x \text{ is irrational} \end{cases}$

$f(x)$ is continuous when $x^2 = -x^2 \Rightarrow x = 0$.

9. $t = \dfrac{1}{x-1}$ is discontinuous at $x = 1$. Also $y = \dfrac{1}{t^2+t-2}$ is discontinuous at $t = -2$ and $t = 1$

when $t = -2$, $\dfrac{1}{x-1} = -2 \Rightarrow x = \dfrac{1}{2}$, when $t = 1$, $\dfrac{1}{x-1} = 1$

$\Rightarrow x = 2$.

So, $y = f(x)$ is discontinuous at three points, $x = 1, \dfrac{1}{2}, 2$.

10. a. Continuity should be checked at the end-points of intervals of each definition i.e. $x = 0, 1, 2$.

b. For $[\sin \pi x]$, continuity should be checked at all values of x at which $\sin \pi x \in I$.

i.e., $x = 0, \dfrac{1}{2}$

c. For $\operatorname{sgn}\left(x - \dfrac{5}{4}\right)\left\{x - \dfrac{2}{3}\right\}$, continuity should be checked when $x - \dfrac{5}{4} = 0$ (as sgn $(g(x))$ is discontinuous at $g(x) = 0$), i.e., $x = \dfrac{5}{4}$ and when $x - \dfrac{2}{3} \in I$, i.e., $x = \dfrac{5}{3}$ (as $\{x\}$ is discontinuous when $x \in I$).

$\therefore$ overall discontinuity should be checked at $x = 0, \dfrac{1}{2}, 1, \dfrac{5}{4}, \dfrac{5}{3}$ and 2 check the discontinuity yourself.

Hence $f(x)$ is discontinuous at $x = \dfrac{1}{2}, 1\ \dfrac{5}{4}, \dfrac{5}{3}$.

At $x = 0$ and $x = 2$, $f(x)$ is continuous as $\lim_{x\to 0^+} f(x) = f(0)$ and $\lim_{x\to 2^-} f(x) = f(2)$.

11. $f(x)$ is continuous if $x^2 = x + a$ or $x^2 - x - a = 0$.

for $f(x)$ to be discontinuous, for all real x, equation must have imaginary roots.

$\therefore D < 0$

$\therefore 1 + 4a < 0$

$\therefore a < -\dfrac{1}{4}$

12. $\operatorname{sgn}(x^2 - 1)$ is discontinuous when $x^2 - 1 = 0$ or $x = \pm 1$

But $\log |x| = 0$ when $x = \pm 1$, hence $f(x)$ is continuous at $x = \pm 1$

Then $f(x)$ is continuous in its entire domain.

13. $f(x) = \lim_{n\to\infty}\left(\cos\dfrac{x\pi}{2}\right)^{2n}$ is discontinuous when

$f(x) = \cos^2\dfrac{x\pi}{2} = 1$

$\dfrac{x\pi}{2} = n\pi \Rightarrow x = 2n, n \in Z$

Hence the only integer where $f(x)$ is discontinuous is $x = 2$

Exercise 3.4

1. $f(x) = |x+1| + |x| + |x-1|$

$|x+1|$, $|x|$, $|x-1|$ are continuous for all x, but non-differentiable at $x = -1, 0, 1$, respectively.

Hence $f(x)$ is non-differentiable at $x = -1, 0, 1$.

$$f(x)=\begin{cases}(-x-1)+(-x)+(1-x), & x<-1\\ (x+1)+(-x)+(1-x), & -1\le x<0\\ (x+1)+(x)+(1-x), & 0\le x<1\\ (x+1)+(x)+(x-1) & x\ge 1\end{cases}$$

$$=\begin{cases}-3x, & x<-1\\ -x+2 & -1\le x<0\\ x+2, & 0\le x<1\\ 3x, & x\ge 1\end{cases}$$

$\therefore$ graph of $f(x)$ is given in Fig. S-3.7.

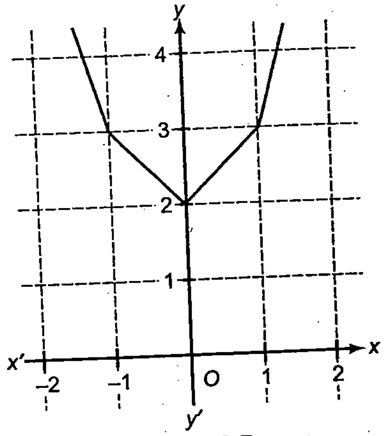


Fig. S-3.7

It is clear from the graph that $f(x)$ is continuous $\forall\ x\in R$ but not differentiable at $x=-1, 0, 1$.

2. Domain of $f(x)$ is $[0, 2]$

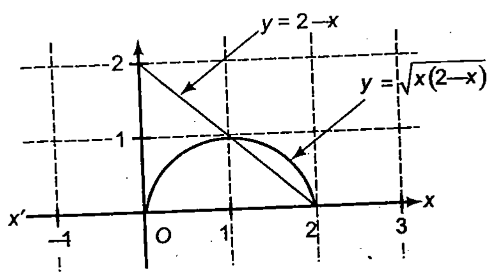


Fig. S-3.8

Clearly from the graph given in Fig. S-3.8, $f(x)$ is non-differentiable at $x=1$.

3. Since x and $|x-x^2|$ are continuous for all x, $f(x)=x-|x-x^2|$ is continuous for all x.
   Also x is differentiable but $|x-x^2|$ is non-differentiable at $x=0$ and 1, hence $f(x)$ is non-differentiable at $x=0$ and 1.
4. We have, $f(x)=|[x]x|$ in $-1<x\le 2$

$$\Rightarrow f(x)=\begin{cases}-x, & -1<x<0\\ 0, & 0\le x<1\\ x, & 1\le x<2\\ 2x, & x=2\end{cases}$$

It is evident from the graph given in Fig. S-3.9 for this function that it is continuous but not differentiable at $x=0$. Also, it is discontinuous at $x=1$ and $x=2$, hence non-differentiable at $x=1$ and $x=2$.

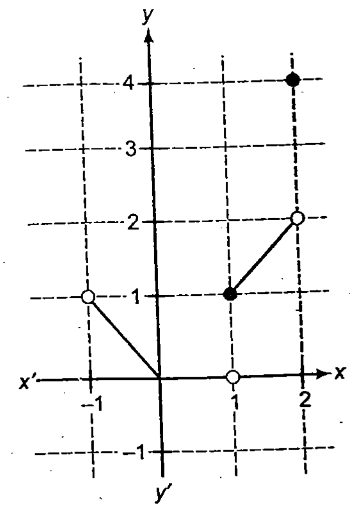


Fig. S-3.9

5.

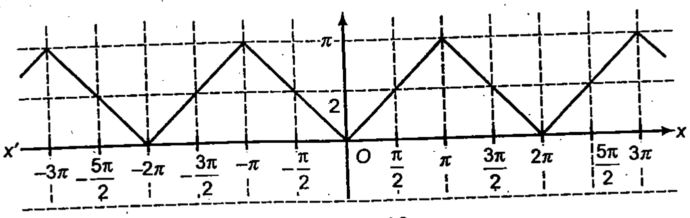


Fig. S-3.10

Clearly from the graph given in Fig. S-3.10, $f(x)$ is non-differentiable at $x=n\pi$; $n\in Z$.

6.

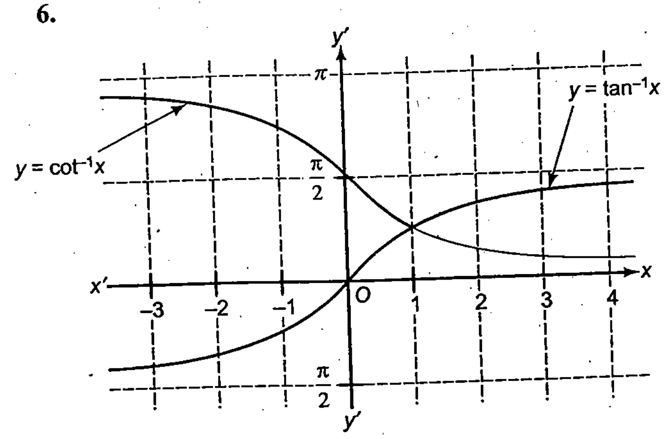


Fig. S-3.11

Clearly from the graph given in Fig. S-3.11, $f(x)$ is non-differentiable at $x=1$.

7. $f(x)=\begin{cases}ax^2+1, & x\le 1\\ x^2+ax+b, & x>1\end{cases} \Rightarrow f'(x)=\begin{cases}2ax, & x\le 1\\ 2x+a, & x>1\end{cases}$

$f(x)$ is differentiable at $x=1$,
then we must have,

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f'(x)=\lim_{x\to 1^-} f'(x)$

$\Rightarrow a+1=1+a+b$ and $2a=2+a$

$\Rightarrow a=2$ and $b=0$

8. $f(x)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\begin{cases}2\tan^{-1}x, & \text{if } 0\le x<\infty\\ -2\tan^{-1}x, & \text{if } -\infty<x\le 0\end{cases}$

$$\Rightarrow f'(x)=\begin{cases}\frac{2}{1+x^2}, & \text{if } 0<x<\infty\\ -\frac{2}{1+x^2}, & \text{if } -\infty<x<0\end{cases}$$

$\Rightarrow f'(0^+)=1$ and $f'(0^-)=-1$

Hence, $f(x)$ is continuous and non-differentiable at x is equal to 0.

Using the shortcut method,

Differentiate $f(x)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ w.r.t. x,

$$\Rightarrow f'(x)=-\frac{1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}\left(\frac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2}\right)$$

$$=\frac{2x}{\sqrt{(1+x^2)^2-(1-x^2)^2}}\frac{1}{1+x^2}$$

$$=\frac{2x}{\sqrt{4x^2}}\frac{1}{1+x^2}$$

$$=\frac{2x}{|x|}\frac{1}{1+x^2}$$

which is discontinuous at $x=0$.

Hence, $f(x)$ is non-differentiable at $x=0$.

9. d. $f(x)=\frac{x-2}{x^2+3}$ is rational function with domain R, which is always differentiable.

$f(x)=\log|x|$ is always differentiable in its domain (draw the graph and verify)

$f(x)=x^3\log x$ is always differentiable as x^3 and $\log x$ are always differentiable

$f(x)=(x-2)^{3/5}\Rightarrow f'(x)=\frac{3}{5(x-2)^{2/5}}$ which does not exist at $x=2$, hence non-differentiable at $x=2$ ($f(x)$ has vertical tangent at $x=2$).

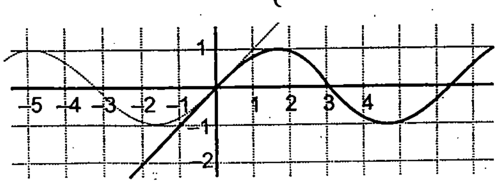
10. $f(x)=||x^2-4|-12|$ is non-differentiable when $x^2-4=0$ and $|x^2-4|-12=0$

or $x=\pm 2$ and $x^2-4=\pm 12$ or $x=\pm 2$ and $x^2=16$ or $x=\pm 2$ and $x=\pm 4$

Hence there are four points of non-differentiability.

11. (i) Graph of $f(x)=\min\{x,\sin x\}$ is as follow.

From the graph $f(x)=\begin{cases}x; & x<0\\ \sin x: & x\ge 0\end{cases}$



$$\Rightarrow f'(x)=\begin{cases}1; & x<0\\ \cos x; & x>0\end{cases}$$

$f'(0+)=f'(0^-)=1$. Hence $f(x)$ is differentiable at $x=0$.

(ii) $f(x)=\begin{cases}0; & x\ge 0\\ x^2; & x<0\end{cases}$

Here $f(x)$ is continuous at $x=0$

Now $f'(x)=\begin{cases}0; & x>0\\ 2x; & x<0\end{cases}$

$f'(0^+)=0$ and $f'(0^-)=0$

Hence $f(x)$ is differentiable at $x=0$

(iii) $f(x)=x^2\operatorname{sgn}(x)=\begin{cases}x^2; & x\ge 0\\ -x^2; & x<0\end{cases}$, which is continuous as well as differentiable at $x=0$

Chapter 4

Exercise 4.1

1. a. Let $f(x)=\sqrt{\sin x}\Rightarrow f(x+h)=\sqrt{\sin(x+h)}$

$$\therefore \frac{d}{dx}[f(x)]=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{\sin(x+h)}-\sqrt{\sin x}}{h}$$

$$=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h(\sqrt{\sin(x+h)}+\sqrt{\sin x})}\text{ (rationalizing)}$$

$$=\lim_{h\to 0}\frac{2\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{h(\sqrt{\sin(x+h)}+\sqrt{\sin x})}$$

$$=\lim_{h\to 0}\frac{(\sin h/2)}{(h/2)}\lim_{h\to 0}\frac{\cos(x+h/2)}{(\sqrt{\sin(x+h)}+\sqrt{\sin x})}$$

$$=\frac{\cos x}{\sqrt{\sin x}+\sqrt{\sin x}}=\frac{\cos x}{2\sqrt{\sin x}}$$

b. Let $f(x)=\cos^2 x$. Then $f(x+h)=\cos^2(x+h)$

$$\therefore \frac{d}{dx}[f(x)]=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cos^2(x+h)-\cos^2 x}{h}$$

$$=\lim_{h\to 0}\frac{\sin^2 x-\sin^2(x+h)}{h}$$

$$=\lim_{h\to 0}\frac{\sin(x+x+h)\sin(x-(x+h))}{h}$$

$$= \lim_{h\to 0}\frac{\sin(2x+h)\sin(-h)}{h}$$

$$= -\lim_{h\to 0}\frac{\sin h}{h}\lim_{h\to 0}\sin(2x+h)$$

$$\Rightarrow \frac{d}{dx}f(x) = -\sin 2x$$

c. Let $f(x) = \tan^{-1}x$. Then $f(x+h) = \tan^{-1}(x+h)$

$$\therefore \frac{d}{dx}[f(x)] = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\tan^{-1}(x+h)-\tan^{-1}x}{h}$$

$$= \lim_{h\to 0}\frac{\tan^{-1}\left(\frac{x+h-x}{1+x(x+h)}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\tan^{-1}\left(\frac{h}{1+x(x+h)}\right)}{\frac{h}{1+x(x+h)}}\times\frac{1}{\{1+x(x+h)\}}$$

$$= \frac{1}{1+x^2}$$

d. Let $f(x) = \log x$. Then, $f(x+h) = \log(x+h)$

$$\therefore \frac{d}{dx}[f(x)] = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\log(x+h)-\log x}{h}$$

$$= \lim_{h\to 0}\frac{\log\left(1+\frac{h}{x}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}}\frac{1}{x}$$

$$= \frac{1}{x}$$

2. $\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h}+g(x)\frac{f(x+h)-f(x)}{h}\right]$$

$$= f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= f(x)\frac{d}{dx}[g(x)]+g(x)\frac{d}{dx}[f(x)]$$

Exercise 4.2

1. Let $y = \left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}\right] = \log e^x + \log\left(\frac{x-2}{x+2}\right)^{3/4}$

$$\Rightarrow y = x+\frac{3}{4}[\log(x-2)-\log(x+2)]$$

$$\Rightarrow \frac{dy}{dx} = 1+\frac{3}{4}\left[\frac{1}{x-2}-\frac{1}{x+2}\right] = 1+\frac{3}{(x^2-4)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2-1}{x^2-4}$$

2. $y = \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)+\sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$

$$= \cos^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)+\sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) = \frac{\pi}{2}$$

$$\Rightarrow \frac{dy}{dx} = 0 \qquad \left\{\because \sin^{-1}x+\cos^{-1}x = \frac{\pi}{2}\right\}$$

3. $y = \tan^{-1}\frac{4x}{1+5x^2}+\tan^{-1}\frac{2+3x}{3-2x}$

$$= \tan^{-1}\frac{5x-x}{1+5x.x}+\tan^{-1}\frac{\frac{2}{3}+x}{1-\frac{2}{3}x}$$

$$= \tan^{-1}5x-\tan^{-1}x+\tan^{-1}\frac{2}{3}+\tan^{-1}x$$

$$= \tan^{-1}5x+\tan^{-1}\frac{2}{3}$$

$$\therefore \frac{dy}{dx} = \frac{5}{1+25x^2}$$

4. Let $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$. Putting $x = \tan\theta$, we get

$$\Rightarrow y = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{1+x^2}\right)$$

5. Let $y = \tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right)$

$$= \tan^{-1}\left(\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right)$$

$$= \tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}(\tan x)$$

$$= \tan^{-1}\left(\frac{a}{b}\right) - x \qquad \left[\because -\frac{\pi}{2} < x < \frac{\pi}{2}\right]$$

$$\Rightarrow \frac{dy}{dx} = 0 - 1 = -1$$

6. Putting $x^2 = \cos 2\theta$, we get,

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta} + \sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta} - \sqrt{2\sin^2\theta}}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right) = \tan^{-1}(\tan(\pi/4 + \theta))$$

$$\Rightarrow y = \frac{\pi}{4} + \theta \qquad \left[\begin{array}{l}\because 0 < x^2 < 1 \Rightarrow 0 < \cos 2\theta < 1 \\ \Rightarrow 0 < 2\theta < \pi/2 \\ \Rightarrow 0 < \theta < \pi/4 \\ \Rightarrow \pi/4 < \pi/4 + \theta < \pi/2\end{array}\right]$$

$$\Rightarrow y = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2} \times \frac{2x}{\sqrt{1-x^4}} = \frac{-x}{\sqrt{1-x^4}}$$

7. From triangular conversions

$$y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

$$= \tan^{-1}x + \tan^{-1}x = 2\tan^{-1}x$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$$

8. Let $y = \tan^{-1}\dfrac{3a^2x - x^3}{a\left(a^2 - 3x^2\right)}$

$$= \tan^{-1}\frac{3(x/a) - (x/a)^3}{1 - 3(x/a)^2}$$

$$= \tan^{-1}\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}, \text{ putting } x/a = \tan\theta$$

$$= \tan^{-1}\tan 3\theta = 3\theta = 3\tan^{-1}(x/a)$$

$$\therefore \frac{dy}{dx} = 3\frac{d}{dx}\tan^{-1}(x/a)$$

$$= 3\frac{1}{1+(x/a)^2} \times \frac{1}{a} = \frac{3a}{a^2 + x^2}$$

9. Putting $x = \sin\theta$, $5 = r\cos\alpha$, and $12 = r\sin\alpha$, so that $r = 13$, $\tan\alpha = 12/5$,

$$y = \sin^{-1}\left[\frac{r\cos\alpha\sin\theta + r\sin\alpha\cos\theta}{13}\right]$$

$$= \sin^{-1}\sin(\theta + \alpha) = \theta + \alpha$$

or $y = \sin^{-1}x + \tan^{-1}(12/5)$

$$\therefore \frac{dy}{dx} = 1/\sqrt{\left(1-x^2\right)}$$

10. $y = \tan^{-1}\left(\dfrac{x}{1+\sqrt{1-x^2}}\right)$

Put $x = \sin\theta$

$$\therefore y = \tan^{-1}\left(\frac{\sin\theta}{1+\sqrt{1-\sin^2\theta}}\right) = \tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

$$= \tan^{-1}\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan^{-1}\tan\frac{\theta}{2} = \frac{\theta}{2}$$

So, $y = \dfrac{\sin^{-1}x}{2} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{1-x^2}}$

Exercise 4.3

1. $\sin^{-1}\sqrt{1-x} = \sin^{-1}\sqrt{1-\left(\sqrt{x}\right)^2} = \cos^{-1}\sqrt{x}$

$$\therefore y = 2\cos^{-1}\sqrt{x} \text{ or } \frac{dy}{dx} = 2 \times \frac{-1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}} = \frac{-1}{\sqrt{x-x^2}}$$

2. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{\sin\sqrt{x}}} \times \cos\sqrt{x} \times \dfrac{1}{2\sqrt{x}}$

3. Let $y = e^{\sin x^2}$

Putting $x^2 = \upsilon$ and $u = \sin x^2 = \sin\upsilon$, we get

$$y = e^u, u = \sin\upsilon \text{ and } \upsilon = x^2$$

$$\therefore \frac{dy}{du} = e^u, \frac{du}{d\upsilon} = \cos\upsilon \text{ and } \frac{d\upsilon}{dx} = 2x$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{d\upsilon} \times \frac{d\upsilon}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^u \cos\upsilon\, 2x = e^{\sin\upsilon}\cos\upsilon\, 2x$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin x^2} \cos x^2\, 2x$$

4. $$\frac{d}{dx}\left[\log\sqrt{\sin\sqrt{e^x}}\right] = \frac{d}{dx}\left[\frac{1}{2}\log\left(\sin\sqrt{e^x}\right)\right]$$

$$= \frac{1}{2}\cot\sqrt{e^x}\ \frac{1}{2\sqrt{e^x}}e^x = \frac{1}{4}e^{x/2}\cot\left(e^{x/2}\right)$$

5. Let $y = a^{\left(\sin^{-1}x\right)^2}$

Using chain rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left\{a^{\left(\sin^{-1}x\right)^2}\right\}$$

$$= a^{\left(\sin^{-1}x\right)^2}\log a\frac{d}{dx}\{(\sin^{-1}x)^2\}$$

$$= a^{\left(\sin^{-1}x\right)^2}\log a\ 2\,(\sin^{-1}x)^1\frac{d}{dx}(\sin^{-1}x)$$

$$= a^{\left(\sin^{-1}x\right)^2}\log a\ 2\sin^{-1}x\ \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{2\log a\sin^{-1}x}{\sqrt{1-x^2}}a^{\left(\sin^{-1}x\right)^2}$$

6. $$y = \log\sqrt{\frac{1+\sin x}{1-\sin x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left\{\frac{1}{1+\sin x}\frac{d}{dx}(1+\sin x) - \frac{1}{1-\sin x}\frac{d}{dx}(1-\sin x)\right\}$$

$$= \frac{1}{2}\left\{\frac{\cos x}{1+\sin x} + \frac{\cos x}{1-\sin x}\right\}$$

$$= \frac{1}{2}\cos x\left(\frac{2}{1-\sin^2 x}\right) = \frac{\cos x}{\cos^2 x} = \sec x$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = \sec\frac{\pi}{3} = 2$$

7. $$\frac{dy}{dx} = 1(1+x^2)(1+x^4)\ldots(1+x^{2^n})$$

$$+2x(1+x)(1+x^4)\ldots(1+x^{2^n})$$

$$+4x^3(1+x)(1+x^2)(1+x^8)\ldots(1+x^{2^n})$$

$$\vdots$$

$$+2^n x^{2^{n-1}}(1+x)(1+x^2)\ldots(1+x^{2^{n-1}})$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 1$$

8. We have, $x^y = e^{x-y}$

$$\Rightarrow e^{y\log x} = e^{x-y} \quad \left[\because x^y = e^{\log x^y} = e^{y\log x}\right]$$

$$\Rightarrow y\log x = x - y$$

$$\Rightarrow y = \frac{x}{1+\log x}$$

On differentiating both the sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x)\times 1 - x\left(0+\frac{1}{x}\right)}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

9. We have $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$$

$$\Rightarrow x^2(1+y) = y^2(1+x) \quad \text{[On squaring both sides]}$$

$$\Rightarrow x^2 - y^2 = y^2x - x^2y$$

$$\Rightarrow (x+y)(x-y) = -xy(x-y)$$

$$\Rightarrow x + y = -xy \quad [\because x-y \neq 0 \text{ as } x = y \text{ does not satisfy the given equation}]$$

$$\Rightarrow y = -\frac{x}{1+x}$$

$$\Rightarrow \frac{dy}{dx} = -\left\{\frac{(1+x)\times 1 - x(0+1)}{(1+x^2)}\right\}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{(1+x^2)}$$

Exercise 4.4

1. Given $x^3 + y^3 - 3axy = 0$ From

$$\frac{dy}{dx} = -\frac{\text{differentiating of } f \text{ w.r.t. } x \text{ keeping } y \text{ as constant}}{\text{differentiating of } f \text{ w.r.t. } y \text{ keeping } x \text{ as constant}}$$

$$\frac{dy}{dx} = -\frac{3x^2 - 3ay}{3y^2 - 3ax} = \frac{ay - x^2}{y^2 - ax}$$

2. Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\left\{\log\left(x^2+y^2\right)\right\} = 2\frac{d}{dx}\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$$

$$\Rightarrow \frac{1}{x^2+y^2}\times\frac{d}{dx}\left(x^2+y^2\right) = 2\frac{1}{1+(y/x)^2}\times\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{1}{x^2+y^2}\left\{\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}\left(y^2\right)\right\} = 2\times\frac{x^2}{x^2+y^2}\left\{\frac{x\frac{dy}{dx} - y\times 1}{x^2}\right\}$$

$$\Rightarrow \frac{1}{x^2+y^2}\left\{2x + 2y\frac{dy}{dx}\right\} = \frac{2}{x^2+y^2}\left\{x\frac{dy}{dx} - y\right\}$$

$$\Rightarrow 2\left\{x + y\frac{dy}{dx}\right\} = 2\left\{x\frac{dy}{dx} - y\right\}$$

$$\Rightarrow x + y\frac{dy}{dx} = x\frac{dy}{dx} - y$$

$$\Rightarrow \frac{dy}{dx}(y - x) = -(x+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$$

3. $y=\sqrt{\sin x+y}$

$\Rightarrow y^2=\sin x+y$

Differentiate w. r. t. x, $2y\dfrac{dy}{dx}=\cos x+\dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{\cos x}{2y-1}$

4. $x=y\sqrt{1-y^2}$

Differentiating with respect to x, we get

$$1=\frac{dy}{dx}\sqrt{1-y^2}+y\times\frac{1}{2\sqrt{1-y^2}}(-2y)\times\frac{dy}{dx}$$

$$\Rightarrow 1=\frac{dy}{dx}\sqrt{1-y^2}-\frac{y^2}{\sqrt{1-y^2}}\times\frac{dy}{dx}$$

$$\Rightarrow 1=\frac{dy}{dx}\left[\frac{1-y^2-y^2}{\sqrt{1-y^2}}\right]$$

$$\Rightarrow 1=\frac{dy}{dx}\left[\frac{1-2y^2}{\sqrt{1-y^2}}\right]$$

$$\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{1-2y^2}$$

5. We have

$$y=b\tan^{-1}\left(\frac{x}{a}+\tan^{-1}\frac{y}{x}\right)$$

$$\Rightarrow \tan\frac{y}{b}=\frac{x}{a}+\tan^{-1}\frac{y}{x}$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{b}\sec^2\left(\frac{y}{b}\right)\frac{dy}{dx}=\frac{1}{a}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\times\frac{x\frac{dy}{dx}-y}{x^2}$$

$$\Rightarrow \frac{1}{b}\sec^2\left(\frac{y}{b}\right)\frac{dy}{dx}=\frac{1}{a}+\frac{x\frac{dy}{dx}-y}{x^2+y^2}$$

$$\Rightarrow \frac{dy}{dx}\left\{\frac{1}{b}\sec^2\left(\frac{y}{b}\right)-\frac{x}{x^2+y^2}\right\}=\frac{1}{a}-\frac{y}{x^2+y^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\frac{1}{a}-\frac{y}{x^2+y^2}}{\frac{1}{b}\sec^2\left(\frac{y}{b}\right)-\frac{x}{x^2+y^2}}$$

6. The given series may be written as $y=\sqrt{\sin x+y}$

$\Rightarrow y^2=\sin x+y$ [Squaring both sides]

$\Rightarrow 2y\dfrac{dy}{dx}=\cos x+\dfrac{dy}{dx}$ [Differentiating both sides w.r.t. x]

$\Rightarrow \dfrac{dy}{dx}(2y-1)=\cos x$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{\cos x}{2y-1}$

Exercise 4.5

1. $$\frac{dx}{dt}=\frac{(1+t^2)2-2t\times 2t}{(1+t^2)^2}=\frac{2-2t^2}{(1+t^2)^2}$$

$$\frac{dy}{dt}=\frac{(1+t^2)(-2t)-(1-t^2)2t}{(1+t^2)^2}=\frac{-4t}{(1+t^2)^2}$$

$$\therefore \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-4t}{2-2t^2}=\frac{2t}{t^2-1}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{t=2}=\frac{4}{3}$$

2. $x=a\cos^3\theta,\ y=b\sin^3\theta$

$$y_1=\frac{dy}{dx}=\frac{3b\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$=-\frac{b}{a}\tan\theta,\ \text{if}\ \sin\theta\neq 0,\ \cos\theta\neq 0$$

$\therefore y_1$ does not exist at $\theta=0$

Hence y_2 and y_3 do not exist at $\theta=0$.

