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Applications of Derivatives

TOPIC 1 Rate of Change of Quantities

1. The position of a moving car at time t is given by $f(t) = at^2 + bt + c, t > 0$, where a, b and c are real numbers greater than 1. Then the average speed of the car over the time interval $[t_1, t_2]$ is attained at the point : **[Sep. 06, 2020 (I)]**
   (a) $(t_2 - t_1)/2$ (b) $a(t_2 - t_1) + b$
   (c) $(t_1 + t_2)/2$ (d) $2a(t_1 + t_2) + b$

2. If the surface area of a cube is increasing at a rate of 3.6 cm^2/sec, retaining its shape; then the rate of change of its volume (in cm^3/sec.), when the length of a side of the cube is 10 cm, is : **[Sep. 03, 2020 (II)]**
   (a) 18 (b) 10
   (c) 20 (d) 9

3. If a function $f(x)$ defined by **[Sep. 02, 2020 (I)]**
$$f(x) = \begin{cases} ae^x + be^{-x}, & -1 \le x < 1 \\ cx^2, & 1 \le x \le 3 \\ ax^2 + 2cx, & 3 < x \le 4 \end{cases}$$
be continuous for some $a, b, c \in \mathbf{R}$ and $f'(0) + f'(2) = e$, then the value of a is :
   (a) $\frac{1}{e^2 - 3e + 13}$ (b) $\frac{e}{e^2 - 3e - 13}$
   (c) $\frac{e}{e^2 + 3e + 13}$ (d) $\frac{e}{e^2 - 3e + 13}$

4. A spherical iron ball of 10 cm radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of ice is 5 cm, then the rate (in cm/min.) at which of the thickness of ice decreases, is: **[Jan. 9, 2020 (I)]**
   (a) $\frac{5}{6\pi}$ (b) $\frac{1}{54\pi}$
   (c) $\frac{1}{36\pi}$ (d) $\frac{1}{18\pi}$

5. A 2 m ladder leans against a vertical wall. If the top of the ladder begins to slide down the wall at the rate 25 cm/sec., then the rate (in cm/sec.) at which the bottom of the ladder slides away from the wall on the horizontal ground when the top of the ladder is 1 m above the ground is: **[April 12, 2019 (I)]**
   (a) $25\sqrt{3}$ (b) $\frac{25}{\sqrt{3}}$
   (c) $\frac{25}{3}$ (d) 25

6. A spherical iron ball of radius 10 cm is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of the ice is 5 cm, then the rate at which the thickness (in cm/min) of the ice decreases, is : **[April 10, 2019 (II)]**
   (a) $\frac{1}{18\pi}$ (b) $\frac{1}{36\pi}$
   (c) $\frac{5}{6\pi}$ (d) $\frac{1}{9\pi}$

7. A water tank has the shape of an inverted right circular cone, whose semi-vertical angle is $\tan^{-1}$. Water is poured into it at a constant rate of 5 cubic meter per minute. Then the rate (in m/min.), at which the level of water is rising at the instant when the depth of water in the tank is 10m; is: **[April 09, 2019 (II)]**
   (a) $1/15\pi$ (b) $1/10\pi$
   (c) $2/\pi$ (d) $1/5\pi$

8. If the volume of a spherical ball is increasing at the rate of 4π cc/sec, then the rate of increase of its radius (in cm/sec), when the volume is 288π cc, **[Online April 19, 2014]**
   (a) $\frac{1}{6}$ (b) $\frac{1}{9}$
   (c) $\frac{1}{36}$ (d) $\frac{1}{24}$

9. Two ships A and B are sailing straight away from a fixed point O along routes such that $\angle AOB$ is always 120°. At a certain instance, OA = 8 km, OB = 6 km and the ship A is sailing at the rate of 20 km/hr while the ship B sailing at the rate of 30 km/hr. Then the distance between A and B is changing at the rate (in km/hr): **[Online April 11, 2014]**

(a) $\frac{260}{\sqrt{37}}$ (b) $\frac{260}{37}$

(c) $\frac{80}{\sqrt{37}}$ (d) $\frac{80}{37}$

10. A spherical balloon is being inflated at the rate of 35cc/min. The rate of increase in the surface area (in cm^2/min.) of the balloon when its diameter is 14 cm, is : **[Online April 25, 2013]**

(a) 10 (b) $\sqrt{10}$

(c) 100 (d) $10\sqrt{10}$

11. If the surface area of a sphere of radius r is increasing uniformly at the rate 8 cm^2/s, then the rate of change of its volume is : **[Online April 9, 2013]**

(a) constant (b) proportional to $\sqrt{r}$

(c) proportional to r^2 (d) proportional to r

12. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is: **[2012]**

(a) $\frac{9}{7}$ (b) $\frac{7}{9}$

(c) $\frac{2}{9}$ (d) $\frac{9}{2}$

13. If a metallic circular plate of radius 50 cm is heated so that its radius increases at the rate of 1 mm per hour, then the rate at which, the area of the plate increases (in cm^2/hour) is **[Online May 26, 2012]**

(a) 5π (b) 10π

(c) 100π (d) 50π

14. The weight W of a certain stock of fish is given by $W = nw$, where n is the size of stock and w is the average weight of a fish. If n and w change with time t as $n = 2t^2 + 3$ and $w = t^2 - t + 2$, then the rate of change of W with respect to t at $t = 1$ is **[Online May 19, 2012]**

(a) 1 (b) 8

(c) 13 (d) 5

15. Consider a rectangle whose length is increasing at the uniform rate of 2 *m/sec*, breadth is decreasing at the uniform rate of 3 *m/sec* and the area is decreasing at the uniform rate of 5 m^2/sec. If after some time the breadth of the rectangle is 2 m then the length of the rectangle is **[Online May 12, 2012]**

(a) 2m (b) 4m

(c) 1m (d) 3m

16. If a circular iron sheet of radius 30 cm is heated such that its area increases at the uniform rate of 6π cm^2/hr, then the rate (in mm/hr) at which the radius of the circular sheet increases is **[Online May 7, 2012]**

(a) 1.0 (b) 0.1

(c) 1.1 (d) 2.0

17. Two points A and B move from rest along a straight line with constant acceleration f and f' respectively. If A takes m sec. more than B and describes 'n' units more than B in acquiring the same speed then **[2005]**

(a) $(f - f')m^2 = ff'n$

(b) $(f + f')m^2 = ff'n$

(c) $\frac{1}{2}(f + f')m = ff'n^2$

(d) $(f' - f)n = \frac{1}{2}ff'm^2$

18. A lizard, at an initial distance of 21 cm behind an insect, moves from rest with an acceleration of $2\ cm/s^2$ and pursues the insect which is crawling uniformly along a straight line at a speed of 20 cm/s. Then the lizard will catch the insect after **[2005]**

(a) 20 s (b) 1 s

(c) 21 s (d) 24 s

19. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of ice is 5 cm, then the rate at which the thickness of ice decreases is **[2005]**

(a) $\frac{1}{36\pi}$ cm/min. (b) $\frac{1}{18\pi}$ cm/min.

(c) $\frac{1}{54\pi}$ cm/min. (d) $\frac{5}{6\pi}$ cm/min

20. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is **[2004]**

(a) $\left(\frac{9}{8}, \frac{9}{2}\right)$ (b) $(2, -4)$

(c) $\left(\frac{-9}{8}, \frac{9}{2}\right)$ (d) $(2, 4)$

TOPIC 2 Increasing & Decreasing Functions

21. The function, $f(x) = (3x - 7)x^{2/3}$, $x \in \mathbf{R}$, is increasing for all x lying in : **[Sep. 03, 2020 (I)]**

(a) $(-\infty, 0) \cup \left(\frac{14}{15}, \infty\right)$ (b) $(-\infty, 0) \cup \left(\frac{3}{7}, \infty\right)$

(c) $\left(-\infty, \frac{14}{15}\right)$ (d) $\left(-\infty, -\frac{14}{15}\right) \cup (0, \infty)$

22. Let f be any function continuous on $[a, b]$ and twice differentiable on (a, b). If for all $x \in (a, b), f2(x) > 0$ and $f''(x) < 0$, then for any $c \in (a, b)$, $\frac{f(c)-f(a)}{f(b)-f(c)}$ is greater than: **[Jan. 9, 2020 (I)]**

(a) $\frac{b+a}{b-a}$ (b) 1

(c) $\frac{b-c}{c-a}$ (d) $\frac{c-a}{b-c}$

23. Let $f(x) = x\cos^{-1}(-\sin|x|)$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then which of the following is true? **[Jan. 8, 2020 (I)]**

(a) f' is increasing in $\left(-\frac{\pi}{2}, 0\right)$ and decreasing in $\left(0, \frac{\pi}{2}\right)$

(b) $f'(0) = -\frac{\pi}{2}$

(c) f' is not differentiable at $x = 0$

(d) f' is decreasing in $\left(-\frac{\pi}{2}, 0\right)$ and increasing in $\left(0, \frac{\pi}{2}\right)$

24. Let $f(x) = e^x - x$ and $g(x) = x^2 - x$, $x \in \mathbf{R}$. Then the set of all $x \in \mathbf{R}$, where the function $h(x) = (fog)(x)$ is increasing, is : **[April 10, 2019 (I)]**

(a) $\left[-1, \frac{-1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$ (b) $\left[0, \frac{1}{2}\right] \cup [1, \infty)$

(c) $[0, \infty)$ (d) $\left[\frac{-1}{2}, 0\right] \cup [1, \infty)$

25. If the function $f : R - \{1, -1\} \to A$ defined by $f(x) = \frac{x^2}{1-x^2}$, is surjective, then A is equal to: **[April 09, 2019 (I)]**

(a) $R - \{-1\}$ (b) [0, “)

(c) $R - [-1, 0)$ (d) $R - (-1, 0)$

26. Let $f: [0 : 2] \to R$ be a twice differentiable function such that $f''(x) > 0$, for all $x \in (0, 2)$. If $\phi(x) = f(x) + f(2 - x)$, then ϕ is : **[April 08, 2019 (I)]**

(a) increasing on (0, 1) and decreasing on (1, 2).
(b) decreasing on (0, 2)
(c) decreasing on (0, 1) and increasing on (1, 2).
(d) increasing on (0, 2)

27. If the function f given by $f(x) = x^3 - 3(a - 2)x^2 + 3ax + 7$, for some $a \in R$ is increasing in (0, 1] and decreasing in [1, 5), then a root of the equation, $\frac{f(x)-14}{(x-1)^2} = 0 (x \neq 1)$ is **[Jan. 12, 2019 (II)]**

(a) –7 (b) 5

(c) 7 (d) 6

28. Let $f(x) = \frac{x}{\sqrt{a^2+x^2}} - \frac{d-x}{\sqrt{b^2+(d-x)^2}}, x \in \mathbf{R}$ where a, b and d are non-zero real constants. Then : **[Jan. 11, 2019 (II)]**

(a) f is an increasing function of x
(b) f is a decreasing function of x
(c) f' is not a continuous function of x
(d) f is neither increasing nor decreasing function of x

29. The function f defined by $f(x) = x^3 - 3x^2 + 5x + 7$, is : **[Online April 9, 2017]**

(a) increasing in R.
(b) decreasing in R.
(c) decreasing in $(0, \infty)$ and increasing in $(-\infty, 0)$.
(d) increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$.

30. Let $f(x) = \sin^4 x + \cos^4 x$. Then f is an increasing function in the interval :

(a) $\left]\frac{5\pi}{8}, \frac{3\pi}{4}\right]$ (b) $\left]\frac{\pi}{2}, \frac{5\pi}{8}\right]$

(c) $\left]\frac{\pi}{4}, \frac{\pi}{2}\right]$ (d) $\left]0, \frac{\pi}{4}\right]$

31. Let f and g be two differentiable functions on R such that $f'(x) > 0$ and $g'(x) < 0$ for all $x \in R$. Then for all x: **[Online April 12, 2014]**

(a) $f(g(x)) > f(g(x-1))$ (b) $f(g(x)) > f(g(x+1))$

(c) $g(f(x)) > g(f(x-1))$ (d) $g(f(x)) < g(f(x+1))$

32. The real number k for which the equation, $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1] **[2013]**

(a) lies between 1 and 2
(b) lies between 2 and 3
(c) lies between .1 and 0
(d) does not exist.

33. **Statement-1:** The function $x^2(e^x + e^{-x})$ is increasing for all $x > 0$.

Statement-2: The functions x^2e^x and x^2e^{-x} are increasing for all $x > 0$ and the sum of two increasing functions in any interval (a, b) is an increasing function in (a, b). **[Online April 22, 2013]**

(a) Statement-1 is false; Statement-2 is true.
(b) Statement-1is true; Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.
(c) Statement-1 is true; Statement-2 is false.
(d) Statement-1is true; Statement-2 is true; Statement-2 is a correct explanation for statement-1.

34. **Statement-1:** The equation $x \log x = 2 - x$ is satisfied by at least one value of x lying between 1 and 2.

Statement-2: The function $f(x) = x \log x$ is an increasing function in [1, 2] and $g(x) = 2 - x$ is a decreasing function in [1, 2] and the graphs represented by these functions intersect at a point in [1, 2] **[Online April 9, 2013]**

(a) Statement-1 is true; Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is true; Statement-2 is true; Statement-2 is **not** correct explanation for Statement-1.
(c) Statement-1 is false, Statement-2 is true.
(d) Statement-1 is true, Statement-2 is false.

35. If $f(x)=xe^{x(1-x)},\ x\in R$, then $f(x)$ is **[Online May 12, 2012]**
(a) decreasing on $[-1/2, 1]$
(b) decreasing on R
(c) increasing on $[-1/2, 1]$
(d) increasing on R

36. For real x, let $f(x)=x^3+5x+1$, then **[2009]**
(a) f is onto R but not one-one
(b) f is one-one and onto R
(c) f is neither one-one nor onto R
(d) f is one-one but not onto R

37. How many real solutions does the equation $x^7+14x^5+16x^3+30x-560=0$ have? **[2008]**
(a) 7 (b) 1
(c) 3 (d) 5

38. The function $f(x)=\tan^{-1}(\sin x+\cos x)$ is an increasing function in **[2007]**
(a) $\left(0,\frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
(c) $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ (d) $\left(-\frac{\pi}{2},\frac{\pi}{4}\right)$

39. A function is matched below against an interval where it is supposed to be increasing. Which of the following pairs is incorrectly matched? **[2005]**

	Interval	*Function*
(a)	$(-\infty,\infty)$	x^3-3x^2+3x+3
(b)	$[2,\infty)$	$2x^3-3x^2-12x+6$
(c)	$\left(-\infty,\frac{1}{3}\right]$	$3x^2-2x+1$
(d)	$(-\infty,-4)$	x^3+6x^2+6

TOPIC 3 Tangents & Normals

40. If the tangent to the curve, $y=f(x)=x\log_e x$, $(x>0)$ at a point (c, f(c)) is parallel to the line segement joining the points (1, 0) and (e, e), then c is equal to: **[Sep. 06, 2020 (II)]**
(a) $\frac{e-1}{e}$ (b) $e^{\left(\frac{1}{e-1}\right)}$
(c) $e^{\left(\frac{1}{1-e}\right)}$ (d) $\frac{e}{e-1}$

41. Which of the following points lies on the tangent to the curve $x^4e^y+2\sqrt{y+1}=3$ at the point (1, 0)? **[Sep. 05, 2020 (II)]**
(a) (2, 2) (b) (2, 6)
(c) (–2, 6) (d) (–2, 4)

42. If the lines $x+y=a$ and $x-y=b$ touch the curve $y=x^2-3x+2$ at the points where the curve intersects the x-axis, then $\frac{a}{b}$ is equal to _____. **[NA Sep. 05, 2020 (II)]**

43. If the tangent to the curve, $y=e^x$ at a point (c, e^c) and the normal to the parabola, $y^2=4x$ at the point (1, 2) intersect at the same point on the x-axis, then the value of c is _____________. **[NA Sep. 03, 2020 (II)]**

44. If $y=\sum_{k=1}^{6}k\cos^{-1}\left\{\frac{3}{5}\cos kx-\frac{4}{5}\sin kx\right\}$, then $\frac{dy}{dx}$ at $x=0$ is ___________. **[NA Sep. 02, 2020 (II)]**

45. Let the normal at a point P on the curve $y^2-3x^2+y+10=0$ intersect the y-axis at $\left(0,\frac{3}{2}\right)$. If m is the slope of the tangent at P to the curve, then $|m|$ is equal to ______. **[NA Jan. 8, 2020 (I)]**

46. The length of the perpendicular from the origin, on the normal to the curve, $x^2+2xy-3y^2=0$ at the point (2, 2) is: **[Jan. 8, 2020 (II)]**
(a) $\sqrt{2}$ (b) $4\sqrt{2}$
(c) 2 (d) $2\sqrt{2}$

47. If the tangent to the curve $y=\frac{x}{x^2-3}$, $x\in R,\left(x\neq\pm\sqrt{3}\right)$, at a point (α, β) (0, 0) on it is parallel to the line $2x+6y-11=0$, then : **[April 10, 2019 (II)]**
(a) $|6\alpha+2\beta|=19$ (b) $|6\alpha+2\beta|=9$
(c) $|2\alpha+6\beta|=19$ (d) $|2\alpha+6\beta|=11$

48. If the tangent to the curve, $y=x^3+ax-b$ at the point (1, –5) is perpendicular to the line, $-x+y+4=0$, then which one of the following points lies on the curve? **[April 09, 2019 (I)]**
(a) (–2, 1) (b) (–2, 2)
(c) (2, –1) (d) (2, –2)

49. Let S be the set of all values of x for which the tangent to the curve $y=f(x)=x^3-x^2-2x$ at (x, y) is parallel to the line segment joining the points (1, f(1)) and (– 1, f(– 1)), then S is equal to: **[April 09, 2019 (I)]**
(a) $\left\{\frac{1}{3},1\right\}$ (b) $\left\{-\frac{1}{3},-1\right\}$
(c) $\left\{\frac{1}{3},-1\right\}$ (d) $\left\{-\frac{1}{3},1\right\}$

50. The tangent and the normal lines at the point ($\sqrt{3}$, 1) to the circle $x^2 + y^2 = 4$ and the x-axis form a triangle. The area of this triangle (in square units) is : **[April 08, 2019 (II)]**

(a) $\frac{4}{\sqrt{3}}$ (b) $\frac{1}{3}$

(c) $\frac{2}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{3}}$

51. The maximum area (in sq. units) of a rectangle having its base on the x-axis and its other two vertices on the parabola, $y = 12 - x^2$ such that the rectangle lies inside the parabola, is: **[Jan. 12, 2019 (I)]**

