Algebraic Identities and Factorisation

- $(x + a)(x + b) = x^2 + (a + b)x + ab$

Example:

Find 206 × 198.

Solution:

We have,

$206 \times 198 = (200 + 6)(200 - 2)$

$= (200)^2 + (6 + (-2)) \times 200 + (6)(-2)$ [Using identity $(x + a)(x + b) = x^2 + (a + b)x + ab$]

$= 40000 + 800 - 12$

$= 40800 - 12$

$= 40788$

- An identity is an equality which is true for all values of the variables in it. It helps us in shortening our calculations.
- Identities for "Square of Sum or Difference of Two Terms" are:
  - $(a + b)^2 = a^2 + 2ab + b^2$
  - $(a - b)^2 = a^2 - 2ab + b^2$

  Example:

  Evaluate $(5x + 2y)^2 - (3x - y)^2$.

  Solution:

  Using identities (i) and (ii), we obtain

  $(5x + 2y)^2 = (5x)^2 + 2(5x)(2y) + (2y)^2$

$= 25x^2 + 20xy + 4y^2$

$(3x - y)^2 = (3x)^2 - 2\,(3x)\,(y) + (y)^2$

$= 9x^2 - 6xy + y^2$

$\therefore (5x + 2y)^2 - (3x - y)^2 = 25x^2 + 20xy + 4y^2 - 9x^2 + 6xy - y^2 = 16x^2 + 26xy + 3y^2$

- **Identity:** $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

We can use this identity to factorize and expand the polynomials.

For example, the given expression can be factorized as follows:

$$2x^2 + 27y^2 + 25z^2 + 6\sqrt{6}xy - 30\sqrt{3}yz - 10\sqrt{2}xz$$
$$= \left(\sqrt{2}x\right)^2 + \left(3\sqrt{3}y\right)^2 + \left(-5z\right)^2 + 2\cdot\left(\sqrt{2}x\right)\left(3\sqrt{3}y\right) + 2\left(3\sqrt{3}y\right)(-5z) + 2\left(\sqrt{2}x\right)(-5z)$$

On comparing the expression with $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$, we get

$$2x^2 + 27y^2 + 25z^2 + 6\sqrt{6}xy - 30\sqrt{3}yz - 10\sqrt{2}xz$$
$$= \left(\sqrt{2}x + 3\sqrt{3}y - 5z\right)^2$$

- **Identities:** $(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$ and $(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)$

Other ways to represent these identities are:

  - $x^3 + y^3 = (x + y)^3 - 3xy\,(x + y)$
  - $x^3 + y^3 = (x + y)\,(x^2 - xy + y^2)$
  - $x^3 - y^3 = (x - y)^3 + 3xy\,(x - y)$
  - $x^3 - y^3 = (x - y)\,(x^2 + xy + y^2)$

Example:

Expand $(3x + 2y)^3 - (3x - 2y)^3$

Solution:

$(3x + 2y)^3 = (3x)^3 + (2y)^3 + 3\,(3x)\,(2y)\,(3x + 2y)$

$= 27x^3 + 8y^3 + 54x^2y + 36xy^2 \qquad \text{... (1)}$

$(3x - 2y)^3 = (3x)^3 - (2y)^3 - 3\,(3x)\,(2y)\,(3x - 2y)$

$= 27x^3 - 8y^3 - 54x^2y + 36xy^2 \qquad \text{... (2)}$

From equations (1) and (2) in given expression, we get

$(3x + 2y)^3 - (3x - 2y)^3 = (27x^3 + 8y^3 + 54x^2y + 36xy^2) - (27x^3 - 8y^3 - 54x^2y + 36xy^2)$

$= 27x^3 + 8y^3 + 54x^2y + 36xy^2 - 27x^3 + 8y^3 + 54x^2y - 36xy^2$

$= 16y^3 + 108x^2y$

- **Factorization** is the decomposition of an algebraic expression into product of factors. Factors of an algebraic term can be numbers or algebraic variables or algebraic expressions.

  For example, the factors of $2a^2b$ are 2, a, a, b, since $2a^2b = 2 \times a \times a \times b$

  The factors, 2, a, a, b, are said to be irreducible factors of $2a^2b$ since they cannot be expressed further as a product of factors.

  Also, $2a^2b = 1 \times 2 \times a \times a \times b$

  Therefore, 1 is also a factor of $2a^2b$. In fact, 1 is a factor of every term. However, we do not represent 1 as a separate factor of any term unless it is specially required.

  For example, the expression, $2x^2(x + 1)$, can be factorized as $2 \times x \times x \times (x + 1)$.

  Here, the algebraic expression $(x + 1)$ is a factor of $2x^2(x + 1)$.

- **Factorization of expressions by the method of common factors**

  This method involves the following steps.

Step 1: Write each term of the expression as a product of irreducible factors.

Step 2: Observe the factors, which are common to the terms and separate them.

Step 3: Combine the remaining factors of each term by making use of distributive law.

Example: Factorize $12p^2q + 8pq^2 + 18\,pq$.

Solution: We have,

$12p^2q = 2 \times 2 \times 3 \times p \times p \times q$

$8pq^2 = 2 \times 2 \times 2 \times p \times q \times q$

$18pq = 2 \times 3 \times 3 \times p \times q$

The common factors are 2, p, and q.

$\therefore 12p^2q + 8pq^2 + 18pq$

$= 2 \times p \times q\,[(2 \times 3 \times p) + (2 \times 2 \times q) + (3 \times 3)]$

$= 2pq\,(6p + 4q + 9)$

- **Factorization by regrouping terms**

Sometimes, all terms in a given expression do not have a common factor. However, the terms can be grouped by trial and error method in such a way that all the terms in each group have a common factor. Then, there happens to occur a common factor amongst each group, which leads to the required factorization.

Example:

Factorize $2a^2 - b + 2a - ab$.

Solution:

$2a^2 - b + 2a - ab = 2a^2 + 2a - b - ab$

The terms,$2a^2$ and $2a$, have common factors, 2 and a.

The terms, $-b$ and $-ab$ have common factors, -1 and b.

Therefore,

$2a^2 - b + 2a - ab = 2a^2 + 2a - b - ab$

$= 2a(a + 1) - b(1 + a)$

$= (a + 1)(2a - b)$ (As the factor, $(1 + a)$, is common to both the terms)

Thus, the factors of the given expression are $(a + 1)$ and $(2a - b)$.

- $(a + b)(a - b) = a^2 - b^2$

Example:

Evaluate 95×105.

Solution:

We have, $95 \times 105 = (100 - 5) \times (100 + 5)$

$= (100)^2 - (5)^2$ [Using identity $(a + b)(a - b) = a^2 - b^2$]

$= 10000 - 25$

$= 9975$

The method to factorize quadratic trinomial of the form $ax^2 + bxy + cy^2$ is same as that of $ax^2 + bx + c$.

For example, $2x^2 + 3xy - 5y^2$ can be factorized as follows:

Here, $2 \times (-5) = (-10)$ or $5 \times (-2) = (-10)$ and $5 - 2 = 3$

$$\begin{aligned}\therefore 2x^2 + 3xy - 5y^2 &= 2x^2 + 5xy - 2xy - 5y^2 \\ &= x(2x + 5y) - y(2x + 5y) \\ &= (x - y)(2x + 5y)\end{aligned}$$