3. $x=\sqrt{a^{\sin^{-1}t}},\ y=\sqrt{a^{\cos^{-1}t}}$

$\Rightarrow x\cdot y=\sqrt{a^{\sin^{-1}t+\cos^{-1}t}}$

$x.y=\sqrt{a^{\pi/2}}$

Differentiating w.r.t. x, we get

$x\dfrac{dy}{dx}+y=0$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{-y}{x}$

4. $x=a\left[\cos t+\log\tan\dfrac{t}{2}\right]$ and $y=a\sin t$

Differentiating w.r.t. t, we get

$$\frac{dx}{dt}=a\left[-\sin t+\frac{1}{\tan t/2}\sec^2\frac{t}{2}\times\frac{1}{2}\right]$$

$$=a\left[-\sin t+\frac{1}{2\sin(t/2)\cos(t/2)}\right]$$

$$=a\left[-\sin t+\frac{1}{\sin t}\right]$$

and $\frac{dy}{dt} = a\cos t$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{\frac{a\cos^2 t}{\sin t}} = \tan t$$

at $x = \pi/4$, $\frac{dy}{dx} = 1$

Exercise 4.6

1. Let $y = x^x$. Then, $y = e^{x\log x}$
   Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{x\log x}\frac{d}{dx}(x\log x)$$

$$\Rightarrow \frac{dy}{dx} = x^x\left(\log x + x\frac{1}{x}\right) \quad [\because e^{x\log x} = x^x]$$

$$\Rightarrow \frac{dy}{dx} = x^x(1+\log x)$$

2. $y^x = x^y$
   $\Rightarrow \log y^x = \log x^y$
   $\Rightarrow x\log y = y\log x$

$$\Rightarrow x\frac{1}{y}\frac{dy}{dx} + \log y \times 1 = \frac{y}{x} + \log x\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{x}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x} - \log y$$

$$\therefore \frac{dy}{dx} = \frac{\frac{y}{x} - \log y}{\frac{x}{y} - \log x} = \frac{y(y - x\log y)}{x(x - y\log x)} = \frac{y(x\log y - y)}{x(y\log x - x)}$$

3. Here $x = e^{y+x}$
   $\Rightarrow \log x = (y+x)$
   $\Rightarrow y = \log x - x$

$$\therefore \frac{dy}{dx} = \frac{1}{x} - 1 = \frac{1-x}{x}$$

4. Taking logarithm of both sides,
   $\log y = (\tan x)^{\tan x}\log\tan x$
   $\therefore \log\log y = [\tan x\log\tan x] + \log\log\tan x.$
   Differentiating w.r.t. x, we get

$$\frac{1}{y\log y}\frac{dy}{dx} = \sec^2 x\log\tan x + \tan x\frac{\sec^2 x}{\tan x} + \frac{1}{\log\tan x}\times\frac{\sec^2 x}{\tan x}$$

$$\therefore \frac{dy}{dx} = y\log y\sec^2 x[\log\tan x + 1 + 1/(\tan x\log\tan x)]$$

At $x = \pi/4$, $y = 1$ and $\log y = 0$

$\Rightarrow (dy/dx)_{x=\pi/4} = 0$

5. We have $y = \frac{\sqrt{1-x^2}(2x+3)^{1/2}}{(x^2+2)^{2/3}}$

   Taking log of both sides, we get

$$\Rightarrow \log y = \frac{1}{2}\log(1-x^2) + \frac{1}{2}\log(2x+3) - \frac{2}{3}\log(x^2+2)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2(1-x^2)}(-2x) + \frac{1}{2(2x+3)}\times 2 - \frac{2}{3}\times\frac{1}{x^2+2}2x$$

$$\therefore \frac{dy}{dx} = y\left[-\frac{x}{1-x^2} + \frac{1}{2x+3} - \frac{4x}{3(x^2+2)}\right]$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = \frac{\sqrt{3}}{\sqrt[3]{4}}\times\frac{1}{3} = \frac{1}{\sqrt[3]{4}\sqrt{3}}$$

Exercise 4.7

1. Let $y = \tan^{-1}\frac{2x}{1-x^2} = 2\tan^{-1}x$

   and $z = \sin^{-1}\frac{2x}{1+x^2} = 2\tan^{-1}x$

$$\Rightarrow \frac{dy}{dz} = 1$$

2. Let $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$ and $z = \sqrt{1-x^2}$

   Put $x = \cos\theta$

   $\therefore y = \sec^{-1}(\sec 2\theta) = 2\theta$ and $z = \sqrt{1-\cos^2\theta} = \sin\theta$

$$\Rightarrow \frac{dy}{d\theta} = 2;\ \frac{dz}{d\theta} = \cos\theta \Rightarrow \frac{dy}{dz} = \frac{2}{\cos\theta} = \frac{2}{x}$$

$$\Rightarrow \text{at } x = \frac{1}{2},\ \frac{dy}{dz} = 4$$

3. We have, $y = f(x^3)$

$$\Rightarrow \frac{dy}{dx} = f'(x^3)\,3x^2 = 3x^2\tan x^3$$

   Also, $z = g(x^5)$

$$\Rightarrow \frac{dz}{dx} = g'(x^5)\,5x^4 = 5x^4\sec x^5$$

$$\Rightarrow \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{3x^2\tan x^3}{5x^4\sec x^5} = \frac{3}{5x^2}\times\frac{\tan x^3}{\sec x^5}$$

$$\Rightarrow \lim_{x\to 0}\frac{(dy/dz)}{x} = \lim_{x\to 0}\frac{3\tan x^3}{5x^3\sec x^5} = \frac{3}{5}$$

Exercise 4.8

1. $$\frac{dy}{dx}=\begin{vmatrix}\cos x & -\sin x & \cos x\\ \cos x & -\sin x & \cos x\\ x & 1 & 1\end{vmatrix}$$

$$+\begin{vmatrix}\sin x & \cos x & \sin x\\ -\sin x & -\cos x & -\sin x\\ x & 1 & 1\end{vmatrix}+\begin{vmatrix}\sin x & \cos x & \sin x\\ \cos x & -\sin x & \cos x\\ 1 & 0 & 0\end{vmatrix}$$

$$=0-\begin{vmatrix}\sin x & \cos x & \sin x\\ \sin x & \cos x & \sin x\\ x & 1 & 1\end{vmatrix}+1\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}$$

$$=0+(\cos^2 x+\sin^2 x)=1$$

2. $$\frac{d^n}{dx^n}[f(x)]=\begin{vmatrix}\frac{d^n}{dx^n}(x^n) & n! & 2\\ \frac{d^n}{dx^n}(\cos x) & \cos\frac{n\pi}{2} & 4\\ \frac{d^n}{dx^n}(\sin x) & \sin\frac{n\pi}{2} & 8\end{vmatrix}$$

$$=\begin{vmatrix}n! & n! & 2\\ \cos\left(x+\frac{n\pi}{2}\right) & \cos\frac{n\pi}{2} & 4\\ \sin\left(x+\frac{n\pi}{2}\right) & \sin\frac{n\pi}{2} & 8\end{vmatrix}$$

$$\Rightarrow \frac{d^n}{dx^n}[f(x)]_{x=0}=0$$

Exercise 4.9

1. $$\frac{d}{dx}\left[e^{2x}+e^{-2x}\right]=2e^{2x}-2e^{-2x}=2^1[e^{2x}-e^{-2x}]$$

$$\frac{d^2}{dx^2}(e^{2x}+e^{-2x})=\frac{d}{dx}2(e^{2x}-e^{-2x})=2^2(e^{2x}+e^{-2x})$$

$$\frac{d^3}{dx^3}(e^{2x}+e^{-2x})=\frac{d}{dx}2^2(e^{2x}+e^{-2x})=2^3(e^{2x}-e^{-2x})$$

$\vdots$

$$\frac{d^n}{dx^n}[e^{2x}+e^{-2x}]=2^n[e^{2x}+(-1)^n e^{-2x}]$$

2. $$\frac{dy}{dx}=\cos(\sin x)\cos x$$

$$\frac{d^2y}{dx^2}=-\cos(\sin x)\sin x+\cos x[-\sin(\sin x)]\cos x$$

$$\therefore \frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x=-\cos(\sin x)\sin x-\cos^2 x\sin(\sin x)$$
$$+\cos(\sin x)\cos x\tan x=-\cos^2 x\sin(\sin x)$$

$$\therefore \frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x+\cos^2 x\sin(\sin x)=0$$

$$\therefore f(x)=\cos^2 x\sin(\sin x)$$

3. $y=\log(1+\sin x)$ (1)

$$y_1=\frac{\cos x}{1+\sin x} \quad (2)$$

$$y_2=\frac{-\sin x(1+\sin x)-\cos x\cos x}{(1+\sin x)^2}$$

$$=\frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$=-\frac{1}{(1+\sin x)} \quad (3)$$

$$y_3=\frac{\cos x}{(1+\sin x)^2}=\frac{\cos x}{1+\sin x}\times\frac{1}{1+\sin x}=-y_1y_2 \quad (4)$$

$$\therefore y_4=-y_2^2-y_1y_3$$

$$\Rightarrow y_4+y_3y_1+y_2^2=0$$

4. $f(x)=(1+x)^n, f(0)=1$

$\Rightarrow f'(x)=n(1+x)^{n-1}, f'(0)=n$

$\Rightarrow f''(x)=n(n-1)(1+x)^{n-2}, f''(0)=n(n-1)$

Similarly proceeding we have $f'''(0)=n(n-1)(n-2)$

$f''''(0)=n(n-1)(n-2)(n-3)$ and so on.

$f^n(0)=n(n-1)(n-2)(n-3)\ldots 1$

$$\Rightarrow f(0)+f'(0)+\frac{f''(0)}{2!}+\frac{f'''(0)}{3!}+\cdots+\frac{f^n(0)}{n!}$$

$$=1+n+\frac{n(n-1)}{2!}+\frac{n(n-1)(n-2)}{3!}+\cdots$$

$$+\frac{n(n-1)(n-2)\ldots 1}{n!}+\cdots$$

$$={}^nC_0+{}^nC_1+{}^nC_2\cdots+{}^nC_{n-1}+{}^nC_n=2^n$$

Exercise 4.10

1. $f(x+y)=f(x)f(y)$ (1)

$$f'(5)=\lim_{h\to 0}\frac{f(5+h)-f(5)}{h}$$

$$=\lim_{h\to 0}\frac{f(5)f(h)-f(5)}{h}$$

$$=f(5)\lim_{h\to 0}\frac{f(h)-1}{h}$$

$$=f(5)\lim_{h\to 0}\frac{f(h)-1}{h}$$

Replace x by 5 and y by 0, $f(5+0)=f(5)\times f(0)$

$\Rightarrow f(0)=1$

$$\Rightarrow f'(5) = f(5)\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$
$$= f(5)f'(0) = 2\times 3 = 6$$

2. $f(xy) = f(x)f(y)$ (1)

$$f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0}\frac{f\left(2\left(1+\frac{h}{2}\right)\right)-f(2)}{h}$$
$$= \lim_{h\to 0}\frac{f(2)f\left(1+\frac{h}{2}\right)-f(2)}{h}$$
$$= \frac{f(2)}{2}\lim_{h\to 0}\frac{f\left(1+\frac{h}{2}\right)-1}{\frac{h}{2}}$$

Replace x and y by 0 in equation (1) $\Rightarrow f(0) = [f(0)]^2$
$\Rightarrow f(0) = 0$

$$\Rightarrow f'(2) = \frac{f(2)}{2}\lim_{h\to 0}\frac{f\left(1+\frac{h}{2}\right)-f(0)}{\frac{h}{2}} = \frac{f(2)f'(1)}{2} = \frac{3}{2}$$

3. Given $f\left(\frac{x+2y}{3}\right) = \frac{f(x)+2f(y)}{3}\ \forall x,\ y \in R$, which satisfies section formula for abscissa on L.H.S. and ordinate on R.H.S. Hence, $f(x)$ must be the linear function (as only straight line satisfies such section formula).
Hence $f(x) = ax + b$.
But $f(0) = 2 \Rightarrow b = 2$, $f'(0) = 1 \Rightarrow a = 1$.
Thus $f(x) = x + 2$.

4. $$f''(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}$$
$$= \lim_{h\to 0}\frac{\lim_{k\to 0}\left[\frac{f(x+h+k)-f(x+h)}{k}-\frac{f(x+k)-f(x)}{k}\right]}{h}$$
Let $k = -h$
$$\Rightarrow f''(x) = -\lim_{h\to 0}\frac{f(x)-f(x+h)-f(x-h)+f(x)}{h^2}$$
$$= \lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}$$

Chapter 5

Exercise 5.1

1. The curve is $3xy^2 - 2x^2y = 1$ (1)

Differentiate w.r.t. x, $\frac{dy}{dx} = \frac{4xy-3y^2}{6xy-2x^2}$

At the point (1, 1), $dy/dx = 1/4$.
Slope of line joining $P(1, 1)$ and $Q(-16/5, -1/20)$ is

$$\frac{1+\frac{1}{20}}{1+\frac{16}{5}} = \frac{21}{84} = \frac{1}{4}.$$

Also point $Q(-16/5, -1/20)$ satisfies the curve.
Hence proved.

2. We have $x = a(1+\cos\theta)$, $y = a\sin\theta$

$\Rightarrow \frac{dx}{d\theta} = -a\sin\theta,\ \frac{dy}{d\theta} = a\cos\theta$

$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$

$\Rightarrow$ slope of normal $= \tan\theta$
$\Rightarrow$ The equation of normal is:
$y - a\sin\theta = \tan\theta(x - a(1+\cos\theta))$
$\Rightarrow y\cos\theta - a\sin\theta\cos\theta = x\sin\theta - a\sin\theta - a\sin\theta\cos\theta$
$\Rightarrow x\sin\theta - y\cos\theta = a\sin\theta$
which clearly passes through $(a, 0)$

3. $y = ax^2 - 6x + b$ passes through (0, 2)
$\Rightarrow 2 = 0 - 0 + b \Rightarrow b = 2$

Again $\frac{dy}{dx} = 2ax - 6$

At $x = \frac{3}{2}$, $\frac{dy}{dx} = 3a - 6$

Since tangent is parallel to x-axis

$\Rightarrow \frac{dy}{dx} = 0 \quad \Rightarrow 3a - 6 = 0 \quad \Rightarrow a = 2$

Hence $a = 2, b = 2$.

4. We have $(1+x^2)y = 2 - x$ or $y = \frac{2-x}{1+x^2}$

This meets x-axis at (2, 0).

Also $\frac{dy}{dx} = \frac{(1+x^2)(-1)-(2-x)2x}{(1+x^2)^2}$

At (2, 0), $\frac{dy}{dx} = \frac{(1+4)(-1)-0}{(1+4)^2} = -\frac{1}{5}$

$\therefore$ the required tangent is $y - 0 = -\frac{1}{5}(x-2)$ or $x + 5y = 2$.

5. $y^2 = ax^3 + b \quad \Rightarrow \frac{dy}{dx} = \frac{3ax^2}{2y}$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(2,3)} = \frac{3a(2)^2}{2\times 3} = 2a = 4 \Rightarrow a = 2$$

Also (2, 3) lies on $y^2 = ax^3 + b \Rightarrow 9 = 8a + b \Rightarrow b = -7$

6. $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2 \quad \Rightarrow \frac{n}{a}\left(\frac{x}{a}\right)^{n-1} + \frac{n}{b}\left(\frac{y}{b}\right)^{n-1}\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{b^n}{a^n}\times\frac{x^{n-1}}{y^{n-1}}$$

At (a, b), $\frac{dy}{dx} = -\frac{b^n}{a^n}\frac{a^{n-1}}{b^{n-1}} = -\frac{b}{a}$

$\therefore$ tangent at (a, b) is $y - b = -\frac{b}{a}(x - a)$.

$$\Rightarrow \quad \frac{y}{b} - 1 = -\frac{x}{a} + 1 \quad \Rightarrow \frac{x}{a} + \frac{y}{b} = 2$$

This shows that $\frac{x}{a} + \frac{y}{b} = 2$ touches the given curve for all n.

7. Let the line is normal to the curve at point $P(x_1, y_1)$ on the curve

$\Rightarrow \quad Ax_1 + By_1 = 1$ (1)

and $a^{n-1} y_1 = x_1^n$ (2)

Differentiating $a^{n-1} y = x^n$ w.r.t. x, we get

$$\Rightarrow \quad \frac{dy}{dx} = \frac{nx^{n-1}}{a^{n-1}} = \frac{nx^n}{xa^{n-1}} = \frac{ny}{x}$$

Now slope of line = slope of normal to the curve at $P(x_1, y_1)$

$$\Rightarrow \quad -\frac{A}{B} = -\frac{x_1}{ny_1} \quad \Rightarrow nAy_1 = Bx_1 \quad (3)$$

From equations (1) and (3), $A(nAy_1)/B + By_1 = 1$

$$(nA^2 + B^2)y_1 = B \quad (4)$$

and $\quad (nA^2 + B^2)(Bx_1/nA) = B$

$$B(nA^2 + B^2)x_1 = nAB \quad (5)$$

Now substitute the values of y_1 and x_1 from equations (4) and (5), respectively, in equation (2).

Exercise 5.2

1. When $x = 0, y = -1$

$$\frac{dy}{dx} = 3x^2 + 6x + 4 \quad \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 4$$

$$\Rightarrow \quad \text{Length of tangent} = \left| y\sqrt{1 + \left(\frac{dx}{dy}\right)^2} \right|$$

$$= \left| -1\sqrt{1 + \frac{1}{16}} \right| = \frac{\sqrt{17}}{4}$$

2. Let point of tangency be (x_1, y_1)

$$m = \left.\frac{dy}{dx}\right|_{x_1} = \frac{2ax_1}{x_1^2 - a^2}$$

tangent + subtangent

$$= y_1\sqrt{1 + \frac{1}{m^2}} + \frac{y_1}{m}$$

$$= y_1\sqrt{1 + \frac{(x_1^2 - a^2)^2}{4a^2x_1^2}} + \frac{y_1(x_1^2 - a^2)}{2ax_1}$$

$$= y_1\frac{\sqrt{x_1^4 + a^4 + 2a^2x_1^2}}{2ax_1} + \frac{y_1(x_1^2 - a^2)}{2ax_1}$$

$$= \frac{y_1(x_1^2 + a^2)}{2ax_1} + \frac{y_1(x_1^2 - a^2)}{2ax_1}$$

$$= \frac{2y_1(x_1^2)}{2ax_1} = \frac{x_1 y_1}{a} \propto x_1 y_1$$

3. $$\left(\frac{\text{length of normal}}{\text{length of tangent}}\right)^2 = \frac{\left(y\sqrt{1 + \frac{dy}{dx}}\right)^2}{\left(y\sqrt{1 + \frac{dx}{dy}}\right)^2}$$

$$= \left(\frac{dy}{dx}\right)^2 = \left(\frac{y\frac{dy}{dx}}{y\frac{dx}{dy}}\right) = \frac{\text{sub-normal}}{\text{sub-tangent}}$$

4. $y = a^{1-n}x^n \Rightarrow \frac{dy}{dx} = a^{1-n}nx^{n-1}$

$$\text{Sub-normal} = \left| y\frac{dy}{dx} \right|$$

$$= \left| ya^{1-n}nx^{n-1} \right|$$

$$= \left| a^{1-n}x^n a^{1-n}nx^{n-1} \right|$$

$$= \left| a^{2-2n}x^{2n-1} \right|$$

which is constant if $2n - 1 = 0$ or $n = 1/2$.

Exercise 5.3

1. Here, the curves are

$$y = a^x \text{ and } y = b^x$$

Solving the curves, $a^x = b^x \quad \Rightarrow \left(\frac{a}{b}\right)^x = 1$

i.e., $\quad x = 0, \quad y = 1$

$\Rightarrow$ Point of intersection is $(0, 1)$.

$$\left(\frac{dy}{dx}\right)_{(0,1)} = (a^x \log a)_{(0,1)} = \log a \qquad (\text{for } y = a^x)$$

$$\left(\frac{dy}{dx}\right)_{(0,1)} = (b^x \log a)_{(0,1)} = \log b \qquad (\text{for } y = b^x)$$

$\Rightarrow$ Angle between the curves;

$$\tan\theta = \tan^{-1}\left|\frac{\log a - \log b}{1 + (\log a)(\log b)}\right|$$

$$\Rightarrow \quad \theta = \tan^{-1}\left|\frac{\log a/b}{1 + \log a \log b}\right|$$

2. The two curves are

$$xy = a^2 \quad (1)$$

$$x^2 + y^2 = 2a^2 \quad (2)$$

Solving equations (1) and (2), the points of intersection are (a, a) and $(-a, -a)$

Diff. equation (1), $dy/dx = -y/x = m_1$ (say)

Diff. equation (2), $dy/dx = -x/y = m_2$ (say)

At both points $m_1 = -1 = m_2$

Hence the two curves touch each other.

3. $\left[\dfrac{dy}{dx}\right]_{x=0} = K^2$

$\Rightarrow \quad \tan\psi = K^2 \Rightarrow \cot\left(\dfrac{\pi}{2} - \psi\right) = K^2$

$\Rightarrow \quad \left(\dfrac{\pi}{2} - \psi\right) = \cot^{-1} k^2$

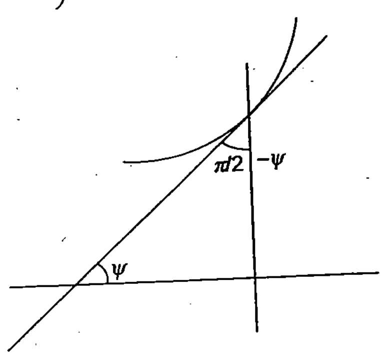


Fig. S-5.1

4. $ay + x^2 = 7$, and $x^3 = y$ cuts orthogonally

Now $\left(\dfrac{dy}{dx}\right) = -\dfrac{2x}{a}$ and $\left(\dfrac{dy}{dx}\right) = 3x^2$

$\Rightarrow \quad \left[\left(-\dfrac{2x}{a}\right)(3x^2)\right]_{(1,1)} = -1$

$\Rightarrow \quad -\dfrac{2}{a} \times 3 = -1 \Rightarrow a = 6$

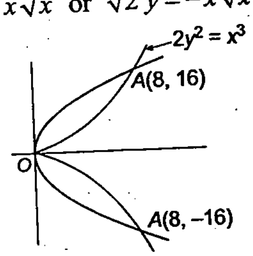
5. $C_1 : 2x - \dfrac{2y}{3}\dfrac{dy}{dx} = 0 \Rightarrow \left.\dfrac{dy}{dx}\right]_{x_1, y_1} = \dfrac{3x_1}{y_1} = m_1$

$C_2 : 3xy^2 \dfrac{dy}{dx} + y^3 = 0 \Rightarrow \left.\dfrac{dy}{dx}\right]_{x_1, y_1} = -\dfrac{y_1}{3x_1} = m_2$

$\therefore \; m_1 \cdot m_2 = -1 \Rightarrow C_1$ and C_2 are orthogonal

6. On solving we get $(0, 0)$; $(8, 16)$ and $(8, -16)$

for $= \sqrt{2}\,y = x\sqrt{x}$ or $\sqrt{2}\,y = -x\sqrt{x}$



For $2y^2 = x^3$ at $(0, 0)$ $\left.\dfrac{dy}{dx}\right|_{(0,0)} = 0$

for $y^2 = 32x$ at $(0, 0)$ $\left.\dfrac{dy}{dx}\right|_{(0,0)} = \infty$

hence angle $= 90°$

At $(8, \pm 16)$ for $2y^2 = x^3$, $\left.\dfrac{dy}{dx}\right|_{I} = \dfrac{3x^2}{4y} = \dfrac{3}{4}\cdot\dfrac{64}{16} = 3$

At $(8, \pm 16)$ for $y^2 = 32x$, $\left.\dfrac{dy}{dx}\right|_{II} = \dfrac{32}{2y} = \dfrac{16}{16} = 1$

$\therefore \; \tan\theta = \dfrac{3-1}{1+3} = \dfrac{2}{4} = \dfrac{1}{2}$

Exercise 5.4

1. Given $s^2 = (at^2 + 2bt + c)$ or $s = \sqrt{(at^2 + 2bt + c)}$ (1)

$\Rightarrow \quad \dfrac{ds}{dt} = \dfrac{(at+b)}{\sqrt{(at^2 + 2bt + c)}}$

$= \dfrac{(at+b)}{s} = V$ (say) [From equation (1)] (2)

Again diff. both sides w.r.t. t,

$\Rightarrow \quad \dfrac{d^2s}{dt^2} = \dfrac{s(a) - (at+b)\dfrac{ds}{dt}}{s^2}$

$= \dfrac{as - (at+b)\dfrac{(at+b)}{s}}{s^2}$ [from equation (2)]

$= \dfrac{as^2 - (at+b)^2}{s^3}$

$= \dfrac{a(at^2 + 2bt + c) - (a^2t^2 + 2abt + b^2)}{s^3}$

$= \dfrac{(ac - b^2)}{s^3}$

$\Rightarrow$ acceleration $\propto \dfrac{1}{s^3}$

2. Given $\dfrac{d(\tan\theta)}{d\theta} = 4$

$\Rightarrow \quad \sec^2\theta = 4$

Now $\dfrac{d(\sin\theta)}{d\theta} = \cos\theta = \dfrac{1}{2}$

3. At time t, the distance z between the cyclists is given by

$$z^2 = (3vt)^2 + (4vt)^2$$

$\therefore \quad z = 5vt \Rightarrow \dfrac{dz}{dt} = 5v$

4. $V = \dfrac{4\pi}{3}(x + 10)^3$, where x is the thickness of ice

$$\therefore \quad \frac{dV}{dt} = 4\pi\,(x+10)^2\,\frac{dx}{dt}$$

$$\therefore \quad \left[\frac{dx}{dt}\right]_{x=5} = \frac{-50}{4\pi(5+10)^2} = \frac{-50}{900\pi} = -\frac{1}{18\pi}$$

Hence, the rate at which thickness decreases

$$= \frac{1}{18\pi}\ \text{cm/s}$$

5. Given x and y are the sides of two squares, thus, the area of two squares is x^2 and y^2

We have to obtain $\dfrac{d(y^2)}{d(x^2)} = \dfrac{2y\dfrac{dy}{dx}}{2x} = \dfrac{y}{x}\times\dfrac{dy}{dx}$ (1)

where the given curve is, $y = x - x^2$

$$\Rightarrow \quad \frac{dy}{dx} = 1 - 2x \qquad (2)$$

Thus, $\dfrac{d(y^2)}{d(x^2)} = \dfrac{y}{x}(1-2x)$ [From equations (1) and (2)]

or $\dfrac{d(y^2)}{d(x^2)} = \dfrac{(x-x^2)(1-2x)}{x}$

$$\Rightarrow \quad \frac{d(y^2)}{d(x^2)} = (2x^2 - 3x + 1)$$

Thus, the rate of change of the area of second square with respect to first square is $(2x^2 - 3x + 1)$.