(a) 36 (b) $20\sqrt{2}$

(c) 32 (d) $18\sqrt{3}$

52. The tangent to the curve $y = x^2 - 5x + 5$, parallel to the line $2y = 4x + 1$, also passes through the point : **[Jan. 12, 2019 (II)]**

(a) $\left(\frac{7}{2}, \frac{1}{4}\right)$ (b) $\left(\frac{1}{8}, -7\right)$

(c) $\left(-\frac{1}{8}, 7\right)$ (d) $\left(\frac{1}{4}, \frac{7}{2}\right)$

53. The shortest distance between the point $\left(\frac{3}{2}, 0\right)$ and the curve $y = \sqrt{x}, (x > 0)$, is: **[Jan. 10, 2019 (I)]**

(a) $\frac{\sqrt{5}}{2}$ (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{3}{2}$ (d) $\frac{5}{4}$

54. The tangent to the curve, $y = xe^{x^2}$ passing through the point (1, e) also passes through the point: **[Jan. 10, 2019 (II)]**

(a) (2, 3e) (b) $\left(\frac{4}{3}, 2e\right)$

(c) $\left(\frac{5}{3}, 2e\right)$ (d) (3, 6e)

55. A helicopter is flying along the curve given by $y - x^{3/2} = 7, (x \geq 0)$. A soldier positioned at the point $\left(\frac{1}{2}, 7\right)$ wants to shoot down the helicopter when it is nearest to him. Then this nearest distance is: **[Jan. 10, 2019 (II)]**

(a) $\frac{\sqrt{5}}{6}$ (b) $\frac{1}{3}\sqrt{\frac{7}{3}}$

(c) $\frac{1}{6}\sqrt{\frac{7}{3}}$ (d) $\frac{1}{2}$

56. If θ denotes the acute angle between the curves, $y = 10 - x^2$ and $y = 2 + x^2$ at a point of their intersection, then $|\tan\theta|$ is equal to: **[Jan. 09, 2019 (I)]**

(a) $\frac{4}{9}$ (b) $\frac{8}{15}$

(c) $\frac{7}{17}$ (d) $\frac{8}{17}$

57. If the curves $y^2 = 6x, 9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is : **[2018]**

(a) $\frac{7}{2}$ (b) 4

(c) $\frac{9}{2}$ (d) 6

58. Let P be a point on the parabola, $x^2 = 4y$. If the distance of P from the centre of the circle, $x^2 + y^2 + 6x + 8 = 0$ is minimum, then the equation of the tangent to the parabola at P, is **[Online April 16, 2018]**

(a) $x + 4y - 2 = 0$ (b) $x + 2y = 0$

(c) $x + y + 1 = 0$ (d) $x - y + 3 = 0$

59. If the tangents drawn to the hyperbola $4y^2 = x^2 + 1$ intersect the co-ordinate axes at the distinct points A and B, then the locus of the mid point of AB is **[Online April 15, 2018]**

(a) $x^2 - 4y^2 + 16x^2y^2 = 0$

(b) $4x^2 - y^2 + 16x^2y^2 = 0$

(c) $4x^2 - y^2 - 16x^2y^2 = 0$

(d) $x^2 - 4y^2 - 16x^2y^2 = 0$

60. If β is one of the angles between the normals to the ellipse, $x^2 + 3y^2 = 9$ at the points $(3\cos\theta, \sqrt{3}\sin\theta)$ and $(-3\sin\theta, \sqrt{3}\cos\theta)$; $\in\left(0, \frac{\pi}{2}\right)$; then $\frac{2\cot\beta}{\sin 2\theta}$ is equal to **[Online April 15, 2018]**

(a) $\sqrt{2}$ (b) $\frac{2}{\sqrt{3}}$

(c) $\frac{1}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{4}$

61. A normal to the hyperbola, $4x^2 - 9y^2 = 36$ meets the co-ordinate axes x and y at A and B, respectively. If the parallelogram $OABP$ (O being the origin) is formed, then the locus of P is **[Online April 15, 2018]**

(a) $4x^2 - 9y^2 = 121$

(b) $4x^2 + 9y^2 = 121$

(c) $9x^2 - 4y^2 = 169$

(d) $9x^2 + 4y^2 = 169$

62. The normal to the curve $y(x-2)(x-3)=x+6$ at the point where the curve intersects the y-axis passes through the point: **[2017]**

(a) $\left(\frac{1}{2},\frac{1}{3}\right)$ (b) $\left(-\frac{1}{2},-\frac{1}{2}\right)$

(c) $\left(\frac{1}{2},\frac{1}{2}\right)$ (d) $\left(\frac{1}{2},-\frac{1}{3}\right)$

63. The eccentricity of an ellipse whose centre is at the origin is $\frac{1}{2}$. If one of its directices is $x=-4$, then the equation of the normal to it at $\left(1,\frac{3}{2}\right)$ is : **[2017]**

(a) $x+2y=4$ (b) $2y-x=2$

(c) $4x-2y=1$ (d) $4x+2y=7$

64. A tangent to the curve, $y=f(x)$ at $P(x,y)$ meets x-axis at A and y-axis at B. If AP : BP = 1 : 3 and $f(a)=1$, then the curve also passes through the point : **[Online April 9, 2017]**

(a) $\left(\frac{1}{3},24\right)$ (b) $\left(\frac{1}{2},4\right)$

(c) $\left(2,\frac{1}{8}\right)$ (d) $\left(3,\frac{1}{28}\right)$

65. The tangent at the point (2, –2) to the curve, $x^2y^2-2x=4(1-y)$ does not pass through the point : **[Online April 8, 2017]**

(a) $\left(4,\frac{1}{3}\right)$ (b) (8, 5)

(c) (–4, –9) (d) (–2, –7)

66. Consider

$$f(x)=\tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x\in\left(0,\frac{\pi}{2}\right).$$ **[2016]**

A normal to y = f(x) at $x=\frac{\pi}{6}$ also passes through the point:

(a) $\left(\frac{\pi}{6},0\right)$ (b) $\left(\frac{\pi}{4},0\right)$

(c) (0, 0) (d) $\left(0,\frac{2\pi}{3}\right)$

67. Let C be a curve given by $y(x)=1+\sqrt{4x-3}, x>\frac{3}{4}$. If P is a point on C, such that the tangent at P has slope $\frac{2}{3}$, then a point through which the normal at P passes, is : **[Online April 10, 2016]**

(a) (1, 7) (b) (3, –4)

(c) (4, –3) (d) (2, 3)

68. If the tangent at a point P, with parameter t, on the curve $x=4t^2+3$, $y=8t^3-1$, $t\in R$, meets the curve again at a point Q, then the coordinates of Q are : **[Online April 9, 2016]**

(a) $(16t^2+3, -64t^3-1)$ (b) $(4t^2+3, -8t^3-2)$

(c) (t^2+3, t^3-1) (d) $(t^2+3, -t^3-1)$

69. The normal to the curve, $x^2+2xy-3y^2=0$, at (1, 1) **[2015]**

(a) meets the curve again in the third quadrant.
(b) meets the curve again in the fourth quadrant.
(c) does not meet the curve again.
(d) meets the curve again in the second quadrant.

70. The equation of a normal to the curve,

$\sin y = x\sin\left(\frac{\pi}{3}+y\right)$ at x = 0, is :

[Online April 11, 2015]

(a) $2x-\sqrt{3}y=0$ (b) $2x+\sqrt{3}y=0$

(c) $2y-\sqrt{3}x=0$ (d) $2y+\sqrt{3}x=0$

71. If the tangent to the conic, $y-6=x^2$ at (2, 10) touches the circle, $x^2+y^2+8x-2y=k$ (for some fixed k) at a point (α,β); then (α,β) is : **[Online April 10, 2015]**

(a) $\left(-\frac{7}{17},\frac{6}{17}\right)$ (b) $\left(-\frac{4}{17},\frac{1}{17}\right)$

(c) $\left(-\frac{6}{17},\frac{10}{17}\right)$ (d) $\left(-\frac{8}{17},\frac{2}{17}\right)$

72. The distance, from the origin, of the normal to the curve,

$x=2\cos t+2t\sin t, y=2\sin t-2t\cos t$ at $t=\frac{\pi}{4}$, is :

[Online April 10, 2015]

(a) 2 (b) 4

(c) $\sqrt{2}$ (d) $2\sqrt{2}$

73. For the curve $y=3\sin\theta\cos\theta$, $x=e^{\theta}\sin\theta$, $0\le\theta\le\pi$, the tangent is parallel to x-axis when θ is:

[Online April 11, 2014]

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

74. If an equation of a tangent to the curve, $y-\cos(x+f), -1\ -1\le x\le 1+\pi$, is $x+2y=k$ then k is equal to : **[Online April 25, 2013]**

(a) 1 (b) 2

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

75. The equation of the normal to the parabola, $x^2=8y$ at $x=4$ is **[Online May 19, 2012]**

(a) $x+2y=0$ (b) $x+y=2$

(c) $x-2y=0$ (d) $x+y=6$

76. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is **[2010]**
(a) $y=1$ (b) $y=2$
(c) $y=3$ (d) $y=0$

77. Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is **[2006]**
(a) π (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

78. The normal to the curve **[2005]**
$x = a(\cos\theta + \theta \sin\theta), y = a(\sin\theta - \theta\cos\theta)$ at any point θ is such that
(a) it passes through the origin
(b) it makes an angle $\frac{\pi}{2} + \theta$ with the x- axis
(c) it passes through $\left(a\frac{\pi}{2}, -a\right)$
(d) It is at a constant distance from the origin

79. The normal to the curve $x = a(1+\cos\theta), y = a\sin\theta$ at 'θ' always passes through the fixed point **[2004]**
(a) (a, a) (b) $(0, a)$
(c) $(0, 0)$ (d) $(a, 0)$

80. A function $y = f(x)$ has a second order derivative $f''(x) = 6(x-1)$. If its graph passes through the point (2,1) and at that point the tangent to the graph is $y = 3x - 5$, then the function is **[2004]**
(a) $(x+1)^2$ (b) $(x-1)^3$
(c) $(x+1)^3$ (d) $(x-1)^2$

TOPIC 4 Approximations, Maxima & Minima

81. Let m and M be respectively the minimum and maximum values of **[Sep. 06, 2020 (I)]**

$$\begin{vmatrix} \cos^2 x & 1+\sin^2 x & \sin 2x \\ 1+\cos^2 x & \sin^2 x & \sin 2x \\ \cos^2 x & \sin^2 x & 1+\sin 2x \end{vmatrix}$$

Then the ordered pair (m, M) is equal to :
(a) $(-3, 3)$ (b) $(-3, -1)$
(c) $(-4, -1)$ (d) $(1, 3)$

82. Let AD and BC be two vertical poles at A and B respectively on a horizontal ground. If AD = 8 m, BC = 11 m and AB = 10 m; then the distance (in meters) of a point M on AB from the point A such that $MD^2 + MC^2$ is minimum is ______.
[NA Sep. 06, 2020 (I)]

83. The set of all real values of λ for which the function $f(x) = (1-\cos^2 x)\cdot(\lambda + \sin x)$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, has exactly one maxima and exactly minima, is:
[Sep. 06, 2020 (II)]
(a) $\left(-\frac{1}{2}, \frac{1}{2}\right) - \{0\}$ (b) $\left(-\frac{3}{2}, \frac{3}{2}\right)$
(c) $\left(-\frac{1}{2}, \frac{1}{2}\right)$ (d) $\left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\}$

84. If $x = 1$ is a critical point of the function $f(x) = (3x^2 + ax - 2 - a)e^x$, then : **[Sep. 05, 2020 (II)]**
(a) $x = 1$ and $x = -\frac{2}{3}$ are local minima of f.
(b) $x = 1$ and $x = -\frac{2}{3}$ are local maxima of f.
(c) $x = 1$ is a local maxima and $x = -\frac{2}{3}$ is a local minima of f.
(d) $x = 1$ is a local minima and $x = -\frac{2}{3}$ is a local maxima of f.

85. The area (in sq. units) of the largest rectangle $ABCD$ whose vertices A and B lie on the x-axis and vertices C and D lie on the parabola, $y = x^2 - 1$ below the x-axis, is :
[Sep. 04, 2020 (II)]
(a) $\frac{2}{3\sqrt{3}}$ (b) $\frac{1}{3\sqrt{3}}$
(c) $\frac{4}{3}$ (d) $\frac{4}{3\sqrt{3}}$

86. Suppose $f(x)$ is a polynomial of degree four, having critical points at –1, 0, 1. If $T = \{x \in \mathbf{R} \mid f(x) = f(0)\}$, then the sum of squares of all the elements of T is :
[Sep. 03, 2020 (II)]
(a) 4 (b) 6
(c) 2 (d) 8

87. Let $f(x)$ be a polynomial of degree 3 such that $f(-1) = 10$, $f(1) = -6$, $f(x)$ has a critical point at $x = -1$ and $f'(x)$ has a critical point at $x = 1$. Then $f(x)$ has a local minima at $x =$ ________. **[NA Jan. 8, 2020 (II)]**

88. Let $f(x)$ be a polynomial of degree 5 such that $x = \pm 1$ are its critical points. If = 4, then which one of the following is not true ? **[Jan. 7, 2020 (II)]**
(a) f is an odd function.
(b) $f(1) - 4f(-1) = 4$.
(c) $x = 1$ is a point of maxima and $x = -1$ is a point of minima of f.
(d) $x = 1$ is a point of minima and $x = -1$ is a point of maxima of f.

89. If m is the minimum value of k for which the function $f(x)=x\sqrt{kx-x^2}$ is increasing in the interval [0,3] and M is the maximum value of f in [0,3] when k = m, then the ordered pair (m, M) is equal to : **[April 12, 2019 (I)]**

(a) $(4,3\sqrt{2})$ (b) $(4,3\sqrt{3})$
(c) $(3,3\sqrt{3})$ (d) $(5,3\sqrt{6})$

90. Let a_1, a_2, a_3, ... be an A. P. with $a_6 = 2$. Then the common difference of this A.P., which maximises the product $a_1 a_4 a_5$, is : **[April 10, 2019 (II)]**

(a) $\frac{3}{2}$ (b) $\frac{8}{5}$
(c) $\frac{6}{5}$ (d) $\frac{2}{3}$

91. If S_1 and S_2 are respectively the sets of local minimum and local maximum points of the function, $f(x) = 9x^4 + 12x^3 - 36x^2 + 25, x \in R$, then : **[April 08, 2019 (I)]**

(a) $S_1 = \{-2\}; S_2 = \{0, 1\}$ (b) $S_1 = \{-2, 0\}; S_2 = \{1\}$
(c) $S_1 = \{-2, 1\}; S_2 = \{0\}$ (d) $S_1 = \{-1\}; S_2 = \{0, 2\}$

92. The height of a right circular cylinder of maximum volume inscribed in a sphere of radius 3 is : **[April 08, 2019 (II)]**

(a) $\sqrt{6}$ (b) $\frac{2}{3}\sqrt{3}$
(c) $2\sqrt{3}$ (d) $\sqrt{3}$

93. The maximum value of $3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right)$ for any real value of θ is: **[Jan. 12, 2019 (I)]**

(a) $\sqrt{19}$ (b) $\frac{\sqrt{79}}{2}$
(c) $\sqrt{34}$ (d) $\sqrt{31}$

94. Let P(4, –4) and Q(9, 6) be two points on the parabola, $y^2 = 4x$ and let this X be any point arc POQ of this parabola, where O is vertex of the parabola, such that the area of ΔPXQ is maximum. Then this minimum area (in sq. units) is: **[Jan. 12, 2019 (I)]**

(a) $\frac{75}{2}$ (b) $\frac{125}{4}$
(c) $\frac{625}{4}$ (d) $\frac{125}{2}$

95. The maximum value of the function $f(x) = 3x^3 - 18x^2 + 27x - 40$ on the set $S = \{x \in R : x^2 + 30 \le 11x\}$ is : **[Jan. 11, 2019 (I)]**

(a) – 122 (b) – 222
(c) 122 (d) 222

96. Let x, y be positive real numbers and m, n positive integers. The maximum value of the expression $\frac{x^m y^n}{(1+x^{2m})(1+y^{2n})}$ is : **[Jan. 11, 2019 (II)]**

(a) 1 (b) $\frac{1}{2}$
(c) $\frac{1}{4}$ (d) $\frac{m+n}{6mn}$

97. The maximum volume (in cu.m) of the right circular cone having slant height 3 m is: **[Jan. 09, 2019 (I)]**

(a) 6π (b) $3\sqrt{3}\,\pi$
(c) $\frac{4}{3}\pi$ (d) $2\sqrt{3}\,\pi$

98. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$, $x \in R - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of h(x) is : **[2018]**

(a) – 3 (b) $-2\sqrt{2}$
(c) $2\sqrt{2}$ (d) 3

99. Let M and m be respectively the absolute maximum and the absolute minimum values of the function, $f(x) = 2x^3 - 9x^2 + 12x + 5$ in the interval [0, 3]. Then $M - m$ is equal to **[Online April 16, 2018]**

(a) 1 (b) 5
(c) 4 (d) 9

100. If a right circularcone having maximum volume, is inscribed in a sphere of radius 3 cm, then the curved surface area (in cm^2) of this cone is **[Online April 15, 2018]**

(a) $8\sqrt{3}\pi$ (b) $6\sqrt{2}\pi$
(c) $6\sqrt{3}\pi$ (d) $8\sqrt{2}\pi$

101. Twenty metres of wire is available for fencing off a flower-bed in the form of a circular sector. Then the maximum area (in sq. m) of the flower-bed, is : **[2017]**

(a) 30 (b) 12.5
(c) 10 (d) 25

102. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then: **[2016]**

(a) x = 2r (b) 2x = r
(c) $2x = (\pi + 4)r$ (d) $(4 - \pi)x = \pi r$

103. The minimum distance of a point on the curve $y = x^2 - 4$ from the origin is : **[Online April 9, 2016]**

(a) $\frac{\sqrt{15}}{2}$ (b) $\sqrt{\frac{19}{2}}$
(c) $\sqrt{\frac{15}{2}}$ (d) $\frac{\sqrt{19}}{2}$

104. Let k and K be the minimum and the maximum values of the function $f(x) = \frac{(1+x)^{0.6}}{1+x^{0.6}}$ in [0, 1] respectively, then the ordered pair (k, K) is equal to : **[Online April 11, 2015]**

(a) $(2^{-0.4}, 1)$ (b) $(2^{-0.4}, 2^{0.6})$
(c) $(2^{-0.6}, 1)$ (d) $(1, 2^{0.6})$

105. From the top of a 64 metres high tower, a stone is thrown upwards vertically with the velocity of 48 m/s. The greatest height (in metres) attained by the stone, assuming the value of the gravitational acceleration g = 32 m s^2, is: **[Online April 11, 2015]**

(a) 128 (b) 88
(c) 112 (d) 100

106. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$ then **[2014]**

(a) $\alpha = 2, \beta = -\frac{1}{2}$ (b) $\alpha = 2, \beta = \frac{1}{2}$
(c) $\alpha = -6, \beta = \frac{1}{2}$ (d) $\alpha = -6, \beta = -\frac{1}{2}$

107. The minimum area of a triangle formed by any tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{81} = 1$ and the co-ordinate axes is: **[Online April 12, 2014]**

(a) 12 (b) 18
(c) 26 (d) 36

108. The volume of the largest possible right circular cylinder that can be inscribed in a sphere of radius = $\sqrt{3}$ is: **[Online April 11, 2014]**

(a) $\frac{4}{3}\sqrt{3}\pi$ (b) $\frac{8}{3}\sqrt{3}\pi$
(c) 4π (d) 2π

109. The cost of running a bus from A to B, is ₹$\left(av + \frac{b}{v}\right)$, where v km/h is the average speed of the bus. When the bus travels at 30 km/h, the cost comes out to be ₹ 75 while at 40 km/h, it is ₹ 65. Then the most economical speed (in km/ h) of the bus is : **[Online April 23, 2013]**

(a) 45 (b) 50
(c) 60 (d) 40

110. The maximum area of a right angled triangle with hypotenuse h is : **[Online April 22, 2013]**

(a) $\frac{h^2}{2\sqrt{2}}$ (b) $\frac{h^2}{2}$
(c) $\frac{h^2}{\sqrt{2}}$ (d) $\frac{h^2}{4}$

111. Let $a, b \in R$ be such that the function f given by $f(x) = ln|x| + bx^2 + ax, x \neq 0$ has extreme values at $x = -1$ and $x = 2$

Statement-1 : f has local maximum at $x = -1$ and *at* $x = 2$.

