

9.12 Line Integral

Scalar functions: $F(x,y,z)$, $F(x,y)$, $f(x)$

Scalar potential: $u(x,y,z)$

Curves: C , C_1 , C_2

Limits of integrations: a , b , α , β

Parameters: t , s

Polar coordinates: r , θ

Vector field: $\vec{F}(P,Q,R)$

Position vector: $\vec{r}(s)$

Unit vectors: \vec{i} , \vec{j} , \vec{k} , $\vec{\tau}$

Area of region: S

Length of a curve: L

Mass of a wire: m

Density: $\rho(x,y,z)$, $\rho(x,y)$

Coordinates of center of mass: \bar{x} , \bar{y} , \bar{z}

First moments: M_{xy} , M_{yz} , M_{xz}

Moments of inertia: I_x , I_y , I_z

Volume of a solid: V

Work: W

Magnetic field: \vec{B}

Current: I

Electromotive force: ε

Magnetic flux: ψ

1117. Line Integral of a Scalar Function

Let a curve C be given by the vector function $\vec{r} = \vec{r}(s)$,

$0 \leq s \leq S$, and a **scalar function** F is defined over the curve C .

Then

$$\int_0^S F(\vec{r}(s))ds = \int_C F(x,y,z)ds = \int_C Fds,$$

where ds is the arc length differential.

$$1118. \int_{C_1 \cup C_2} F \, ds = \int_{C_1} F \, ds + \int_{C_2} F \, ds$$

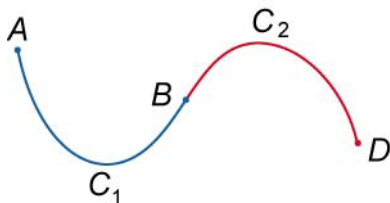


Figure 203.

- 1119.** If the smooth curve C is parametrized by $\vec{r} = \vec{r}(t)$, $\alpha \leq t \leq \beta$, then

$$\int_C F(x, y, z) \, ds = \int_{\alpha}^{\beta} F(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt.$$

- 1120.** If C is a smooth curve in the xy -plane given by the equation $y = f(x)$, $a \leq x \leq b$, then

$$\int_C F(x, y) \, ds = \int_a^b F(x, f(x)) \sqrt{1 + (f'(x))^2} \, dx.$$

- 1121.** Line Integral of Scalar Function in Polar Coordinates

$$\int_C F(x, y) \, ds = \int_{\alpha}^{\beta} F(r \cos \theta, r \sin \theta) \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} \, d\theta,$$

where the curve C is defined by the polar function $r(\theta)$.

- 1122.** Line Integral of Vector Field

Let a curve C be defined by the vector function $\vec{r} = \vec{r}(s)$, $0 \leq s \leq S$. Then

$$\frac{d\vec{r}}{ds} = \vec{\tau} = (\cos \alpha, \cos \beta, \cos \gamma)$$

is the unit vector of the tangent line to this curve.

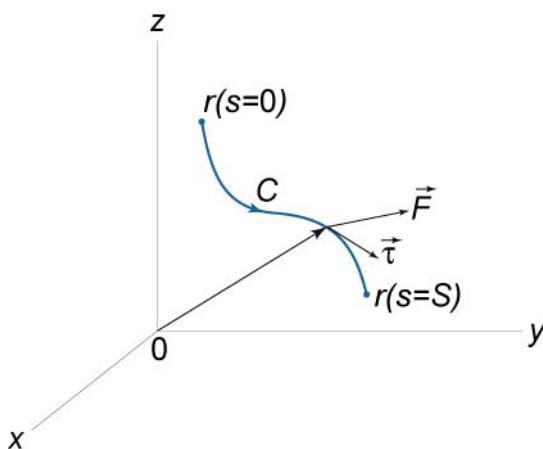


Figure 204.

Let a **vector field** $\vec{F}(P, Q, R)$ is defined over the curve C . Then the line integral of the vector field \vec{F} along the curve C is

$$\int_C P dx + Q dy + R dz = \int_0^s (P \cos \alpha + Q \cos \beta + R \cos \gamma) ds.$$

1123. Properties of Line Integrals of Vector Fields

$$\int_{-C} (\vec{F} \cdot d\vec{r}) = - \int_C (\vec{F} \cdot d\vec{r}),$$

where $-C$ denote the curve with the opposite orientation.

$$\int_C (\vec{F} \cdot d\vec{r}) = \int_{C_1 \cup C_2} (\vec{F} \cdot d\vec{r}) = \int_{C_1} (\vec{F} \cdot d\vec{r}) + \int_{C_2} (\vec{F} \cdot d\vec{r}),$$

where C is the union of the curves C_1 and C_2 .