6. Let R and S be the positions of men P and Q at any time t. Since velocities are same

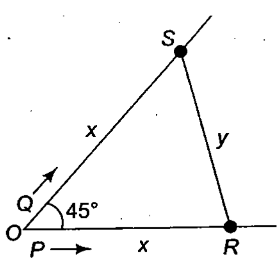


Fig. S-5.2

$\Rightarrow$ $OR = OS = x$ (say) and given $\dfrac{dx}{dt} = v$ (1)

and let $SR = y$

Now in triangle ORS, applying cosine rule then

$$y^2 = x^2 + x^2 - 2x \times x \cos 45° = 2x^2 - x^2\sqrt{2}$$

$$\Rightarrow \quad y = x\sqrt{(2-\sqrt{2})}$$

$$\Rightarrow \quad \frac{dy}{dt} = \left[\sqrt{(2-\sqrt{2})}\right]\frac{dx}{dt} = u\sqrt{(2-\sqrt{2})}$$

rom equation (1)]

Hence, the required rate at which they are being separated is $u\sqrt{(2-\sqrt{2})}$.

Exercise 5.5

1. Let $x = 3$ and $\Delta x = 0.02$. Then

$$f(3.02) = f(x+\Delta x) = 3(x+\Delta x)^2 + 5(x+\Delta x) + 3$$

Now $f(x+\Delta x) = f(x) + \Delta y$

$$\approx f(x) + f'(x)\Delta x \qquad (\text{as } dx = \Delta x)$$

$\Rightarrow$ $f(3.02) \approx (3x^2 + 5x + 3) + (6x+5)\Delta x$

$= (3(3)^2 + 5(3) + 3) + (6(3)+5)(0.02)$

(as $x = 3, \Delta x = 0.02$)

$= (27 + 15 + 3) + (18 + 5)(0.02)$

$= 45 + 0.46 = 45.46$

Hence, approximate value of $f(3.02)$ is 45.46.

2. Let r be the radius of the sphere and Δr be the error in measuring the radius.

Then $r = 9$ cm and $\Delta r = 0.03$ cm. Now, the volume V of the sphere is given by

$$V = \frac{4}{3}\pi r^3$$

or $\dfrac{dV}{dr} = 4\pi r^2$

Therefore, $dV = \left(\dfrac{dV}{dr}\right)\Delta r = (4\pi r^2)\,\Delta r$

$= 4\pi(9)^2(0.03) = 9.72\ \text{cm}^3$

Thus, the approximate error in calculating the volume is $9.72\,\pi\text{cm}^3$.

3. Consider function $y = x^6$

Let $x = 2$ and $\Delta x = -0.001$

Then $\Delta y = (x+\Delta x)^6 - x^6$

Now dy is approximately equal to Δy and is given by

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=2}\Delta x = 6(2)^5(-0.001)$$

$\Rightarrow$ $f(1.999) = f(2) - 6 \times 32 \times 0.001$

$= 64 - 64 \times 0.003$

$= 63.808$ (approx.)

4. Let $y = \cos x$ $\quad \therefore \quad \dfrac{dy}{dx} = -\sin x$

Then $\Delta y = \cos(x+\Delta x) - \cos x$

$= \cos(60°\ 1') - \cos 60°$

Now dy is approximately equal to Δy and is given by

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=60°}\Delta x = -\frac{\sqrt{3}}{2}\times 1' = -\frac{\sqrt{3}}{2}\times\frac{\alpha}{60}$$

$$\Rightarrow \quad \cos 60°\ 1' = \frac{1}{2} - \frac{\alpha\sqrt{3}}{120}$$

Exercise 5.6

1. Consider the function $f(x) = x^3 + 2ax^2 + bx$

Obviously $f(x)$ being a polynomial function is continuous in $[0, 1]$ and differentiable in $(0, 1)$.

Also $f(0) = 0$. If $f(1) = 0$, then all the three conditions of Rolle's theorem will be satisfied.

$\Rightarrow f'(c)=0$, for at least one c in $(0, 1)$. Hence $f'(x)=3x^2+4ax+b=0$ at least once in $(0, 1)$.
i.e., the equation $3x^2+4ax+b=0$ has at least one root in $(0, 1)$ if $f(1)=0$, i.e., $1+2a+b=0$ (1)

2. Given $f(x)=3x^2+5x+7$

$\Rightarrow f(1)=3+5+7=15$ and

$f(3)=27+15+7=49$

Again $f'(x)=6x+5$.
Here $a=1, b=3$.
Now from Lagrange's mean value theorem

$$f'(c)=\frac{f(b)-f(a)}{b-a} \Rightarrow 6c+5=\frac{f(3)-f(1)}{3-1}$$

$$=\frac{49-15}{2}=17 \text{ or } c=2$$

3. As $f(x)$ and $g(x)$ are continuous and differentiable in $[0, 2]$, then there exists at least one c such that

$$\frac{f'(c)}{g'(c)}=\frac{f(2)-f(0)}{g(2)-g(0)} \Rightarrow \frac{8-2}{g(2)-1}=3$$

$\Rightarrow g(2)-1=2 \Rightarrow g(2)=3$

4. Let $f(x)=x^5-a_0x^4+3ax^3+bx^2+cx+d=0$

$f'(x)=5x^4-4a_0x^3+9ax^2+2bx+c$

$f''(x)=20x^3-12a_0x^2+18ax+2b$

$f'''(x)=60x^2-24a_0x+18a$

or $f'''(x)=6(10x^2-4a_0x+3a)$

Now, discriminant $=16a_0^2-4\times10\times3a$.

$\Rightarrow D=8(2a_0^2-15a)<0$ [as $2a_0^2-15a<0$ given]

Hence, the roots of $f'''(x)=0$ cannot be real. Therefore, all the roots of the $f(x)=0$ will not be real.

5. We have to prove

$(b^3-a^3)f'(c)-[f(b)-f(a)](3c^2)=0$

Let us assume a function

$F(x)=(b^3-a^3)f(x)-[f(b)-f(a)]x^3$

which is continuous in $[a, b]$ and differentiable in (a, b) as both $f(x)$ and x are continuous.

Also $F(a)=b^3f(a)-a^3f(b)=F(b)$

So, according to Rolle's theorem, there exists at least one $c\in(a, b)$ such that $F(c)=0$, which proves the required result.

6. Let $f(x)$ is equal to $\log_e x, x\in[a, b]$, and $0<a<b$
Clearly, $f(x)$ is continuous and differentiable.
Hence according to LMVT there exists at least one $c\in(a, b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \frac{1}{c}=\frac{\log b-\log a}{b-a}=\frac{\log\left(\frac{b}{a}\right)}{b-a}$$

now $a<c<b$

$$\Rightarrow \frac{1}{a}>\frac{1}{c}>\frac{1}{b}$$

$$\Rightarrow \frac{1}{a}>\frac{\log\left(\frac{b}{a}\right)}{b-a}>\frac{1}{b}$$

$$\Rightarrow \frac{b-a}{b}<\log\left(\frac{b}{a}\right)<\frac{b-a}{a}$$

7. Consider $\phi(x)=f(x)-g(x)$

$\Rightarrow \phi'(x)=f'(x)-g'(x)$

$\phi(x)$ is also continuous and derivable in $[x_0, x]$ using LMVT for $Q(x)$ in $[x_0, x]$

$$\phi'(x)=\frac{\phi(x)-\phi(x_0)}{x-x_0}$$

Since $\phi'(x)=f'(x)-g'(x)$ are $f'(x)-g'(x)>0$

$\therefore \phi'(x)>0$

Hence $\phi(x)-\phi(x_0)>0$

$\phi(x)>\phi(x_0)$

$f(x)-g(x)>0$

$[\because \phi(x_0)=f(x_0)-g(x_0)=0]$

8. Since $f(x)$ and $g(x)$ are continuous and differentiable functions.

Now let $H(x)=\begin{vmatrix} f(a) & f(x) \\ g(a) & g(x)\end{vmatrix}$ (1)

then $H(a)=0$ and $H(b)=\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b)\end{vmatrix}$

So, $H(x)$ satisfies the condition of mean value theorem

$\Rightarrow \frac{H(b)-H(a)}{b-a}=H'(c)$, where $a<c<b$

or $H'(c)=\frac{1}{(b-a)}\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b)\end{vmatrix}$ (2)

From (1),

$$H'(x)=\begin{vmatrix} 0 & f(x) \\ 0 & g(x)\end{vmatrix}+\begin{vmatrix} f(a) & f'(x) \\ g(a) & g'(x)\end{vmatrix}=\begin{vmatrix} f(a) & f'(x) \\ g(a) & g'(x)\end{vmatrix}$$

$\Rightarrow H'(c)=\frac{1}{b-a}\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b)\end{vmatrix}=\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c)\end{vmatrix}$ (3)

From equations (2) and (3), we get

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b)\end{vmatrix}=(b-a)\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c)\end{vmatrix}$$

Chapter 6

Exercise 6.1

1.

a. $f(x)=\cot^{-1}x+x$,
Differentiating w.r.t. x, we get,

$$f'(x) = \frac{-1}{1+x^2} + 1 = \frac{-1+1+x^2}{1+x^2} = \frac{x^2}{1+x^2}$$

Clearly, $f'(x) \geq 0$ for all x.

So, $f(x)$ increases in $(-\infty, \infty)$.

b. $f(x) = \log(1+x) - \dfrac{2x}{2+x}$

$$\Rightarrow f'(x) = \frac{1}{1+x} - \frac{2(2+x)-2x}{(2+x)^2}$$

$$\Rightarrow \quad f'(x) = \frac{x^2}{(x+1)(x+2)^2}$$

Obviously, $f'(x) > 0$ for all $x > -1$

Hence, $f(x)$ is increasing on $(-1, \infty)$.

2.

a. $f(x) = -2x^3 - 9x^2 - 12x + 1$

$\Rightarrow \quad f'(x) = -6x^2 - 18x - 12$

$= -6(x+2)(x+1)$

$\Rightarrow \quad f'(x) > 0$, if $x \in (-2, -1)$

and $f'(x) < 0$, if $x \in (-\infty, -2) \cup (-1, -\infty)$

Thus, $f(x)$ is increasing for $x \in (-2, -1)$ and

$f(x)$ is decreasing for $x \in (-\infty, -2) \cup (-1, -\infty)$.

b. Let $y = f(x) = x^2 e^{-x}$

$$\Rightarrow \quad \frac{dy}{dx} = 2xe^{-x} - x^2 e^{-x}$$

$= e^{-x}(2x - x^2)$

$= e^{-x} x(2-x)$

$f(x)$ is increasing if $f'(x) > 0 \Rightarrow x(2-x) > 0 \Rightarrow x \in (0, 2)$

$f(x)$ is decreasing if $f'(x) < 0 \Rightarrow x(2-x) < 0$

$\Rightarrow \quad x \in (-\infty, 0) \cup (2, \infty)$

c. We have, $f'(x) = \cos x - \sin x$

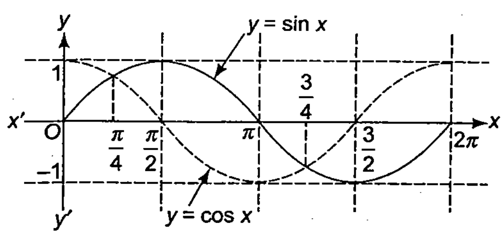


Fig. S-6.1

$f(x)$ is increasing if $f'(x) > 0 \Rightarrow \cos x > \sin x$

$\Rightarrow \quad x \in (0, \pi/4) \cup \left(\dfrac{5\pi}{4}, 2\pi\right)$ (see the graph)

$f(x)$ is decreasing if $f'(x) < 0 \Rightarrow \cos x < \sin x$

$\Rightarrow \quad x \in \left(\pi/4, \dfrac{5\pi}{4}\right)$

d. Given, $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$

$\Rightarrow \quad f'(x) = 12\cos^3 x(-\sin x) + 30\cos^2 x(-\sin x) + 12\cos x(-\sin x)$

$= -3\sin 2x\,(2\cos^2 x + 5\cos x + 2)$

$= -3\sin 2x\,(2\cos x + 1)(\cos x + 2)$

when $f'(x) = 0 \Rightarrow \sin 2x = 0 \Rightarrow x = 0, \dfrac{\pi}{2}, \pi$

or $2\cos x + 1 = 0 \Rightarrow x = \dfrac{2\pi}{3}$

as $\cos x + 2 \neq 0$.

Sign scheme of $f'(x)$

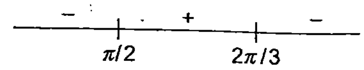


Fig. 6.2

So, $f(x)$ decreases on $\left(0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{2\pi}{3}, \pi\right)$ and increases on $\left(\dfrac{\pi}{2}, \dfrac{2\pi}{3}\right)$.

3. $f'(x) = \dfrac{\sin x - x\cos x}{\sin^2 x} = \dfrac{\cos x(\tan x - x)}{\sin^2 x}$

$0 < x \leq 1 \Rightarrow x \in$ first quadrant $\Rightarrow \tan x > x$, $\cos x > 0$

$\Rightarrow \quad f'(x) > 0$ for $0 < x \leq 1$

$\Rightarrow \quad f(x)$ is an increasing function.

$$g'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x} = \frac{\sin x\cos x - x}{\sin^2 x} = \frac{\sin 2x - 2x}{2\sin^2 x}$$

now $0 < 2x \leq 2$, for which $\sin 2x < 2x$

$\Rightarrow \quad g'(x) < 0$

$\Rightarrow \quad g'(x) < 0 \Rightarrow g(x)$ is decreasing.

4. $x = \dfrac{1}{1+t^2}$ and $y = \dfrac{1}{t(1+t^2)}$

$$\Rightarrow \quad \frac{dx}{dt} = -\frac{2t}{(1+t^2)^2}, \quad \frac{dy}{dt} = -\frac{1+3t^2}{t^2(1+t^2)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1+3t^2}{2t^3}$$

$\dfrac{dy}{dx} > 0$, if $t > 0 \Rightarrow x = \dfrac{1}{1+t^2} \in (0, 1)$

Hence, $f(x)$ is an increasing function.

5. If $f(x) = (a+2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically for all $x \in R$, then $f'(x) \leq 0$ for all $x \in R$

$\Rightarrow \quad a + 2 < 0$ and $3(a+2)x^2 - 6ax + 9a \leq 0$ for all $x \in R$

$\Rightarrow \quad a + 2 < 0$ and discriminant ≤ 0

$\Rightarrow \quad a < -2$ and $-8a^2 - 24a \leq 0$

$\Rightarrow \quad a < -2$ and $a(a+3) \geq 0$

$\Rightarrow \quad a < -2$ and $a \leq -3$ or $a \geq 0$

$\Rightarrow \quad a \leq -3$.

6. Since $f(x) = \sqrt{3}\,\sin x - \cos x - 2ax + b$ is decreasing for all real values of x, therefore $f'(x) \leq 0$ all x.

$\Rightarrow \quad \sqrt{3}\,\cos x + \sin x - 2a \leq 0$ for all x

$\Rightarrow \quad \dfrac{\sqrt{3}}{2}\cos x + \dfrac{1}{2}\sin x \leq a$ for all x

$\Rightarrow \quad \sin\left(x + \dfrac{\pi}{3}\right) \leq a$ for all x

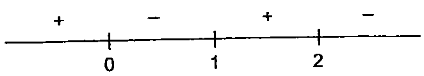
$\Rightarrow \quad a \geq 1 \qquad \left[\because \sin\left(x + \dfrac{\pi}{3}\right) \leq 1\right]$

7. $f(x) = 2\log|x-1| - x^2 + 2x + 3$

$$\Rightarrow f'(x) = \frac{2}{x-1} - 2x + 2 = 2\ 2\left[\frac{1-(x-1)^2}{x-1}\right]$$

$$= \frac{-2x(x-2)}{x-1}$$

Sign scheme for $\dfrac{-2x(x-2)}{(x-1)}$



$f'(x) > 0$ if $x \in (-\infty, 0)$ or $x \in (1, 2)$

$\therefore$ $f(x)$ is increasing in the interval $(-\infty, 0) \cup (1, 2)$ and decreases if $x\ (0, 1) \cup (2, \infty)$

8. $g(x) = f(\log x) + f(2 - \log x)$

$\Rightarrow g'(x) = [f'(\log x) - f'(2 - \log x)]/x$

$g(x)$ increases if $g'(x) > 0$, now $x > 0$

$\Rightarrow f'(\log x) - f'(2 - \log x) > 0$

$\Rightarrow f'(\log x) > f'(2 - \log x)$

$\Rightarrow \log x < 2 - \log x$ ($f''(x) < 0, f'(x)$ is decreasing)

$\Rightarrow \log x < 1$

$\Rightarrow 0 < x < e$

Exercise 6.2

1. Let $f(x) = \ln(1+x) - \dfrac{x}{1+x}$

$$\therefore f'(x) = \frac{1}{(1+x)} - \frac{1+x-x}{(1+x)^2} = \frac{x}{(1+x)^2} > 0 \qquad (\because x > 0)$$

$\Rightarrow f(x)$ is an increasing function.

$\because x > 0 \Rightarrow f(x) > f(0)$

$$\Rightarrow \ln(1+x) - \frac{x}{1+x} > 0 \qquad (\because f(0) = 0)$$

$$\Rightarrow \frac{x}{1+x} < \ln(1+x).$$

2. Let $f(x) = \sin x - x$

$\therefore f'(x) = \cos x - 1 = -(1 - \cos x) = -2\sin^2 x/2 < 0$

$\therefore f(x)$ is a decreasing function.

Now $x > 0$

$\Rightarrow f(x) < f(0) \Rightarrow \sin x - x < 0$ $[\because f(0) = 0]$

$\Rightarrow \sin x < x$ (1)

Now, let $g(x) = x - \dfrac{x^3}{6} - \sin x$

$$\therefore g'(x) = 1 - \frac{x^2}{2} - \cos x$$

To find sign of $g'(x)$, we consider

$$\phi(x) = 1 - \frac{x^2}{2} - \cos x$$

$\therefore \phi'(x) = -x + \sin x < 0$ From equation (1)

$\therefore \phi(x)$ is a decreasing function

$\Rightarrow g'(x) < 0$

$\Rightarrow g(x)$ is decreasing function

$\because x > 0 \Rightarrow g(x) < g(0)$

$$\Rightarrow x - \frac{x^3}{6} - \sin x < 0 \qquad [\because g(0) = 0]$$

$$\Rightarrow x - \frac{x^3}{6} < \sin x \qquad (2)$$

Combining equations (1) and (2), we get

$$x - \frac{x^3}{6} < \sin x < x$$

3. Let us assume $f(x) = \tan^{-1} x - \dfrac{3x}{x^2+3}$

$$\Rightarrow f'(x) = \frac{1}{1+x^2} - \left[\frac{3(x^2+3) - 3x(2x)}{(x^2+3)^2}\right]$$

$$= \frac{x^4 + 6x^2 + 9 - (3x^2 + 9 - 6x^2)(1+x^2)}{(1+x^2)(x^2+3)^2}$$

$$= \frac{4x^4}{(1+x^2)(x^2+3)^2}$$

Hence $f(x)$ is increasing throughout

Also $f(0) = 0$

Hence, $f(x) > 0, \forall x > 0$

$$\Rightarrow \tan^{-1} x > \frac{3x}{3+x^2}$$

4. Let $f(x) = \dfrac{\sin x}{x}$

$$\therefore f'(x) = \frac{(x\cos x - \sin x)}{x^2} = \frac{\cos x(x - \tan x)}{x^2}$$

To find sign of $f'(x)$, we consider

$g(x) = x - \tan x,\ 0 < x < \dfrac{\pi}{2}$

$\therefore g'(x) = 1 - \sec^2 x < 0$ $(\because \sec x > 1)$

$\therefore g(x)$ is a decreasing function

$\Rightarrow g(x) < g(0)$

$\Rightarrow x - \tan x < 0$

$\Rightarrow f'(x) < 0$

$\Rightarrow f(x)$ is a decreasing function.

Also, $0 < x < \pi/2$

$$\Rightarrow f(\pi/2) < f(x) < \lim_{x\to 0} f(x)$$

$$\Rightarrow \frac{2}{\pi} < \frac{\sin x}{x} < 1$$

Exercise 6.3

1. $f(x) = 2x^3 - 3x^2 - 12x + 5$

$\Rightarrow f'(x) = 6x^2 - 6x - 12$

$f'(x) = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = -1, 2$

Here, $f(4) = 128 - 48 - 48 + 5 = 37$

$f(-1) = -2 - 3 + 12 + 5 = 12$

$f(2) = 16 - 12 - 24 + 5 = -15$

$f(-2) = -16 - 12 + 24 + 5 = 1$

Therefore, the global maximum value of function is 37 at $x = 4$ and global minimum value is -15 at $x = 2$.

Hence, range of $f(x)$ is $[-15, 37]$.

2. $f(x) = 1 + 2\sin x + 3\cos^2 x,\ 0 \le x \le 2\pi/3$

$\Rightarrow f'(x) = 2\cos x - 6\sin x\cos x$

$= 2\cos x(1 - 3\sin x)$

If $f'(x) = 0 \Rightarrow \cos x = 0$ or $1 - 3\sin x = 0$

$\Rightarrow \quad x = \pi/2$ or $\sin x = 1/3$

$f''(x) = -2\sin x - 6\cos 2x$

$$f''\left(\frac{\pi}{2}\right) = -2\sin\frac{\pi}{2} - 6\cos\left(2\times\frac{\pi}{2}\right) = -2+6 = 4 > 0$$

Hence, $x = \pi/2$ is point of minima.

$f''(\sin^{-1} 1/3) = -2\,(1/3) - 6(1 - 2\times 1/9) = -2/3 - 14/3 < 0$

Hence, $x = \sin^{-1} 1/3$ is point of maximum.

$$f_{\min} = f\left(\frac{\pi}{2}\right) = 1 + 2\sin\frac{\pi}{2} + 3\cos^2\frac{\pi}{2} = 1+2 = 3$$

$$f_{\max} = f\left(\sin^{-1}\frac{1}{3}\right) = 1 + 2\left(\frac{1}{3}\right) + 3\left(1 - \frac{1}{9}\right) = \frac{5}{3} + \frac{8}{3} = \frac{13}{3}$$

3. $f(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x,\ 0 \le x \le \pi$

$f'(x) = \cos x + \cos 2x + \cos 3x = 2\cos 2x\cos x + \cos 2x$

$= \cos 2x\,(2\cos x + 1)$

Let $f'(x) = 0$

$\Rightarrow \quad \cos 2x = 0$ or $2\cos x + 1 = 0$

$\Rightarrow \quad 2x = \pi/2, 3\pi/2$ or $\cos x = -1/2$

$\Rightarrow \quad x = \pi/4, 3\pi/4$ or $x = 2\pi/3$

Sign scheme of $f'(x)$

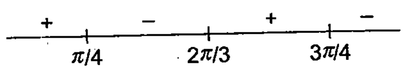


Fig. S-6.3

Hence, $x = \pi/4, 3\pi/4$ are points of maxima.

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{3\sqrt{2}} = \frac{4}{3\sqrt{2}} + \frac{1}{2} = \left(\frac{4\sqrt{2}+3}{6}\right)$$

$$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} + \frac{1}{2}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{9\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{1}{3}\frac{1}{\sqrt{2}} = \frac{4\sqrt{2}-3}{6}$$

$$f\left(\frac{2\pi}{3}\right) = \sin^2\frac{2\pi}{3} + \frac{1}{2}\sin\frac{4\pi}{3} + \frac{1}{3}\sin\left(\frac{9\pi}{3}\right) = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}$$

$$= \frac{\sqrt{3}}{4}$$

Thus $x = \dfrac{\pi}{4}$ is the point of global maxima.

$x = \dfrac{3\pi}{4}$ is the pont of local maxima

$x = \dfrac{2\pi}{3}$ is the point of local minima.

4. $f(\theta) = \sin^p\theta\cos^q\theta, p, q > 0, 0 < \theta < \pi/2$

$f'(\theta) = p\sin^{p-1}\theta\cos\theta\cos^q\theta - q\cos^{q-1}\theta\sin\theta\sin^p\theta$

$= \sin^{p-1}\theta\cos^{q-1}\theta\,[p\cos^2\theta - q\sin^2\theta]$

Let $f'(\theta) = 0$

$\Rightarrow \quad \sin\theta = 0$ or $\cos\theta = 0$ (not possible)

or $p\cos^2\theta - q\sin^2\theta = 0 \Rightarrow \tan^2\theta = p/q \Rightarrow \tan\theta = \sqrt{\dfrac{p}{q}}$

$\left(\tan\theta \ne -\sqrt{\dfrac{p}{q}}\right.$, as $\left.0 < \theta < \pi/2\right)$

Check for extremum

When $\theta \to 0, f(\theta) \to 0$ (as $\sin\theta \to 0$)

When $\theta \to \pi/2, f(\theta) \to 0$ (as $\cos\theta \to 0$)

also, for $\theta \in (0, \pi/2), f(\theta)$ is + ve

Hence, the only point of extremum is point of maxima.

$\Rightarrow \quad \theta = \tan^{-1}\sqrt{\dfrac{p}{q}}$ is point of maxima

when $\tan\theta = \sqrt{\dfrac{p}{q}}$, $\cos\theta = \dfrac{\sqrt{q}}{\sqrt{p+q}}$ and $\sin\theta = \dfrac{\sqrt{p}}{\sqrt{p+q}}$

Hence, maximum value, $f_{\max} = \left(\dfrac{\sqrt{q}}{\sqrt{p+q}}\right)^q \left(\dfrac{\sqrt{p}}{\sqrt{p+q}}\right)^p$

$$= \left(\frac{p^p\,q^q}{(p+q)^{p+q}}\right)^{1/2}$$

5. $f(x) = \log_e(3x^4 - 2x^3 - 6x^2 + 6x + 1), x \in (0, 2)$

$$f'(x) = \frac{12x^3 - 6x^2 - 12x + 6}{(3x^4 - 2x^3 - 6x^2 + 6x + 1)}$$

$$= \frac{6(2x^3 - x^2 - 2x + 1)}{(3x^4 - 2x^3 - 6x^2 + 6x + 1)}$$

$$= \frac{6(x^2 - 1)(2x - 1)}{(3x^4 - 2x^3 - 6x^2 + 6x + 1)}$$

Sign scheme of $f'(x)$

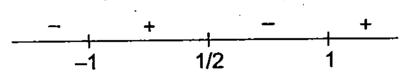


Fig. S-6.4

But $x \in (0, 2)$

Hence, $x = 1/2$ is point of maxima,

and $x = 1$ is point of minima.

Hence, $f_{\min} = f(1) = \ln 2$

and $f_{\max} = f(1/2) = \ln(39/16)$.

6. $f(x) = -\sin^3 x + 3\sin^2 x + 5$

$\Rightarrow \quad f'(x) = -3\cos x\sin^2 x + 6\sin x\cos x$

$= -3\sin x\cos x\,(\sin x - 2)$

Now, $\sin x - 2 < 0\ \forall\ x \in \left[0, \dfrac{\pi}{2}\right]$

$\sin x, \cos x \ge 0\ \forall\ x \in R$

$\Rightarrow \quad f'(x) \ge 0\ \forall\ x \in R$

$\Rightarrow \quad f(x)$ is a strictly increasing function $\forall\ x \in [0, \pi/2]$

Hence, $f(x)$ is minimum when $x = 0$

and maximum when $x = \pi/2$.