Statement-2 : $a = \frac{1}{2}$ and $b = \frac{-1}{4}$ **[2012]**

(a) Statement-1 is false, Statement-2 is true.
(b) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.
(c) Statement-1 is true, statement-2 is true; statement-2 is **not** a correct explanation for Statement-1.
(d) Statement-1 is true, statement-2 is false.

112. A line is drawn through the point (1,2) to meet the coordinate axes at P and Q such that it forms a triangle OPQ, where O is the origin. If the area of the triangle OPQ is least, then the slope of the line PQ is : **[2012]**

(a) $-\frac{1}{4}$ (b) -4
(c) -2 (d) $-\frac{1}{2}$

113. Let $f: (-\infty, \infty) \to (-\infty, \infty)$ be defined by $f(x) = x^3 + 1$. **[Online May 26, 2012]**

Statement 1: The function f has a local extremum at $x = 0$

Statement 2: The function f is continuous and differentiable on $(-\infty, \infty)$ and $f'(0) = 0$

(a) Statement 1 is true, Statement 2 is false.
(b) Statement 1 is true, Statement 2 is true, Statement 2 is a correct explanation for Statement 1.
(c) Statement 1 is true, Statement 2 is true, Statement 2 is not the correct explanation for Statement 1.
(d) Statement 1 is false, Statement 2 is true.

114. Let f be a function defined by - **[2011RS]**

$$f(x) = \begin{cases} \frac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Statement - 1 : $x = 0$ is point of minima of f

Statement - 2 : $f'(0) = 0$.

(a) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for statement-1.
(b) Statement-1 is true, statement-2 is true; statement-2 is NOT a correct explanation for statement-1.
(c) Statement-1 is true, statement-2 is false.
(d) Statement-1 is false, statement-2 is true.

115. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t} \sin t \, dt$. Then f has **[2011]**

(a) local minimum at π and 2π
(b) local minimum at π and local maximum at 2π
(c) local maximum at π and local minimum at 2π
(d) local maximum at π and 2π

116. Let f : R $\to$ R be a continuous function defined by

$$f(x)=\frac{1}{e^x+2e^{-x}}$$ **[2010]**

Statement -1 : $f(c)=\frac{1}{3}$, for some $c\in$ R.

Statement -2 : $0<f(x)\le\frac{1}{2\sqrt{2}}$, for all $x\in$ R

(a) Statement -1 is true, Statement -2 is true ; Statement -2 is **not** a correct explanation for Statement -1.
(b) Statement -1 is true, Statement -2 is false.
(c) Statement -1 is false, Statement -2 is true .
(d) Statement - 1 is true, Statement 2 is true ; Statement -2 is a correct explanation for Statement -1.

117. Let f: R $\to$ R be defined by

$$f(x)=\begin{cases}k-2x, & if\ \ x\le -1\\ 2x+3, & if\ \ x>-1\end{cases}$$

If f has a local minimum at $x=-1$, then a possible value of k is **[2010]**

(a) 0 (b) $-\frac{1}{2}$
(c) –1 (d) 1

118. Given P(x) = x^4 + ax^3 + bx^2 + cx + d such that x = 0 is the only real root of P' (x) = 0. If P(–1) < P(1), then in the interval [–1, 1] : **[2009]**

(a) P(–1) is not minimum but P(1) is the maximum of P
(b) P(–1) is the minimum but P(1) is not the maximum of P
(c) Neither P(–1) is the minimum nor P(1) is the maximum of P
(d) P(–1) is the minimum and P(1) is the maximum of P

119. Suppose the cubic x^3-px+q has three distinct real roots where $p>0$ and $q>0$. Then which one of the following holds? **[2008]**

(a) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$
(b) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$
(c) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$
(d) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

120. The function $f(x)=\frac{x}{2}+\frac{2}{x}$ has a local minimum at

(a) $x=2$ (b) $x=-2$ **[2006]**
(c) $x=0$ (d) $x=1$

121. The real number x when added to its inverse gives the minimum value of the sum at x equal to

(a) –2 (b) 2 **[2003]**
(c) 1 (d) –1

122. If the function $f(x)=2x^3-9ax^2+12a^2x+1$, where $a>0$, attains its maximum and minimum at p and q respectively such that $p^2=q$, then a equals **[2003]**

(a) $\frac{1}{2}$ (b) 3
(c) 1 (d) 2

123. The maximum distance from origin of a point on the curve

$x=a\sin t-b\sin\left(\frac{at}{b}\right)$, $y=a\cos t-b\cos\left(\frac{at}{b}\right)$, both $a, b>0$ is

(a) $a-b$ (b) $a+b$ **[2002]**
(c) $\sqrt{a^2+b^2}$ (d) $\sqrt{a^2-b^2}$

Hints & Solutions

1. **(c)** Average speed $= f'(t) = \dfrac{f(t_2) - f(t_1)}{t_2 - t_1}$

$2at + b = a(t_1 + t_2) + b \Rightarrow t = \dfrac{t_1 + t_2}{2}$

2. **(d)** Let the side of cube be a.

$S = 6a^2 \Rightarrow \dfrac{dS}{dt} = 12a \cdot \dfrac{da}{dt} \Rightarrow 3.6 = 12a \cdot \dfrac{da}{dt}$

$\Rightarrow 12(10)\dfrac{da}{dt} = 3.6 \Rightarrow \dfrac{da}{dt} = 0.03$

$V = a^3 \Rightarrow \dfrac{dV}{dt} = 3a^2 \cdot \dfrac{da}{dt} = 3(10)^2 \cdot \left(\dfrac{3}{100}\right) = 9$

3. **(d)** Since, function $f(x)$ is continuous at $x = 1, 3$

$\therefore f(1) = f(1^+)$

$\Rightarrow ae + be^{-1} = c$ ...(i)

$f(3) = f(3^+)$

$\Rightarrow 9c = 9a + 6c \Rightarrow c = 3a$ ...(ii)

From (i) and (ii),

$b = ae(3 - e)$ ...(iii)

$$f'(x) = \begin{cases} ae^x - be^{-x} & -1 < x < 1 \\ 2cx & 1 < x < 3 \\ 2ax + 2c & 3 < x < 4 \end{cases}$$

$f'(0) = a - b,\ f'(2) = 4c$

Given, $f'(0) + f'(2) = e$

$a - b + 4c = e$ ...(iv)

From eqs. (i), (ii), (iii) and (iv),

$a - 3ae + ae^2 + 12a = e$

$\Rightarrow 13a - 3ae + ae^2 = e$

$\Rightarrow a = \dfrac{e}{e^2 - 3e + 13}$

4. **(d)** Let the thickness of ice layer be $= x$ cm

Total volume $V = \dfrac{4}{3}\pi(10 + x)^3$

$\dfrac{dV}{dt} = 4\pi(10 + x)^2 \dfrac{dx}{dt}$ ...(i)

Since, it is given that

$\dfrac{dV}{dt} = 50\text{ cm}^3/\text{min}$ ...(ii)

From (i) and (ii), $50 = 4\pi(10 + x)$

$\Rightarrow 50 = 4\pi(10 + 5)^2 \dfrac{dx}{dt}$ [$\because$ thickness of ice $x = 5$]

$\Rightarrow \dfrac{dx}{dt} = \dfrac{1}{18\pi}\text{cm/min}$

5. **(b)** According to the question,

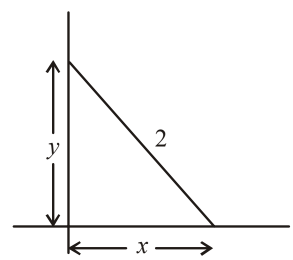
$\dfrac{dy}{dt} = -25$ at $y = 1$

By Pythagoras theorem, $x^2 + y^2 = 4$ ...(i)

When $y = 1 \Rightarrow x = \sqrt{3}$

Diff. equation (i) w. r. t. t,

$2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$



$\Rightarrow x\dfrac{dx}{dt} + y\dfrac{dy}{dt} = 0 \Rightarrow \sqrt{3}\dfrac{dx}{dt} + (-25) = 0$

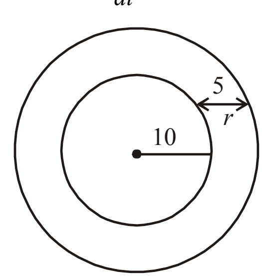
$\Rightarrow \dfrac{dx}{dt} = \dfrac{25}{\sqrt{3}}$ cm/s

6. **(a)** Given that ice melts at a rate of 50 cm³/min.

$\therefore \dfrac{dV_{ice}}{dt} = 50$

$V_{ice} = \dfrac{4}{3}\pi(10 + r)^3 - \dfrac{4}{3}\pi(10)^3$

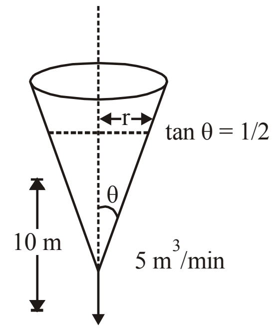
$\Rightarrow \dfrac{dV}{dt} = \dfrac{4}{3}\pi 3(10 + r)^2 \dfrac{dr}{dt} = 4\pi(10 + r)^2 \dfrac{dr}{dt}$



Substitute r = 5,

$50 = 4\pi(225)\dfrac{dr}{dt} \Rightarrow \dfrac{dr}{dt} = \dfrac{50}{4\pi(225)} = \dfrac{1}{18\pi}$cm/min

7. **(d)**



Given that water is poured into the tank at a constant rate of 5 m³/minute.

$\therefore \frac{dv}{dt} = 5\text{m}^3 / \text{min}$

Volume of the tank is,

$V = \frac{1}{3}\pi r^2 h$ ...(i)

where r is radius and h is height at any time.

By the diagram,

$\tan\theta = \frac{r}{h} = \frac{1}{2}$

$\Rightarrow h = 2r \Rightarrow \frac{dh}{dt} = \frac{2dr}{dt}$ ...(ii)

Differentiate eq. (i) w.r.t. 't', we get

$\frac{dV}{dt} = \frac{1}{3}\left(\pi 2r\frac{dr}{dt}h + \pi r^2 \frac{dh}{dt}\right)$

Putting $h = 10$, $r = 5$ and $\frac{dV}{dt} = 5$ in the above equation.

$5 = \frac{75\pi}{3}\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{5\pi}\text{m/min.}$

8. **(c)** Volume of sphere $V = \frac{4}{3}\pi r^3$ ...(i)

$\frac{dv}{dt} = \frac{4}{3}.3\pi r^2.\frac{dr}{dt}$

$4\pi = 4\pi r^2.\frac{dr}{dt}$

$\frac{1}{r^2} = \frac{dr}{dt}$

Since, V = 288π, therefore from (i), we have

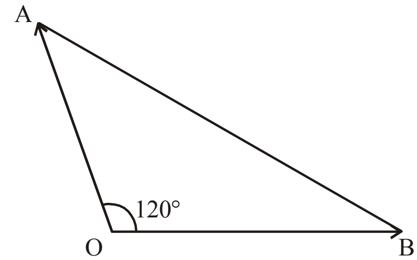
$288\pi = \frac{4}{3}\pi(r^3) \Rightarrow \frac{288\times 3}{4} = r^3$

$\Rightarrow 216 = r^3$

$\Rightarrow r = 6$

Hence, $\frac{dr}{dt} = \frac{1}{36}$.

9. **(a)**



Let OA = x km, OB = y km, AB = R

$(AB)^2 = (OA)^2 + (OB)^2 - 2\,(OA)\,(OB)\cos 120°$

$R^2 = x^2 + y^2 - 2xy\left(-\frac{1}{2}\right) = x^2 + y^2 + xy$ ...(i)

R at $x = 6$ km, and $y = 8$ km

$R = \sqrt{6^2 + 8^2 + 6\times 8} = 2\sqrt{37}$

Differentiating equation (i) with respect to t

$2R\frac{dR}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + \left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$

$= \frac{1}{2R}[2\times 8\times 20 + 2\times 6\times 30 + (8\times 30 + 6\times 20)]$

$\frac{dR}{dt} = \frac{1}{2\times 2\sqrt{37}}[1040] = \frac{260}{\sqrt{37}}$

10. **(a)** Volume of sphere $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}.\pi.3r^2.\frac{dr}{dt}$

$35 = 4\pi r^2.\frac{dr}{dt}$ or $\frac{dr}{dt} = \frac{35}{4\pi r^2}$ ...(i)

Surface area of sphere = S = $4\pi r^2$

$\frac{dS}{dt} = 4\pi\times 2r\times\frac{dr}{dt} = 8\pi r.\frac{dr}{dt}$

$\frac{dS}{dt} = \frac{70}{r}$ (By using (i))

Now, diameter = 14 cm, $r = 7$

$\therefore \frac{dS}{dt} = 10$

11. **(d)** $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2.\frac{dr}{dt}$ ...(i)

$S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r.\frac{dr}{dt}$

$\Rightarrow 8 = 8\pi r\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{\pi r}$

Putting the value of $\frac{dr}{dt}$ in (i), we get

$$\frac{dV}{dt} = 4\pi r^2 \times \frac{1}{\pi r} = 4r$$

$\Rightarrow \frac{dV}{dt}$ is proportional to r.

12. **(c)** Volume of spherical balloon = $V = \frac{4}{3}\pi r^3$

Differentiate both the side, w.r.t 't' we get,

$$\frac{dV}{dt} = 4\pi r^2 \left(\frac{dr}{dt}\right) \quad ...(i)$$

$\therefore$ After 49 min,

Volume = $(4500 - 49 \times 72)\pi = (4500 - 3528)\pi = 972\,\pi\,m^3$

$\Rightarrow V = 972\,\pi\,m^3$

$\therefore \quad 972\pi = \frac{4}{3}\pi r^3$

$\Rightarrow r^3 = 3 \times 243 = 3 \times 3^5 = 3^6 = (3^2)^3$

$\Rightarrow r = 9$

Given $\frac{dV}{dt} = 72\pi$

Putting $\frac{dV}{dt} = 72\pi$ and $r = 9$, we get

$\therefore \quad 72\pi = 4\pi \times 9 \times 9\left(\frac{dr}{dt}\right)$

$\Rightarrow \frac{dr}{dt} = \left(\frac{2}{9}\right)$

13. **(b)** Let $A = \pi r^2$ be area of metalic circular plate of $r = 50$ cm.

Also, given $\frac{dr}{dt} = 1\text{mm} = \frac{1}{10}\text{cm}$

$\therefore A = \pi r^2$

$\Rightarrow \frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi.50.\frac{1}{10} = 10\pi$

Hence, area of plate increases in 10π cm^2/hour.

14. **(c)** Let W = nw

$\Rightarrow \frac{dW}{dt} = n\frac{dw}{dt} + w.\frac{dn}{dt} \quad ...(i)$

Given : $w = t^2 - t + 2$ and $n = 2t^2 + 3$

$\Rightarrow \frac{dw}{dt} = 2t - 1$ and $\frac{dn}{dt} = 4t$

$\therefore$ Equation (i)

$\Rightarrow \frac{dw}{dt} = (2t^2 + 3)(2t - 1) + (t^2 - t + 2)(4t)$

Thus, $\left.\frac{dW}{dt}\right|_{t=1} = (2 + 3)(2 - 1) + (2)(4)$

$= 5(1) + 8 = 13$

15. **(d)** Let A be the area, b be the breadth and ℓ be the length of the rectangle.

Given : $\frac{dA}{dt} = -5$, $\frac{d\ell}{dt} = 2$, $\frac{db}{dt} = -3$

We know, $A = \ell \times b$

$\Rightarrow \frac{dA}{dt} = \ell.\frac{db}{dt} + b.\frac{d\ell}{dt} = -3\ell + 2b$

$\Rightarrow -5 = -3\ell + 2b$.

When $b = 2$, we have

$-5 = -3\ell + 4 \Rightarrow \ell = \frac{9}{3} = 3m$

16. **(b)** Let $A = \pi r^2$.

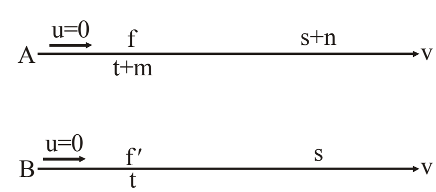
$\frac{dA}{dt} = 2\pi r.\frac{dr}{dt}$

$6\pi = 2\pi(30).\frac{dr}{dt}$

$\Rightarrow \frac{3}{30} = \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{10} = 0.1$

Thus, the rate at which the radius of the circular sheet increases is 0.1

17. **(d)**



As per question if point B moves s distance in t time then point A moves $(s + n)$ distance in time $(t + m)$ after which both have same velocity v.