1124. If the curve C is parameterized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $\alpha \leq t \leq \beta$, then

$$\int P dx + Q dy + R dz =$$

$$\int_C^c \left(P(x(t), y(t), z(t)) \frac{dx}{dt} + Q(x(t), y(t), z(t)) \frac{dy}{dt} + R(x(t), y(t), z(t)) \frac{dz}{dt} \right) dt$$

1125. If C lies in the xy -plane and given by the equation $y = f(x)$, then

$$\int_C P dx + Q dy = \int_a^b \left(P(x, f(x)) + Q(x, f(x)) \frac{df}{dx} \right) dx.$$

1126. Green's Theorem

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_C P dx + Q dy,$$

where $\vec{F} = P(x, y)\vec{i} + Q(x, y)\vec{j}$ is a continuous vector function with continuous first partial derivatives $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$ in a some domain R , which is bounded by a closed, piecewise smooth curve C .

1127. Area of a Region R Bounded by the Curve C

$$S = \iint_R dx dy = \frac{1}{2} \oint_C x dy - y dx$$

1128. Path Independence of Line Integrals

The line integral of a vector function $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is said to be **path independent**, if and only if P , Q , and R are continuous in a domain D , and if there exists some scalar function $u = u(x, y, z)$ (a **scalar potential**) in D such that

$$\vec{F} = \text{grad } u, \text{ or } \frac{\partial u}{\partial x} = P, \frac{\partial u}{\partial y} = Q, \frac{\partial u}{\partial z} = R.$$

Then

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_C P dx + Q dy + R dz = u(B) - u(A).$$

1129. Test for a Conservative Field

A vector field of the form $\vec{F} = \text{grad } u$ is called a **conservative field**. The line integral of a vector function $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is path independent if and only if

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{0}.$$

If the line integral is taken in xy-plane so that

$$\int_C Pdx + Qdy = u(B) - u(A),$$

then the test for determining if a vector field is conservative can be written in the form

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

1130. Length of a Curve

$$L = \int_C ds = \int_{\alpha}^{\beta} \left| \frac{d\vec{r}}{dt}(t) \right| dt = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2} dt,$$

where C is a piecewise smooth curve described by the position vector $\vec{r}(t)$, $\alpha \leq t \leq \beta$.

If the curve C is two-dimensional, then

$$L = \int_C ds = \int_{\alpha}^{\beta} \left| \frac{d\vec{r}}{dt}(t) \right| dt = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt.$$

If the curve C is the graph of a function $y = f(x)$ in the xy-plane ($a \leq x \leq b$), then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx.$$

1131. Length of a Curve in Polar Coordinates

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} d\theta,$$

where the curve C is given by the equation $r = r(\theta)$,
 $\alpha \leq \theta \leq \beta$ in polar coordinates.

1132. Mass of a Wire

$$m = \int_C \rho(x, y, z) ds,$$

where $\rho(x, y, z)$ is the mass per unit length of the wire.

If C is a curve parametrized by the vector function
 $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, then the mass can be computed by
the formula

$$m = \int_{\alpha}^{\beta} \rho(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt.$$

If C is a curve in xy -plane, then the mass of the wire is given
by

$$m = \int_C \rho(x, y) ds,$$

or

$$m = \int_{\alpha}^{\beta} \rho(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \text{ (in parametric form).}$$

1133. Center of Mass of a Wire

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m},$$

where

$$M_{yz} = \int_C x \rho(x, y, z) ds,$$

$$M_{xz} = \int_C y\rho(x, y, z)ds,$$

$$M_{xy} = \int_C z\rho(x, y, z)ds.$$

1134. Moments of Inertia

The moments of inertia about the x-axis, y-axis, and z-axis are given by the formulas

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z)ds,$$

$$I_y = \int_C (x^2 + z^2)\rho(x, y, z)ds,$$

$$I_z = \int_C (x^2 + y^2)\rho(x, y, z)ds.$$

1135. Area of a Region Bounded by a Closed Curve

$$S = \oint_C xdy = -\oint_C ydx = \frac{1}{2} \oint_C xdy - ydx.$$

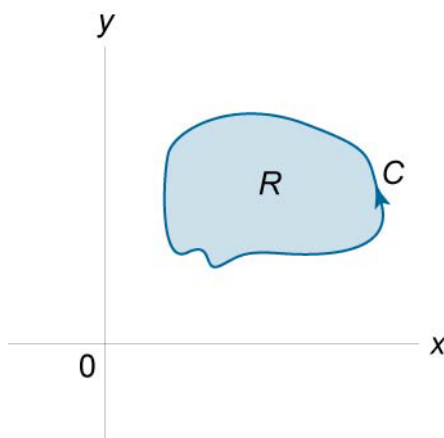


Figure 205.