$f_{\min} = f(0) = 5$

$f_{\max} = f(\pi/2) = 7$

7. $f(x) = \dfrac{1}{3}\left(x + \dfrac{1}{x}\right)$

$\Rightarrow \quad f'(x) = \dfrac{1}{3}\left(1 - \dfrac{1}{x^2}\right)$

Let $f'(x) = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

Also, $f''(x) = \frac{1}{3}\left(\frac{2}{x^3}\right) \Rightarrow f''(1) > 0$ and $f''(-1) < 0$

$\Rightarrow$ $x = 1$ is point of minima and $x = -1$ is point of maxima.

Here, $f(1) = \frac{2}{3}$ and $f(-1) = -\frac{2}{3}$.

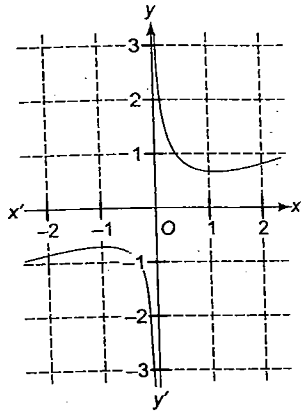


Fig. S-6.5

Thus, local maximum value is less than local minimum value.

8. $f(x) = x(x^2-4)^{-1/3}$

$f'(x) = (x^2-4)^{-1/3} - \frac{1}{3}(x^2-4)^{-4/3}(2x)\,x$

$= \frac{1}{(x^2-4)^{1/3}} - \frac{2x^2}{3(x^2-4)^{4/3}}$

$= \frac{3(x^2-4)-2x^2}{3(x^2-4)^{4/3}}$

$= \frac{(x^2-12)}{3(x^2-4)^{4/3}}$

Sign scheme of $f'(x)$

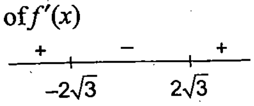


Fig. S-6.6

$\Rightarrow$ $x = 2\sqrt{3}$ is point of minima and $x = -2\sqrt{3}$ is point of maxima.

$f_{max} = f(-2\sqrt{3}) = (-2\sqrt{3})\,(12-4)^{-1/3} = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$

$f_{min} = f(2\sqrt{3}) = (2\sqrt{3})(12-4)^{-1/3} = \sqrt{3}$

Here, maximum value is less than minimum value.

This is because $f(x)$ is discontinuous at $x = \pm 2$.

Since $f(x)$ is unbounded function both the extreme values are local.

9. For $x < 1$, $f'(x) = 3x^2 - 2x + 10 > 0$

$\Rightarrow$ $f(x)$ is an increasing function for $x < 1$

For $x > 1$, $f'(x) = -2$.

$\Rightarrow$ $f(x)$ is a decreasing function for $x > 1$. Now $f(x)$ will have greatest value at $x = 1$.

If $\lim\limits_{x\to(1^+)} f(x) \le f(1)$

$\Rightarrow$ $-2 + \log_2(b^2-2) \le 5$

$\Rightarrow$ $0 < b^2 - 2 \le 128 \Rightarrow 2 \le b^2 \le 130$

$\Rightarrow$ $b \in \left[-\sqrt{130}, -\sqrt{2}\right] \cup \left[\sqrt{2}, \sqrt{130}\right]$

10. $f(x) = \begin{cases} \tan^{-1}\alpha - 5x^2, & 0 < x < 1 \\ -6x, & x \ge 1 \end{cases}$

$f(1) = -6$

For maximum at $x = 1$

$\lim\limits_{x\to1^-} f(x) = \tan^{-1}\alpha - 5 < -6$

$\Rightarrow$ $\tan^{-1}\alpha < -1 \Rightarrow \alpha < -\tan 1$

11. Clearly from the graph give in Fig S-6.7 $x = \sqrt{2}$ is point of minima

$x = \sqrt{3}$ is not a point of extremum

$x = 2\sqrt{3}$ is also not a point of extremum

$x = 0$ is point of maxima

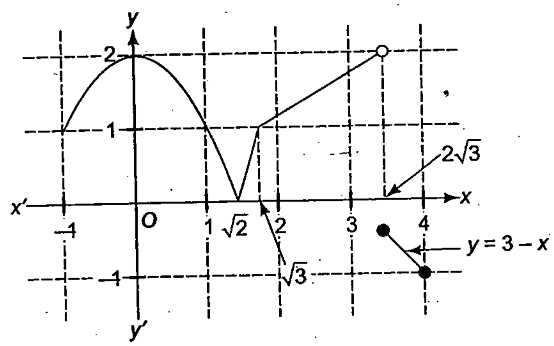


Fig. S-6.7

12. $f(x) = |x| + \left|x + \frac{1}{2}\right| + |x - 3| + \left|x - \frac{5}{2}\right|$

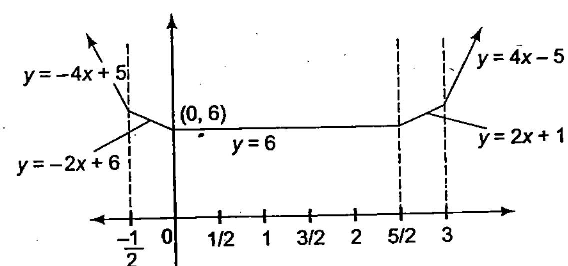


Fig. S-6.8

From the graph, minimum value is 6.

13. $f(0) = 1$

$f(0^+) = \lim\limits_{x\to0^+}(x^2 - x + 1) \to 1^-$ as $x^2 - x + 1$ is decreasing for $(-\infty, 1/2)$

$f(0^-) \to \lim\limits_{x\to0^-}(1 + \sin x) \to 1^-$

Thus, $f(0^-) < f(0)$ and $f(0) > f(0^+)$.

Then at $x = 0$, $f(x)$ is the point of maxima.

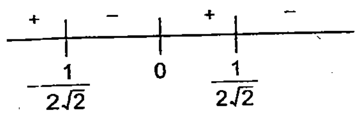
14. $f(x) = x^{2/3} - x^{4/3}$,

$f'(x) = (2/3)x^{-1/3} - (4/3)x^{1/3}$

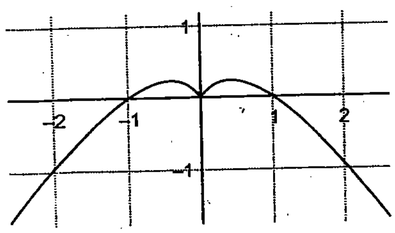
$= \frac{2}{3}\frac{(1-2x^{2/3})}{x^{1/3}}$

Critical points, $f'(x) = 0$ at $x = \pm\frac{1}{2\sqrt{2}}$

Also $f'(x)$ does not exists at $x = 0$.
As $f(x)$ is continuous at $x = 0$ so it is also a critical point.
Sign scheme of $f'(x)$



Thus $x = 0$ is point of minima and $x = \pm\frac{1}{2\sqrt{2}}$ are points of maxima



Also range of $f(x)$ is $\left(-\infty, f\left(\pm\frac{1}{2\sqrt{2}}\right)\right]$

15. $f(x) = \frac{x^2+ax+b}{x-10}$, has a stationary point at (4, 1) so it must lie on the curve.

$\therefore\ 16+4+b=-6$ (1)

Also $\left(\frac{dy}{dx}\right)_{(x=4)} = 0$

$\Rightarrow \left(\frac{x^2-20x-10a-b}{(x-10)^2}\right)_{(x=4)} = 0$

$\Rightarrow 10a+b=-64$ (2)

From (1) and (2) we have $a=-7, b=6$
Also for these values of x,

$y=\frac{x^2-7x+6}{x-10}$ (3)

$\therefore\ \frac{dy}{dx}=\frac{(x-4)(x-16)}{(x-10)^2}$

From $x = 4 - h$ to $x = 4 + h$, $\frac{dy}{dx}$ changes its sign from +ve to −ve.
Hence $x = 4$ is point of maxima.

Exercise 6.4

1. Let the additional number of subscribers be x, so the number of subscribers becomes 725 + x, and then the profit per subscriber is ₹$(12 - x/100)$.
If P is the total profit in Rs., then

$$P = (725+x)\left(12-\frac{x}{100}\right)$$

$$= -\frac{x^2}{100}+\frac{19}{4}x+8700$$

$$= \frac{1}{100}\left[870000-\left(\frac{475}{2}\right)^2-\left(x-\frac{475}{2}\right)^2\right]$$

P is maximum (greatest) when $x - 475/2 = 0$
i.e., $x = 237.5$. But x is + ve integer, so x can be taken as 237 or 238.
Since $P(237) = P(238)$, for maximum profit, total number of subscribers should be $725 + 237 = 962$
or $725 + 238 = 963$.

2.

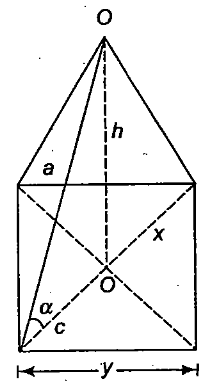


Fig. S-6.9

$h = a \sin\alpha$ and $x = a\cos\alpha$; $x^2 + h^2 = a^2$

$V=\frac{1}{3}y^2h = \frac{1}{3}2x^2h$ (note : $4x^2 = 2y^2 \Rightarrow y^2 = 2x^2$)

$V(\alpha) = \frac{2}{3}a^2\cos^2\alpha\, a \sin\alpha = \frac{2}{3}a^3 \sin\alpha\cos^2\alpha$

now $V'(\alpha)=0 \Rightarrow \tan\alpha = \frac{1}{\sqrt{2}}$; $\Rightarrow V_{max} = \frac{4\sqrt{3}a^3}{27}$

3. Equation of the curve is $y = x^2 + 1$.
Tangent at $P(a, b)$ is $y - b = 2a(x - a)$
i.e., $y-(a^2+1)=2a(x-a)$
$x = 0 \Rightarrow y = 1 - a^2$ which is positive for $0 < a < 1$ and
$x = 1 \Rightarrow y = 1 + 2a - a^2$

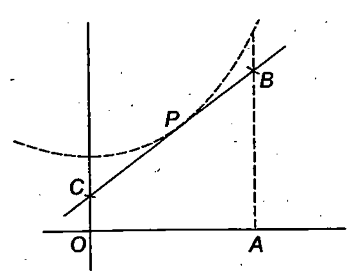


Fig. S-6.10

$\therefore\ OC = 1 - a^2$ and $AB = 1 + 2a - a^2$

Z = area of trapezium $OABC$

$$= \frac{1}{2}(OC + AB)OA = 1 + a - a^2, 0 < a < 1$$

$$\frac{dZ}{da} = [1 - 2a] = 0 \Rightarrow a = 1/2$$

and $\frac{d^2Z}{da^2} = -4 < 0$

$\therefore$ at $a = 1/2$, area of trapezium is maximum (greatest).

Thus, the required point is (1/2, 5/4).

4. Let $AD = x$ be the height of the cone ABC inscribed in a sphere of radius a.

$\therefore$ $OD = x - a$

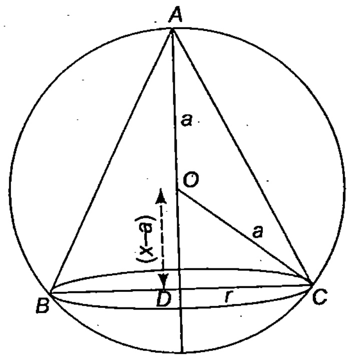


Fig. S-6.11

Then radius of its base $r = CD = \sqrt{(OC^2 - OD^2)}$

$$= \sqrt{[a^2 - (x-a)^2]} = \sqrt{(2ax - x^2)}$$

$\Rightarrow$ Volume V of the cone is given by

$$V = \frac{1}{3}\pi r^2 x = \frac{1}{3}\pi(2ax - x^2)x = \frac{1}{3}\pi(2ax^2 - x^3)$$

$$\Rightarrow \frac{dV}{dx} = \frac{1}{3}\pi(4ax - 3x^2) \text{ and } \frac{d^2V}{dx} = \frac{1}{3}\pi(4a - 6x)$$

For max. or min. of V, $dV/dx = 0 \Rightarrow x = 4a/3$ $(\because x \neq 0)$

For this value of V, $\frac{d^2V}{dx^2} = -(4\pi a/3) = (-\text{ve})$

$\Rightarrow$ V is max. (i.e., greatest)

when $x = 4a/3 = (2/3)(2a)$, i.e., when the height of cone is (2/3)rd of the diameter of sphere.

5. Here,

$f(x) = e^x \cos x$

$\Rightarrow f'(x) = e^x \cos x - e^x \sin x$

$= e^x (\cos x - \sin x)$

where, $f'(x)$ is slope of tangent (i.e., to be minimised)

So let $f'(x) = g(x)$

$\therefore$ $g(x) = e^x (\cos x - \sin x)$

$\Rightarrow g'(x) = e^x \{\cos x - \sin x\} + e^x \{-\sin x - \cos x\}$

$= e^x \{-2 \sin x\}$

which is + ve when $x \in [\pi, 2\pi]$ and

$-$ve when $x \in [0, \pi]$

i.e., $g(x)$ is decreasing in $(0, \pi)$ and

$g(x)$ is increasing in $(\pi, 2\pi)$

So, at $x = \pi$. Slope of tangent of the function $f(x)$ attains minima.

6. Let C be the centre of the circular plot of lawn of with a diameter of 100 m.

i.e., $CA = CB = 50$ m.

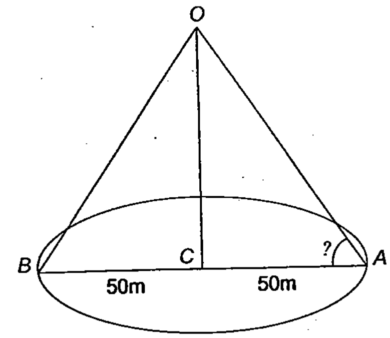


Fig. S-6.12

And let O be the position of light which is directly above C. Also let

$\angle OAC = \theta, (0 < \theta < \pi/2)$

Then according to the question, the intensity I of the light at the circumcentre of the plot is given by

$$I = \frac{k \sin\theta}{(OA)^2} = \frac{k \sin\theta}{(50 \sec\theta)^2}$$

or $I = [k/(2500)] \sin\theta \cos^2\theta$

$\therefore$ $dI/d\theta = [k/(2500)] (\cos^3\theta - 2\sin^2\theta\cos\theta)$

$= (k/2500) \cos\theta (\cos^2\theta - 2\sin^2\theta)$

For max. or min. of I, $dI/d\theta = 0$, $\tan\theta = 1/\sqrt{2}$

Now we have

$d^2I/d\theta^2 = (k/2500)(-7\sin\theta\cos^2\theta + 2\sin^3\theta)$

$= (k/2500) \sin\theta\cos^2\theta(-7 + 2\tan^2\theta)$

When $\tan\theta = 1/\sqrt{2}$, $d^2I/d\theta^2$ is $-$ve

Hence, I(Intensity of light) is max., when $\tan\theta = 1/\sqrt{2}$, which is the only point of extrema so gives the greatest intensity.

$\therefore$ The required height of the light

$= OC = AC \tan\theta = 50/\sqrt{2} = 25\sqrt{2}$ m.

Chapter 7

Exercise 7.1

1. $\int (\sec x + \tan x)^2 \, dx$

$$= \int \left(\sec^2 x + \tan^2 x + 2\sec x \tan x\right) dx$$

$$= \int \left(2\sec^2 x - 1 + 2\sec x \tan x\right) dx$$

$$= 2(\sec x + \tan x) - x + C$$

2. $\int (1 - \cos x) \operatorname{cosec}^2 x \, dx$

$= \int \operatorname{cosec}^2 x\, dx - \int \operatorname{cosec} x \cot x\, dx$

$= -\cot x + \operatorname{cosec} x + C$

$= \dfrac{1-\cos x}{\sin x} + C$

$= \dfrac{2\sin^2 \frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} + C$

$= \tan\dfrac{x}{2} + C$

3. $I = \int a^{mx} b^{nx}\, dx$

$= \int \left(a^m b^n\right)^x dx$

$= \dfrac{\left(a^m b^n\right)^x}{\log\left(a^m b^n\right)} + C$

4. $\int \dfrac{\tan x}{(\sec x + \tan x)} dx$

$= \int \dfrac{\tan x(\sec x - \tan x)}{(\sec x + \tan x)(\sec x - \tan x)} dx$

$= \int \dfrac{\tan x(\sec x - \tan x)}{\left(\sec^2 x - \tan^2 x\right)} dx$

$= \int \left(\sec x \tan x - \tan^2 x\right) dx$

$= \int \sec x \tan x\, dx - \int \left(\sec^2 x - 1\right) dx$

$= \int \sec x \tan x\, dx - \int \sec^2 x\, dx + \int 1\, dx$

$= \sec x - \tan x + x + C$

5. $\int \dfrac{x^4 dx}{x + x^5}$

$= \int \dfrac{(x^4+1)\, dx}{x + x^5} - \int \dfrac{dx}{x + x^5}$

$= \int \dfrac{(x^4+1)\, dx}{x\,(1+x^4)} - \int \dfrac{dx}{x\,(x^4+1)}$

$= \int \dfrac{dx}{x} - \int \dfrac{dx}{x + x^5}$

$= \log x - f(x) + C$

$= \log x - f(x) + C$

6. $\int \dfrac{(x^3+8)(x-1)}{x^2-2x+4} dx = \int \dfrac{(x^3+2^3)(x-1)}{x^2-2x+4} dx$

$= \int \dfrac{(x+2)(x^2-2x+4)(x-1)}{x^2-2x+4} dx$

$= \int (x+2)(x-1) dx = \int (x^2+x-2) dx$

$= \int x^2 dx + \int x dx - 2\int x^0 dx = \dfrac{x^3}{3} + \dfrac{x^2}{2} - 2x + C$

7. $\int \dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx$

$= \int \dfrac{\sin^3 x}{\sin^2 x \cos^2 x} dx + \int \dfrac{\cos^3 x}{\sin^2 x \cos^2 x} dx$

$= \int \dfrac{\sin x}{\cos^2 x} dx + \int \dfrac{\cos x}{\sin^2 x} dx$

$= \int \tan x \sec x\, dx + \int \cot x \operatorname{cosec} x\, dx = \sec x - \operatorname{cosec} x + C$

8. $= \int \tan^{-1}(\sec x + \tan x) dx$

$= \int \tan^{-1}\left(\dfrac{1+\sin x}{\cos x}\right) dx$

$= \int \tan^{-1}\left\{\dfrac{1-\cos\left(\frac{\pi}{2}+x\right)}{\sin\left(\frac{\pi}{2}+x\right)}\right\} dx$

$= \int \tan^{-1}\left\{\dfrac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right\} dx$

$= \int \tan^{-1}\left\{\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\right\} dx$

$= \int \dfrac{\pi}{4} + \dfrac{x}{2} dx = \dfrac{\pi}{4}\int 1.dx + \dfrac{1}{2}\int x dx = \dfrac{\pi}{4}x + \dfrac{x^2}{4} + C$

Exercise 7.2

1. $\int \dfrac{dx}{\sqrt{2\,ax - x^2}}$

$= \int \dfrac{dx}{\sqrt{a^2 - (x-a)^2}}$

$= \sin^{-1}\left(\dfrac{x-a}{a}\right) + C$

2. $I=\int \frac{e^{3x}+e^{5x}}{e^{x}+e^{-x}}$

$= \int \frac{e^{4x}+e^{6x}}{e^{2x}+1}dx$

$= \int \frac{e^{4x}(e^{2x}+1)}{e^{2x}+1}dx$

$= \int e^{4x}dx = \frac{e^{4x}}{4}+C$

3. $\int \tan^2 x \sin^2 x\, dx$

$= \int \frac{\sin^4 x}{\cos^2 x}dx$

$= \int \frac{\sin^2 x(1-\cos^2 x)}{\cos^2 x}dx$

$= \int\left(\tan^2 x-\sin^2 x\right)dx$

$= \int\left(\sec^2 x-1-\frac{1-\cos 2x}{2}\right)dx$

$= \tan x-\frac{3}{2}x+\frac{\sin 2x}{4}+C$

4. $\int \frac{\cos x-\sin x}{\cos x+\sin x}(2+2\sin 2x)\, dx$

$=2\int \frac{(\cos x-\sin x)(\cos x+\sin x)^2}{\cos x+\sin x}dx$

$=2\int(\cos x-\sin x)(\cos x+\sin x)\, dx$

$= \int(\cos^2 x-\sin^2 x)\, dx$

$=2\int \cos 2x\, dx$

$=\sin 2x+C$

5. $I=\int \operatorname{cosec}^4 x\, dx = \int \operatorname{cosec}^2 x \operatorname{cosec}^2 x\, dx$

$= \int \operatorname{cosec}^2 x(1+\cot^2 x)\, dx$

$= \int \operatorname{cosec}^2 x\, dx + \int \cot^2 x \operatorname{cosec}^2 x\, dx$

$=-\cot x-\frac{\cot^3 x}{3}+C$

6. Let $I=\int \frac{\sin 2x}{(a+b\cos x)^2}dx=\int \frac{2\sin x\cos x}{(a+b\cos x)^2}dx.$

Putting $a+b\cos x=t$, then $-b\sin x\, dx=dt$

$I=-\frac{2}{b}\int \frac{1}{t^2}\left(\frac{t-a}{b}\right)dt$

$\left[\because a+b\cos x=t, \therefore \cos x=\frac{t-a}{b}\right]$

$=-\frac{2}{b^2}\int\left(\frac{1}{t}-\frac{a}{t^2}\right)dt=-\frac{2}{b^2}\left[\log|t|+\frac{a}{t}\right]+C$

$=-\frac{2}{b^2}\left[\log|a+b\cos x|+\frac{a}{a+b\cos x}\right]+C$

7. $I=\int \sin x\cos x\cos 2x\cos 4x\cos 8x\, dx$

$=\frac{1}{2}\int \sin 2x\cos 2x\cos 4x\cos 8x\, dx$

$=\frac{1}{4}\int \sin 4x\cos 4x\cos 8x\, dx=\frac{1}{8}\int \sin 8x\cos 8x\, dx$

$=\frac{1}{16}\int \sin 16x\, dx=\frac{-1}{256}\cos 16x+C$

8. $\int \frac{(1+\ln x)^5}{x}dx = \int(1+\ln x)^5 d(1+\ln x)$

$=\frac{1}{6}(1+\ln x)^6+C$

9. $\int \frac{\cos 2x-\cos 2\theta}{\cos x-\cos\theta}d\theta = \int \frac{2\cos^2 x-1-(2\cos^2\theta-1)}{\cos x-\cos\theta}d\theta$

$=2\int \frac{\cos^2 x-\cos^2\theta}{\cos x-\cos\theta}d\theta$

$=2\int(\cos x+\cos\theta)d\theta$

$=2\cos x+2x\cos\theta+C$

10. $\frac{x^3}{x+1}=\frac{x^3+1-1}{x+1}=\frac{-1}{(x+1)}+(x^2+1-x)$

Thus the given integral is,

$=\int\left(x^2+1-x-\frac{1}{1+x}\right)dx=\frac{x^3}{3}+x-\frac{x^2}{2}-\ln|x+1|+C$

11. $\int \frac{dx}{\sqrt{x}+\sqrt{x-2}}$

$= \int \frac{(\sqrt{x} - \sqrt{x-2})dx}{x-(x-2)}$ (rationalizing)

$= \frac{1}{2}\int \left(\sqrt{x} - \sqrt{x-2}\right)dx$

$= \frac{1}{3}\left\{x^{3/2} - (x-2)^{3/2}\right\} + C$

12. $\int \left(1 + 2x + 3x^2 + 4x^3 + \cdots\right)dx$

$= \int (1-x)^{-2}\, dx$

$= (1-x)^{-1} + C$

13. $I = \int \frac{\ln(\ln x)}{x \ln x} dx$, let $t = \ln(\ln x)$

$\Rightarrow \quad \frac{dt}{dx} = \frac{1}{\ln x} \times \frac{1}{x}$

$\Rightarrow \quad I = \int t\,dt = \frac{1}{2}t^2 + C = \frac{1}{2}[\ln(\ln x)]^2 + C$

14. $\int \frac{dx}{x + x\log x}$

$= \int \frac{\frac{1}{x}dx}{(1+\log x)}$

$= \int \frac{(1+\log x)' dx}{(1+\log x)}$

$= \log(1 + \log x) + C$

15. Put $\sec x = t \Rightarrow \sec x \tan x\, dx = dt$, therefore

$\int \sec^p x \tan x\, dx = \int t^{p-1}\, dt = \frac{t^p}{p} + C = \frac{\sec^p x}{p} + C$

16. $\int \frac{\sin^6 x}{\cos^8 x}\, dx$

$= \int \frac{\sin^6 x}{\cos^6 x} \times \frac{1}{\cos^2 x}\, dx$

$= \int \tan^6 x \sec^2 x\, dx$

$= \frac{\tan^7 x}{7} + C$

17. Let $z = \tan x - x$ then $dz = (\sec^2 x - 1)dx = \tan^2 x\, dx$

Now, $\int (\tan x - x)\tan^2 x\, dx = \int z\,dz = \frac{z^2}{2} + C = \frac{(\tan x - x)^2}{2} + C$

Exercise 7.3

1. $\int \frac{dx}{(1+\sin x)^{1/2}} = \int \frac{dx}{\cos\frac{x}{2} + \sin\frac{x}{2}}$

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)}$

$= \frac{1}{\sqrt{2}} \int \operatorname{cosec}\left(\frac{x}{2} + \frac{\pi}{4}\right)dx$

$= \frac{1}{\sqrt{2}} \frac{\log\left|\tan\left(\frac{x}{4} + \frac{\pi}{8}\right)\right|}{\frac{1}{2}} + C$

$= \sqrt{2}\log\left|\tan\left(\frac{x}{4} + \frac{\pi}{x}\right)\right| + C$

2. $I = \int \frac{dx}{\cos x - \sin x}$

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\cos x.\frac{1}{\sqrt{2}} - \sin x\frac{1}{\sqrt{2}}}$

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sin\left(\frac{\pi}{4} - x\right)}$

$= \frac{1}{\sqrt{2}} \int \operatorname{cosec}\left(\frac{\pi}{4} - x\right)dx$

$= \frac{1}{\sqrt{2}} \log\left|\tan\left(\frac{\pi}{8} - \frac{x}{2}\right)\right| + C$

$= \frac{1}{\sqrt{2}} \log\left|\tan\left(\frac{x}{2} - \frac{\pi}{8}\right)\right| + C$

3. $\int \frac{\sin x}{\sin(x-a)}dx$

$= \int \frac{\sin(x - a + a)}{\sin(x-a)}dx$

$= \int \frac{\sin(x-a)\cos a + \cos(x-a)\sin a}{\sin(x-a)}dx$

$= \cos a \int dx + \sin a \int \frac{\cos(x-a)}{\sin(x-a)}dx$

$= (\cos a)\, x + \sin a \log \sin (x-a) + C$
$= (x-a) \cos a + \sin a \log \sin (x-a) + C$

4. $I = \int \tan^3 x\, dx$

$= \int \tan^2 x \tan x\, dx$

$= \int (\sec^2 x - 1) \tan x\, dx$

$\Rightarrow \quad I = \int \tan x \sec^2 x\, dx - \int \tan x\, dx$

$\Rightarrow \quad I = I_1 - \log|\sec x| + C$, where $I_1 = \int \tan x \sec^2 x\, dx$