Then using equation $v = u + at$ we get

$$v = f(t+m) = f't \Rightarrow t = \frac{fm}{f' - f} \quad ....(i)$$

Using equation $v^2 = u^2 + 2$, as we get

$$v^2 = 2f(s+n) = 2f's \quad \Rightarrow s = \frac{fn}{f' - f} \quad ....(ii)$$

Also for point B using the eqn $s = ut + \frac{1}{2}at^2$, we get

$$s = \frac{1}{2}f't^2$$

Substituting values of t and s from equations (i) and (ii) in the above relation, we get

$$\frac{fn}{f' - f} = \frac{1}{2}f'\frac{f^2m^2}{(f' - f)^2}$$

$$\Rightarrow (f' - f)n = \frac{1}{2}ff'm^2$$

18. (c) Let the lizard catches the insect after time t then distance covered by lizard = 21cm + distance covered by insect

$$\Rightarrow \frac{1}{2}ft^2 = 4 \times t + 21$$

$$\Rightarrow \frac{1}{2} \times 2 \times t^2 = 20 \times t + 21$$

$$\Rightarrow t^2 - 20t - 21 = 0 \Rightarrow t = 21 \text{ sec}$$

19. (b) Given that

Total radius $r = 10 + 5 = 15$ cm

$$\frac{dv}{dt} = 50 \text{ cm}^3/\text{min} \Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 50$$

$$\Rightarrow 4\pi r^2 \frac{dr}{dt} = 50$$

$$\Rightarrow \frac{dr}{dt} = \frac{50}{4\pi(15)^2} = \frac{1}{18\pi} \text{ cm/min}$$

20. (a) Given $y^2 = 18x \Rightarrow 2y\frac{dy}{dx} = 18 \Rightarrow \frac{dy}{dx} = \frac{9}{y}$

ATQ $\frac{dy}{dt} = \frac{2dx}{dt} \Rightarrow \frac{dy}{dx} = 2$

$$\Rightarrow \frac{9}{y} = 2 \Rightarrow y = \frac{9}{2}$$

Putting in $y^2 = 18x \Rightarrow x = \frac{9}{8}$

$\therefore$ Required point is $\left(\frac{9}{8}, \frac{9}{2}\right)$

21. (a) $f(x) = (3x-7)\cdot x^{2/3}$

$$f'(x) = 3x^{2/3} + (3x-7)\cdot\frac{2}{3}x^{-1/3}$$

$$= \frac{15x-14}{3x^{1/3}}$$

+ × – × +
0, $\frac{14}{15}$

For increasing function

$f'(x) > 0$ then $x \in (-\infty, 0) \cup \left(\frac{14}{15}, \infty\right)$

22. (d) Since, function $f(x)$ is twice differentiable and continuous in $x \in [a, b]$. Then, by LMVT for $x \in [a, c]$

$$\frac{f(c)-f(a)}{c-a} = f'(\alpha), \alpha \in (a,c)$$

Again by LMVT for $x \in [c, b]$

$$\frac{f(b)-f(c)}{b-c} = f'(\beta), \beta \in (c,b)$$

$\because f''(x) < 0 \Rightarrow f'(x)$ is decreasing

$$f'(\alpha) > f'(\beta) \Rightarrow \frac{f(c)-f(a)}{c-a} > \frac{f(b)-f(c)}{b-c}$$

$$\Rightarrow \frac{f(c)-f(a)}{f(b)-f(c)} > \frac{c-a}{b-c} \quad (\because f(x) \text{ is increasing})$$

23. (d) $f'(x) = x(\pi - \cos^{-1}(\sin|x|))$

$$= x\left(\pi - \left(\frac{\pi}{2} - \sin^{-1}(\sin|x|)\right)\right) = x\left(\frac{\pi}{2} + |x|\right)$$

$$f(x) = \begin{cases} x\left(\frac{\pi}{2}+x\right), & x \ge 0 \\ x\left(\frac{\pi}{2}-x\right), & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} \frac{\pi}{2}+2x, & x \ge 0 \\ \frac{\pi}{2}-2x, & x < 0 \end{cases}$$

Hence, $f'(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$ and decreasing in $\left(\frac{-\pi}{2}, 0\right)$.

24. (b) Given functions are, $f(x) = e^x - x$ and $g(x) = x^2 - x$

$$f(g(x)) = e^{(x^2-x)} - (x^2-x)$$

Given $f(g(x))$ is increasing function.

$$\therefore (f(g(x)))' = e^{(x^2-x)} \times (2x-1) - 2x + 1$$

$$= (2x-1)e^{(x^2-x)} + 1 - 2x = (2x-1)[e^{(x^2-x)} - 1] \ge 0$$

For $(f(g(x)))' \ge 0$,

$(2x-1)$ & $[e^{(x^2-x)} - 1]$ are either both positive or negative

– | +ve | –ve | +
0, $\frac{1}{2}$, 1

$$x \in \left[0, \frac{1}{2}\right] \cup [1, \infty)$$

25. **(c)** $f(x)=\dfrac{x^2}{1-x^2}$

$\Rightarrow f(-x)=\dfrac{x^2}{1-x^2}=f(x)$

$f'(-x)=\dfrac{2x}{(1-x^2)^2}$

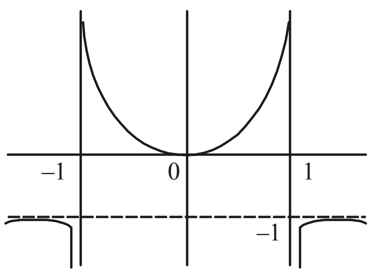
f(x) increases in $x \in (10, \infty)$

Also f(0) = 0 and

$\lim\limits_{x\to\pm\infty} f(x)=-1$ and f(x) is even function

Set A = R – [–1, 0)

And the graph of function f(x) is



26. **(c)** $f(x)=f(x)+f(2-x)$

Now, differentiate w.r.t. x,

$f'(x)=f'(x)-f'(2-x)$

For $f(x)$ to be increasing $f'(x)>0$

$\Rightarrow f'(x)-f'(2-x)>0$

$\Rightarrow f'(x)>f'(2-x)$

But $f''(x)>0 \Rightarrow f'(x)$ is an increasing function

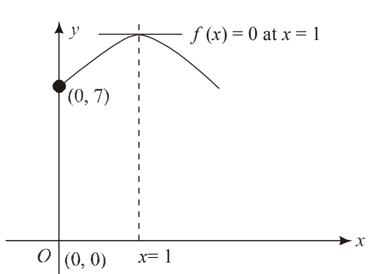
Then, $f'(x)>f'(2-x)>0$

$\Rightarrow x>2-x$

$\Rightarrow x>1$

Hence, $f(x)$ is increasing on (1, 2) and decreasing on (0, 1).

27. **(c)** $f(x) = x^3 - 3(a-2)x^2 + 3ax + 7, f(0) = 7$



$\Rightarrow f'(x) = 3x^2 - 6(a-2)x + 3a$

$f'(1) = 0$

$\Rightarrow 1 - 2a + 4 + a = 0$

$\Rightarrow a = 5$

Then, $f(x) = x^3 - 9x^2 + 15x + 7$

Now,

$\dfrac{f(x)-14}{(x-1)^2}=0$

$\Rightarrow \dfrac{x^3-9x^2+15x+7-14}{(x-1)^2}=0$

$\Rightarrow \dfrac{(x-1)^2(x-7)}{(x-1)^2}=0 \Rightarrow x=7$

28. **(a)** $f(x)=\dfrac{x}{\sqrt{a^2+x^2}}-\dfrac{(d-x)}{\sqrt{b^2+(d-x)^2}}$

$=\dfrac{x}{\sqrt{a^2+x^2}}+\dfrac{(x-d)}{\sqrt{b^2+(x-d)^2}}$

$$f'(x)=\frac{\sqrt{a^2+x^2}-\dfrac{x(2x)}{2\sqrt{a^2+x^2}}}{(a^2+x^2)}+\frac{\sqrt{b^2+(x-d)^2}-\dfrac{(x-d)2(x-d)}{2\sqrt{b^2+(x-d)^2}}}{\left(b^2+(x-d)^2\right)}$$

$=\dfrac{a^2+x^2-x^2}{(a^2+x^2)^{3/2}}+\dfrac{b^2+(x-d)^2-(x-d)^2}{\left(b^2+(x-d)^2\right)^{3/2}}$

$=\dfrac{a^2}{\left(a^2+x^2\right)^{3/2}}+\dfrac{b^2}{\left(b^2+(x-d)^2\right)^{3/2}}>0$

$\Rightarrow f'(x)>0, \square\, x\in R$

$\Rightarrow f(x)$ is increasing function.

Hence, $f(x)$ is increasing function.

29. **(a)** $f(x)=x^3-3x^2+5x+7$

For increasing

$f'(x)=3x^2-6x+5>0$

$\Rightarrow x\in R$

For decreasing

$f'(x)=3x^2-6x+5<0$

30. **(c)** $f(x) = \sin^4 x + \cos^4 x$

$f'(x) = 4\sin^3 x \cos x + 4\cos^3 x(-\sin x)$

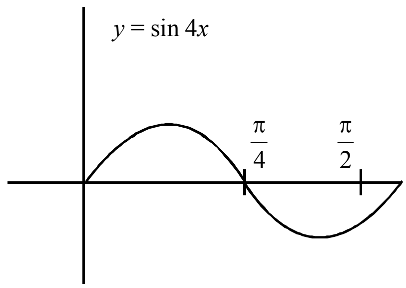
$= 4\sin x \cos x(\sin^2 x - \cos^2 x)$

$= -2\sin 2x \cos 2x = -\sin 4x$

$f(x)$ is increasing when $f'(x) > 0$

$\Rightarrow\ -\sin 4x > 0 \Rightarrow\ \sin 4x < 0$

$\Rightarrow\ x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$



31. **(b)** Since $f'(x) > 0$ and $g'(x) < 0$, therefore $f(x)$ is increasing function and $g(x)$ is decreasing function.

$\Rightarrow f(x+1) > f(x)$ and $g(x+1) < g(x)$

$\Rightarrow g[f(x+1)] < g[f(x)]$ and $f[g(x+1)] < f[g(x)]$

Hence option (b) is correct.

32. **(d)** $f(x) = 2x^3 + 3x + k$

$f'(x) = 6x^2 + 3 > 0\ \forall x \in R\quad (\because\ x^2 > 0)$

$\Rightarrow\ f(x)$ is strictly increasing function

$\Rightarrow\ f(x) = 0$ has only one real root, so two roots are not possible.

33. **(c)** Let $y = x^2 . e^{-x}$

For increasing function,

$\frac{dy}{dx} > 0\ \Rightarrow\ x[(2-x)e^{-x}] > 0$

$\because\ x > 0,\ \therefore\ (2-x)e^{-x} > 0$

$\Rightarrow\ (2-x)\frac{1}{e^x} > 0$

For $0 < x < 2$, $(2-x) < 0$

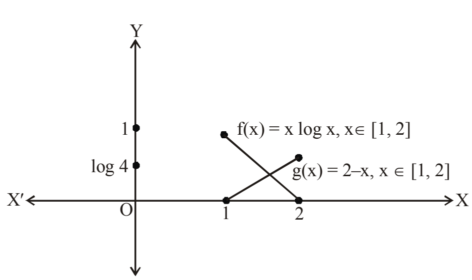
$\therefore\ \frac{1}{e^x} < 0$, but it is not possible

Hence the statement-2 is false.

34. **(a)** $f(x) = x\log x, f(1) = 0, f(2) = 4$

$g(x) = 2 - x, g(1) = 1, g(2) = 0$

$\log 10 > \log 4\ \Rightarrow\ 1 > \log 4$



Thus statement -1 and 2 both are true and statement-2 is a correct explanation of statement 1.

35. **(c)** $f(x) = xe^{x(1-x)}, x \in R$

$f'(x) = e^{x(1-x)}.\left[1 + x - 2x^2\right]$

$= -e^{x(1-x)}.\left[2x^2 - x - 1\right]$

$= -2e^{x(1-x)}.\left[\left(x + \frac{1}{2}\right)(x-1)\right]$

$f'(x) = -2e^{x(1-x)}.A$

where $A = \left(x + \frac{1}{2}\right)(x-1)$

Now, exponential function is always +ve and $f'(x)$ will be opposite to the sign of A which is –ve in $\left[-\frac{1}{2}, 1\right]$

Hence, $f'(x)$ is +ve in $\left[-\frac{1}{2}, 1\right]$

$\therefore\ f(x)$ is increasing on $\left[-\frac{1}{2}, 1\right]$

36. **(b)** Given that $f(x) = x^3 + 5x + 1$

$\therefore\ f'(x) = 3x^2 + 5 > 0,\ \forall x \in R$

$\Rightarrow f(x)$ is strictly increasing on R

$\Rightarrow\ f(x)$ is one one

$\therefore$ Being a polynomial $f(x)$ is continuous and increasing.

on R with $\lim_{x\to\infty} f(x) = -\infty$

and $\lim_{x\to\infty} f(x) = \infty$

$\therefore$ Range of $f = (-\infty, \infty) = R$

Hence f is onto also. So, f is one one and onto R.

37. **(b)** Let $f(x) = x^7 + 14x^5 + 16x^3 + 30x - 560$

$\Rightarrow f'(x) = 7x^6 + 70x^4 + 48x^2 + 30 > 0, \forall\ x \in R$ ...(i)

$\Rightarrow f$ is an increasing function on R

Also $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$ ...(ii)

From (i) and (ii) clear that the curve $y = f(x)$ crosses x-axis only once.

$\therefore f(x) = 0$ has exactly one real root.

38. **(d)** Given that $f(x) = \tan^{-1}(\sin x + \cos x)$

Differentiate w.r. to x

$$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2}.(\cos x - \sin x)$$

$$= \frac{\sqrt{2}.\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)}{1+(\sin x+\cos x)^2}$$

$$= \frac{\sqrt{2}\left(\cos\frac{\pi}{4}.\cos x - \sin\frac{\pi}{4}.\sin x\right)}{1+(\sin x+\cos x)^2}$$

$$\therefore \quad f'(x) = \frac{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}{1+(\sin x+\cos x)^2}$$

Given that $f(x)$ is increasing

$\therefore f'(x) > 0 \Rightarrow \cos\left(x+\frac{\pi}{4}\right) > 0$

$\Rightarrow -\frac{\pi}{2} < x+\frac{\pi}{4} < \frac{\pi}{2}$

$\Rightarrow -\frac{3\pi}{4} < x < \frac{\pi}{4}$

Hence, $f(x)$ is increasing when

$n \in \left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$

39. **(c)** From option (c), $f(x) = 3x^2 - 2x + 1$ is increasing when $f'(x) = 6x - 2 \geq 0$

$\Rightarrow \quad x \in [1/3, \infty)$

$\therefore f(x)$ is incorrectly matched with $\left(-\infty, \frac{1}{3}\right]$

40. **(b)** The given tangent to the curve is,

$y = x\log_e x \qquad (x > 0)$

$\Rightarrow \frac{dy}{dx} = 1 + \log_e x$

$\Rightarrow \left.\frac{dy}{dx}\right]_{x=c} = 1 + \log_e c \qquad$ (slope)

$\because$ The tangent is parallel to line joining (1, 0), (*e, e*)

$\therefore 1 + \log_e c = \frac{e-0}{e-1}$

$\Rightarrow \log_e c = \frac{e}{e-1} - 1 \Rightarrow \log_e c = \frac{1}{e-1}$

$\Rightarrow c = e^{\frac{1}{e-1}}$

41. **(c)** The given curve is, $x^4 \cdot e^y + 2\sqrt{y+1} = 3$

Differentiating w.r.t. x, we get

$(4x^3 + x^4 \cdot y')e^y + \frac{y'}{\sqrt{1+y}} = 0$

$$\Rightarrow \left(\frac{dy}{dx}\right) = \frac{-4x^3e^y}{\left(\frac{1}{\sqrt{y+1}} + e^y x^4\right)}$$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = -2$

$\therefore$ Equation of tangent;

$y - 0 = -2(x-1) \Rightarrow 2x + y = 2$

Only point (–2, 6) lies on the tangent.

42. **(0.50)**

The given curve $y = (x-1)(x-2)$, intersects the x-axis at $A(1, 0)$ and $B(2, 0)$.

$\therefore \frac{dy}{dx} = 2x - 3;\ \left(\frac{dy}{dx}\right)_{(x=1)} = -1$ and $\left(\frac{dy}{dx}\right)_{(x=2)} = 1$

Equation of tangent at $A(1, 0)$,

$y = -1(x-1) \Rightarrow x + y = 1$

Equation of tangent at $B(2, 0)$,

$y = 1(x-2) \Rightarrow x - y = 2$

So $a = 1$ and $b = 2$

$\Rightarrow \frac{a}{b} = \frac{1}{2} = 0.5.$

43. **(4)**

For (1, 2) of $y^2 = 4x \Rightarrow t = 1, a = 1$

Equation of normal to the parabola

$\Rightarrow tx + y = 2at + at^3$

$\Rightarrow x + y = 3$ intersect x-axis at (3, 0)

$y = e^x \Rightarrow \frac{dy}{dx} = e^x$

Equation of tangent to the curve

$\Rightarrow y - e^c = e^c(x - c)$

$\because$ Tangent to the curve and normal to the parabola intersect at same point.