If the closed curve C is given in parametric form

$\vec{r}(t) = \langle x(t), y(t) \rangle$, then the area can be calculated by the formula

$$S = \int_{\alpha}^{\beta} x(t) \frac{dy}{dt} dt = - \int_{\alpha}^{\beta} y(t) \frac{dx}{dt} dt = \frac{1}{2} \int_{\alpha}^{\beta} \left(x(t) \frac{dy}{dt} - y(t) \frac{dx}{dt} \right) dt.$$

1136. Volume of a Solid Formed by Rotating a Closed Curve about the x-axis

$$V = -\pi \oint_C y^2 dx = -2\pi \oint_C xy dy = -\frac{\pi}{2} \oint_C 2xy dy + y^2 dx$$

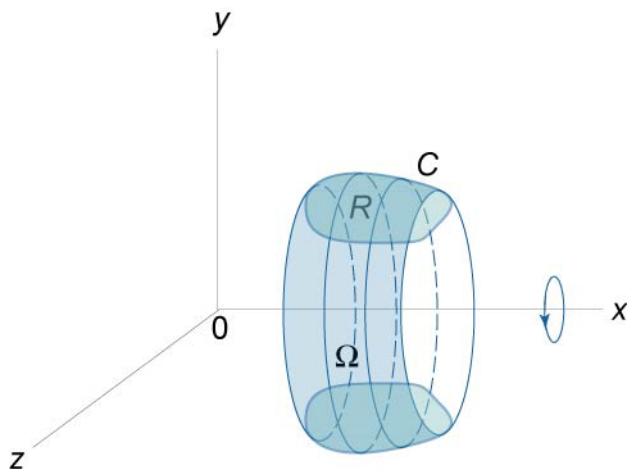


Figure 206.

1137. Work

Work done by a force \vec{F} on an object moving along a curve C is given by the line integral

$$W = \int_C \vec{F} \cdot d\vec{r},$$

where \vec{F} is the vector force field acting on the object, $d\vec{r}$ is the unit tangent vector.

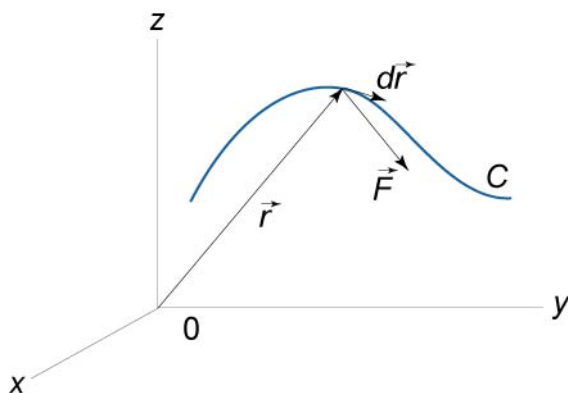


Figure 207.

If the object is moved along a curve C in the xy -plane, then

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy,$$

If a path C is specified by a parameter t (t often means time), the formula for calculating work becomes

$$W = \int_{\alpha}^{\beta} \left[P(x(t), y(t), z(t)) \frac{dx}{dt} + Q(x(t), y(t), z(t)) \frac{dy}{dt} + R(x(t), y(t), z(t)) \frac{dz}{dt} \right] dt,$$

where t goes from α to β .

If a vector field \vec{F} is conservative and $u(x, y, z)$ is a scalar potential of the field, then the work on an object moving from A to B can be found by the formula

$$W = u(B) - u(A).$$

1138. Ampere's Law

$$\oint_C \vec{B} \cdot d\vec{r} = \mu_0 I.$$

The line integral of a magnetic field \vec{B} around a closed path C is equal to the total current I flowing through the area bounded by the path.

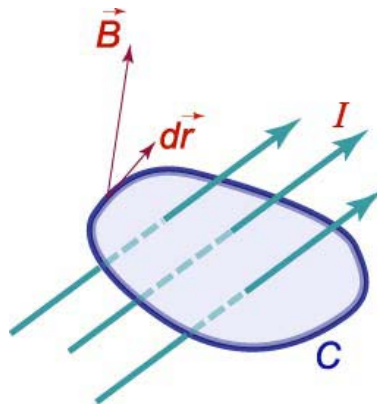


Figure 208.

1139. Faraday's Law

$$\varepsilon = \oint_C \vec{E} \cdot d\vec{r} = -\frac{d\psi}{dt}$$

The electromotive force (emf) ε induced around a closed loop C is equal to the rate of the change of magnetic flux ψ passing through the loop.

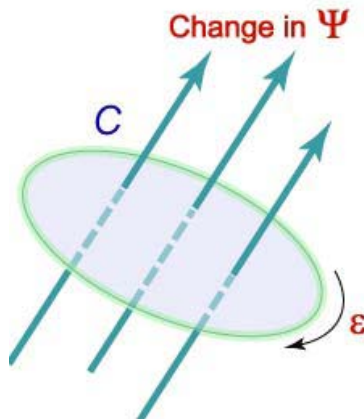


Figure 209.