Putting $\tan x = t$ and $\sec^2 x\, dx = dt$ in I_1,

we get $I_1 = \int t\, dt = \dfrac{t^2}{2} = \dfrac{1}{2} \tan^2 x + C$

Hence, $I = \dfrac{1}{2} \tan^2 x - \log|\sec x| + C.$

Exercise 7.4

1. $\int \dfrac{x^2 \tan^{-1} x^3}{1+x^6} dx$

$= \dfrac{1}{3} \int \tan^{-1} x^3 \dfrac{3x^2}{1+x^6} dx$

$= \dfrac{1}{3} \int \tan^{-1} x^3 (\tan^{-1} x^3)'\, dx$

$= \dfrac{1}{6} (\tan^{-1} x^3)^2 + C$

2. Put $x = t^2 \Rightarrow dx = 2t\, dt$

$\Rightarrow \quad \int \dfrac{\sqrt{x}\, dx}{1+x}$

$= 2\int \dfrac{t^2 dt}{1+t^2}$

$= 2\int \left(1 - \dfrac{1}{1+t^2}\right) dt$

$= 2(t - \tan^{-1} t) + C$

$= 2\sqrt{x} - 2 \tan^{-1} \sqrt{x} + C$

3. $\int \dfrac{\cot x}{\sqrt{\sin x}} dx = \int \dfrac{\cos x}{(\sin x)^{3/2}} dx$

$= \int \dfrac{dz}{z^{3/2}}$, where $z = \sin x$

$= \dfrac{z^{-1/2}}{-\dfrac{1}{2}} + C = \dfrac{-2}{\sqrt{z}} + C$

$= -\dfrac{2}{\sqrt{\sin x}} + C$

4. $I = \int \dfrac{dx}{x + \sqrt{x}} = \int \dfrac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)}$. Put $\sqrt{x} = z$

$\therefore \quad \dfrac{1}{2\sqrt{x}} dx = dz$

$\Rightarrow \quad I = \int \dfrac{2\, dz}{z+1}$

$= 2 \log|z+1| + C$

$= 2 \log\left(\sqrt{x}+1\right) + C$

5. $\int \dfrac{dx}{9 + 16 \sin^2 x}$

$= \int \dfrac{dx}{9\cos^2 x + 25 \sin^2 x} = \int \dfrac{\sec^2 x\, dx}{9 + 25 \tan^2 x}$

$= \int \dfrac{dz}{9 + 25 z^2} = \dfrac{1}{25} \int \dfrac{dz}{z^2 + \left(\dfrac{3}{5}\right)^2}$ $(z = \tan x)$

$= \dfrac{1}{25} \times \dfrac{1}{3/5} \tan^{-1} \dfrac{z}{3/5} + C$

$= \dfrac{1}{15} \tan^{-1} \dfrac{5z}{3} + C = \dfrac{1}{15} \tan^{-1} \dfrac{5 \tan x}{3} + C$

6. $\int \dfrac{e^{2x} - 2e^x}{e^{2x}+1} dx$

$= \dfrac{1}{2} \int \dfrac{2e^{2x}}{e^{2x}+1} dx - 2 \int \dfrac{e^x\, dx}{(e^x)^2 + 1}$

$= \dfrac{1}{2} \log(e^{2x}+1) - 2 \int \dfrac{dz}{z^2+1}$, where $z = e^x$

$= \dfrac{1}{2} \log(e^{2x}+1) - 2\tan^{-1}(e^x) + C$

7. $I = \int \dfrac{ax^3 + bx}{x^4 + c^2} dx$

$= \int \left[\dfrac{ax^3}{x^4+c^2} + \dfrac{bx}{x^4+c^2}\right] dx$

$= \dfrac{a}{4} \int \dfrac{4x^3}{x^4+c^2} dx + \dfrac{b}{2} \int \dfrac{2x}{x^4+c^2} dx$

$\quad I_1 \quad + \quad I_2$

$= \dfrac{a}{4} \log\left(x^4 + c^2\right) + \dfrac{b}{2} \int \dfrac{dt}{t^2+c^2}$ (in I_2, put $x^2 = t$)

$$= \frac{a}{4}\log\left(x^4+c^2\right)+\frac{b}{2}\frac{1}{c}\tan^{-1}\frac{x}{c}+k$$

8. $I=\int \frac{dx}{x^{2/3}(1+x^{2/3})}$,

let $t^3 = x \Rightarrow dx = 3t^2\,dt$

$$\Rightarrow \quad I=\int \frac{3t^2 dt}{t^2(1+t^2)}$$

$$= 3\int \frac{dt}{1+t^2} = 3\tan^{-1}(t)+C = 3\tan^{-1}(x^{1/3})+C$$

9. $\int e^{3\log x}(x^4+1)^{-1}dx$

$$= \int e^{\log x^3}\frac{dx}{x^4+1}$$

$$= \int \frac{x^3 dx}{x^4+1}$$

$$= \frac{1}{4}\int \frac{4x^3}{x^4+1}dx$$

$$= \frac{1}{4}\log(x^4+1)+C$$

10. $\int \frac{\sec x\,dx}{\sqrt{\cos 2x}} = \int \frac{\sec x}{\sqrt{\cos^2 x - \sin^2 x}}dx$

$$= \int \frac{\sec^2 x\,dx}{\sqrt{1-\tan^2 x}}$$

$$= \sin^{-1}(\tan x)+C$$

11. [Here power of $\sin x$ is odd positive integer, therefore, put $z = \cos x$.]

Let $z = \cos x$, then $dz = -\sin x\,dx$

Now $\int \sin^3 x\cos^2 x\,dx = \int \sin^2 x\cos^2 x\sin x\,dx$

$$= \int (1-\cos^2 x)\cos^2 x\sin x\,dx$$

$$= \int (1-z^2)z^2(-dz)$$

$$= -\int (z^2 - z^4)dz$$

$$= -\left(\frac{z^3}{3}-\frac{z^5}{5}\right)+C = -\frac{\cos^3 x}{3}+\frac{\cos^5 x}{5}+C$$

Exercise 7.5

1. $\int \frac{1}{2x^2+x-1}dx$

$$= \frac{1}{2}\int \frac{1}{x^2+\frac{x}{2}-\frac{1}{2}}dx$$

$$= \frac{1}{2}\int \frac{1}{(x+1/4)^2-(3/4)^2}dx$$

$$= \frac{1}{2}\times\frac{1}{2\,(3/4)}\log\left|\frac{x+1/4-3/4}{x+1/4+3/4}\right|+C$$

$$= \frac{1}{3}\log\left|\frac{x-1/2}{x+1}\right|+C = \frac{1}{3}\log\left|\frac{2x-1}{2\,(x+1)}\right|+C$$

2. $I = \int \frac{x}{x^4+x^2+1}dx = \int \frac{x}{\left(x^2\right)^2+x^2+1}dx$

Let $x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow dx = \frac{dt}{2x}$

$$\therefore \quad I=\int \frac{x}{t^2+t+1}\times\frac{dt}{2x}$$

$$= \frac{1}{2}\int \frac{1}{t^2+t+1}dt$$

$$= \frac{1}{2}\int \frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt$$

$$= \frac{1}{2}\times\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)+C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+C$$

3. $I=\int \frac{4x+1}{x^2+3x+2}dx$

$$= \int \frac{2(2x+3)-5}{x^2+3x+2}dx$$

$$= 2\int \frac{2x+3}{x^2+3x+2}dx - 5\int \frac{1}{x^2+3x+2}dx$$

$$= 2\log|x^2+3x+2| - 5\int \frac{1}{x^2+3x+(9/4)-(9/4)+2}dx$$

$$= 2\log|x^2+3x+2| - 5\int \frac{1}{(x+3/2)^2-(1/2)^2}dx$$

$$= 2\log|x^2+3x+2| - 5\times\frac{1}{2(1/2)}\log\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right| + C$$

$$= 2\log|x^2+3x+2| - 5\log\left|\frac{x+1}{x+2}\right| + C$$

4. $\int \frac{x^3+x+1}{x^2-1}dx$

$$= \int\left(x+\frac{2x+1}{x^2-1}\right)dx$$

$$= \int x\,dx + \int\frac{2x}{x^2-1}dx + \int\frac{1}{x^2-1}dx$$

$$= \frac{x^2}{2} + \log\left|x^2-1\right| + \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$$

5. $I = \int \frac{x^2-1}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)}dx$

$$= \int\frac{1-\frac{1}{x^2}}{\left(x^2+\frac{1}{x^2}+3\right)\tan^{-1}\left(x+\frac{1}{x}\right)}dx$$

Put $x+\frac{1}{x}=t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$ and $x^2+\frac{1}{x^2}+2=t^2$

$$\therefore \quad I = \int\frac{dt}{(t^2+1)\tan^{-1}t} = \ln|\tan^{-1}t| + C$$

$$= \ln\left|\tan^{-1}\left(x+\frac{1}{x}\right)\right| + C$$

6. $I = \int\frac{1}{x^4+1}dx$

$$= \int\frac{\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \frac{1}{2}\int\frac{\frac{2}{x^2}}{x^2+\frac{1}{x^2}}dx$$

$$= \frac{1}{2}\int\left(\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} - \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\right)dx$$

$$= \frac{1}{2}\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx - \frac{1}{2}\int\frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$$

$$= \frac{1}{2}\int\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx - \frac{1}{2}\int\frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}$$

Putting $x-\frac{1}{x}=u$ in 1st integral and $x+\frac{1}{x}=v$ in 2nd integral, we get

$$I = \frac{1}{2}\int\frac{du}{u^2+\left(\sqrt{2}\right)^2} - \frac{1}{2}\int\frac{dv}{v^2-\left(\sqrt{2}\right)^2}$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2}\times\frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x+1/x-\sqrt{2}}{x+1/x+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}x+1}{x^2+x\sqrt{2}+1}\right| + C$$

7. $I = \int\frac{1}{\sin^4 x+\cos^4 x}dx = \int\frac{1/\cos^4 x}{\frac{\sin^4 x+\cos^4 x}{\cos^4 x}}dx$

$$= \int\frac{\sec^4 x}{\tan^4 x+1}dx = \int\frac{\sec^2 x\sec^2 x}{\tan^4 x+1}dx$$

$$= \int\left(\frac{1+\tan^2 x}{1+\tan^4 x}\right)\sec^2 x\,dx$$

Putting $\tan x = t$ and $\sec^2 x\,dx = dt$, we get

$$I = \int\frac{1+t^2}{1+t^4}dt = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right) + C$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x-1}{\sqrt{2}\tan x}\right) + C$$

Exercise 7.6

1. $I = \int\frac{x^2}{\sqrt{1-x^6}}dx = \int\frac{x^2}{\sqrt{1^2-\left(x^3\right)^2}}dx$

Let $x^3 = t \Rightarrow 3x^2\,dx = dt \Rightarrow dx = \frac{dt}{3x^2}$

$$\Rightarrow I = \frac{1}{3}\int\frac{dt}{\sqrt{1-t^2}} = \frac{1}{3}\sin^{-1}(t) + C = \frac{1}{3}\sin^{-1}\left(x^3\right) + C$$

2. $I=\int\sqrt{\frac{x}{a^3-x^3}}\,dx=\int\frac{\sqrt{x}}{\sqrt{\left(a^{3/2}\right)^2-\left(x^{3/2}\right)^2}}\,dx$

Let $x^{3/2}=t\Rightarrow \frac{3}{2}x^{1/2}\,dx=dt\Rightarrow dx=\frac{2}{3\sqrt{x}}dt$

$\therefore\quad I=\int\frac{2/3\,dt}{\sqrt{\left(a^{3/2}\right)^2-t^2}}dt=\frac{2}{3}\int\frac{dt}{\sqrt{\left(a^{3/2}\right)^2-t^2}}$

$=\frac{2}{3}\sin^{-1}\left(\frac{t}{a^{3/2}}\right)+C=\frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right)+C$

3. $I=\int\frac{1}{\sqrt{1-e^{2x}}}dx=\int\frac{1}{\sqrt{1-\frac{1}{e^{-2x}}}}dx=\int\frac{e^{-x}}{\sqrt{e^{-2x}-1}}dx$

$=\int\frac{e^{-x}}{\sqrt{\left(e^{-x}\right)^2-1^2}}dx$

Let $e^{-x}=t\Rightarrow -e^{-x}\,dx=dt$

$\therefore\quad I=-\int\frac{dt}{\sqrt{t^2-1^2}}=-\log\left|t+\sqrt{t^2-1}\right|+C$

$=-\log\left|e^{-x}+\sqrt{e^{-2x}-1}\right|+C$

4. $I=\int\frac{2x+3}{\sqrt{x^2+4x+1}}dx=\int\frac{(2x+4)-1}{\sqrt{x^2+4x+1}}dx$

$=\int\frac{2x+4}{\sqrt{x^2+4x+1}}dx-\int\frac{1}{\sqrt{x^2+4x+1}}dx$

$=\int\frac{dt}{\sqrt{t}}-\int\frac{1}{\sqrt{(x+2)^2-\left(\sqrt{3}\right)^2}}dx$, where $t=x^2+4x+1$

$=2\sqrt{t}-\log\left|(x+2)+\sqrt{x^2+4x+1}\right|+C$

$=2\sqrt{x^2+4x+1}-\log\left|x+2+\sqrt{x^2+4x+1}\right|+C$

5. Put $x^{7/2}=t$

$\therefore\quad \frac{7}{2}x^{5/2}dx=dt$

$\therefore\quad I=\int\frac{2}{7}\frac{dt}{\sqrt{1+t^2}}=\frac{2}{7}\log\left(t+\sqrt{1+t^2}\right)+C$

$=\frac{2}{7}\log(x^{7/2}+\sqrt{1+x^7})+C$

6. $I=\int x^3 d(\tan^{-1}x)=\int\frac{x^3}{1+x^2}dx$

$=\int\left(x-\frac{x}{1+x^2}\right)dx=\frac{x^2}{2}-\frac{1}{2}\ln(1+x^2)+C$

Exercise 7.7

1. $\int x\sin^2 x\,dx$

$=\int x\left\{\frac{1-\cos 2x}{2}\right\}dx=\frac{1}{2}\int x\,dx-\frac{1}{2}\int \underset{\text{I}}{x}\,\underset{\text{II}}{\cos 2x}\,dx$

$=\frac{1}{2}\left(\frac{x^2}{2}\right)-\frac{1}{2}\left[x\left\{\int\cos 2x\,dx\right\}\right.$

$\left.-\int\left\{\frac{d}{dx}(x)\int\cos 2x\,dx\right\}dx\right]$

$=\frac{1}{4}x^2-\frac{1}{2}\left\{\frac{x}{2}\sin 2x-\int 1\times\frac{\sin 2x}{2}dx\right\}$

$=\frac{x^2}{4}-\frac{1}{2}\left\{\frac{x}{2}\sin 2x-\frac{1}{2}\int\sin 2x\,dx\right\}$

$=\frac{x^2}{4}-\frac{1}{2}\left\{\frac{x}{2}\sin 2x-\frac{1}{2}\left(-\frac{1}{2}\cos 2x\right)\right\}+C$

$=\frac{1}{4}x^2-\frac{x}{4}\sin 2x-\frac{1}{8}\cos 2x+C$

2. $\int f(x)dx=g(x)$

$I=\int f^{-1}(x).1dx$

$=f^{-1}(x)\int dx-\int\left\{\frac{d}{dx}f^{-1}(x)\int dx\right\}dx$

$=xf^{-1}(x)-\int x\frac{d}{dx}f^{-1}(x)\,dx$

$=xf^{-1}(x)-\int xd\,\{f^{-1}(x)\}$

Let $f^{-1}(x)=t\Rightarrow x=f(t)$ and $d\{f^{-1}(x)\}=dt$

$\Rightarrow\quad I=xf^{-1}(x)-\int f(t)\,dt=xf^{-1}(x)-g(t)$

$=xf^{-1}(x)-g\{f^{-1}(x)\}+C$

3. $\int g(x)\{f(x)+f'(x)\}dx=\int g(x)f(x)dx+\int g(x)f'(x)dx$

$=f(x)\left(\int g(x)dx\right)-\int\left(f'(x)\int g(x)dx\right)dx+\int g(x)f'(x)dx$

$=f(x)\,g(x)-\int g(x)f'(x)dx+\int g(x)f'(x)dx+C$

$=f(x)\,g(x)+C\qquad\left[\because\int g(x)dx=g(x)\right]$

4. Put $\sqrt{x}=t\Rightarrow\frac{1}{2\sqrt{x}}dx=dt\Rightarrow dx=2t\,dt,$

$\Rightarrow\quad\int\cos\sqrt{x}\,dx$

$= \int 2t \cos t \, dt$

$= 2\left[t \sin t - \int \sin t \, dt\right]$

$= 2t \sin t + 2 \cos t + C$

$= 2[\sqrt{x} \sin\sqrt{x} + \cos\sqrt{x}] + C$

5. Putting $\sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$, we get

$$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = \int t \sin t \, dt = -t \cos t + \sin t + C$$

$$= -\sin^{-1} x \cos(\sin^{-1} x) + \sin(\sin^{-1} x) + C$$

$$= x - \sin^{-1} x \sqrt{1-x^2} + C$$

6. $\int \tan^{-1} \sqrt{x} .1 \, dx$

$$= (\tan^{-1} \sqrt{x}) x - \int \frac{1}{1+x} \frac{1}{2\sqrt{x}} x \, dx$$

(integrating by parts)

$$= x \tan^{-1} \sqrt{x} - \frac{1}{2} \int \frac{\sqrt{x} \, dx}{1+x}$$

$$= x \tan^{-1} \sqrt{x} - \frac{1}{2}\left[2\left(\sqrt{x} - \tan^{-1}\sqrt{x}\right)\right] + C$$

$$= x \tan^{-1} \sqrt{x} - \sqrt{x} + \tan^{-1} \sqrt{x} + C$$

$$= (x+1) \tan^{-1} \sqrt{x} - \sqrt{x} + C$$

7. $\int \underset{\text{II}}{\cos} x \log\left(\underset{\text{I}}{\tan} \frac{x}{2}\right) dx$

$$= \log\left(\tan\frac{x}{2}\right) \sin x - \int \frac{1}{\tan\frac{x}{2}} \sec^2 \frac{x}{2} \times \frac{1}{2} \sin x \, dx + C$$

(integrating by parts)

$$= \sin x \log\left(\tan \frac{x}{2}\right) - \int \frac{1}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \sin x \, dx + C$$

$$= \sin x \log\left(\tan\frac{x}{2}\right) - \int dx + C$$

$$= \sin x \log\left(\tan \frac{x}{2}\right) - x + C$$

8. Put $\log x = t$, i.e., $x = e^t$ so that $dx = e^t dt$

$$\therefore \int \left(\frac{(t-1)}{1+t^2}\right)^2 e^t \, dt$$

$$= \int e^t \left[\frac{1}{1+t^2} - \frac{2t}{\left(1+t^2\right)^2}\right] dt$$

$$= e^t \frac{1}{1+t^2} + C$$

$$= \frac{x}{1+(\log x)^2} + C$$

9. $\int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}} dx$

$$= \int \frac{e^x(1-x^2+1)}{(1-x)\sqrt{1-x^2}} dx$$

$$= \int e^x \left[\frac{(1-x^2)}{(1-x)\sqrt{1-x^2}} + \frac{1}{(1-x)\sqrt{1-x^2}}\right] dx$$

$$= \int e^x \left[\frac{1+x}{\sqrt{1-x^2}} + \frac{1}{(1-x)\sqrt{1-x^2}}\right] dx$$

$$= \int e^x \left[\sqrt{\frac{1+x}{1-x}} + \frac{d}{dx}\left(\sqrt{\frac{1+x}{1-x}}\right)\right] dx$$

$$= e^x \sqrt{\frac{1+x}{1-x}} + C$$

10. $\int e^x \left(1 + \tan x + \tan^2 x\right) dx$

$$= \int e^x \left(\tan x + \sec^2 x\right) dx$$

$$= e^x \tan x + C$$

11. $I = \int \sin^2(\log x) dx$, let $t = \log x \Rightarrow dt = \frac{dx}{x}$

$\Rightarrow \quad dx = e^t \, dt$

$\Rightarrow \quad I = \int e^t \sin^2 t \, dt = \frac{1}{2} \int e^t (1 - \cos 2t) \, dt$

$\Rightarrow \quad 2I = e^t - \int e^t \cos 2t \, dt$

$$= e^t - \frac{e^t}{5}(2 \sin 2t + \cos 2t) + C$$

$\Rightarrow \quad I = \frac{1}{10} x(5 - 2\sin(2\log x) - \cos(2\log x)) + C$

12. $\int [f(x)g''(x) - f''g(x)] dx$

$$= \int f(x)g''(x)dx - \int f''(x)g(x)dx$$

$$= \left(f(x)g'(x) - \int f'(x)g'(x)dx\right)$$
$$-\left(g(x)f'(x) - \int g'(x)f'(x)dx\right)$$
$$= f(x)g'(x) - f'(x)g(x) + C$$

13. $I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$

Let $x = \cos 2\theta \Rightarrow dx = -2\sin 2\theta\, d\theta$

$$\Rightarrow \quad I = \int \tan^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}(-2\sin 2\theta)\,d\theta$$
$$= -2\int \tan^{-1}(\tan\theta)\sin 2\theta\, d\theta$$
$$= -2\int \theta \sin 2\theta\, d\theta$$
$$= -2\left[-\frac{\theta\cos 2\theta}{2} + \int \frac{\cos 2\theta}{2}d\theta\right]$$
$$= \theta\cos 2\theta - \frac{\sin 2\theta}{2} + C$$

Exercise 7.8

1. Let $I = \int \frac{1}{(x^2-4)\sqrt{x+1}}\,dx$

Putting $x + 1 = t^2$ and $dx = 2t\,dt$, we get

$$I = \int \frac{2t\,dt}{\left[(t^2-1)^2 - 4\right]\sqrt{t^2}}$$
$$= 2\,\frac{dt}{(t^2-1-2)(t^2-1+2)}$$
$$= 2\int \frac{dt}{(t^2-3)(t^2+1)}$$
$$= \frac{2}{4}\int\left(\frac{1}{t^2-3} - \frac{1}{t^2+1}\right)dt$$
$$= \frac{1}{4\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| - \frac{1}{2}\tan^{-1}t + C$$

where $t = \sqrt{x+1}$

2. $\int \frac{x^2+1}{x(x^2-1)}\,dx$

$$= \int \frac{x^2+1}{x(x-1)(x+1)}\,dx$$
$$= \int\left(\frac{-1}{x} + \frac{1}{x-1} + \frac{1}{x+1}\right)dx$$
$$= \log|x-1| + \log|x+1| - \log|x| + C$$
$$= \log\left|\frac{x^2-1}{x}\right| + C$$

3. $I = \int \frac{\sin x}{\sin 4x}\,dx = \int \frac{\sin x}{2\sin 2x\cos 2x}\,dx$

$$= \int \frac{\sin x}{4\sin x\cos x\,\cos 2x}\,dx$$
$$= \frac{1}{4}\int \frac{1}{\cos x\cos 2x}\,dx = \frac{1}{4}\int \frac{\cos x}{\cos^2 x\cos 2x}\,dx$$
$$= \frac{1}{4}\int \frac{\cos x}{(1-\sin^2 x)(1-2\sin^2 x)}\,dx$$

Putting $\sin x = t$ and $\cos x\,dx = dt$, we get

$$I = \frac{1}{4}\int \frac{dt}{(1-t^2)(1-2t^2)}$$
$$= \frac{1}{4}\int\left(\frac{2}{1-2t^2} - \frac{1}{1-t^2}\right)dt$$
$$\Rightarrow \quad I = -\frac{1}{4}\cdot\frac{1}{2}\log\left|\frac{1+t}{1-t}\right| + \frac{1}{2}\cdot\frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}\,t}{1-\sqrt{2}\,t}\right| + C$$
$$\Rightarrow \quad I = -\frac{1}{8}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + C$$

4. Here, the degree of numerator is greater than that of denominator. So, we divide the numerator by denominator to obtain

$$\int \frac{x^3}{(x-1)(x-2)}\,dx$$
$$= \int\left(x + 3 + \frac{7x-6}{(x-1)(x-2)}\right)dx$$
$$= \frac{x^2}{3} + 3x + \int\left(\frac{-1}{(x-1)} + \frac{8}{(x-2)}\right)dx$$
$$= \frac{x^2}{3} + 3x - \log|x-1| + 8\log|x-2| + C$$

5. $\int \frac{dx}{\sin x(3+\cos^2 x)}$

$$= \int \frac{\sin x\,dx}{\sin^2 x(3+\cos^2 x)}$$
$$= \int \frac{\sin x\,dx}{(1-\cos^2 x)(3+\cos^2 x)}$$

$$= \int \frac{dy}{(y^2-1)(y^2+3)} \quad \text{(Putting } \cos x = y\text{)}$$

$$= \frac{1}{4}\int\left[\frac{1}{y^2-1} - \frac{1}{y^2+3}\right]dy$$

$$= \frac{1}{4}\log\left|\frac{y-1}{y+1}\right| - \frac{1}{4\sqrt{3}}\tan^{-1}\frac{y}{\sqrt{3}} + C$$

6. $I = \int \frac{\cos 2x \sin 4x\, dx}{\cos^4 x(1+\cos^2 2x)}$

$$= \int \frac{2\cos^2 2x \sin 2x\, dx}{\left(\frac{1+\cos 2x}{2}\right)^2 (1+\cos^2 2x)}$$

Let $\cos 2x = t \Rightarrow dt = -2\sin 2x\, dx$

$$= -\int \frac{t^2 dt}{\left(\frac{1+t}{2}\right)^2 (1+t^2)}$$

$$= -4\int \frac{t^2 dt}{(1+t)^2(1+t^2)}$$

Now $\frac{t^2}{(1+t)^2(1+t^2)} = \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{Ct+D}{1+t^2}$

$\Rightarrow t^2 = A(1+t)(1+t^2) + B(1+t^2) + (Ct+D)(1+t)^2$

Put $t = -1 \Rightarrow B = 1/2$

Put $t = 0 \Rightarrow 0 = A + 1/2 + D$ (1)

Put $t = 1 \Rightarrow 1 = 4A + 1 + 4C + 4D \Rightarrow A + C + D = 0$ (2)

From equations (1) and (2), $C = -1/2$

Compare co-efficient of t^3, $A + C = 0$ (3)

$\Rightarrow A = 1/2$

From equations (2) and (3), $D = 0$

Hence, $I = \int\left(\frac{1/2}{1+t} + \frac{1/2}{(1+t)^2} - \frac{(1/2)t}{1+t^2}\right)dt$

$$= \frac{1}{2}\log|t| - \frac{1}{2(1+t)} - \frac{1}{4}\log(1+t^2) + C$$

$$= \frac{1}{2}\log|t| - \frac{1}{2(1+t)} - \frac{1}{4}\log(1+t^2) + C, \text{ where } t = \cos 2x$$