$\therefore 0 - e^c = e^c(3 - c) \Rightarrow c = 4.$

44. **(91)**

$$y = \sum_{k=1}^{6} k\cos^{-1}\left\{\frac{3}{5}\cos kx - \frac{4}{5}\sin kx\right\}$$

Let $\cos a = \frac{3}{5}$ and $\sin a = \frac{4}{5}$

$\therefore y=\sum_{k=1}^{6} k\cos^{-1}\{\cos a\cos kx-\sin a\sin kx\}$

$=\sum_{k=1}^{6} k\cos^{-1}(\cos(kx+a))$

$=\sum_{k=1}^{6} k(kx+a)=\sum_{k=1}^{6}(k^2x+ak)$

$\therefore \frac{dy}{dx}=\sum_{k=1}^{6} k^2=\frac{6(7)(13)}{6}=91.$

45. **(4.0)** $P\equiv(x_1, y_1)$

$2yy'-6x+y'=0$

$\Rightarrow \quad y'=\left(\frac{6x_1}{1+2y_1}\right)$

$\left(\frac{\frac{3}{2}-y_1}{-x_1}\right)=-\left(\frac{1+2y_1}{6x_1}\right)$

[By point slope form, $y-y_1=m(x-x_1)$]

$\Rightarrow \quad 9-6y_1=1+2y_1$

$\Rightarrow \quad y_1=1$

$\therefore \quad x_1=\pm 2$

$\therefore$ Slope of tangent $(m)=\left(\frac{\pm 12}{3}\right)=\pm 4$

$\therefore \quad |m|=4$

46. **(d)** Given equation of curve is

$x^2+2xy-3y^2=0$

$\Rightarrow \quad 2x+2y+2xy'-6yy'=0$

$\Rightarrow \quad x+y+xy'-3yy'=0$

$\Rightarrow \quad y'(x-3y)=-(x+y)$

$\Rightarrow \quad \frac{dy}{dx}=\frac{x+y}{3y-x}$

Slope of normal = $\frac{-dx}{dy}=\frac{x-3y}{x-3y}$

Normal at point (2, 2) = $\frac{2-6}{2+2}=-1$

Equation of normal to curve = $y-2=-1(x-2)$

$\Rightarrow \quad x+y=4$

$\therefore$ Perpendicular distance from origin

$=\left|\frac{0+0-4}{\sqrt{2}}\right|=2\sqrt{2}$

47 **(a)** Given curve is, $y=\frac{x}{x^2-3}$

$\Rightarrow \frac{dy}{dx}=\frac{(x^2-3)-x(2x)}{(x^2-3)^2}=\frac{-x^2-3}{(x^2-3)^2}$

$\left.\frac{dy}{dx}\right|_{(\alpha,\beta)}=\frac{\alpha^2-3}{(\alpha^2-3)^2}=-\frac{2}{6}=-\frac{1}{3}$

$3(\alpha^2+3)=(\alpha^2-3)^2 \Rightarrow \alpha^2=9$

And, $\beta=\frac{x}{a^2-3} \Rightarrow \alpha^2-3=\frac{\alpha}{\beta} \Rightarrow \frac{\alpha}{\beta}=6$

$\Rightarrow a=\pm 3, \beta=\pm\frac{1}{2}$

These values of α and β satisfies $|6\alpha+2\beta|=19$

48. **(d)** $y=x^3+ax-b$

Since, the point (1, –5) lies on the curve.

$\Rightarrow 1+a-b=-5$

$\Rightarrow a-b=-6$ ...(i)

$\frac{dy}{dx}=3x^2+a$

$\left(\frac{dy}{dx}\right)_{at\ x=1}=3+a$

Since, required line is perpendicular to $y=x-4$, then slope of tangent at the point P (1, –5) = –1

$3+a=-1$

$a=-4$

$b=2$

the equation of the curve is $y=x^3-4x-2$

(2, –2) lies on the curve

49. **(d)** $y=f(x)=x^3-x^2-2x$

$\frac{dy}{dx}-3x^2-2x-2$

$f(1)=1-1-2=-2,\quad f(-1)=-1-1+2=0$

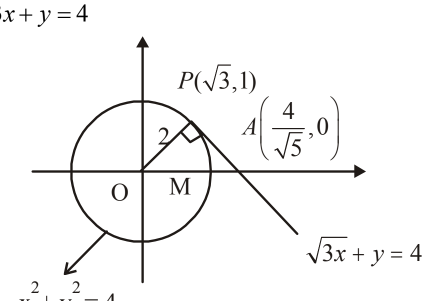
Since the tangent to the curve is parallel to the line segment joining the points (1, –2) (–1, 0)

Since their slopes are equal

$\Rightarrow 3x^2-2x-2=\frac{-2-0}{2} \quad \Rightarrow x=1,\frac{-1}{3}$

Hence, the required set $S=\left\{\frac{-1}{3},1\right\}$

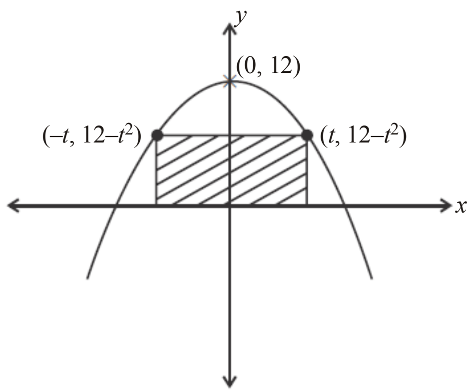
50. **(c)** Equation of tangent to circle at point $(\sqrt{3},1)$ is $\sqrt{3}x+y=4$



coordinates of the point $A=\left(\frac{4}{\sqrt{3}},0\right)$

Area = $\frac{1}{2}\times OA\times PM=\frac{1}{2}\times\frac{4}{\sqrt{3}}\times 1=\frac{2}{\sqrt{3}}$ sq. units

51. **(c)** Given, the equation of parabola is,
$x^2 = 12 - y$



Area of the rectangle = $(2t)(12 - t^2)$
$A = 24t - 2t^3$

$\frac{dA}{dt} = 24 - 6t^2$

Put $\frac{dA}{dt} = 0 \Rightarrow 24 - 6t^2 = 0$

$\Rightarrow\quad t = \pm 2$

$$\begin{array}{ccccc} - & & + & & - \\ & -2 & & +2 & \end{array}$$

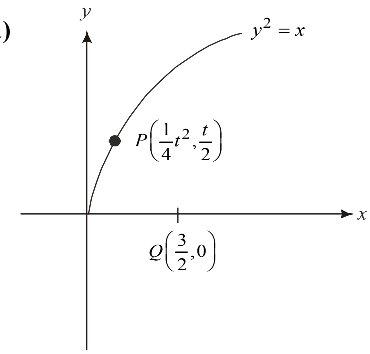
At $t = 2$, area is maximum = $24(2) - 2(2)^3$
= 48 – 16 = 32 sq. units

52. **(b)** $\because$ Tangent to the given curve is parallel to line $2y = 4x + 1$
$\therefore$ Slope of tangent (m) = 2
Then, the equation of tangent will be of the form
$y = 2x + c$ ...(i)
$\because$ Line (i) and curve $y = x^2 - 5x + 5$ has only one point of intersection.
$\therefore\quad 2x + c = x^2 - 5x + 5$
$x^2 - 7x + (5 - c) = 0$
$\therefore\quad D = 49 - 4(5 - c) = 0$

$\Rightarrow\quad c=-\frac{29}{4}$

Hence, the equation of tangent: $y = 2x - \frac{29}{4}$

53. **(a)**



Here the curve is parabola with $a=\frac{1}{4}$.

Let P(at^2, 2at) or $P\left(\frac{t^2}{4},\frac{t}{2}\right)$ be a point on the curve.

Now, $y^2 = x$

$\Rightarrow 2y\frac{dy}{dx}=1=\frac{dy}{dx}=\frac{1}{2\sqrt{x}}$

$\Rightarrow\left(\frac{dy}{dx}\right)_{at\ p}=\frac{1}{t}$

$\therefore$ equation of normal at P to $y^2 = x$ is,

$\left(y-\frac{t}{2}\right)=-t\left(x-\frac{1}{4}t^2\right)$

$\Rightarrow y=-tx+\frac{1}{2}t+\frac{1}{4}t^3$ ...(i)

For minimum PQ, (i) passes through $Q\left(\frac{3}{2},0\right)$

$\frac{-3}{2}t+\frac{t}{2}+\frac{t^3}{4}=0\Rightarrow -4t+t^3=0$

$\Rightarrow\quad t(t^2-4)=0\Rightarrow t=-2, 0, 2$

$\because\quad t\geq 0\Rightarrow t=0, 2$

If $t=0$, $P(0, 0)\Rightarrow AP=\frac{3}{2}$

If $t=2$, $P(1, 1)\Rightarrow AP=\frac{\sqrt{5}}{2}$

Shortest distance $\left(\frac{3}{2},0\right)$ and $y=\sqrt{x}$ is $\frac{\sqrt{5}}{2}$

54. **(b)** The equation of curve $y = xe^{x^2}$

$\Rightarrow\quad \frac{dy}{dx}=e^{x^2}.1+x.e^{x^2}.2x$

Since $(1, e)$ lies on the curve $y = xe^{x^2}$, then equation of tangent at $(1, e)$ is

$$y - e = \left(e^{x^2}(1+2x^2)\right)_{x=1}(x-1)$$

$y - e = 3e(x - 1)$

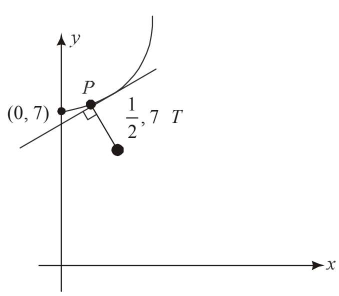
$3ex - y = 2e$

So, equation of tangent to the curve passes through the point $\left(\frac{4}{3}, 2e\right)$

55. **(c)** $f(x) = y = x^{3/2} + 7$

$\Rightarrow \frac{dy}{dx} \Rightarrow \frac{3}{2}\sqrt{x} > 0$

$\Rightarrow$ $f(x)$ is increasing function $\forall\, x > 0$



Let $P\left(x_1, x_1^{3/2} + 7\right)$

$m_{TP} = m_{\text{at P}} = -1$

$$\Rightarrow \left(\frac{x_1^{3/2}}{x_1 - \frac{1}{2}}\right) \times \frac{3}{2} x_1^{\frac{1}{2}} = -1$$

$$\Rightarrow -\frac{2}{3} = \frac{x_1^2}{x_1 - \frac{1}{2}}$$

$\Rightarrow -3x_1^2 = 2x_1 - 1 \Rightarrow 3x_1^2 + 2x_1 - 1 = 0$

$\Rightarrow 3x_1^2 + 3x_1 - x_1 - 1 = 0$

$\Rightarrow 3x_1(x_1 + 1) - 1(x_1 + 1) = 0$

$\Rightarrow x_1 = \frac{1}{3}$ $\quad (\because x_1 > 0)$

$$\Rightarrow P\left(\frac{1}{3}, 7 + \frac{1}{3\sqrt{3}}\right)$$

$$TP = \sqrt{\frac{1}{27} + \frac{1}{36}} = \frac{1}{6}\sqrt{\frac{7}{3}}$$

56. **(b)** Since, the equation of curves are

$y = 10 - x^2$ ...(i)

$y = 2 + x^2$ ...(ii)

Adding eqn (i) and (ii), we get

$2y = 12 \Rightarrow y = 6$

Then, from eqn (i)

$x = \pm 2$

Differentiate equation (i) with respect to x

$$\frac{dy}{dx} = -2x \Rightarrow \left(\frac{dy}{dx}\right)_{(2,6)} = -4 \text{ and } \left(\frac{dy}{dx}\right)_{(-2,6)} = 4$$

Differentiate equation (ii) with respect to x

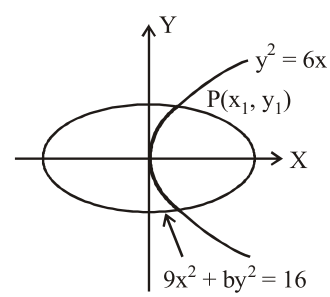
$$\frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)_{(2,6)} = 4 \text{ and } \left(\frac{dy}{dx}\right)_{(-2,6)} = -4$$

$$\text{At } (2, 6) \tan\theta = \left(\frac{(-4)-(4)}{1+(-4)\times(4)}\right) = \frac{8}{15}$$

$$\text{At } (-2, 6), \tan\theta = \frac{(4)-(-4)}{1+(4)(-4)} = \frac{8}{-15} \Rightarrow |\tan\theta| = \frac{8}{15}$$

$$\therefore |\tan\theta| = \frac{8}{15}$$

57. **(c)** Let curve intersect each other at point $P(x_1, y_1)$



Since, point of intersection is on both the curves, then

$y_1^2 = 6x_1$ ...(i)

and $9x_1^2 + by_1^2 = 16$ ...(ii)

Now, find the slope of tangent to both the curves at the point of intersection $P(x_1, y_1)$

For slope of curves:

Curve (i):

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = m_1 = \frac{3}{y_1}$$

Curve (ii):

and $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = m_2 = -\frac{9x_1}{by_1}$

Since, both the curves intersect each other at right angle then,

$m_1m_2 = -1 \Rightarrow \frac{27x_1}{by_1^2} = 1 \Rightarrow b = 27\frac{x_1}{y_1^2}$

$\therefore$ from equation (i), $b = 27 \times \frac{1}{6} = \frac{9}{2}$

58. **(c)** Let $P(2t, t^2)$ be any point on the parabola.
Centre of the given circle $C = (-g, -f) = (-3, 0)$
For PC to be minimum, it must be the normal to the parabola at P.

Slope of line $PC = \frac{y_2 - y_1}{x_2 - x_1} = \frac{t^2 - 0}{2t + 3}$

Also, slope of tangent to parabola at $P = \frac{dy}{dx} = \frac{x}{2} = t$

$\therefore$ Slope of normal $= \frac{-1}{t}$

$\therefore \frac{t^2 - 0}{2t + 3} = \frac{-1}{t}$

$\Rightarrow t^3 + 2t + 3 = 0$

$\Rightarrow (t + 1)(t^2 - t + 3) = 0$

$\therefore$ Real roots of above equation is

$t = -1$

Coordinate of $P = (2t, t^2) = (-2, 1)$

Slope of tangent to parabola at $P = t = -1$

Therefore, equation of tangent is:

$(y - 1) = (-1)(x + 2)$

$\Rightarrow x + y + 1 = 0$

59. **(d)** Equation of hyperbola is :

$4y^2 = x^2 + 1$

$\Rightarrow -x^2 + 4y^2 = 1$

$\Rightarrow -\frac{x^2}{1^2} + \frac{y^2}{\left(\frac{1}{2}\right)^2} = 1$

$\therefore a = 1, b = \frac{1}{2}$

Now, tangent to the curve at point (x_1, y_1) is given by

$4 \times 2y_1\frac{dy}{dx} = 2x_1$

$\Rightarrow \frac{dy}{dx} = \frac{2x_1}{8y_1} = \frac{x_1}{4y_1}$

Equation of tangent at (x_1, y_1) is

$y = mx + c$

$\Rightarrow y = \frac{x_1}{4y_1} \cdot x + c$

As tangent passes through (x_1, y_1)

$\therefore y_1 = \frac{x_1 x_1}{4y_1} + c$

$\Rightarrow C = \frac{4y_1^2 - x_1^2}{4y_1} = \frac{1}{4y_1}$

Therefore, $y = \frac{x_1}{4y_1}x + \frac{1}{4y_1} \quad \Rightarrow 4y_1y = x_1x + 1$

which intersects x axis at $A\left(\frac{-1}{x_1}, 0\right)$ and y axis at $B\left(0, \frac{1}{4y_1}\right)$

Let midpoint of AB is (h, k)

$\therefore h = \frac{-1}{2x_1}$

$\Rightarrow x_1 = \frac{-1}{2h}$ & $y_1 = \frac{1}{8k}$

Thus, $4\left(\frac{1}{8k}\right)^2 = \left(\frac{-1}{2h}\right)^2 + 1$

$\Rightarrow \frac{1}{16k^2} = \frac{1}{4h^2} + 1$

$\Rightarrow 1 = \frac{16k^2}{4h^2} + 16k^2$

$\Rightarrow h^2 = 4k^2 + 16h^2k.$

So, required equation is

$x^2 - 4y^2 - 16x^2y^2 = 0$

60. **(b)** Since, $x^2 + 3y^2 = 9$

$\Rightarrow 2x + 6y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{3y}$

Slope of normal is $-\frac{dx}{dy} = \frac{3y}{x}$

$\Rightarrow \left(-\dfrac{dx}{dy}\right)_{(3\cos\theta, \sqrt{3}\sin\theta)} = \dfrac{3\sqrt{3}\sin\theta}{3\cos\theta} = \sqrt{3}\tan\theta = m_1$

& $\left(-\dfrac{dx}{dy}\right)_{(-3\sin\theta, \sqrt{3}\cos\theta)}$

$= \dfrac{3\sqrt{3}\cos\theta}{-3\sin\theta} = -\sqrt{3}\cot\theta = m_2$

As, β is the anagle between the normals to the given ellipse then

$\tan\beta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\dfrac{\sqrt{3}\tan\theta + \sqrt{3}\cot\theta}{1 - 3\tan\theta\cot\theta}\right| = \left|\dfrac{\sqrt{3}\tan\theta + \sqrt{3}\cot\theta}{1-3}\right|$

So, $\tan\beta = \dfrac{\sqrt{3}}{2}|\tan\theta + \cot\theta|$

$\Rightarrow \dfrac{1}{\cot\beta} = \dfrac{\sqrt{3}}{2}\left|\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}\right|$

$\Rightarrow \dfrac{1}{\cot\beta} = \dfrac{\sqrt{3}}{2}\left|\dfrac{1}{\sin\theta\cos\theta}\right|$

$\Rightarrow \dfrac{1}{\cot\beta} = \dfrac{\sqrt{3}}{\sin 2\theta} \Rightarrow \dfrac{2\cot\beta}{\sin 2\theta} = \dfrac{2}{\sqrt{3}}$

61. **(c)** Given, $4x^2 - 9y^2 = 36$
After differentiating w.r.t. x, we get

$4.2.x - 9.2.y.\dfrac{dy}{dx} = 0$

$\Rightarrow$ Slope of tangent $= \dfrac{dy}{dx} = \dfrac{4x}{9y}$

So, slope of normal $= \dfrac{-9y}{4x}$

Now, equation of normal at point (x_0, y_0) is given by

$y - y_0 = \dfrac{-9y_0}{4x_0}(x - x_0)$

As normal intersects X axis at A, Then

$A \equiv \left(\dfrac{13x_0}{9}, 0\right)$ and $B \equiv \left(0, \dfrac{13y_0}{4}\right)$

As $OABP$ is a parallelogram

$\therefore$ midpoint of $OB \equiv \left(0, \dfrac{13y_0}{8}\right) \equiv$ Midpoint of AP

So, $P(x, y) \equiv \left(\dfrac{-13x_0}{9}, \dfrac{13y_0}{4}\right)$ ...(i)

$\because (x_0, y_0)$ lies on hyperbola, therefore

$4(x_0)^2 - 9(y_0)^2 = 36$ ...(ii)

From equation (i): $x_0 = \dfrac{-9x}{13}$ and $y_0 = \dfrac{4y}{13}$

From equation (ii), we get
$9x^2 - 4y^2 = 169$
Hence, locus of point P is : $9x^2 - 4y^2 = 169$

62. **(c)** We have $y = \dfrac{x+6}{(x-2)(x-3)}$

At y-axis, $x = 0 \Rightarrow y = 1$
On differentiating, we get

$\dfrac{dy}{dx} = \dfrac{(x^2-5x+6)(1) - (x+6)(2x-5)}{(x^2-5x+6)^2}$

$\dfrac{dy}{dx} = 1$ at point (0, 1)

$\therefore$ Slope of normal = – 1
Now equation of normal is $y - 1 = -1(x - 0)$
$\Rightarrow y - 1 = -x$
$x + y = 1$

$\therefore \left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ satisfy it.