Exercise 7.9

1. Let $I = \int \frac{1}{(x+1)\sqrt{x^2-1}}dx$

Putting $x + 1 = \frac{1}{t}$ and $dx = -\frac{1}{t^2}dt$, we get

$$\therefore \quad I = \int \frac{1}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2 - 1}}\left(-\frac{1}{t^2}\right)dt$$

$$= -\int \frac{dt}{\sqrt{1-2t}} = -\int (1-2t)^{-1/2}dt$$

$$= -\frac{(1-2t)^{1/2}}{(-2)\left(\frac{1}{2}\right)} + C = \sqrt{1-2t} + C$$

$$= \sqrt{1-\frac{2}{x+1}} + C = \sqrt{\frac{x-1}{x+1}} + C$$

2. $I = \int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}}dx$

$$= \int \frac{x^2(1-1/x^2)}{x^2(x+1/x)\sqrt{x^2+1/x^2}}dx$$

$$= \int \frac{(1-1/x^2)dx}{(x+1/x)\sqrt{(x+1/x)^2-2}}$$

Putting $x + 1/x = t$, we have

$$I = \int \frac{dt}{t\sqrt{t^2-2}}$$

Again putting $t^2 - 2 = y^2$, $2t\, dt = 2y\, dy$,

$$I = \int \frac{y dy}{(y^2+2)y} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{y}{\sqrt{2}} + C$$

$$= \frac{1}{2}\tan^{-1}\frac{\sqrt{x^2+1/x^2}}{\sqrt{2}} + C$$

3. Let $I = \int \sec^3 x\, dx$

$$= \int \sec x \sec^2 x\, dx$$

$$= \int \sqrt{1+\tan^2 x}\sec^2 x\, dx$$

Put $\tan x = z$, $\therefore \sec^2 x\, dx = dz$

$$\Rightarrow \quad I = \int \sqrt{1+z^2}\,dz$$

$$= \frac{z\sqrt{z^2+1}}{2} + \frac{1}{2}\log|z + \sqrt{z^2+1}| + C$$

$$= \frac{\tan x \sec x}{2} + \frac{1}{2}\log(\tan x + \sec x) + C$$

$$= \frac{1}{2}[\sec x \tan x + \log(\sec x + \tan x)] + C$$

4. $I = \int \frac{x+1}{(x-1)\sqrt{x+2}} dx$

Let $x + 2 = t^2$

$\Rightarrow \quad dx = 2t\, dt$

$$\Rightarrow \quad I = \int \frac{t^2-1}{(t^2-3)t} 2t\, dt$$

$$= 2\int \frac{t^2-3+2}{(t^2-3)} dt$$

$$= 2\int \left(1 + \frac{2}{(t^2-3)}\right) dt$$

$$= 2t + \frac{2}{\sqrt{3}} \log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + C$$

$$= 2\sqrt{x+2} + \frac{2}{\sqrt{3}} \log\left|\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right| + C$$

5. $I = \int \frac{x}{(x^2+4)\sqrt{x^2+1}} dx$

Let $x^2 + 1 = t^2 \Rightarrow x\, dx = t\, dt$

$$\Rightarrow \quad I = \int \frac{t\, dt}{(t^2+3)t}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} + C = \frac{1}{\sqrt{3}} \tan^{-1} \sqrt{\frac{x^2+1}{3}} + C$$

6. $I = \int \frac{1}{(x+1)\sqrt{x^2+x+1}} dx$

Let $x + 1 = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2} dt$

$$\Rightarrow \quad I = \int \frac{1}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2 + \left(\frac{1}{t}-1\right) + 1}} \left(-\frac{1}{t^2}\right) dt$$

$$= -\int \frac{dt}{\sqrt{(1-t)^2 + (t-t^2) + t^2}}$$

$$= -\int \frac{dt}{\sqrt{t^2-t+1}} = -\int \frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2 + \frac{3}{4}}}$$

$$= -\log\left|\left(t-\frac{1}{2}\right) + \sqrt{t^2-t+1}\right| + C$$

$$= -\log\left|\frac{1}{x+1} - \frac{1}{2} + \frac{\sqrt{x^2+x+1}}{x+1}\right| + C$$

7. $I = \int \frac{x^3+1}{\sqrt{x^2+x}} dx$

$$= \int \frac{x^3+x+1-x}{\sqrt{x^2+x}} dx = \int x\sqrt{x^2+x}\, dx - \int \frac{x-1}{\sqrt{x^2+x}} dx$$

$$= \frac{1}{2}\left[\int (2x+1)\sqrt{x^2+x}\, dx - \int \sqrt{x^2+x}\, dx\right]$$

$$-\frac{1}{2}\int \frac{2x+1-3}{\sqrt{x^2+x}} dx$$

$$= \frac{1}{2}\left[\int (2x+1)\sqrt{x^2+x}\, dx - \int \sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}}\, dx\right]$$

$$-\frac{1}{2}\left[\int \frac{2x+1}{\sqrt{x^2+x}} dx - 3\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}}} dx\right]$$

$$= \frac{1}{2}\left[\frac{2(x^2+x)^{3/2}}{3} - \frac{x+\frac{1}{2}}{2}\sqrt{x^2+x} + \frac{1}{4}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{x^2+x}\right|\right]$$

$$-\frac{1}{2}\left[2\sqrt{x^2+x} - 3\log\left|\left(x+\frac{1}{2}\right) + \sqrt{x^2+x}\right|\right] + C$$

Exercise 7.10

1. $I = \int \frac{dx}{x^2(1+x^5)^{4/5}} = \int \frac{dx}{x^6\left(\frac{1}{x^5}+1\right)^{4/5}}$

Let $t = 1 + \frac{1}{x^5} \Rightarrow dt = -\frac{5dx}{x^6}$

$$\Rightarrow \quad I = -\frac{1}{5}\int \frac{dt}{t^{4/5}} = -t^{1/5} + C = -\left(1+\frac{1}{x^5}\right)^{1/5} + C$$

$$= -\frac{(1+x^5)^{1/5}}{x} + C$$

2. $I = \int \frac{1+x^4}{(1-x^4)^{3/2}} dx = \int \frac{x^3(x+1/x^3)\, dx}{(1-x^4)^{3/2}}$

$$= \int \frac{(x+1/x^3)\, dx}{\left(\frac{1}{x^2} - x^2\right)^{3/2}}$$

Let $\frac{1}{x^2} - x^2 = t \Rightarrow \left(\frac{-2}{x^3} - 2x\right) dx = dt$

$\Rightarrow \left(x + \frac{1}{x^3}\right) dx = -\frac{1}{2} dt$

$\Rightarrow I = -\frac{1}{2}\int \frac{dt}{t^{3/2}} = \frac{1}{\sqrt{t}} + C = \frac{1}{\sqrt{\frac{1}{x^2} - x^2}} + C$

3. $\int \frac{1}{x^2(x^4+1)^{3/4}} dx = \int \frac{1}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}} dx$

$= -\frac{1}{4}\int \frac{1}{t^{3/4}} dt = -\frac{1}{4}\frac{t^{1/4}}{1/4} + C = -t^{1/4} + C$, where

$t = 1 + \frac{1}{x^4}$

$= -\left(1+\frac{1}{x^4}\right)^{1/4} + C$

4. $\int \frac{(x^4-x)^{1/4}}{x^5} dx = \int \frac{1}{x^4}\left(1-\frac{1}{x^3}\right)^{1/4} dx,$

Putting $1 - \frac{1}{x^3} = t$, we get

$I = \frac{1}{3}\int t^{1/4} dt = \frac{4}{15} t^{5/4} + C = \frac{4}{15}\left(1-\frac{1}{x^3}\right)^{5/4} + C$

5. $I = \int \frac{(x-1)dx}{(x+1)\sqrt{x^3+x^2+x}}$

$= \int \frac{(x^2-1)}{(x^2+2x+1)\sqrt{x^3+x^2+x}}$

$= \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}$

Putting $x + \frac{1}{x} + 1 = u^2$, $I = \int \frac{2u\,du}{(u^2+1)u} = 2\tan^{-1} u + C$

$= 2\tan^{-1}\sqrt{\left(x+\frac{1}{x}+1\right)} + C$

6. $I = \int x^x (\ln ex)\, dx = \int x^x (1 + \ln x)\, dx$

let $t = x^x = e^{x\ln x} \Rightarrow \frac{dt}{dx} = x^x\left(x \times \frac{1}{x} + \ln x\right)$

$\Rightarrow dt = x^x(1+\ln x)\,dx \Rightarrow I = \int dt = t + C = x^x + C$

7. $\sqrt{\frac{x-q}{x-p}} = t$

$\therefore \frac{1}{2}\left(\frac{x-p}{x-q}\right)^{1/2} \frac{(x-p)1-(x-q)1}{(x-p)^2} dx = dt.$

$\Rightarrow \frac{1}{2} \frac{q-p}{\sqrt{x-q}\,(x-p)^{3/2}} dx = dt$

$\Rightarrow \frac{dx}{(x-p)^{3/2}\sqrt{(x-q)}} = \frac{2\,dt}{q-p}$

$\Rightarrow I = -\int \frac{2\,dt}{p-q}\quad t = -\frac{2}{p-q}\sqrt{\frac{x-q}{x-p}} + C$

8. Let $(\sqrt{1+x^2} + x)^n = z$

$\Rightarrow n(\sqrt{1+x^2} + x)^{n-1}\left(\frac{x}{\sqrt{1+x^2}} + 1\right)dx = dz$

$\Rightarrow \frac{(\sqrt{1+x^2}+x)^n}{\sqrt{1+x^2}} dx = \frac{dz}{n}$

$\therefore$ given integral $= \int \frac{dz}{n} = \frac{1}{n} z + C$

$= \frac{1}{n}(\sqrt{1+x^2} + x)^n + C$

9. $\int \frac{dx}{\cos^3 x\sqrt{\sin 2x}}$

$= \frac{1}{\sqrt{2}}\int \frac{dx}{\cos^{7/2} x \sin^{1/2} x}$

$= \frac{1}{\sqrt{2}}\int \frac{dx}{\cos^4 x \tan^{1/2} x}$

$= \frac{1}{\sqrt{2}}\int \frac{(1+\tan^2 x)\sec^2 x\,dx}{\tan^{1/2} x}$

$= \frac{1}{\sqrt{2}}\int \frac{(1+t^2)dt}{t^{1/2}}$

$= \frac{1}{\sqrt{2}}\int \left(t^{-1/2} + t^{3/2}\right)dt$

$= \frac{1}{\sqrt{2}}\left[\frac{t^{1/2}}{1/2} + \frac{t^{5/2}}{5/2}\right] + C$, where $t = \tan x$

10. $\int \sec^5 x \operatorname{cosec}^3 x\, dx$

$$= \int \frac{dx}{\cos^5 x \sin^3 x}$$

$$= \int \frac{dx}{\cos^8 x \ \tan^3 x}$$

$$= \int \frac{\sec^6 x \sec^2 x\, dx}{\tan^3 x}$$

$$= \int \frac{(1+\tan^2 x)^3 \sec^2 x\, dx}{\tan^3 x}$$

$$= \int \frac{(1+t^2)^3\, dt}{t^3}$$

$$= \int \left(t + \frac{1}{t}\right)^3 dt$$

$$= \int \left(t^3 + \frac{1}{t^3} + 3t + \frac{3}{t}\right) dt$$

$$= \frac{t^4}{4} + \frac{t^{-2}}{-2} + 3\frac{t^2}{2} + 3\log t + C, \text{ where } t = \tan^{1/2} x.$$

Chapter 8

Exercise 8.1

1. a. Here $f(x) = \cos x$

$$\int_a^b \cos x\, dx$$

$$= \lim_{n\to\infty} h\left[\cos a + \cos(a+h) + \cdots + \cos\{a+(n-1)h\}\right],$$

where $nh = b - a$

$$= \lim_{n\to\infty} h \frac{\cos\left\{a + \frac{1}{2}(n-1)h\right\} \sin\left(\frac{1}{2}nh\right)}{\sin\left(\frac{1}{2}h\right)}$$

$$= \lim_{n\to\infty} \left[2\cos\left\{a + \left(\frac{1}{2}nh - \frac{1}{2}h\right)\right\} \sin\left(\frac{1}{2}nh\right)\left(\frac{\frac{1}{2}h}{\sin\left(\frac{1}{2}h\right)}\right)\right]$$

$$= 2\cos\left\{a + \frac{1}{2}(b-a) - 0\right\} . \sin\left(\frac{1}{2}(b-a)\right) \times 1$$

$$= 2\cos\frac{1}{2}(a+b) \sin\frac{1}{2}(b-a)$$

$$= \sin b - \sin a$$

b. Here $f(x) = x^3$

$$\therefore \int_a^b x^3\, dx = \lim_{n\to\infty} h \sum_{r=0}^{n-1} (a+rh)^3$$

where $nh = b - a$

$$= \lim_{n\to\infty}\left[nha^3 + 3a^2h^2 \sum_{r=1}^{n-1} r + 3ah^3 \sum_{r=1}^{n-1} r^2 + h^4 \sum_{r=1}^{n-1} r^3\right]$$

$$= \lim_{n\to\infty}\left[nha^3 + 3a^2h^2\left\{\frac{(n-1)n}{2}\right\} + 3ah^3\left\{\frac{n(n-1)(2n-1)}{6}\right\} + h^4\left\{\frac{(n-1)^2 n^2}{4}\right\}\right]$$

$$= \lim_{n\to\infty}\left[(nh)a^3 + 3a^2\left\{\frac{(nh-h)(nh)}{2}\right\} + 3a\left\{\frac{(nh)(nh-h)(2nh-h)}{6}\right\} + \left\{\frac{(nh-h)^2(nh)^2}{4}\right\}\right]$$

$$= \left[(b-a)a^3 + 3a^2\left\{\frac{(b-a-0)(b-a)}{2}\right\} + 3a\left\{\frac{(b-a)(b-a-0)(2(b-a)-0)}{6}\right\} + \left\{\frac{(b-a-0)^2(b-a)^2}{4}\right\}\right]$$

$[\because$ as $n \to \infty, h \to 0, nh \to b-a]$

$$= \frac{1}{4}(b-a)\left[4a^3 + 6a^2(b-a) + 4a(b-a)^2 + (b-a)^3\right]$$

$$= \frac{1}{4}(b-a)\left(a^3 + a^2b + ab^2 + b^3\right)$$

$$= \frac{1}{4}(b-a)(b+a)\left(b^2+a^2\right) = \frac{1}{4}\left(b^4 - a^4\right)$$

2. a. Given limit

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{n}{\sqrt{4n^2-1}} + \frac{n}{\sqrt{4n^2-2^2}} + \cdots + \frac{n}{\sqrt{4n^2-n^2}}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{\sqrt{4-\left(\frac{1}{n}\right)^2}} + \frac{1}{\sqrt{4-\left(\frac{2}{n}\right)^2}} + \cdots \frac{1}{\sqrt{4-\left(\frac{n}{n}\right)^2}}\right]$$

$$= \int_0^1 \frac{dx}{\sqrt{4-x^2}}$$

$$= \left|\sin^{-1}\frac{x}{2}\right|_0^1$$

$$= \sin^{-1}\frac{1}{2} - 0 = \frac{\pi}{6}$$

b. $$\lim_{n\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1}{n^2} + \frac{2}{n^2}\sec^2\frac{4}{n^2} + \cdots + \frac{1}{n}\sec^2 1\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{r}{n}\sec^2\left(\frac{r}{n}\right)^2$$

$$= \int_0^1 x\sec^2 x^2\, dx$$

Put $x^2 = t$ so that $2x\,dx = dt$

When $x = 0, t = 0$. When $x = 1, t = 1$

$$\therefore \text{ the required limit} = \frac{1}{2}\int_0^1 \sec^2 t\, dt$$

$$= \frac{1}{2}[\tan t]_0^1 = \frac{1}{2}[\tan 1 - 0]$$

$$= \frac{1}{2}\tan 1$$

c. $$\lim_{n\to\infty}\sum_{K=1}^{n}\frac{K}{n^2+K^2}$$

$$= \lim_{n\to\infty}\sum_{K=1}^{n}\frac{1}{n^2}\times\frac{K}{1+\left(\frac{K}{n}\right)^2}$$

$$= \lim_{n\to\infty}\sum_{K=1}^{n}\frac{1}{n}\times\frac{K/n}{1+\left(\frac{K}{n}\right)^2}$$

$$= \int_0^1\frac{x}{1+x^2}dx$$

$$= \frac{1}{2}\log(1+x^2)\Big|_0^1$$

$$= \frac{1}{2}\log 2$$

d. $$\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\sqrt{r}\sum_{r=1}^{n}\frac{1}{\sqrt{r}}}{\sum_{r=1}^{n} r}$$

$$\Rightarrow \text{Limit} = \lim_{n\to\infty}\frac{\left(\frac{1}{n}\sum_{r=1}^{n}\sqrt{\frac{r}{n}}\right)\left(\frac{1}{n}\sum_{r=1}^{n}\sqrt{\frac{n}{r}}\right)}{\frac{1}{n}\sum_{r=1}^{n}\frac{r}{n}}$$

($1/n$ is properly adjusted and a function of $\frac{r}{n}$ is created at all three places)

$$= \frac{\int_0^1\sqrt{x}\,dx\int_0^1\frac{dx}{\sqrt{x}}}{\int_0^1 x\,dx} = \frac{8}{3}$$

e. Let $A = \lim_{n\to\infty}\left[\frac{n!}{n^n}\right]^{1/n}$

$$= \lim_{n\to\infty}\left(\frac{1\times2\times3\ldots n}{n\times n\times n\ldots n}\right)^{1/n}$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\times\frac{2}{n}\times\frac{3}{n}\cdots\frac{n}{n}\right)^{1/n}$$

$$\therefore \log A = \lim_{n\to\infty}\frac{1}{n}\left[\log\frac{1}{n} + \log\frac{2}{n} + \cdots + \log\frac{n}{n}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\frac{r}{n}$$

$$= \int_0^1 \log x\,dx = [x\log x]_0^1 - \int_0^1 x\times 1/x\,dx$$

$$= 0 - \int_0^1 dx = 0 - [x]_0^1 = -1$$

$$\therefore A = e^{-1} = 1/e$$

Exercise 8.2

1. Here, the mistake lies in the substitution $\tan\frac{1}{2}x = t$, because $\tan\frac{1}{2}x$ is discontinuous at $x = \pi$ which is a point in the interval $[0, 2\pi]$.

2. $$\int_0^\pi\frac{dx}{1+\sin x} = \int_0^\pi\frac{1-\sin x}{1-\sin^2 x}dx = \int_0^\pi\frac{1-\sin x}{\cos^2 x}dx$$

$$= \int_0^\pi(\sec^2 x - \sec x\tan x)\,dx$$

$$= |\tan x - \sec x|_0^\pi$$

$$= (\tan\pi - \sec\pi) - (\tan 0 - \sec 0)$$

$$= 0 - (-1) - (0 - 1) = 1 + 1 = 2$$

3. $$I = \int_1^\infty\frac{dx}{(ee^x + e^3e^{-x})}$$

$$= \int_1^\infty\frac{e^x\,dx}{e(e^{2x}+e^2)}$$ (multiply N^r and D^r by e^x)

put $e^x = t \Rightarrow e^x\,dx = dt$

$$\Rightarrow I = \frac{1}{e}\int_e^\infty\frac{dt}{t^2+e^2}$$

$$= \frac{1}{e^2}\tan^{-1}\frac{t}{e}\Big|_e^\infty$$

$$= \frac{1}{e^2}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{\pi}{4e^2}$$

4. Put $x=\sin\theta$, $\therefore dx=\cos\theta\, d\theta$

When $x=0$, $\theta=0$, when $x=\frac{1}{\sqrt{2}}$, $\theta=\frac{\pi}{4}$

$\therefore$ the given integral

$$=\int_0^{\pi/4}\frac{\sin^{-1}(\sin\theta)\cos\theta\, d\theta}{(1-\sin^2\theta)^{3/2}}$$

$$=\int_0^{\pi/4}\frac{\theta\cos\theta}{\cos^3\theta}d\theta=\int_0^{\pi/4}\underset{\text{I}}{\theta}\,.\underset{\text{II}}{\sec^2\theta}\, d\theta$$

$$=|\theta\tan\theta|_0^{\pi/4}-\int_0^{\pi/4}1.\tan\theta\, d\theta$$

$$=\frac{\pi}{4}\tan\frac{\pi}{4}+\log\cos\theta|_0^{\pi/4}$$

$$=\frac{\pi}{4}+\log\cos\frac{\pi}{4}-\log\cos 0=\frac{\pi}{4}+\log\frac{1}{\sqrt{2}}$$

$$=\frac{\pi}{4}+\log 1-\log(2)^{1/2}=\frac{\pi}{4}-\frac{1}{2}\log 2$$

5. Put $x=\sin\theta$, $\therefore dx=\cos\theta\, d\theta$

$\therefore$ the given integral

$$=\int_0^{\pi/2}\frac{(2-\sin^2\theta)\cos\theta\, d\theta}{(1+\sin\theta)\cos\theta}$$

$$=\int_0^{\pi/2}\left(1-\sin\theta+\frac{1}{1+\sin\theta}\right)d\theta$$

$$=|\theta+\cos\theta|_0^{\pi/2}+\int_0^{\pi/2}\frac{d\theta}{1+\sin\theta}$$

$$=\frac{\pi}{2}-1+\int_0^{\pi/2}\frac{1-\sin\theta}{\cos^2\theta}d\theta$$

$$=\frac{\pi}{2}-1+\int_0^{\pi/2}(\sec^2\theta-\sec\theta\tan\theta)d\theta$$

$$=\frac{\pi}{2}-1+|\tan\theta-\sec\theta|_0^{\pi/2}$$

$$=\frac{\pi}{2}-1+\lim_{\theta\to\pi/2}\frac{\sin\theta-1}{\cos\theta}-\frac{\sin 0-1}{\cos 0}$$

$$=\frac{\pi}{2}-1+\lim_{\theta\to\pi/2}\frac{\cos\theta}{-\sin\theta}+1$$

$$=\frac{\pi}{2}$$

6. $$I=\int_0^{\pi/2}\frac{dx}{a^2\cos^2x+b^2\sin^2x}=\int_0^{\pi/2}\frac{\sec^2x\, dx}{a^2+b^2\tan^2x}$$

$$=\frac{1}{b^2}\int_0^{\pi/2}\frac{\sec^2x\, dx}{\left(\frac{a}{b}\right)^2+\tan^2x}$$

Put $\tan x=z$, $\therefore \sec^2x\, dx=dz$

when $x=0, z=0, x\to\frac{\pi}{2}, z\to\infty$

$$\Rightarrow I=\frac{1}{b^2}\int_0^{\infty}\frac{dz}{\left(\frac{a}{b}\right)^2+z^2}=\frac{1}{b^2}\frac{1}{\frac{a}{b}}\left|\tan^{-1}\frac{z}{a/b}\right|_0^{\infty}$$

$$=\frac{1}{ab}[\tan^{-1}\infty-\tan^{-1}0]=\frac{1}{ab}\frac{\pi}{2}=\frac{\pi}{2ab}$$

Exercise 8.3

1. $$I=\int_a^b xf(x)dx=\int_a^b x\, f(a+b-x)\, dx$$

$$=\int_a^b(a+b-x)\, f((a+b)-(a+b-x))\, dx$$

$$=\int_a^b(a+b-x)\, f(x)\, dx=(a+b)\int_a^b f(x)\, dx-I$$

$$\Rightarrow 2I=(a+b)\int_a^b f(x)\, dx\Rightarrow I=\frac{a+b}{2}\int_a^b f(x)\, dx$$

2. $$I=\int_3^6\frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}}dx \quad (1)$$

$$\therefore I=\int_3^6\frac{\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}}dx \quad (2)$$

Adding equations (1) and (2), $2I=\int_3^6 1.dx=[x]_3^6=6-3=3$

Hence $I=\frac{3}{2}$.

3. Let $$I=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \quad (1)$$

$$=\int\frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx$$

$$=\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \quad (2)$$

Adding equations (1) and (2), we get

$$2I=\int_0^{\pi/2}1.dx=[x]_0^{\pi/2}=\frac{\pi}{2}-0=\frac{\pi}{2}$$

Hence $I=\frac{\pi}{4}$.

4. Using $\int_0^a f(x)\, dx=\int_0^a f(a-x)\, dx$

$I=-I\Rightarrow=0$

5. $I=\int_0^1(1-x)x^n\, dx$ (replacing x by $1-x$)

$$=\int_0^1(x^n-x^{n+1})\, dx$$

$$\left(\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right)_0^1$$

$$= \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)}$$

6. $f(x) f(a-x) = 1 \Rightarrow f(a-x) = \frac{1}{f(x)}$

Now, $I = \int_0^a \frac{dx}{1+f(x)}$

$$= \int_0^a \frac{dx}{1+f(a-x)}$$

$$= \int_0^a \frac{dx}{1+\frac{1}{f(x)}}$$

$$= \int_0^a \frac{f(x)dx}{1+f(x)}$$

$$\Rightarrow 2I = \int_0^a \frac{1+f(x)}{1+f(x)} dx = a \Rightarrow I = a/2$$

7. $I = \int_0^{\pi/2} \sin 2x \log \tan x \, dx$

$$= \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \log \tan\left(\frac{\pi}{2} - x\right) dx$$

$$= -\int_0^{\pi/2} \sin 2x \log \tan x \, dx = -I$$

$\Rightarrow 2I = 0$

$\Rightarrow I = 0$

8. Let $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx, \ a > 0$ (1)

$$\therefore \ I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^{-x}} dx$$

$$\Rightarrow I = \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{1+a^x} dx \quad (2)$$

Adding equations (1) and (2), we get $2I = 4\int_0^{\pi/2} \cos^2 x \, dx$

$$= 4\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \pi$$

$$\Rightarrow I = \frac{\pi}{2}$$

9. Let $I = \int_0^{\pi} \frac{x \sin x \, dx}{1+\cos^2 x}$ (1)

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x) \sin x \, dx}{1+\cos^2 x} \quad (2)$$

Adding (1) and (2), we get $2I = \pi \int_0^{\pi} \frac{\sin x \, dx}{1+\cos^2 x}$

or $I = -\frac{\pi}{2} \int_1^{-1} \frac{dt}{1+t^2} = -\frac{\pi}{2} \left[\tan^{-1} t\right]_1^{-1}$

[Putting $\cos x = t, -\sin x \, dx = dt$]

$$= -\frac{1}{2} \pi \left[\tan^{-1}(-1) - \tan^{-1} 1\right] = \pi^2/4$$

10. $I_1 = \int_0^{\pi} (\pi - x) f(\sin^3 x + \cos^2 x) \, dx$

Adding $2I = \pi \int_0^{\pi} f(\sin^3 x + \cos^2 x) \, dx$

$$= 2\pi \int_0^{\pi/2} f(\sin^3 x + \cos^2 x) \, dx$$

$$\Rightarrow I_1 = \pi \int_0^{\pi/2} f(\sin^3 x + \cos^2 x) \, dx = \pi I_2$$

11. $I = \int_0^{\pi} \log(1+\cos x) dx = \int_0^{\pi} \log\left(2\cos^2 \frac{x}{2}\right) dx$

$$= \int_0^{\pi} \left(\log 2 + 2\log\cos\frac{x}{2}\right) dx = \pi \log 2 + 2\int_0^{\pi} \log\cos\frac{x}{2} dx$$

$= \pi \log 2 + 2 \times 2 \int_0^{\pi/2} \log \cos t \, dt$, where $t = \frac{x}{2}$ and $dx = 2\,dt$

$$= \pi \log 2 + 4 \times \left(-\frac{\pi}{2} \log 2\right) = -\pi \log 2$$

12. $\int_0^1 \{(\sin^{-1} x)/x\} dx$

$$= \left[(\sin^{-1} x)(\log x)\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} \log x \, dx$$

$$= 0 - \lim_{x \to 0} \frac{\sin^{-1} x}{x} (x \log x)$$

$$- \int_0^{\pi/2} \frac{1}{\sqrt{1-\sin^2\theta}} \log \sin\theta \cos\theta d\theta$$

$$= -\lim_{x\to 0} x \log x - \int_0^{\pi/2} \log \sin\theta d\theta = \frac{\pi}{2} \log 2$$

Exercise 8.4

1. Let $I = \int_{-\pi/2}^{\pi/2} \sin^2 x \cos^2 x (\sin x + \cos x) dx$

$$= \int_{-\pi/2}^{\pi/2} \sin^3 x \cos^2 x \, dx - \int_{-\pi/2}^{\pi/2} \sin^2 x \cos^3 x \, dx \quad (1)$$

Since $\sin^3 x \cos^2 x$ is an odd function and $\sin^2 x \cos^3 x$ is an even function, therefore $\int_{-\pi/2}^{\pi/2} \sin^3 x \cos^2 x \, dx = 0$

and, $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^3 x \, dx = 2\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx.$

Therefore, $I = 2\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx$

$$= 2\int_0^1 t^2(1-t^2) dt$$

$$= 2\int_0^1 (t^2 - t^4) dt = 2\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{4}{15}$$

2. $I = \int_{-1}^{1} \frac{x^3 + |x| + 1}{x^2 + 2|x| + 1} dx$

$$= \int_{-1}^{1} \frac{x^3}{x^2+2|x|+1} dx + \int_{-1}^{1} \frac{|x|+1}{x^2+2|x|+1} dx$$

$$= 0 + 2\int_0^1 \frac{(|x|+1)}{(|x|+1)^2} dx = 2\int_0^1 \frac{dx}{1+x}$$

$$= 2\ln(1+x)\Big|_0^1 = 2\ln 2$$

3. Value = 0 $\because (1-x^2)\sin x\cos^2 x$ is an odd function of x.

4. $\int_{-1}^{1} \frac{\sin x - x^2}{3-|x|} dx$

$$= \int_{-1}^{1} \frac{\sin x}{3-|x|} dx - \int_{-1}^{1} \frac{x^2}{3-|x|}$$

$$= 0 - 2\int_0^1 \frac{x^2}{3-|x|} dx$$

$$\left[\because \frac{\sin x}{3-|x|} \text{ is odd and } \frac{x^2}{3-|x|} \text{ is even}\right]$$

$$= -2\int_0^1 \frac{x^2\, dx}{3-|x|} = 2\int_0^1 \frac{x^2}{x-3} dx = \int_0^1 \left(x+3+\frac{9}{x-3}\right) dx$$

$$= \left[x^2+3x+9\log|x-3|\right]_0^1$$

$$= \left[4 + 9\log\frac{2}{3}\right]$$

5. $I = \int_{-\pi/2}^{\pi/2} \sqrt{\cos^{2n-1} x - \cos^{2n+1} x}\, dx$

$$= 2\int_0^{\pi/2} \cos^{(2n-1)/2} x \sin x\, dx$$

$$= 2\frac{u^{(2n+1)/2}}{(2n+1)/2}\Bigg|_0^1 = \frac{4}{2n+1}$$

6. Since $\cos x \log \frac{1-x}{1+x}$ is an odd function of x

$$\therefore \int_{-1/2}^{1/2} \cos x \log \frac{1-x}{1+x} dx = 0.$$

7. Let $I = \int_{-3\pi/2}^{-\pi/2} [(x+\pi)^3 + \cos^2(x+3\pi)]\, dx$

Put $x + \pi = t$, so that $dx = dt$

When $x = -\frac{3\pi}{2}$, then $t = -\frac{\pi}{2}$

When $x = -\frac{\pi}{2}$, then $t = \frac{\pi}{2}$

$$\therefore I = \int_{-\pi/2}^{\pi/2} \left[t^3 + \cos^2(t+2\pi)\right] dt$$

$$= \int_{-\pi/2}^{\pi/2} \left[t^3 + \cos^2 t\right] dt$$

$$= 0 + 2\int_0^{\pi/2} \cos^2 t\, dt = \int_0^{\pi/2} (1+\cos 2t)\, dt$$

$$= \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

Exercise 8.5

1. We have $f(x) = \sqrt{(1-\cos 2x)} = \sqrt{2\sin^2 x} = \sqrt{2}|\sin x|$

Now $f(x+\pi) = \sqrt{2}|\sin(x+\pi)| = \sqrt{2}|\sin x| = f(x)$

i.e., $f(x)$ is periodic function with period π

$$\int_0^{100\pi} \sqrt{(1-\cos 2x)}\, dx$$

$$= \int_0^{100\pi} \sqrt{2}|\sin x|\, dx$$

$$= 100\sqrt{2}\int_0^{\pi} |\sin x|\, dx$$

$$= 100\sqrt{2}\int_0^{\pi} \sin x\, dx$$

$$= 100\sqrt{2}\left[-\cos x\right]_0^{\pi} = 200\sqrt{2}$$

2. Since $\cos^2 x$ is a periodic function with period π. Therefore, so is $f(\cos^2 x)$.

Hence, $\int_0^{n\pi} f(\cos^2 x)dx = n\int_0^{\pi} f(\cos^2 x)dx \Rightarrow k = n.$

3. Let, $I = \int_0^{n\pi+t} (|\cos x| + |\sin x|)dx$

$$= \int_0^{n\pi} (|\cos x| + |\sin x|)dx + \int_{n\pi}^{n\pi+t} (|\cos x| + |\sin x|)dx$$

$$= 2n\int_0^{\pi/2} (|\cos x| + |\sin x|)dx + \int_0^{t} (|\cos x| + |\sin x|)dx$$

$$= 2n\int_0^{\pi/2} (\cos x + \sin x)dx + \int_0^{t} (\cos x + \sin x)dx$$

$$= 4n + \sin t - \cos t + 1$$

4. $\int_0^{10} e^{2x-[2x]}\, d(x-[x])$

$$= \int_0^{10} e^{\{2x\}}\, dx$$

$$= 20\int_0^{1/2} e^{\{2x\}}\, dx \ (\{2x\} \text{ has period } 1/2)$$

$$= 20\int_0^{1/2} e^{2x}\, dx, [\text{for } x \in (0, 1/2), \{2x\} = 2x]$$

$$= 10\left(e^{2x}\right)_0^{1/2}$$

$$= 10(e-1)$$

5. $f(x+a) + f(x) = 0$

$\Rightarrow f(x+2a) + f(x+a) = 0$

$\Rightarrow f(x) = f(x+2a)$

$\Rightarrow f(x)$ is periodic with period $2a$.

Since $\int_b^{c+b} f(x)dx$ is independent of b, then c must be $k(2a)$ where $k \in N$.
Hence, least positive value of c is $2a$.

Exercise 8.6

1. $L = \lim_{x\to 4}\int_4^x \frac{(4t-f(t))}{(x-4)}dt = \lim_{x\to 4}\frac{\int_4^x (4t-f(t))dt}{x-4}$ (0/0 form, using L' Hopital's Rule)

$\Rightarrow L = \lim_{x\to 4}\frac{4x-f(x)}{1} = 16-f(4)$

2. Given limit is of the form $\frac{0}{0}$

Then by L'Hopital's Rule

Given limit $\text{Lim}_{x\to 0}\frac{\int_0^x \cos t^2\,dt}{x} = \text{Lim}_{x\to 0}\frac{\cos x^2}{1} = 1.$

3. $f(x) = \int_0^x t(t-1)(t-2)dt$

$\Rightarrow f'(x) = x(x-1)(x-2) = 0$
$\Rightarrow x = 0, 1,$ or 2
At $x = 0$ and 2, $f'(x)$ changes sign from $-$ve to $+$ve.
Hence $x = 0$ and 2 are points of minima.

4. $g(x) = \int_2^x \frac{t\,dt}{1+t^4} \Rightarrow g'(x) = \frac{x}{1+x^4} \Rightarrow g'(2) = \frac{2}{17}$
Now $f(x) = e^{g(x)} \Rightarrow f'(x) = e^{g(x)}g'(x) \Rightarrow f'(2) = e^{g(2)}g'(2)$

$\Rightarrow f'(2) = e^0 \times \frac{2}{17} = \frac{2}{17}$ as $g(2) = 0$

5. $f(x) = \sin x \int_{\pi^2/16}^{x^2} \frac{\sin\sqrt{\theta}}{1+\cos^2\sqrt{\theta}}d\theta$

$\Rightarrow f'(x)$

$= \sin x\left[\frac{\sin x}{1+\cos^2 x}2x - 0\right] + \left(\int_{\pi^2/16}^{x^2}\frac{\sin\sqrt{\theta}}{1+\cos^2\sqrt{\theta}}d\theta\right)\cos x$

$\Rightarrow f'\left(\frac{\pi}{2}\right) = \pi$

6. $y|_{x=1} = 0$, $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^6}}3x^2 - \frac{1}{\sqrt{1+x^4}}2x$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{\sqrt{2}}$

$\Rightarrow$ Required equation is $y\sqrt{2} = x - 1$.

7. We have $\int_{\pi/3}^x \sqrt{(3-\sin^2 t)}\,dt + \int_0^y \cos t\,dt = 0$
Differentiating both sides w.r.t. x then

$\frac{d}{dx}\int_{\pi/3}^x \sqrt{(3-\sin^2 t)}\,dt + \frac{d}{dx}\int_0^y \cos t\,dt = 0$

$\Rightarrow \sqrt{(3-\sin^2 x)} + \cos y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{(3-\sin^2 x)}}{\cos y}$

Exercise 8.7

1. Since $1 \le x \le 3$
$\Rightarrow 1 \le x^2 \le 9$
$\Rightarrow 4 \le x^2 + 3 \le 12$
$\Rightarrow 2 \le \sqrt{3+x^2} \le 2\sqrt{3}$
$\Rightarrow 2(3-1) \le \int_1^3 \sqrt{3+x^2}\,dx \le 2\sqrt{3}(3-1)$
$\Rightarrow 4 \le \int_1^3 \sqrt{3+x^2} \le 4\sqrt{3}$

2. For $0 < x < 1$, $x^2 > x^3$
$\Rightarrow 2^{x^2} > 2^{x^3}$
$\Rightarrow \int_0^1 2^{x^2}dx > \int_0^1 2^{x^3}dx$
Hence, $I_1 > I_2$
Also, $1 < x < 2$, $x^2 < x^3$
$\Rightarrow 2^{x^2} < 2^{x^3}$
$\Rightarrow \int_1^2 2^{x^2}dx < \int_1^2 2^{x^3}dx$
$\Rightarrow I_3 < I_4$

3. $I_1 = \int_0^{\pi/2}\cos(\sin x)dx = \int_0^{\pi/2}\cos(\cos x)dx$

$I_2 = \int_0^{\pi/2}\sin(\cos x)dx$

$I_3 = \int_0^{\pi/2}\cos x\,dx$
Let $f_1(x) = \cos(\cos x)$
$f_2(x) = \sin(\cos x)$
$f_3(x) = \cos x$

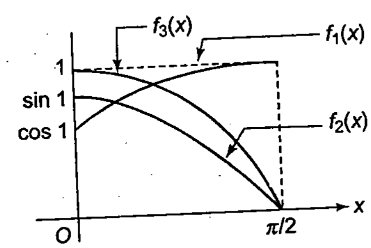


Fig. S-8.1

From Fig. S-8.1 it is clear that the area under $f_1(x)$ is the largest and that under $f_2(x)$ is the least.
$\therefore I_1 > I_3 > I_2$.

4. $\because 0 < x^3 < x^2$
$\Rightarrow x^2 < x^2 + x^3 < 2x^2$
$\Rightarrow -2x^2 < -x^2 - x^3 < -x^2$

$$\Rightarrow 4-2x^2<4-x^2-x^3<4-x^2$$

$$\Rightarrow \sqrt{4-2x^2}<\sqrt{4-x^2-x^3}<\sqrt{4-x^2}$$

$$\Rightarrow \frac{1}{\sqrt{4-x^2}}<\frac{1}{\sqrt{4-x^2-x^3}}<\frac{1}{\sqrt{4-2x^2}}$$

$$\Rightarrow \int_0^1\frac{1}{\sqrt{4-x^2}}dx<\int_0^1\frac{1}{\sqrt{4-x^2-x^3}}dx<\int_0^1\frac{1}{\sqrt{4-2x^2}}dx$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{2}\right)\Big]_0^1<\int_0^1\frac{dx}{\sqrt{4-x^2-x^3}}<\frac{1}{\sqrt{2}}\sin^{-1}\frac{x}{\sqrt{2}}\Big]_0^1$$

$$\Rightarrow \frac{\pi}{6}<\int_0^1\frac{dx}{\sqrt{4-x^2-x^3}}<\frac{\pi}{4\sqrt{2}}$$

Exercise 8.8

1. $\int_{-1}^{1}[x^2+\{x\}]dx$

$$=\int_{-1}^{0}[x^2+x+1]dx+\int_0^1[x^2+x]dx$$

$$=0+\int_0^{\frac{\sqrt{5}-1}{2}}0dx+\int_{\frac{\sqrt{5}-1}{2}}^{1}1dx$$

$$=\frac{3-\sqrt{5}}{2}$$

2. Let $x=n+f\ \forall\ n\in I$ and $0\le f<1\ \therefore [x]=n$ (1)

$$\int_0^x [t]\,dt=\int_0^1 [t]\,dt+\int_1^2 [t]\,dt+\int_2^3 [t]\,dt+\cdots+\int_n^{n+f}[t]\,dt$$

$$=0+1\int_1^2 dt+2\int_2^3 dt+\cdots+n\int_n^{n+f}dt$$

$$=(2-1)+2(3-2)+\cdots+n(n+f-n)$$

$$=1+2+3+\cdots+(n-1)+nf$$

$$=\frac{(n-1)\,n}{2}+nf$$

$$=\frac{[x]([x]-1)}{2}+[x](x-[x])$$ [from equation (1)]

3. $\forall\ x\in[0,\infty), ne^{-x}\in(0,n]$

If $0<ne^{-x}<1\Rightarrow x\in(\ln n,\infty)$,

If $1\le ne^{-x}<2\Rightarrow x\in(\ln n/2,\ln n]$

If $2\le ne^{-x}<3\Rightarrow x\in(\ln n/3,\ln n/2]$

...

If $n-1\le ne^{-x}<n\Rightarrow x\in\left(0,\ln\frac{n}{n-1}\right]$

$$\Rightarrow \int_0^\infty[ne^{-x}]dx=\int_0^{\ln\frac{n}{n-1}}(n-1)dx+\int_{\ln\frac{n}{n-1}}^{\ln\frac{n}{n-2}}(n-2)dx+\cdots+\int_{\ln\frac{n}{2}}^{\ln n}1dx+\int_{\ln n}^{\infty}0\,dx$$

$$=(n-1)\left(\ln\frac{n}{n-1}\right)+(n-2)\left[\ln\left(\frac{n}{n-2}\right)-\ln\left(\frac{n}{n-1}\right)\right]+\cdots+1\left[\ln n-\ln\frac{n}{2}\right]=\ln\left(\frac{n^n}{n!}\right).$$

4.

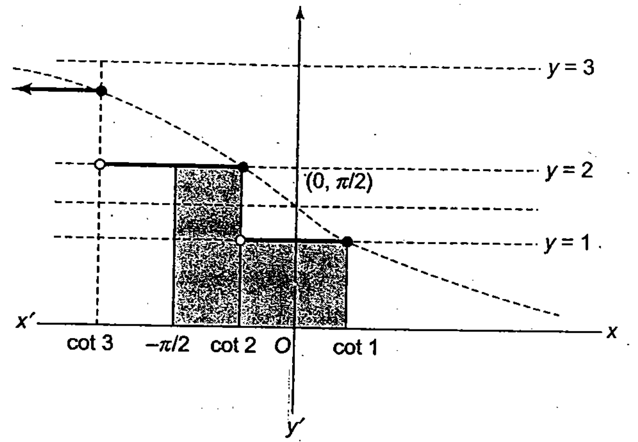


Fig. S-8.2

$$\int_{-\frac{\pi}{2}}^{2\pi}[\cot^{-1}x]dx$$

$$=\int_{-\frac{\pi}{2}}^{\cot 2}[\cot^{-1}x]dx+\int_{\cot 2}^{\cot 1}[\cot^{-1}x]dx+\int_{\cot 1}^{2\pi}[\cot^{-1}x]dx$$

(verify that $\cot 2>-\pi/2$)

$$=2\int_{-\frac{\pi}{2}}^{\cot 2}dx+\int_{\cot 2}^{\cot 1}dx+0$$

$$=2\left(\cot 2+\frac{\pi}{2}\right)+(\cot 1-\cot 2)=\pi+\cot 1+\cot 2$$

5. Given $\int_{b-1}^{b}\frac{e^{-t}dt}{t-b-1}$, put $t-b-1=-y-1\Rightarrow dt=-dy$

$$\Rightarrow \int_{b-1}^{b}\frac{e^{-t}dt}{t-b-1}=\int_1^0\frac{e^{y-b}}{-y-1}(-dy)=-e^{-b}\int_0^1\frac{e^y}{y+1}dy$$

$$=-ae^{-b}$$

6. Given $f(x)=\int_1^x\frac{\log t}{1+t+t^2}\,dt\Rightarrow f\left(\frac{1}{x}\right)=\int_1^{1/x}\frac{\log t}{1+t+t^2}\,dt$

Let $y=\frac{1}{t}\Rightarrow dy=-\frac{dt}{t^2}\Rightarrow f\left(\frac{1}{x}\right)=\int_1^x\frac{\log\frac{1}{y}}{1+\frac{1}{y}+\frac{1}{y^2}}\left(-\frac{1}{y^2}dy\right)$

$$=\int_1^x\frac{\log y}{1+y+y^2}dy=f(x)$$

7. We have

$$f\left(\frac{1}{x}\right)+x^2\,f(x)=0\Rightarrow f(x)=-\frac{1}{x^2}\,f\left(\frac{1}{x}\right)$$

$\therefore I=\int_{\sin\theta}^{\text{cosec}\,\theta} f(x)dx=\int_{\sin\theta}^{\text{cosec}\,\theta}-\frac{1}{x^2}f\left(\frac{1}{x}\right)dx,$

put $\frac{1}{x}=t \Rightarrow -\frac{1}{x^2}dx=dt$

$\Rightarrow I=\int_{\text{cosec}\,\theta}^{\sin\theta} f(t)dt=-\int_{\sin\theta}^{\text{cosec}\,\theta} f(t)dt=-I \Rightarrow 2I$

$=0 \Rightarrow I=0$

8. Put $x+1=t$ in first integral

$$\Rightarrow \int_1^e \frac{e\,e^{\frac{t^2-2}{2}}}{t}dt+\int_1^e x\log x e^{\frac{x^2-2}{2}}dx$$

$$=\int_1^e \frac{e\,e^{\frac{x^2-2}{2}}}{x}dt+\int_1^e x\log x\, e^{\frac{x^2-2}{2}}dx$$

$$=\left[e^{\frac{x^2-2}{2}}\log x\right]_1^e-\int_1^e xe^{\frac{x^2-2}{2}}\log x\,dx+\int_1^e x\log x\,e^{\frac{x^2-2}{2}}dx$$

$$=e^{\frac{e^2-2}{2}}$$

9. $I_n=\int_0^\infty (x^2)^n x\,e^{-x^2}dx$

put $x^2=t \Rightarrow x\,dx=dt/2$

$$\Rightarrow I_n=\frac{1}{2}\int_0^\infty t^n e^{-t}\,dt$$

$$=\frac{1}{2}\left[\left[t^n e^{-t}\right]_0^\infty+n\int_0^\infty t^{n-1}e^{-t}\,dt\right]$$

$$=\frac{1}{2}\left[0+n\int_0^\infty t^{n-1}e^{-t}\,dt\right]$$

$$=\frac{n}{2}\int_0^\infty t^{n-1}e^{-t}\,dt=nI_{n-1}$$

$\Rightarrow I_{n-1}=(n-1)I_{n-2}$

$\Rightarrow I_n=n(n-1)(n-2)\ \ldots\ 1I_0$

$$\Rightarrow I_n=n!\,I_0=n!\,\frac{1}{2}\int_0^\infty e^{-t}\,dt=n!\,\frac{1}{2}\left[-e^{-t}\right]_0^\infty=\frac{n!}{2}$$

10. $I_{m,n}=\int_0^{\frac{\pi}{2}}\sin^{m-1}x\left(\sin x\cos^n x\right)dx$

$$=\left[-\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}\right]_0^{\frac{\pi}{2}}$$

$$+\int_0^{\frac{\pi}{2}}\frac{\cos^{n+1}x}{n+1}(m-1)\sin^{m-2}\cos x\,dx$$

$$=\left(\frac{m-1}{n+1}\right)\int_0^{\frac{\pi}{2}}\sin^{m-2}x\cos^n\cos^2 x\,dx$$

$$=\left(\frac{m-1}{n+1}\right)\int_0^{\frac{\pi}{2}}\left(\sin^{m-2}x\cos^n x-\sin^m x\cos^n x\right)dx$$

$$=\left(\frac{m-1}{n+1}\right)I_{m-2,n}-\left(\frac{m-1}{n+1}\right)I_{m,n}$$

$$\Rightarrow\left(1+\frac{m-1}{n+1}\right)I_{m,n}=\left(\frac{m-1}{n+1}\right)I_{m-2,n}$$

$$\Rightarrow I_{m,n}=\left(\frac{m-1}{m+n}\right)I_{m-2,n}$$

$$\Rightarrow I_{m,n}=\left(\frac{m-1}{m+n}\right)\left(\frac{m-3}{m+n-2}\right)\left(\frac{m-5}{m+n-4}\right)\ldots I_{0,n}\text{ or }I_{1,n}$$

according as m is even or odd

$$I_{0,n}=\int_0^{\frac{\pi}{2}}\cos^n x\,dx\text{ and }I_{1,n}=\int_0^{\frac{\pi}{2}}\sin x\cos^n x\,dx=\frac{1}{n+1}$$

$$\Rightarrow I_{m,n}=\begin{cases}\dfrac{(m-1)(m-3)(m-5)\ldots(n-1)(n-3)(n-5)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots}\cdot\dfrac{\pi}{2} & \text{when both } m,n \text{ are even}\\[2ex] \dfrac{(m-1)(m-3)(m-5)\ldots(n-1)(n-3)(n-5)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} & \text{otherwise}\end{cases}$$

Chapter 9

Exercise 9.1

1. The line $y=4x$ meets $y=x^3$ at $4x=x^3$.

$\therefore x=0,2,-2 \Rightarrow y=0,8,-8$

$$\Rightarrow A=\int_0^2\left(4x-x^3\right)=\left(2x^2-\frac{x^4}{4}\right)_0^2=4\text{ sq. units}$$

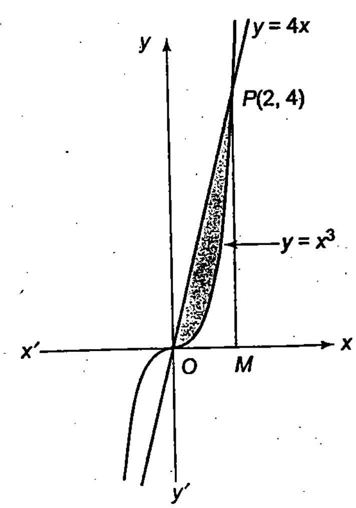


Fig. S-9.1

2.

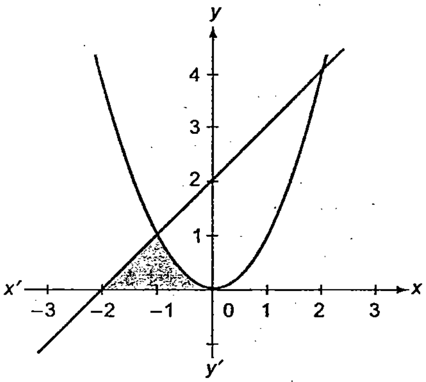


Fig. S-9.2

Required area $= \int_{-2}^{-1}(x+2)\,dx + \int_{-1}^{0} x^2\,dx$

$= \left|\frac{x^2}{2} + 2x\right|_{-2}^{-1} + \left|\frac{x^3}{3}\right|_{-1}^{0}$

$= \left(\frac{1}{2} - 2\right) - (2-4) + \left(0 + \frac{1}{3}\right)$

$= \frac{5}{6}$ sq. units.

3. The given curve is

$$y = \begin{cases} \sqrt{(4-x^2)}, & 0 \le x < 1 \\ \sqrt{(3x)}, & 1 \le x \le 3 \end{cases}$$

Obviously, the curve is the arc of the circle $x^2 + y^2 = 4$ (1)

between $0 \le x < 1$ and the arc of parabola $y^2 = 3x$ (2)

between $1 \le x \le 3$

Required area = shaded area

= Area $OABCO$ + Area $CBDEC$

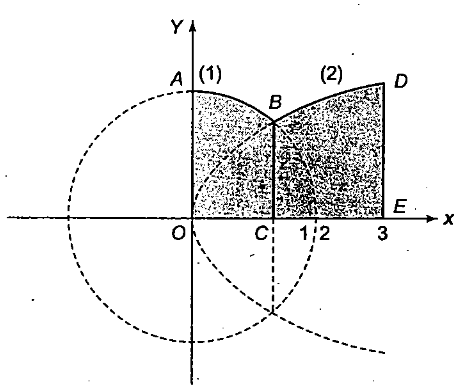


Fig. S-9.3

$= \left|\int_0^1 \sqrt{(4-x^2)}\,dx\right| + \left|\int_1^3 \sqrt{3x}\,dx\right|$

$= \left|\left[\frac{1}{2}x\sqrt{(4-x^2)} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1\right| + \left|\sqrt{3}\left[\frac{2}{3}x^{3/2}\right]_1^3\right|$

$= \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right) + \frac{2}{3}(9 - \sqrt{3})$

$= \frac{1}{6}(2\pi - \sqrt{3} + 36)$ sq. units.

4.

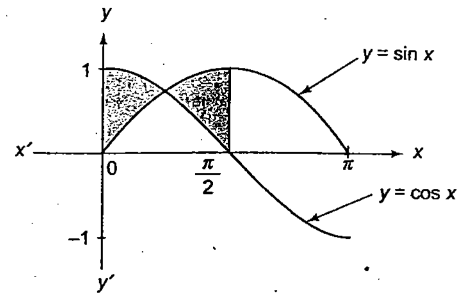


Fig. S-9.4

Required area $= \int_0^{\pi/2} |\sin x - \cos x|\,dx$

$= 2\int_0^{\pi/4} (\cos x - \sin x)\,dx$

$= 2\,|\sin x + \cos x|_0^{\pi/4}$

$= 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right)$

$= 2\left(\sqrt{2} - 1\right)$ sq. units

5.

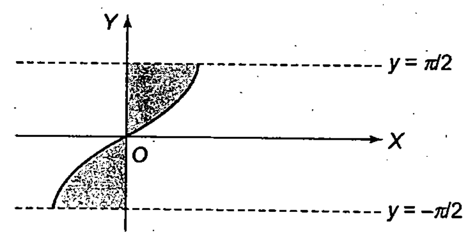


Fig. S-9.5

The required area is shown by shaded portion in the figure.

The required area is $A = \int_{-\pi/2}^{\pi/2} |\sin y|\,dy = 2\int_0^{\pi/2} \sin y\,dy$

$= 2$ sq. units.

6.

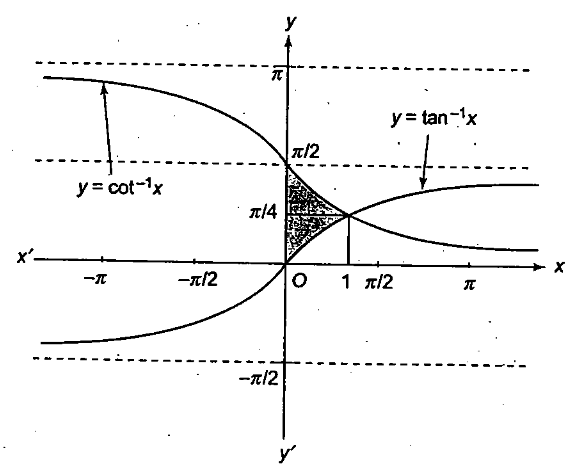


Fig. S-9.6

Integrating along x-axis.

$$A=\int_0^1\left(\cot^{-1}x-\tan^{-1}x\right)dx$$

$$=\int_0^1\left(\frac{\pi}{2}-2\tan^{-1}x\right)dx$$

Integrating along y-axis, we get

$$A=2\int_0^{\pi/4}x\,dy=2.\int_0^{\pi/4}\tan y\,dy=\left[\log(\sec y)\right]_0^{\pi/4}$$

$$=\log\sqrt{2}\text{ sq. units}$$

7. Common area = area of circle – area of ellipse

$$=\pi a^2-\pi ab$$

$$=\pi a(a-b)\text{ sq. units}$$

which is clearly an area of ellipse whose semi-axis are a and $a-b$.