63. **(c)** Eccentricity of ellipse $= \dfrac{1}{2}$

Now, $-\dfrac{a}{e} = -4 \Rightarrow a = 4 \times \dfrac{1}{2} = 2 \Rightarrow a = 2$

We have $b^2 = a^2(1 - e^2) = a^2\left(1 - \dfrac{1}{4}\right)$

$= 4 \times \dfrac{3}{4} = 3$

$\therefore$ Equation of ellipse is

$\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

Now differentiating, we get

$\Rightarrow \dfrac{x}{2} + \dfrac{2y}{3} \times y' = 0 \Rightarrow y' = -\dfrac{3x}{4y}$

$y'\Big|_{(1,3/2)} = -\dfrac{3}{4} \times \dfrac{2}{3} = -\dfrac{1}{2}$

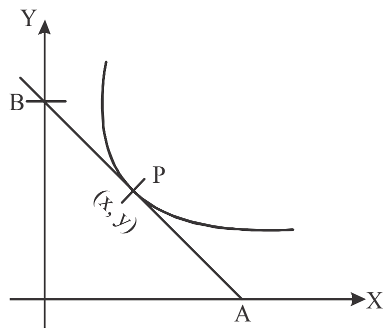
Slope of normal = 2

$\therefore$ Equation of normal at $\left(1, \frac{3}{2}\right)$ is

$$y - \frac{3}{2} = 2(x - 1) \Rightarrow 2y - 3 = 4x - 4$$

$\therefore \quad 4x - 2y = 1$

64. **(c)**



Let y = f (x) be a curve
slope of tangent = f ′ (x)
Equation of tangent (Y – y) = f ′ (x) (X – x)
Put Y = 0

$$\Rightarrow \quad X = \left(x - \frac{y}{f'(x)}\right)$$

Put X = 0

$$\Rightarrow \quad Y = y - x f'(x)$$

$$\Rightarrow \quad A = \left(x - \frac{y}{f'(x)}, 0\right)$$

and B = (0, y – x f ′ (x))
$\therefore \quad$ AP : PB = 1 : 3

$$\Rightarrow \quad x = \frac{3}{4}\left(x - \frac{y}{f'(x)}\right)$$

$$\Rightarrow \quad x = \frac{-3y}{f'(x)} \Rightarrow \frac{dy}{dx} = \frac{-3y}{x}$$

$$\frac{dy}{y} = \frac{-3\,dx}{x} \Rightarrow y = \frac{C}{x^3}$$

$\because \quad f(a) = 1 \Rightarrow C = 1$

$\therefore \quad y = \frac{1}{x^3}$ is required curve and $\left(2, \frac{1}{8}\right)$ passing through $y = \frac{1}{x^3}$

65. **(d)** $x^2y^2 - 2x = 4 - 4y$
Differentiate w.r.t. 'x'

$$2xy^2 + 2y \,.\, x^2 \,.\, \frac{dy}{dx} - 2 = -4 \,.\, \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}(2y \,.\, x^2 + 4) = 2 - 2x \,.\, y^2$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{2,-2} = \frac{2 - 2\times2\times4}{2(-2)\times4+4} = \frac{-14}{-12} = \frac{7}{6}$$

$\therefore \quad$ Equation of tangent is

$$(y+2) = \frac{7}{6}(x-2) \text{ or } 7x - 6y = 26$$

$\therefore$ (–2, –7) does not passes through the required tangent.

66. **(d)** $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$

$$= \tan^{-1}\left(\sqrt{\frac{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{\left(\sin\frac{x}{x}-\cos\frac{x}{2}\right)^2}}\right) = \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)$$

$$\Rightarrow \quad y = \frac{\pi}{4}+\frac{x}{2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}$$

$$\text{Slope of normal} = \frac{-1}{\left(\frac{dy}{dx}\right)} = -2$$

Equation of normal at $\left(\frac{\pi}{6}, \frac{\pi}{4}+\frac{\pi}{12}\right)$

$$y - \left(\frac{\pi}{4}+\frac{\pi}{12}\right) = -2\left(x - \frac{\pi}{6}\right)$$

$$y - \frac{4\pi}{12} = -2x + \frac{2\pi}{6}$$

$$y - \frac{\pi}{3} = -2x + \frac{\pi}{3}$$

$$y = -2x + \frac{2\pi}{3}$$

This equation is satisfied only by the point $\left(0, \frac{2\pi}{3}\right)$

67. **(a)** $\frac{dy}{dx} = \frac{1}{2\sqrt{4x-3}} \times 4 = \frac{2}{3}$

$\Rightarrow \quad 4x - 3 = 9$
$\Rightarrow \quad x = 3$

So, $y = 4$

Equation of normal at $P(3, 4)$ is

$$y - 4 = -\frac{3}{2}(x - 3)$$

i.e. $2y - 8 = -3x + 9$

$\Rightarrow \quad 3x + 2y - 17 = 0$

This line is satisfied by the point (1, 7)

68. **(d)** $P(4t^2 + 3, 8t^3 - 1)$

$$\frac{dy/dt}{dt/dt} = \frac{dy}{dx} = 3t \text{ (slope of tangent at } P)$$

Let $Q = (4\lambda^2 + 3, 8\lambda^3 - 1)$

slope of PQ = $3t$

$$\frac{8t^3 - 8\lambda^3}{4t^2 - 4\lambda^2} = 3t$$

$\Rightarrow \quad t^3 - 3\lambda^2 t + 2\lambda^3 = 0$

$(t - \lambda) . (t^2 + t\lambda - 2\lambda^2) = 0$

$(t - \lambda)^2 . (t + 2\lambda) = 0$

$$t = \lambda \text{ (or) } \lambda = \frac{-t}{2}$$

$\therefore \quad Q\,[t^2 + 3, -t^3 - 1].$

69. **(b)** Given curve is

$x^2 + 2xy - 3y^2 = 0$ ...(i)

Differentiatew.r.t. x

$$2x + 2x\frac{dy}{dx} + 2y - 6y\frac{dy}{dx} = 0$$

$$\left(\frac{dy}{dx}\right)_{(1,1)} = 1$$

Equation of normal at (1, 1) is

$y = 2 - x$ ...(ii)

Solving eqs. (i) and (ii), we get

$x = 1, 3$

Point of intersection (1, 1), (3, –1)

Normal cuts the curve again in 4th quadrant.

70. **(b)** Given curve is $\sin y = x \sin\left(\frac{\pi}{3} + y\right)$

Diff with respect to x, we get

$$\cos y \frac{dy}{dx} = \sin\left(\frac{\pi}{3} + y\right) + x\cos\left(\frac{\pi}{3} + y\right)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin\left(\frac{\pi}{3} + y\right)}{\cos y - x\cos\left(\frac{\pi}{3} + y\right)}$$

$$\frac{dy}{dx} \text{ at } (0, 0) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \text{Equation of normal is } y - 0 = -\frac{2}{\sqrt{3}}(x - 0)$$

$$\Rightarrow 2x + \sqrt{3}\,y = 0$$

71. **(d)** $x^2 - y + 6 = 0$

$$2x - \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = 2x$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,10)} = 4$$

equation of tangent

$y - 10 = 4(x - z)$

$4x - y + z = 0$

tangent passes through (α, β)

$4\alpha - \beta + z = 0 \Rightarrow \beta = 4\alpha + z$ ...(i)

and $2x + 2yy' + 8 - 2y' = 0$

$$y' = \frac{2x + 8}{2 - 2y} = \frac{2\alpha + 8}{2 - 2\beta} = 4 \quad \text{...(ii)}$$

from (i) and (ii)

$$\alpha = \frac{-8}{17}, \beta = \frac{2}{17}$$

$$\left(\frac{-8}{17}, \frac{2}{17}\right)$$

72. **(a)** Given that

$x = 2\cos t + 2t\sin t$

$$\text{so, } \frac{dx}{dt} = -2\sin t + 2[t\cos t + \sin t]$$

$$\frac{dy}{dt} = 2\cos t - 2[-t\sin t + \cos t]$$

$$\frac{dy}{dx} = 2t\sin t$$

$$\frac{dy}{dx} = \frac{2t\sin t}{2t\cos t}$$

$$\frac{dy}{dx} = \tan t$$

$$\left(\frac{dy}{dx}\right)_{t=\pi/4} = 1$$

so the slope of the normal is – 1

At $t = \pi/4\, x = \sqrt{2} + \frac{\pi}{2\sqrt{2}}$ and

$y = \sqrt{2} - \pi/2\sqrt{2}$

the equation of normal is

$$\left[y - \left(\sqrt{2} - \pi/2\sqrt{2}\right)\right] = -1\left[\left(x - \left(\sqrt{2} + \pi/2\sqrt{2}\right)\right)\right]$$

$$y - \sqrt{2} + \frac{\pi}{2\sqrt{2}} = -x + \sqrt{2} + \pi/2\sqrt{2}$$

$x + y = 2\sqrt{2}$, so the distance from the origin is 2

73. **(c)** Given, $y = 3\sin\theta.\cos\theta$

$$\frac{dy}{d\theta} = 3[\sin\theta(-\sin\theta)+\cos\theta(\cos\theta)]$$

$$\frac{dy}{d\theta} = 3[\cos^2\theta-\sin^2\theta] = 3\cos 2\theta \quad ...(i)$$

and $x = e^{\theta}\sin\theta$

$$\frac{dx}{d\theta} = e^{\theta}\cos\theta + \sin\theta\, e^{\theta}$$

$$\frac{dx}{d\theta} = e^{\theta}(\sin\theta + \cos\theta) \quad ...(ii)$$

Dividing (i) by (i)

$$\frac{dy}{dx} = \frac{3\cos 2\theta}{e^{\theta}(\sin\theta+\cos\theta)} = \frac{3(\cos^2\theta-\sin^2\theta)}{e^{\theta}(\sin\theta+\cos\theta)}$$

$$\frac{dy}{dx} = \frac{3(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)}{e^{\theta}(\sin\theta+\cos\theta)}$$

$$\frac{dy}{dx} = \frac{3(\cos\theta-\sin\theta)}{e^{\theta}}$$

Given tangent is parallel to x-axis then $\frac{dy}{dx} = 0$

$$0 = \frac{3(\cos\theta-\sin\theta)}{e^{\theta}}$$

or $\cos\theta - \sin\theta = 0 \Rightarrow \cos\theta = \sin\theta$

$$\Rightarrow \tan\theta = 1 \Rightarrow \tan\theta = \frac{\tan\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$$

74. **(d)** Let $y = \cos(x + y)$

$$\Rightarrow \frac{dy}{dx} = -\sin(x+y)\left(1+\frac{dy}{dx}\right) \quad ...(i)$$

Now, given equation of tangent is

$x + 2y = k$

$$\Rightarrow \text{Slope} = \frac{-1}{2}$$

So, $\frac{dy}{dx} = \frac{-1}{2}$ put this value in (i), we get

$$\frac{-1}{2} = -\sin(x+y)\left(1-\frac{1}{2}\right)$$

$\Rightarrow \sin(x + y) = 1$

$$\Rightarrow x+y = \frac{\pi}{2} \Rightarrow y = \frac{\pi}{2} - x$$

Now, $\frac{\pi}{2} - x = \cos(x + y)$

$$\Rightarrow x = \frac{\pi}{2} \text{ and } y = 0$$

Thus $x + 2y = k \Rightarrow \frac{\pi}{2} = k$

75. **(d)** $x^2 = 8y$ ...(i)

When, $x = 4$, then $y = 2$

Now $\frac{dy}{dx} = \frac{2x}{8} = \frac{x}{4}, \left.\frac{dy}{dx}\right]_{x=4} = 1$

Slope of normal $= -\frac{1}{\frac{dy}{dx}} = -1$

Euqation of normal at $x = 4$ is

$y - 2 = -1(x - 4)$

$\Rightarrow y = -x + 4 + 2 = -x + 6$

$\Rightarrow x + y = 6$

76. **(c)** Since the tangent is parallel to x-axis,

$$\therefore \frac{dy}{dx} = 0 \Rightarrow 1 - \frac{8}{x^3} = 0 \Rightarrow x = 2 \Rightarrow y = 3$$

Equation of the tangent is $y - 3 = 0(x - 2)$

$\Rightarrow y = 3$

77. **(b)** $\frac{dy}{dx} = 2x - 5 \therefore m_1 = (2x-5)_{(2,0)} = -1$,

$m_2 = (2x-5)_{(3,0)} = 1 \Rightarrow m_1 m_2 = -1$

i.e. the tangents are perpendicular to each other.

78. **(d)** Given $x = a(\cos\theta + \theta\sin\theta)$

$$\Rightarrow \frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = a\theta\cos\theta \quad .....(i)$$

$y = a(\sin\theta - \theta\cos\theta)$

$$\frac{dy}{d\theta} = a[\cos\theta - \cos\theta + \theta\sin\theta]$$

$$\Rightarrow \frac{dy}{d\theta} = a\theta\sin\theta \quad .....(ii)$$

From equations (i) and (ii) we get

$\frac{dy}{dx} = \tan\theta \Rightarrow$ Slope of normal $= -\cot\theta$

Equation of normal at 'θ' is

$y - a(\sin\theta - \theta\cos\theta) = -\cot\theta(x - a(\cos\theta + \theta\sin\theta))$

$\Rightarrow y\sin\theta - a\sin^2\theta + a\,\theta\cos\theta\sin\theta$

$= -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$

$\Rightarrow x\cos\theta + y\sin\theta = a$

Clearly this is an equation of straight line which is at a constant distance 'a' from origin.

79. **(d)** Since, $x = a(1 + \cos\theta)$

$$\Rightarrow \frac{dx}{d\theta} = -a\sin\theta \text{ and } y = a\sin\theta$$

$$\Rightarrow \frac{dy}{d\theta} = a\cos\theta$$

$$\therefore \frac{dy}{dx} = -\cot\theta.$$

$\therefore$ The slope of the normal at $\theta = \tan\theta$

$\therefore$ The equation of the normal at θ is

$$y - a\sin\theta = \tan\theta(x - a - a\cos\theta)$$

$$\Rightarrow y\cos\theta - a\sin\theta\cos\theta = x\sin\theta - a\sin\theta - a\sin\theta\cos\theta$$

$$\Rightarrow x\sin\theta - y\cos\theta = a\sin\theta$$

$$\Rightarrow y = (x - a)\tan\theta$$

which always passes through $(a, 0)$

80. **(b)** $f''(x) = 6(x-1)$. Inegrating, we get

$$f'(x) = 3x^2 - 6x + c$$

Slope at (2, 1) $= f'(2) = c = 3$

[$\because$ slope of tangent at (2,1) is 3]

$$\therefore f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$$

Inegrating again, we get $f(x) = (x-1)^3 + D$

The curve passes through (2, 1)

$$\Rightarrow 1 = (2-1)^3 + D \Rightarrow D = 0$$

$$\therefore f(x) = (x-1)^3$$

81. **(b)** $C_1 \to C_1 + C_2$

$$\text{Let } f(x) = \begin{vmatrix} 2 & 1+\sin^2 x & \sin 2x \\ 2 & \sin^2 x & \sin 2x \\ 1 & \sin^2 x & 1+\sin 2x \end{vmatrix}$$

$R_1 \to R_1 - 2R_3;\ R_2 \to R_2 - 2R_3$

$$= \begin{vmatrix} 0 & \cos^2\theta & -(2+\sin 2x) \\ 0 & -\sin^2 x & -(2+\sin 2x) \\ 1 & \sin^2 x & 1+\sin 2x \end{vmatrix} = -2 - 2\sin 2x$$

$$f'(x) = -2\cos 2x = 0$$

$$\Rightarrow \cos 2x = 0 \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$f''(x) = 4\sin 2x$$

So, $f''\left(\frac{\pi}{4}\right) = 4 > 0$ (minima)

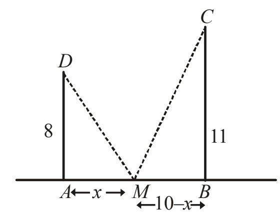
$$m = f\left(\frac{\pi}{4}\right) = -2 - 1 = -3$$

$f''\left(\frac{3\pi}{4}\right) = -4 < 0$ (maxima)

$$M = f\left(\frac{3\pi}{4}\right) = -2 + 1 = -1$$

So, $(m, M) = (-3, -1)$

82. **(5)**



Let $AM = x$ m

$$\therefore (MD)^2 + (MC)^2 = 64 + x^2 + 121 + (10-x)^2 = f(x)$$ (say)

$$f'(x) = 2x - 2(10-x) = 0$$

$$\Rightarrow 4x = 20 \Rightarrow x = 5$$

$$f''(x) = 2 - 2(-1) > 0$$

$\therefore f(x)$ is minimum at $x = 5$ m.

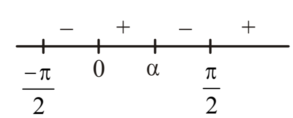
83. **(d)** $f(x) = (1-\cos^2 x)(\lambda + \sin x) = \sin^2 x(\lambda + \sin x)$

$$\Rightarrow f(x) = \lambda\sin^2 x + \sin^3 x \quad ...(i)$$

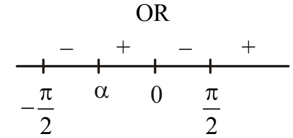
$$\Rightarrow f'(x) = \sin x\cos x[2\lambda + 3\sin x] = 0$$

$$\Rightarrow \sin x = 0 \text{ and } \sin x = -\frac{2\lambda}{3} \Rightarrow x = \alpha \text{ (let)}$$

So, $f(x)$ will change its sign at $x = 0, \alpha$ because there is exactly one maxima and one minima in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$



OR



Now, $\sin x = -\frac{2\lambda}{3}$

$$\Rightarrow -1 \le -\frac{2\lambda}{3} \le 1 \Rightarrow -\frac{3}{2} \le \lambda \le \frac{3}{2} - \{0\}$$

$\because$ If $\lambda = 0 \Rightarrow f(x) = \sin^3 x$ (from (i))

Which is monotonic, then no maxima/minima

So, $\lambda \in \left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\}$

84. **(d)** The given function

$f(x)=(3x^2+ax-2-a)e^x$

$f'(x)=(6x+a)e^x+(3x^2+ax-2-a)e^x$

$f'(x)=[3x^2+(a+6)x-2]e^x$

$\because$ $x = 1$ is critical point :

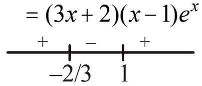
$\therefore f'(1)=0$

$\Rightarrow (3+a+6-2)\cdot e=0$

$\Rightarrow a=-7$ $\qquad (\because e>0)$

$\therefore f'(x)=(3x^2-x-2)e^x$

$=(3x+2)(x-1)e^x$



$\therefore x=-\frac{2}{3}$ is point of local maxima.

and $x = 1$ is point of local minima.

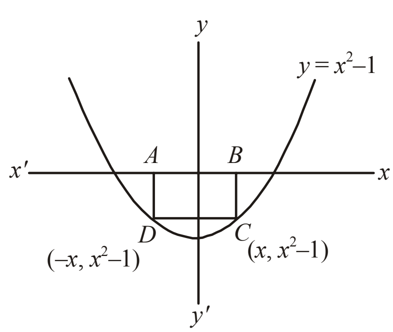
85. **(d)** Area of rectangle $ABCD$

$A=2x\cdot(x^2-1)=2x^3-2x$

$\therefore \quad \frac{dA}{dx}=6x^2-2$

For maximum area $\frac{dA}{dx}=0 \Rightarrow x=\pm\frac{1}{\sqrt{3}}$

$\frac{d^2A}{dx^2}=(12x)\Rightarrow\left(\frac{d^2A}{dx^2}\right)_{x=\frac{-1}{\sqrt{3}}}=\frac{-12}{\sqrt{3}}<0$



$\therefore$ Maximum area $=\left|\frac{2}{3\sqrt{3}}-\frac{2}{\sqrt{3}}\right|=\frac{4}{3\sqrt{3}}$

86. **(a)** $\because$ The critical points are –1, 0, 1

$\therefore f'(x)=k\cdot x(x+1)(x-1)=k(x^3-x)$

$\Rightarrow f(x)=k\left(\frac{x^4}{4}-\frac{x^2}{2}\right)+C$

$\Rightarrow f(0)=C$

$\because f(x)=f(0)$

$\Rightarrow k\frac{(x^4-2x^2)}{4}+C=C$

$\Rightarrow x^2(x^2-2)=0$

$\Rightarrow x=0,\sqrt{2},-\sqrt{2}$

$\Rightarrow T=\{0,\sqrt{2},-\sqrt{2}\}$

87. **(3)** Let $f(x) = ax^3 + bx^2 + cx + d$

$f(-1) = 10$ and $f(1) = -6$

$-a + b - c + d = 10$ ...(i)

$a + b + c + d = -6$ ...(ii)

Solving equations (i) and (ii), we get

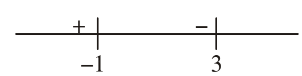
$a=\frac{1}{4}, d=\frac{35}{4}$

$b=\frac{-3}{4}, c=-\frac{9}{4}$

$\Rightarrow \quad f(x) = a(x^3 - 3x^2 - 9x) + d$

$f'(x)=\frac{3}{4}(x^2-2x-3)=0$

$\Rightarrow \quad x = 3, -1$



Local minima exist at $x = 3$

88. **(d)** $f(x) = ax^5 + bx^4 + cx^3$

$\lim\limits_{x\to 0}\left(2+\frac{ax^5+bx^4+cx^3}{x^3}\right)=4$

$\Rightarrow \quad 2 + c = 4 \quad \Rightarrow \quad c = 2$

$f'(x) = 5ax^4 + 4bx^3 + 6x^2$

$= x^2(5ax^2 + 4bx + 6)$

Since, $x = \pm 1$ are the critical points,

$\therefore \quad f'(1) = 0 \quad \Rightarrow \quad 5a + 4b + 6 = 0$ ...(i)

$f'(-1) = 0 \quad \Rightarrow \quad 5a - 4b + 6 = 0$ ...(ii)

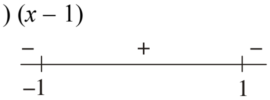
From eqns. (i) and (ii),

$b = 0$ and $a=-\frac{6}{5}$

$f(x)=\frac{-6}{5}x^5+2x^3$

$f'(x) = -6x^4 + 6x^2 = 6x^2(-x^2 + 1)$

$= -6x^2(x + 1)(x - 1)$



$\therefore \quad f(x)$ has minima at $x = -1$ and maxima at $x = 1$

89. **(b)** Given function $f(x) = x\sqrt{kx - x^2} = \sqrt{kx^3 - x^4}$

Differentiating w. r. t. x,

$$f'(x) = \frac{(3kx^2 - 4x^3)}{2\sqrt{kx^3 - x^4}} \geq 0 \text{ for } x \in [0, 3]$$

[$\because f(x)$ is increasing in [0, 3]]

$\Rightarrow 3k - 4x \geq 0 \Rightarrow 3k \geq 4x$

i.e., $3k \geq 4x$ for $x \in [0, 3]$

$\therefore k \geq 4$ i.e., $m = 4$

Putting $k = 4$ in the function, $f(x) = x\sqrt{4x - x^2}$

For max. value, $f'(x) = 0$

i.e. $\frac{12x^2 - 4x^3}{2\sqrt{4x^3 - x^4}} = 0 \Rightarrow x = 3$

$y = 3\sqrt{3}$ i.e., $M = 3\sqrt{3}$

90. **(b)** $a_6 = a + 5d = 2$

Here, a is first term of A.P and d is common difference

Let $A = a_1 a_4 a_5 = a(a + 3d)(a + 4d)$

$= a(2 - 2d)(2 - d)$

$A = (2 - 5d)(4 - 6d + 2d^2)$

By $\frac{dA}{dd} = 0$

$(2 - 5d)(-6 + 4d) + (4 - 6d + 2d^2)(-5) = 0$

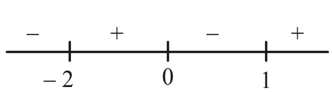
$-15d^2 + 34d - 16 = 0 \Rightarrow d = \frac{8}{5}, \frac{2}{3}$

For $d = \frac{8}{5}, \frac{d^2A}{dd^2} < 0$.

Hence $d = \frac{8}{5}$

91. **(c)** $f(x) = 9x^4 + 12x^3 - 36x^2 + 25$

$f'(x) = 36[x^3 + x^2 - 2x] = 36x(x - 1)(x + 2)$



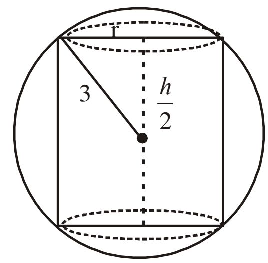
Here at –2 & 1, $f'(x)$ changes from negative value to positive value.

$\Rightarrow$ –2 & 1 are local minimum points. At 0, $f'(x)$ changes from positive value to negative value.

$\Rightarrow$ 0 is the local maximum point.

Hence, $S_1 = \{-2, 1\}$ and $S_2 = \{0\}$

92. **(c)** Let radius of base and height of cylinder be r and h respectively.



$$\therefore r^2 + \frac{h^2}{4} = 9 \quad \text{...(i)}$$

Now, volume of cylinder, $V = \pi r^2 h$

Substitute the value of r^2 from equation (i),

$$V = \pi h\left(9 - \frac{h^2}{4}\right) \Rightarrow V = 9\pi h - \frac{\pi}{4}h^3$$

Differentiating w.r.t. h,

$$\frac{dV}{dh} = 9\pi - \frac{3}{4}\pi h^2$$

For maxima/minima,

$$\frac{dV}{dh} = 0 \Rightarrow h = \sqrt{12}$$

and $\frac{d^2V}{dh^2} = -\frac{3}{2}\pi h$

$$\left(\frac{d^2V}{dh^2}\right)_{h=\sqrt{12}} < 0$$

Volume is maximum when $h = 2\sqrt{3}$

93. **(a)** Let, the functions is,

$$f(\theta) = 3\cos\theta + 5\sin\theta \cdot \cos\frac{\pi}{6} - 5\sin\frac{\pi}{6}\cos\theta$$

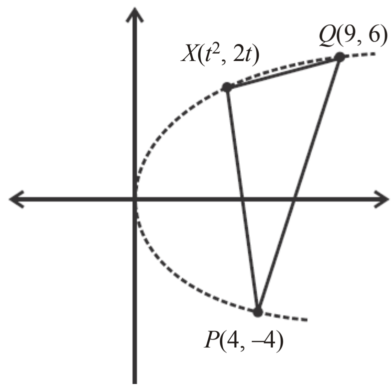
$$= 3\cos\theta + 5 \times \frac{\sqrt{3}}{2}\sin\theta - 5 \times \frac{1}{2}\cos\theta$$

$$= \left(3 - \frac{5}{2}\right)\cos\theta + 5 \times \frac{\sqrt{3}}{2}\sin\theta$$

$$= \frac{1}{2}\cos\theta + \frac{5\sqrt{3}}{2}\sin\theta$$

$$\max f(\theta) = \sqrt{\frac{1}{4} + \frac{25}{4} \times 3} = \sqrt{\frac{76}{4}} = \sqrt{19}$$

94. (b)



Parametric equations of the parabola $y^2 = 4x$ are, $x = t^2$ and $y = 2t$.

$$\text{Area } \Delta PXQ = \frac{1}{2}\begin{vmatrix} t^2 & 2t & 1 \\ 4 & -4 & 1 \\ 9 & 6 & 1 \end{vmatrix}$$

$= -5t^2 + 5t + 30$

$= -5(t^2 - t - 6)$

$$= -5\left[\left(t-\frac{1}{2}\right)^2 - \frac{25}{4}\right]$$

For maximum area $t = \frac{1}{2}$

$$\therefore \text{ maximum area } = 5\left(\frac{25}{4}\right) = \frac{125}{4}$$

95. (c) Consider the function,

$f(x) = 3x(x - 3)^2 - 40$

Now $S = \{x \in R : x^2 + 30 \le 11x\}$

So $x^2 - 11x + 30 \le 0 \Rightarrow x \in [5, 6]$

$\therefore f(x)$ will have maximum value for $x = 6$

The maximum value of function is,

$f(6) = 3 \times 6 \times 3 \times 3 - 40 = 122.$

96. (c) $A = \dfrac{x^m y^n}{(1+x^{2m})(1+y^{2n})} = \dfrac{1}{(x^{-m}+x^m)(y^{-n}+y^n)}$

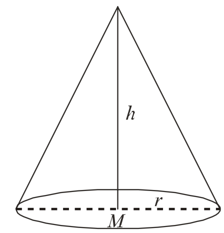
$$\frac{x^m + y^{-m}}{2} \ge (x^m . x^{-m})^{\frac{1}{2}} \Rightarrow x^m + x^{-m} \ge 2$$

In the same way, $y^{-n} + y^n \ge 2$

Then, $(x^m + x^{-m})(y^{-n} + y^n) \ge 4$

$$\Rightarrow \frac{1}{(x^m + x^{-m})(y^{-n}+y^n)} \le \frac{1}{4}$$

97. (d)



$h^2 + r^2 = \ell^2 = 9$ ...(i)

Volume of cone

$V = \frac{1}{3}\pi r^2 h$ ...(ii)

From (i) and (ii),

$\Rightarrow V = \frac{1}{3}\pi(9 - h^2)h$

$\Rightarrow V = \frac{1}{3}\pi(9h - h^3) \Rightarrow \frac{dv}{dh} = \frac{1}{3}\pi(9 - 3h^2)$

For maxima/minima,

$\frac{dV}{dh} = 0 \Rightarrow \frac{1}{3}\pi(9 - 3h^2) = 0$

$\Rightarrow h = \pm\sqrt{3} \Rightarrow h = \sqrt{3}$ $\quad (\because h > 0)$

Now; $\frac{d^2V}{dh^2} = \frac{1}{3}\pi(-6h)$

Here, $\left(\frac{d^2V}{dh^2}\right)_{\text{at } h=\sqrt{3}} < 0$

Then, $h = \sqrt{3}$ is point of maxima

Hence, the required maximum volume is,

$V = \frac{1}{3}\pi(9-3)\sqrt{3} = 2\sqrt{3}\pi$

98. (c) Here, $h(x) = \dfrac{x^2 + \frac{1}{x^2}}{x - \frac{1}{x}} = \left(x - \frac{1}{x}\right) + \dfrac{2}{x - \frac{1}{x}}$

When $x - \frac{1}{x} < 0$

$$\therefore \quad x - \frac{1}{x} + \frac{2}{x - \frac{1}{x}} \le -2\sqrt{2}$$

Hence, $-2\sqrt{2}$ will be local maximum value of h(x).

When $x - \frac{1}{x} > 0$

$\therefore \quad x-\frac{1}{x}+\frac{2}{x-\frac{1}{x}} \geq 2\sqrt{2}$

Hence, $2\sqrt{2}$ will be local minimum value of h(x).

99. **(a)** Here, $f(x) = 2x^3 - 9x^2 + 12x + 5$

$\Rightarrow \quad f'(x) = 6x^2 - 18x + 12 = 0$

For maxima or minima put $f'(x) = 0$

$\Rightarrow x^2 - 3x + 2 = 0$

$\Rightarrow x = 1$ or $x = 2$

Now, $f''(x) = 12x - 18$

$\Rightarrow \quad f''(1) = 12(1) - 18 = -6 < 0$

Hence, $f(x)$ has maxima at $x = 1$

$\therefore$ maximum value $= M = f(1) = 2 - 9 + 12 + 5 = 10.$

And, $f''(2) = 12(2) - 18 = 6 > 0.$

Hence, $f(x)$ has minima at $x = 2.$

$\therefore$ minimum value $= m = f(2)$

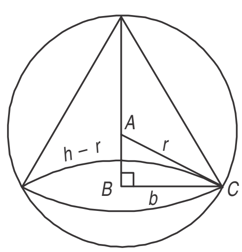
$= 2(8) - 9(4) + 12(2) + 5 = 9$

$\therefore M - m = 10 - 9 = 1$

100. **(a)** Sphere of radius $r = 3$ *cm*

Let b, h be base radius and height of cone respectively.

So, volume of cone $= \frac{1}{2}\pi b^2 h$



In right angled ΔABC by Pythagoras theorem

$(h - r)^2 + b^2 = r^2$ ...(i)

$\Rightarrow b^2 = r^2 - (h - r)^2 = r^2 - (h^2 - 2hr + r^2) = 2hr - h^2$

$\therefore$ Volume $(v) = \frac{1}{3}\pi h\left[2hr - h^2\right] = \frac{1}{3}\left[2h^2 r - h^3\right]$

$\frac{dv}{dh} = \frac{1}{3}\left[4hr - 3h^2\right] = 0 \Rightarrow h(4r - 3h) = 0$

$\frac{d^2v}{dh^2} = \frac{1}{3}[4r - 6h]$

At $h = \frac{4r}{3}, \frac{d^2v}{dh^2} = \frac{1}{3}\left[4r - \frac{4r}{3}\times 6\right] = \frac{1}{3}[4r - 8r] < 0$

$\Rightarrow$ maximum volume ocurs at $h = \frac{4r}{3} = \frac{4}{3}\times 3 = 4$ *cm*

As from (i),

$(h - r)^2 + b^2 = r^2$

$\Rightarrow b^2 = 2hr - h^2 = 2\cdot\frac{4r}{3}r - \frac{16r^2}{9} = \frac{8r^2}{3} - \frac{16r^2}{9}$

$= \frac{(24-16)r^2}{9} = \frac{8r^2}{9}$

$\Rightarrow b = \frac{2\sqrt{2}}{3}r = 2\sqrt{2}$ *cm*

Therefore curved surface area $= \pi bl$

$= \pi b\sqrt{h^2 + r^2} = \pi 2\sqrt{2}\sqrt{4^2 + 8} = 8\sqrt{3}\pi\ cm^2$

101. **(d)** We have

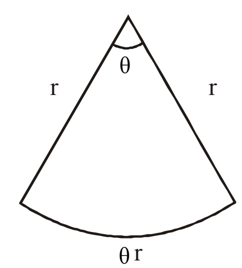
Total length = r + r + rθ = 20

$\Rightarrow 2r + r\theta = 20$

$\Rightarrow \theta = \frac{20 - 2r}{r}$ ...(i)

$A = \text{Area} = \frac{\theta}{2\pi}\times \pi r^2$

$= \frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\frac{20-2r}{r}\right)$



$A = 10r - r^2$

For A to be maximum

$\frac{dA}{dr} = 0 \Rightarrow 10 - 2r = 0$

$\Rightarrow r = 5$

$\frac{d^2A}{dr^2} = -2 < 0$

$\therefore$ For r = 5 A is maximum

From (i)

$\theta = \frac{20 - 2(5)}{5} = \frac{10}{5} = 2$

$A = \frac{2}{2\pi}\times\pi(5)^2 = 25$ sq. m

102. **(a)** $4x + 2\pi r = 2 \quad \Rightarrow 2x + \pi r = 1$

$S = x^2 + \pi r^2$

$S = \left(\frac{1-\pi r}{2}\right)^2 + \pi r^2$

$\frac{dS}{dr} = 2\left(\frac{1-\pi r}{2}\right)\left(\frac{-\pi}{2}\right) + 2\pi r$

$\Rightarrow \frac{-\pi}{2} + \frac{\pi^2 r}{2} + 2\pi r = 0 \quad \Rightarrow r = \frac{1}{\pi + 4}$

$\Rightarrow x = \frac{2}{\pi + 4} \quad \Rightarrow x = 2r$

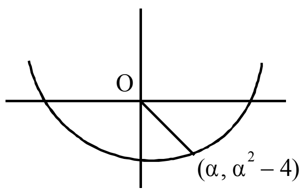
103. (a) $D = \sqrt{\alpha^2 + (\alpha^2 - 4)^2}$

$D^2 = \alpha^2 + \alpha^4 + 16 - 8\alpha^2 = \alpha^4 - 7\alpha^2 + 16$

$\frac{dD^2}{d\alpha} = 4\alpha^3 - 14\alpha = 0$

$2\alpha(2\alpha^2 - 7) = 0$

$\alpha^2 = \frac{7}{2}$



$D^2 = \frac{49}{4} - \frac{49}{2} + 16 = -\frac{49}{4} + 16 = \frac{15}{4}$

$D = \frac{\sqrt{15}}{2}$

104. (a) Let $f(x) = \frac{(1+x)^{\frac{3}{5}}}{1+x^{\frac{3}{5}}}$ and $x \in [0, 1]$

$\Rightarrow f'(x) = \frac{(1+x^{\frac{3}{5}})\frac{3}{5}(1+x)^{-\frac{2}{5}} - \frac{3}{5}(1+x)^{\frac{3}{5}}(x^{\frac{-2}{5}})}{(1+x^{\frac{3}{5}})^2}$

$= \frac{3}{5}\left[\left(1+x^{\frac{3}{5}}\right)(1+x)^{-\frac{2}{5}} - (1+x)^{\frac{3}{5}}x^{\frac{-2}{5}}\right]$

$= \frac{3}{5}\left[\frac{1+x^{\frac{3}{5}}}{(1+x)^{\frac{2}{5}}} - \frac{(1+x)^{\frac{3}{5}}}{x^{\frac{2}{5}}}\right]$

$= \frac{x^{\frac{2}{5}} + x - 1 - x}{x^{\frac{2}{5}}(1+x)^{\frac{2}{5}}} = \frac{x^{\frac{2}{5}} - 1}{x^{\frac{2}{5}}(1+x)^{\frac{2}{5}}} < 0$

Also, $f(0) = 1 \Rightarrow f(x) \in [2^{-0.4}, 1]$
$f(a) = 2^{-0.4}$

105. (d) Let 'u' be the velocity
$\therefore$ u = 4 8 m/s, Given, g = 32
At maximum height v = 0
Now, we know $v^2 = u^2 - 2gh$
$\Rightarrow 0 = (48)^2 - 2(32)h \Rightarrow h = 36$
Maximum height = 36 + 64 = 100 mt

106. (a) Let $f(x) = \alpha \log |x| + \beta x^2 + x$
Differentiate both side,

$f'(x) = \frac{\alpha}{x} + 2\beta x + 1$

Since $x = -1$ and $x = 2$ are extreme points therefore $f'(x) = 0$ at these points.