8.

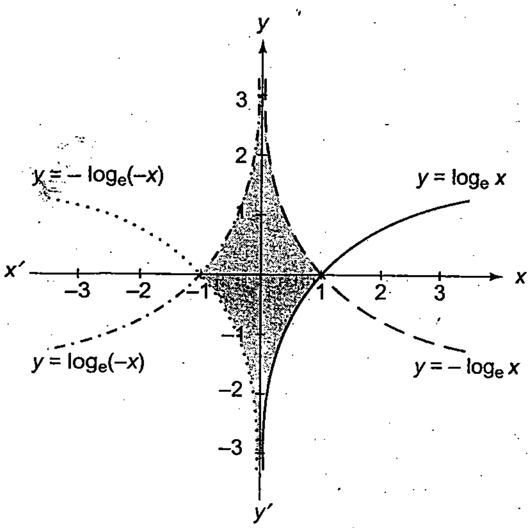


Fig. S-9.7

From the figure, required area = area of shaded region = 1 + 1 + 1 + 1 = 4 sq. units.

Chapter 10

Exercise 10.1

1. $\frac{d^2y}{dx^2}=\left\{1+\left(\frac{dy}{dx}\right)^4\right\}^{5/3}$

$$\Rightarrow\left(\frac{d^2y}{dx^2}\right)^3=\left\{1+\left(\frac{dy}{dx}\right)^4\right\}^5$$

Hence, the order is 2 and the degree is 3.

2. $\frac{d^3y}{dx^3}=x\ln\left(\frac{dy}{dx}\right)$

Clearly, the order is 3 and the degree is not defined due to $\ln\left(\frac{dy}{dx}\right)$ term.

3. $\left(\frac{d^4y}{dx^4}\right)^3+3\left(\frac{d^2y}{dx^2}\right)^6+\sin x=2\cos x$

Clearly, order is 4 and degree is 3.

4. We have $\left(\frac{d^3y}{dx^3}\right)^{2/3}+4-3\frac{d^2y}{dx^2}+5\frac{dy}{dx}=0$

$$\Rightarrow\left(\frac{d^3y}{dx^3}\right)^2=\left(3\frac{d^2y}{dx^2}-5\frac{dy}{dx}-4\right)^3$$

Clearly, it is a differential equation of degree 2 and order 3.

5. The given equation when expressed as a polynomial in derivative is

$$a^2\left(\frac{d^2y}{dx^2}\right)^2=\left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

Clearly, it is a second order differential equation of degree 2.

Exercise 10.2

1. Equation of such parabolas is given by $y=ax^2+bx+c$. Here, we have three effective constants, so it is required to differentiate three times.

$$y=ax^2+bx+c$$

$$\Rightarrow\frac{dy}{dx}=2ax+b$$

$$\Rightarrow\frac{d^2y}{dx^2}=2a$$

$$\Rightarrow\frac{d^3y}{dx^3}=0\text{, which is the required differential equation.}$$

2. $y=Ae^{2x}+Be^{-2x}$

$$\Rightarrow\frac{dy}{dx}=2Ae^{2x}-2Be^{-2x}$$

$$\Rightarrow\frac{d^2y}{dx^2}=4Ae^{2x}+4Be^{-2x}=4(Ae^{2x}+Be^{-2x})$$

$$\Rightarrow\frac{d^2y}{dx^2}=4y\text{, which is the required differential equation.}$$

3. All such lines are given by $y=mx+c$. Here, we have two effective constants m and c, so it is required to differentiate twice.

$$y=mx+c$$

$$\Rightarrow\frac{dy}{dx}=m$$

$$\Rightarrow\frac{d^2y}{dx^2}=0$$

4. Equation of such ellipses is given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ (1)

Here we have two effective constants.
Diff. equation (1) w.r.t. x, we get

$$\frac{2x}{a^2}+\frac{2yy'}{b^2}=0$$

or $$\frac{x}{a^2}+\frac{yy'}{b^2}=0 \quad (2)$$

Diff. equation (2) w.r.t. x, we get

or $$\frac{1}{a^2}+\frac{yy''+y'^2}{b^2}=0 \quad (3)$$

Eliminating a^2 and b^2 from equations (2) and (3), we get

$$x=\frac{yy'}{yy''+y'^2}$$

or $x(yy''+y'^2)=yy'$

5. Differentiating the given equation, we get

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \quad (1)$$

$$\frac{2x}{a^2+\lambda}+\frac{2y\frac{dy}{dx}}{b^2+\lambda}=0$$

$$\Rightarrow \frac{x^2}{a^2+\lambda}+\frac{xy\frac{dy}{dx}}{b^2+\lambda}=0 \quad (2)$$

(1) – (2) gives

$$\frac{y^2-xy\frac{dy}{dx}}{b^2+\lambda}=1$$

$$\Rightarrow b^2+\lambda=y^2-xy\frac{dy}{dx}$$

$$\therefore a^2+\lambda=\frac{x^2\frac{dy}{dx}-xy}{\frac{dy}{dx}}$$

Eliminating λ, we get

$$a^2-b^2=\frac{x^2\frac{dy}{dx}-xy}{\frac{dy}{dx}}-y^2+xy\frac{dy}{dx}$$

6. Putting $x=\tan A$, and $y=\tan B$ in the given relation, we get

$$\cos A+\cos B=\lambda(\sin A-\sin B)$$

$$\Rightarrow \tan\left(\frac{A-B}{2}\right)=\frac{1}{\lambda} \Rightarrow \tan^{-1}x-\tan^{-1}y=2\tan^{-1}\left(\frac{1}{\lambda}\right)$$

Differentiating w.r.t. to x, we get

$$\frac{1}{1+x^2}-\frac{1}{1+y^2}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{1+y^2}{1+x^2}$$

Clearly, it is a differential equation of degree 1.

Exercise 10.3

1. The given equation can be rewritten as

$$\int\frac{\sec^2 x}{\tan x}dx+\int\frac{\sec^2 y}{\tan y}dy=\log c$$

$\Rightarrow$ $\log\tan x+\log\tan y=\log c$; where c is an arbitrary positive constant

$\Rightarrow$ $\tan x\tan y=c$.

2. $$y-x\frac{dy}{dx}=a\left(y^2+\frac{dy}{dx}\right)$$

$$\Rightarrow \int\frac{dx}{a+x}=\int\frac{dy}{y-ay^2}=\int\left(\frac{1}{y}+\frac{a}{1-ay}\right)dy$$

[By partial fractions]

Integrating, we get
$\log(a+x)+\log c=\log y-\log(1-ay)$
where c is an arbitrary positive constant.

Thus, the solution can be written as $\frac{y}{1-ay}=c(a+x)$

3. Given $\frac{dy}{dx}=\frac{x+y+1}{x+y-1}$

Putting $x+y=v$, we have

$$1+\frac{dy}{dx}=\frac{dv}{dx}$$

Hence, the given equation transforms to

$$-1+\frac{dv}{dx}=\frac{v+1}{v-1}$$

$$\Rightarrow \frac{dv}{dx}=\frac{2v}{v-1}$$

$$\Rightarrow \int\frac{v-1}{v}dv=\int 2dx$$

$\Rightarrow$ $2x=v-\log v+\log c$

$\Rightarrow$ $\log(v/c)=v-2x$

$\Rightarrow$ $v=ce^{v-2x}$

$\Rightarrow$ $x+y=ce^{y-x}$, where c is an arbitrary constant.

4. $$\frac{dy}{dx}+yf'(x)=f(x)f'(x)$$

$$\Rightarrow \frac{dy}{dx}=[f(x)-y]f'(x)$$

Put $f(x)-y=t$

$$\Rightarrow f'(x)-\frac{dy}{dx}=\frac{dt}{dx}$$

Then the given equation transforms to

$$\Rightarrow f'(x)-\frac{dt}{dx}=tf'(x)$$

$$\Rightarrow (1-t)f'(x)=\frac{dt}{dx}$$

$\Rightarrow \int \frac{dt}{1-t} = \int f'(x)dx$

$\Rightarrow -\log(1-t) = f(x) + c$

$\Rightarrow \log[1 + y - f(x)] + f(x) + c = 0$

5. $\frac{dy}{dx} = \cos(x+y) - \sin(x+y)$

Putting $x + y = t$, we get $\frac{dy}{dx} = \frac{dt}{dx} - 1$

Therefore, $\frac{dt}{dx} - 1 = \cos t - \sin t$

$$\Rightarrow \frac{dt}{1 + \cos t - \sin t} = dx \Rightarrow \frac{\sec^2 \frac{t}{2}\, dt}{2\left(1 - \tan\frac{t}{2}\right)} = dx$$

$$\Rightarrow -\ln\left|1 - \tan\frac{x+y}{2}\right| = x + c.$$

Exercise 10.4

1. Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

We get $xv + x^2\frac{dv}{dx} = vx + 2x\sqrt{v^2-1}$

$\Rightarrow \int \frac{dv}{2\sqrt{v^2-1}} = \int \frac{dx}{x}$, integrating, we get

$$\Rightarrow \frac{1}{2}\ln\left(v + \sqrt{v^2-1}\right) = \ln cx$$

$$\Rightarrow \frac{1}{2}\ln\left(\frac{y + \sqrt{y^2 - x^2}}{x}\right) = \ln cx$$

2. $x(dy/dx) = y(\log y - \log x + 1)$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left[\log\frac{y}{x} + 1\right]$$

Putting $y = vx$, we get $\frac{dy}{dx} = v + x\frac{dv}{dx}$

And the given equation transforms to

$$v + x\frac{dv}{dx} = v[\log v + 1]$$

$$\Rightarrow \quad x\frac{dv}{dx} = v\log v$$

$$\Rightarrow \int \frac{dv}{v\log v} = \int \frac{dx}{x}$$

$\Rightarrow \log\log v = \log x + \log c, c > 0$

$\Rightarrow cx = \log(y/x)$

$\Rightarrow y = xe^{cx}; c > 0$

3. Given equation is $\frac{dy}{dx} = \frac{x + y\sin(y/x)}{x\sin(y/x)}$

or $\frac{dy}{dx} = \frac{1 + (y/x)\sin(y/x)}{\sin(y/x)}$

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$,

And the given equation transforms to

$$v + x\frac{dv}{dx} = \operatorname{cosec} v + v$$

$\Rightarrow \sin v\, dv = dx/x$

Integrating and replacing v by y/x, we get

$\cos(y/x) + \log|x| = c, c \in R$

4. $y^3 dy + (x + y^2)dx = 0$

$$\Rightarrow y\frac{dy}{dx} = \frac{x + y^2}{y^2} \qquad (1)$$

Let $y^2 = t \Rightarrow 2y\frac{dy}{dx} = \frac{dt}{dx}$

Equation (1) transforms to $\frac{dt}{dx} = 2\frac{x+t}{t}$

$\Rightarrow \frac{dt}{dx} = 2\left(\frac{x}{t} + 1\right)$, which is homogeneous.

5. Here, $\frac{dy}{dx} = \frac{2x - y + 1}{x + 2y - 3}$

Cross multiplying, we get

$xdy + (2y - 3)dy = (2x + 1)dx - ydx$

$\Rightarrow (xdy + ydx) + (2y - 3)dy = (2x + 1)dx$

$\Rightarrow d(xy) + (2y - 3)dy = (2x + 1)dx$

Integrating, we get

$xy + y^2 - 3y = x^2 + x + c.$

Exercise 10.5

1. Given equation is linear and

$P = \cot x, Q = \sin x$

$\therefore$ I.F. $= e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$

Hence, the solution is

$$y\sin x = \int \sin x \sin x\, dx + c$$

$$= \frac{1}{2}\int (1 - \cos 2x)dx + c$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + c$$

$$\therefore \quad y\sin x = \frac{1}{4}[2x - \sin 2x] + c$$

2. The given equation can be rewritten as

$$\frac{dx}{dy} - 1x = (y+1)$$

[linear, y as independent variable]

Here $P = -1, Q = (y+1)$

$$\text{I.F.} = e^{-\int dy} = e^{-y}$$

Therefore, the solution is

$$xe^{-y} = \int (y+1)e^{-y}\,dy + c$$
$$= -(y+1)\,e^{-y} - e^{-y} + c$$

or $x = ce^{y} - y - 2$

3. We have

$$\frac{dy}{dx} + \frac{2x}{1-x^2}y = \frac{x}{\sqrt{1-x^2}}$$

Here, $P = \frac{2x}{1-x^2}$ and $Q = \frac{x}{\sqrt{1-x^2}}$

$$\text{I.F.} = e^{\int \frac{2x}{1-x^2}dx} = e^{-\log(1-x^2)} = \frac{1}{1-x^2}$$

Therefore, the solution is

$$\frac{y}{1-x^2} = \int \frac{x}{\sqrt{(1-x^2)}} \times \frac{1}{(1-x^2)}\,dx + c$$

$$= \frac{1}{\sqrt{1-x^2}} + c$$

$$\Rightarrow \quad y = \sqrt{1-x^2} + c\,(1-x^2)$$

4. Given equation is $\frac{dx}{dy} = \frac{2y\ln y + y - x}{y}$

or $\frac{dx}{dy} + \frac{1}{y}x = (2\ln y + 1)$

I.F. $= y$ and solution is $xy = \int (2\ln y + 1)\,y\,dy + c$

$$\Rightarrow \quad xy = y^2 \ln y + c$$

ercise 10.6

1. Dividing by e^{y}, we get

$$e^{-y}\frac{dy}{dx} + e^{-y}\frac{1}{x} = \frac{1}{x^2}$$

Putting $e^{-y} = v$, we get

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x^2}$$

or $\frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x^2}$ (linear)

$$\text{I.F.} = e^{-\int (1/x)dx} = e^{-\log x} = 1/x$$

Therefore, solution is

$$\frac{v}{x} = \int -\frac{1}{x^2}\frac{1}{x}dx + c$$

$$\Rightarrow \quad \frac{v}{x} = \frac{x^{-2}}{2} + c$$

$$\Rightarrow \quad \frac{e^{-x}}{x} = \frac{x^{-2}}{2} + c$$

2. The given equation can be expressed as

$$\sec^2 y\,\frac{dy}{dx} + 2x\tan y = x^3,$$

Put $\tan y = z$ so that $\sec^2 y\frac{dy}{dx} = \frac{dz}{dx}$

Given equation transforms to

$\frac{dz}{dx} + 2xz = x^3$, which is linear in z.

$$\text{I.F.} = e^{2\int x\,dx} = e^{x^2}$$

Therefore, solution is given by

$$z\,e^{x^2} = \int x^3 e^{x^2}\,dx + c$$

$$\Rightarrow \quad \tan y\, e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c$$

(substitute for $x^2 = t$ and then integrate by parts)

3. $\frac{dy}{dx} + \frac{xy}{(1-x^2)} = x\sqrt{y}$

Dividing by $\sqrt{y}$, we have

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} + \frac{x}{(1-x^2)}\sqrt{y} = x \qquad (1)$$

Putting $\sqrt{y} = v$, so $\frac{1}{2\sqrt{y}}\frac{dy}{dx} = \frac{dv}{dx}$,

Then given equation transforms to

$$\frac{dv}{dx} + \frac{x}{2(1-x^2)}v = \frac{1}{2}x, \qquad (2)$$

$$\text{I.F.} = e^{\frac{1}{2}\int [x/(1-x^2)]dx}$$

$$= e^{-\frac{1}{4}\log(1-x^2)}$$

$$= 1/(1-x^2)^{1/4}$$

Therefore, the solution is

$$v/(1-x^2)^{1/4} = \frac{1}{2}\int [x/(1-x^2)^{1/4}]dx + c$$

$$= -\frac{1}{4}\int [(-2x)/(1-x^2)^{1/4}]dx + c$$

$$= -\frac{1}{4}(4/3)\left(1-x^2\right)^{3/4} + c$$

Hence, the required solution is

$$\sqrt{y}/(1-x^2)^{1/4} = -\frac{1}{3}\left(1-x^2\right)^{3/4} + c.$$

Exercise 10.7

1. $y\,dx - x\,dy + 3x^2 y^2 e^{x^3}\,dx = 0$

$$\Rightarrow \frac{y\,dx - x\,dy}{y^2} + 3x^2 e^{x^3}\,dx = 0$$

$$\Rightarrow \int d\left(\frac{x}{y}\right) + \int d(e^{x^3}) = c$$

$$\Rightarrow \frac{x}{y} + e^{x^3} = c$$

2. $\dfrac{dy}{dx} = \dfrac{2xy}{x^2 - 1 - 2y}$

$$\Rightarrow x^2\,dy - (1+2y)\,dy = 2xy\,dx$$

$$\Rightarrow 2xy\,dx - x^2\,dy = -(1+2y)\,dy$$

$$\Rightarrow \frac{y\,d(x^2) - x^2\,dy}{y^2} = -\left(\frac{1}{y^2} + \frac{2}{y}\right)dy$$

$$\Rightarrow d\left(\frac{x^2}{y}\right) = -\left(\frac{1}{y^2} + \frac{2}{y}\right)dy$$

Integrating, we get $\dfrac{x^2}{y} = \dfrac{1}{y} - 2\log y + c$

3. $y\,dx + (x + x^2 y)dy = 0$

$$\Rightarrow (x\,dy + y\,dx) + x^2 y\,dy = 0$$

$$\Rightarrow d(xy) + x^2 y\,dy = 0$$

$$\Rightarrow \frac{d(xy)}{(xy)^2} + \frac{1}{y}\,dy = 0$$

Integrating, we get $-\dfrac{1}{xy} + \log y = c$.

4. The given equation is $xy^4\,dx + y\,dx - x\,dy = 0$

Dividing by y^4, we get

$$x\,dx + \frac{y\,dx - x\,dy}{y^4} = 0 \qquad (1)$$

$$\Rightarrow x^3\,dx + \left(\frac{x}{y}\right)^2 d(x/y) = 0 \qquad (2)$$

Integrating equation (2), we get $\dfrac{x^4}{4} + \dfrac{1}{3}\left(\dfrac{x}{y}\right)^3 = c$

$\Rightarrow$ $3x^4 y^3 + 4x^3 = cy^3$, which is the required solution.

Exercise 10.8

1. Since subnormal is $y\dfrac{dy}{dx}$

we have, $y\dfrac{dy}{dx} = ky^2$

$$\Rightarrow \frac{dy}{y} = k\,dx$$

Integrating, we get

$\log y = kx + \log c$ or $y = ce^{kx}$.

2. Length of normal $= y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

and radius vector $= \sqrt{x^2+y^2}$

$$\therefore y^2\left[1+\left(\frac{dy}{dx}\right)^2\right] = x^2 + y^2$$

$$\Rightarrow y\frac{dy}{dx} = \pm x$$

$$\Rightarrow y\,dy \pm x\,dx = 0$$

$$\Rightarrow y^2 \pm x^2 = c$$

3.

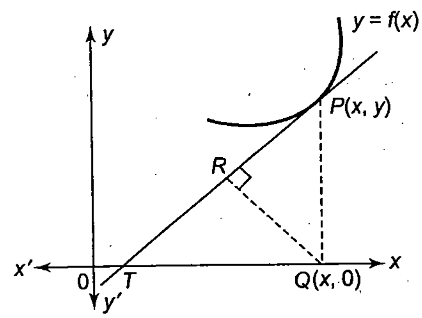


Fig. S-10.1

Equation of the tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

or $\dfrac{dy}{dx}X - Y + \left(y - x\dfrac{dy}{dx}\right) = 0$

Length of perpendicular QR upon the tangent from the foot of ordinate $Q(x, 0)$ is

$$\left|\frac{x\dfrac{dy}{dx} - 0 + y - x\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}\right| = k$$

$$\Rightarrow \quad 1+\left(\frac{dy}{dx}\right)^2=\frac{y^2}{k^2} \text{ or } \frac{dy}{dx}=\pm\sqrt{\frac{y^2-k^2}{k^2}}$$

$$\Rightarrow \quad \int\frac{dy}{\sqrt{y^2-k^2}}=\pm\int\frac{1}{k}\,dx$$

$$\Rightarrow \quad \log[y+\sqrt{y^2-k^2}]=\pm\frac{x}{k}+\log c$$

$$\Rightarrow \quad y+\sqrt{y^2-k^2}=ce^{\pm x/k}$$

4. Area of $OBPO$: area of $OPAP=m:n$

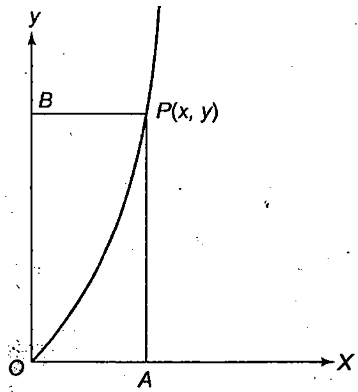


Fig. S-10.2

$$\Rightarrow \quad \frac{xy-\int_0^x y\,dx}{\int_0^x y\,dx}=\frac{m}{n}$$

$$\Rightarrow \quad nxy=(m+n)\int_0^x y\,dx$$

Differential w.r.t.x, we get

$$n\left(x\frac{dy}{dx}+y\right)=(m+n)\,y$$

$$\Rightarrow \quad nx\frac{dy}{dx}=my \Rightarrow \frac{m}{n}\frac{dx}{x}=\frac{dy}{y}$$

$$\Rightarrow \quad y=cx^{m/n}$$

$x^2+y^2=cx$ (1)

Differentiating w.r.t. x, we get $2x+2y\dfrac{dy}{dx}=c$ (2)

Eliminating c between equations (1) and (2)

$$2x+2y\frac{dy}{dx}=\frac{x^2+y^2}{x} \quad \Rightarrow \frac{dy}{dx}=\frac{y^2-x^2}{2xy}$$

Replace $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$, we get $\dfrac{dy}{dx}=\dfrac{2xy}{x^2-y^2}$.

This equation is homogeneous, and its solution gives the orthogonal trajectories as $x^2+y^2=ky$.

$y^2=4ax$ (1)

$2y\dfrac{dy}{dx}=4a$ (2)

Eliminating a from equations (1) and (2), we get

$$y^2=2y\frac{dy}{dx}x$$

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$, we get

$$y=2\left(-\frac{dx}{dy}\right)x$$

$$2x\,dx+y\,dy=0$$

Integrating each term, we get

$$x^2+\frac{y^2}{2}=c$$

$$2x^2+y^2=2c$$

which is the required orthogonal trajectories.

Exercise 10.9

1. Let $N(t)$ denote the balance in the account at any time t. Initially, $N(0)=500$.

For the first four years, $k=0.085$. Therefore,

$$\frac{dN}{dt}-0.085\,N=0$$

Its solution is $N(t)=ce^{0.085t}$ $(0\le t\le 4)$ (1)

At $t=0$, $N(0)=500$, then from (1) $5000=ce^{0.085(0)}=c$

and equation (1) becomes $N(t)=5000\,e^{0.085t}(0\le t\le 4)$ (2)

Substituting $t=4$ into equation (2),

we find the balance after four years to be

$N(4)=5000\,e^{0.085(4)}=5000\,(1.404948)=7024.74$.

This amount also represents the beginning balance for the last three-year period.

Over the last three years, the interest rate is 9.25%

$$\therefore \quad \frac{dN}{dt}-0.925\,N=0 \qquad (4\le t\le 7)$$

Its solution is $N(t)=ce^{0.0925t}$ $(4\le t\le 7)$ (3)

At $t=4$, $N(4)=7024.74$, then from equation (3)

$7024.74=ce^{0.0925(4)}=c\,(1.447735)$ or $c=4852.23$

then from equation (3)

$N(7)=4852.23\,e^{0.0925(7)}=4852.23(1.910758)$

Exercise 10.10

1.

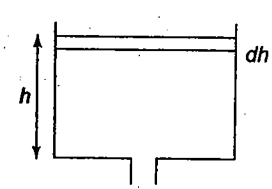


Fig. S-10.3

Let us allow the water to flow for dt time.

We suppose that in this time the height of the water level reduces by dh. Therefore,

$$\pi(2.5)^2\,dh = 2.5\sqrt{h}\pi(0.025)^2\,dt$$

or $\dfrac{dh}{dt} = -2.5\times10^{-4}\sqrt{h}$

(negative signs denotes that the rate of flow will decrease as t increases)

$$\int\frac{dh}{\sqrt{h}} = -2.5\times10^{-4}\int dt$$

$$\Rightarrow\ 2\sqrt{h} = -2.5\times10^{-4}t + c$$

At $t=0, h=3 \Rightarrow c = 2\sqrt{3}$

Hence, for $h=0$, $t = \dfrac{2\sqrt{3}}{2.5\times10^{-4}} = 8000\sqrt{3}$ s.

2. Let x denote the population at a time t in years.

Then $\dfrac{dx}{dt} \propto x \Rightarrow \dfrac{dx}{dt} = kx$, where k is a constant of proportionality

Solving $\dfrac{dx}{dt} = kx$, we get $\displaystyle\int\frac{dx}{x} = \int k\,dt$

$\Rightarrow \log x = kt + c \Rightarrow x = e^{kt+c} \Rightarrow x = x_0 e^{kt}$,

where x_0 is the population at time $t = 0$.

Since it doubles in 50 years, at $t = 50$, we must have $x = 2x_0$

Hence, $2x_0 = x_0 e^{50k} \Rightarrow 50k = \log 2$

$\Rightarrow k = \dfrac{\log 2}{50}$, so that $x = x_0 e^{\frac{\log 2}{50}t}$

To find t, when it triples, i.e., $x = 3x_0$

i.e., $3x_0 = x_0 e^{\frac{\log 2}{50}t}$

$$\Rightarrow \quad \log 3 = \frac{\log 2}{50}t$$

$$\Rightarrow \quad t = \frac{50\log 3}{\log 2} = 50\log_2 3$$

3. Let T be the temperature of the substance at a time t.

$$-\frac{dT}{dt} = \alpha(T-290)$$

$$\Rightarrow \quad \frac{dT}{dt} = -k(T-290)$$

(Negative sign because $\dfrac{dT}{dt}$ is rate of cooling)

$$\Rightarrow \quad \int\frac{dT}{T-290} = -k\int dt \qquad (1)$$

Integrating the L.H.S. between the limits, we get $T = 370$ to $T = 330$ and the R.H.S. between the limits $t = 0$ to $t = 10$, we get

$$\int_{370}^{330}\frac{dT}{T-290} = -k\int_0^{10} dt$$

$$\Rightarrow \log(T-290)\Big|_{370}^{330} = -kt\Big|_0^{10}$$

$$\Rightarrow \log 40 - \log 80 = -k\times 10$$

$$\Rightarrow \log 2 = 10\kappa$$

$$\Rightarrow k = \frac{\log 2}{10} \qquad (2)$$

Now integrating equation (1) between $T = 370$ and $T = 295$ and $t = 0$ and $t = t$

$$\int_{370}^{295}\frac{dT}{T-290} = -k\int_0^{1} dt$$

$$\Rightarrow \log(T-290)\Big|_{370}^{295} = -kt$$

$$\Rightarrow \log 5 - \log 80 = -kt$$

$$\Rightarrow -\log 16 = -kt$$

$$\Rightarrow t = \frac{\log 16}{k}$$

Hence from equation (2), we get

$$t = \frac{\log 16}{\log 2}\times 10 = 40$$

i.e., after 40 min.