Put $x = -1$ and $x = 2$ in $f'(x)$, we get

$-\alpha - 2\beta + 1 = 0 \Rightarrow \alpha + 2\beta = 1$ ...(i)

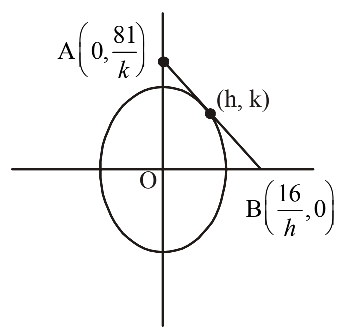
$\frac{\alpha}{2} + 4\beta + 1 = 0 \Rightarrow \alpha + 8\beta = -2$ ...(ii)

On solving (i) and (ii), we get

$6\beta = -3 \Rightarrow \beta = -\frac{1}{2}$

$\therefore \alpha = 2$

107. (d)



Let (h, k) be the point on ellipse through which tangent is passing.

Equation of tangent at (h, k) = $\frac{xh}{16} + \frac{yk}{81} = 1$

at $y = 0$, $x = \frac{16}{h}$

at $x = 0$, $y = \frac{81}{k}$

Area of AOB = $\frac{1}{2} \times \left(\frac{16}{h}\right) \times \left(\frac{81}{k}\right) = \frac{648}{hk}$

$A^2 = \frac{(648)^2}{h^2k^2}$ ...(i)

(h, k) must satisfy equation of ellipse

$\frac{h^2}{16} + \frac{k^2}{81} = 1$

$h^2 = \frac{16}{81}(81 - k^2)$

Putting value of h^2 in equation (i)

$$A^2 = \frac{81(648)^2}{16 \times k^2(81-k^2)} = \frac{\alpha}{81k^2 - k^4}$$

differentiating w.r. to k

$$2AA' = \alpha\left(\frac{-1}{81k^2 - k^4}\right)(162k - 4k^3)$$

$2AA' = -2A\ (81k - 4k^3) \Rightarrow A' = -\ 81k - 4k^3$

Put $A' = 0$

$\Rightarrow\ 162k - 4k^3 = 0,\ k\ (162 - 4k^2) = 0$

$$\Rightarrow\quad k = 0,\ k = \pm\frac{9}{\sqrt{2}}$$

$A'' = -\ (81 - 12k^2)$

For both value of k, $A'' = 405 > 0$

Area will be minimum for $k = \pm\frac{9}{\sqrt{2}}$

$$h^2 = \frac{16}{81}(81-k^2) = 8$$

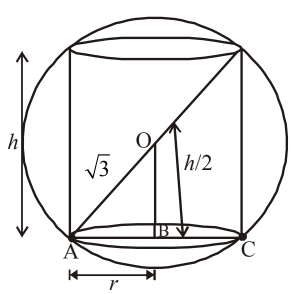
$$h = \pm 2\sqrt{2}$$

Area of triangle AOB = $\frac{648 \times \sqrt{2}}{2\sqrt{2} \times 9}$ = 36 sq unit

108. (c) Given, radius of sphere = $\sqrt{3}$

Now, In ΔOAB, by Pythagoras theorem

$(OA)^2 = (OB)^2 + (AB)^2$



$$(\sqrt{3})^2 = \left(\frac{h}{2}\right)^2 + r^2$$

$$3 = \frac{h^2}{4} + r^2 \Rightarrow \boxed{r^2 = 3 - \frac{h^2}{4}} \quad \text{...(i)}$$

Now, volume of cylinder = $\pi r^2 h$

$$V = \pi\left(3 - \frac{h^2}{4}\right)h \quad \text{(using eq. (i))}$$

$$V = 3\pi h - \frac{\pi h^3}{4} \quad \text{...(ii)}$$

Now, for largest possible right circular cylinder the volume must be maximum

$\therefore$ For maximum volume, $\frac{dV}{dh} = 0$

Now, Differentiating eq. (2) w.r.t. h

$$\frac{dV}{dh} = 3\pi - \frac{3}{4}\pi h^2$$

or $3\pi - \frac{3}{4}\pi h^2 = 0 \Rightarrow 3\pi = \frac{3}{4}\pi h^2$

$\Rightarrow h^2 = 4 \Rightarrow h = 2$

Now, volume (V) of the cylinder

$$= \pi\left(3 - \frac{h^2}{4}\right)h = \pi(6-2) = 4\pi$$

109. (c) Let cost $C = av + \frac{b}{v}$

According to given question,

$$30a + \frac{b}{30} = 75 \quad \text{... (i)}$$

$$40a + \frac{b}{40} = 65 \quad \text{... (ii)}$$

On solving (i) and (ii), we get

$a = \frac{1}{2}$ and $b = 1800$

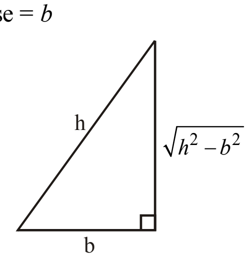
Now, $C = av + \frac{b}{v}$

$$\Rightarrow \frac{dC}{dv} = a - \frac{b}{v^2}$$

$$\frac{dC}{dv} = 0 \Rightarrow a - \frac{b}{v^2} = 0$$

$$\Rightarrow v = \sqrt{\frac{b}{a}} = \sqrt{3600} \Rightarrow v = 60 \text{ kmph}$$

110. (d) Let base = b



Altitude (or perpendicular) = $\sqrt{h^2 - b^2}$

Area, A = $\frac{1}{2}$ × base × altitude = $\frac{1}{2} \times b \times \sqrt{h^2 - b^2}$

$$\Rightarrow \frac{dA}{db} = \frac{1}{2}\left[\sqrt{h^2 - b^2} + b \cdot \frac{-2b}{2\sqrt{h^2 - b^2}}\right]$$

$= \frac{1}{2}\left[\frac{h^2 - 2b^2}{\sqrt{h^2 - b^2}}\right]$

Put $\frac{dA}{db} = 0$, $\Rightarrow \quad b = \frac{h}{\sqrt{2}}$

Maximum area = $\frac{1}{2} \times \frac{h}{\sqrt{2}} \times \sqrt{h^2 - \frac{h^2}{2}} = \frac{h^2}{4}$

111. **(b)** Given that, $f(x) = \ln|x| + bx^2 + ax$

$\therefore \quad f'(x) = \frac{1}{x} + 2bx + a$

At $x = -1$, $f'(x) = -1 - 2b + a = 0$

$\Rightarrow \quad a - 2b = 1$ ...(i)

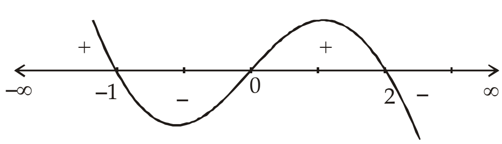
At $x = 2$, $f'(x) = \frac{1}{2} + 4b + a = 0$

$\Rightarrow \quad a + 4b = -\frac{1}{2}$ ...(ii)

On solving (i) and (ii) we get $a = \frac{1}{2}, b = -\frac{1}{4}$

Thus, $f'(x) = \frac{1}{x} - \frac{x}{2} + \frac{1}{2} = \frac{2 - x^2 + x}{2x}$

$= \frac{-x^2 + x + 2}{2x} = \frac{-(x^2 - x - 2)}{2x} = \frac{-(x+1)(x-2)}{2x}$



So maxima at $x = -1, 2$

112. **(c)** Equation of a line passing through (x_1, y_1) having slope m is given by $y - y_1 = m(x - x_1)$

Since the line PQ is passing through (1,2) therefore its equation is $(y - 2) = m(x - 1)$

where m is the slope of the line PQ.

Now, point $P(x,0)$ will also satisfy the equation of PQ

$\therefore \quad y - 2 = m(x - 1) \Rightarrow 0 - 2 = m(x - 1)$

$\Rightarrow \quad -2 = m(x - 1) \Rightarrow \quad x - 1 = \frac{-2}{m}$

$\Rightarrow \quad x = \frac{-2}{m} + 1$

Also, $OP = \sqrt{(x-0)^2 + (0-0)^2} = x = \frac{-2}{m} + 1$

Similarly, point $Q(0,y)$ will satisfy equation of PQ

$\therefore \quad y - 2 = m(x - 1)$

$\Rightarrow \quad y - 2 = m(-1)$

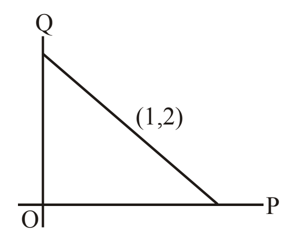
$\Rightarrow y = 2 - m$ and $OQ = y = 2 - m$

Area of $\Delta POQ = \frac{1}{2}(OP)(OQ) = \frac{1}{2}\left(1 - \frac{2}{m}\right)(2 - m)$

($\because$ Area of $\Delta = \frac{1}{2} \times \text{base} \times \text{height}$)

$= \frac{1}{2}\left[2 - m - \frac{4}{m} + 2\right] = \frac{1}{2}\left[4 - \left(m + \frac{4}{m}\right)\right]$

$= 2 - \frac{m}{2} - \frac{2}{m}$



Let Area $= f(m) = 2 - \frac{m}{2} - \frac{2}{m}$

Now, $f'(m) = \frac{-1}{2} + \frac{2}{m^2}$

Put $f'(m) = 0$

$\Rightarrow m^2 = 4 \Rightarrow m = \pm 2$

Now, $f''(m) = \frac{-4}{m^3}$

$f''(m)\big|_{m=2} = -\frac{1}{2} < 0$

$f''(m)\big|_{m=-2} = \frac{1}{2} > 0$

Area will be least at $m = -2$

Hence, slope of PQ is -2.

113. **(d)** Let $f: (-\infty, \infty) \to (-\infty, \infty)$ be defined by $f(x) = x^3 + 1$.

Clearly, $f(x)$ is symmetric along $y = 1$ and it has neither maxima nor minima.

$\therefore$ Statement-1 is false.

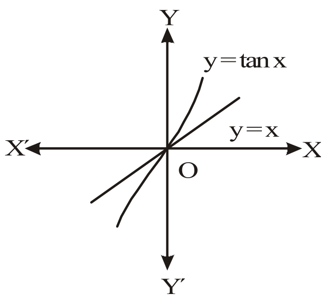
Hence, option (d) is correct.

114. **(b)** $f(x) = \begin{cases} \frac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

For $x > 0$

$\tan x > x$

$\frac{\tan x}{x} > 1$



For $x < 0 \Rightarrow \tan x < x$

$\Rightarrow \frac{\tan x}{x} > 1$

$f(0) = 1$ at $x = 0$

$\Rightarrow \quad x = 0$ is the point of minima

So, Statement 1 is true. Statement 2 is also true.

115. (c) $f'(x) = \sqrt{x}\sin x$

$f'(x) = 0$

$\Rightarrow x = 0$ or $\sin x = 0$

$\Rightarrow x = 2\pi, \pi$

$f''(x) = \sqrt{x}\cos x + \frac{1}{2\sqrt{x}}\sin x$

$= \frac{1}{2\sqrt{x}}(2x\cos x + \sin x)$

At $x = \pi$, $f''(x) < 0$

Hence, local maxima at $x = \pi$

At $x = 2\pi$, $f''(x) > 0$

Hence local minima at $x = 2\pi$

116. (d) Given $f(x) = \frac{1}{e^x + 2e^{-x}} = \frac{e^x}{e^{2x} + 2}$

$f'(x) = \frac{(e^{2x}+2)e^x - 2e^{2x}.e^x}{(e^{2x}+2)^2}$

$f'(x) = 0 \Rightarrow e^{2x} + 2 = 2e^{2x}$

$\Rightarrow e^{2x} = 2 \quad \Rightarrow e^x = \sqrt{2}$

$\because f''\left(\sqrt{2}\right) = +ve$

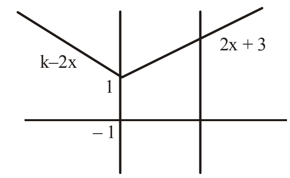
$\therefore$ Maximum values of $f(x) = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}}$

$\Rightarrow 0 < f(x) \le \frac{1}{2\sqrt{2}} \quad \forall x \in R$

Since, $0 < \frac{1}{3} < \frac{1}{2\sqrt{2}}$

$\Rightarrow$ for some $c \in R$, $f(c) = \frac{1}{3}$

117. (c) $f(x) = \begin{cases} k - 2x, & \text{if } x \le -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$



Clear that $f(x)$ is minimum at (–1, 1)

$\therefore f(-1) = 1$

$1 = k + 2 \Rightarrow k = -1$

118. (a) Given that $P(x) = x^4 + ax^3 + bx^2 + cx + d$

$\Rightarrow \quad P'(x) = 4x^3 + 3ax^2 + 2bx + c$

But given $P'(0) = 0 \Rightarrow c = 0$

$\therefore \quad P(x) = x^4 + ax^3 + bx^2 + d$

Again given that $P(-1) < P(1)$

$\Rightarrow \quad 1 - a + b + d < 1 + a + b + d$

$\Rightarrow \quad a > 0$

Now $P'(x) = 4x^3 + 3ax^2 + 2bx = x(4x^2 + 3ax + 2b)$

As $P'(x) = 0$, there is only one solution $x = 0$, therefore $4x^2 + 3ax + 2b = 0$ should not have any real roots i.e. $D < 0$

$\Rightarrow \quad 9a^2 - 32b < 0 \Rightarrow b > \frac{9a^2}{32} > 0$

Hence $a, b > 0$

$\Rightarrow \quad P'(x) = 4x^3 + 3ax^2 + 2bx > 0 \qquad \forall x > 0$

$\therefore \quad P(x)$ is an increasing function on (0,1)

$\therefore \quad P(0) < P(a)$

Similarly we can prove $P(x)$ is decreasing on (– 1, 0)

$\therefore \quad P(-1) > P(0)$

So we can conclude that

Max $P(x) = P(1)$ and Min $P(x) = P(0)$

$\Rightarrow P(-1)$ is not minimum but $P(1)$ is the maximum of P.

119. (a) Let $y = x^3 - px + q \Rightarrow \frac{dy}{dx} = 3x^2 - p$

For maxima and minima

$\frac{dy}{dx} = 0 \Rightarrow 3x^2 - p = 0 \Rightarrow x = \pm\sqrt{\frac{p}{3}}$

$\frac{d^2y}{dx^2} = 6x \quad \left.\frac{d^2y}{dx^2}\right|_{x=\sqrt{\frac{p}{3}}} = +ve$ and $\left.\frac{d^2y}{dx^2}\right|_{x=-\sqrt{\frac{p}{3}}} = -ve$

$\therefore y$ has minimum at $x = \sqrt{\frac{p}{3}}$ and maximum at

$x = -\sqrt{\frac{p}{3}}$

120. (a) Given $f(x) = \frac{x}{2} + \frac{2}{x} \Rightarrow f'(x) = \frac{1}{2} - \frac{2}{x^2} = 0$

$\Rightarrow x^2 = 4 \Rightarrow x = 2, -2;$

Now, $f''(x) = \frac{4}{x^3}$

$f''(x)]_{x=2} = +ve \Rightarrow f(x)$ has local min at $x = 2$.

121. (c) ATQ, $y = x + \frac{1}{x} \Rightarrow \frac{dy}{dx} = 1 - \frac{1}{x^2}$

For maxima. or minima.,

$1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1$

$\frac{d^2y}{dx^2} = \frac{2}{x^3} \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=1} = 2 > 0$

$\therefore y$ is minimum at $x = 1$

122. (d) $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$

$f'(x) = 6x^2 - 18ax + 12a^2;$

For maxima or minima.

$6x^2 - 18ax + 12a^2 = 0 \Rightarrow x^2 - 3ax + 2a^2 = 0$

$\Rightarrow x = a$ or $x = 2a$.

$f''(x) = 12x - 18a$

$f''(a) = -6a < 0 \therefore f(x)$ is max. at $x = a$,

$f''(2a) = 6a > 0$

$\therefore f(x)$ is min. at $x = 2a$

$\therefore p = a$ and $q = 2a$

ATQ, $p^2 = q$

$\therefore a^2 = 2a \Rightarrow a = 2$ or $a = 0$

but $a > 0$, therefore, $a = 2$.

123. (b) We know that distance of origin from

$(x, y) = \sqrt{x^2 + y^2}$

$= \sqrt{a^2 + b^2 - 2ab\cos\left(t - \frac{at}{b}\right)};$

$\leq \sqrt{a^2 + b^2 + 2ab}$

$\left[\left\{\cos\left(t - \frac{at}{b}\right)\right\}_{min} = -1\right] = a + b$

$\therefore$ Maximum distance from origin = $a + b$