Standard X

MATHEMATICS

Part-1

Government of Kerala
Department of Education

State Council of Educational Research and Training (SCERT)
2016

THE NATIONAL ANTHEM

Jana-gana-mana adhinayaka, jaya he
Bharatha-bhagya-vidhata.
Punjab-Sindh-Gujarat-Maratha
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchala-Jaladhi-taranga
Tava subha name jage,
Tava subha asisa mage,
Gahe tava jaya gatha.
Jana-gana-mangala-dayaka jaya he
Bharatha-bhagya-vidhata.
Jaya he, jaya he, jaya he,
Jaya jaya jaya, jaya he!

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give respect to my parents, teachers and all elders and treat everyone with courtesy.

I pledge my devotion to my country and my people. In their well-being and prosperity alone lies my happiness.

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Dear children,

Mathematics starts in counting and measuring. In the age of agriculture, it becomes the second degree equations of areas; rises to astronomy for weather prediction. Grows into the branch of mathematics called trigonometry. In Renaissance Europe, trigonometry forms the foundation of navigation. It becomes the basis of locating places using satellites in today's world. The mathematical principles which seventeenth century mathematicians developed as purely mathematical operations of numbers are now used to make security systems in e-transactions. I wish all of you would recognize the innumerable applications of mathematics and revel in its theoretical rhythms.

With love and regards

Dr. P. A. Fathima
Director, SCERT

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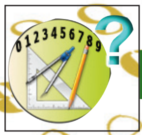

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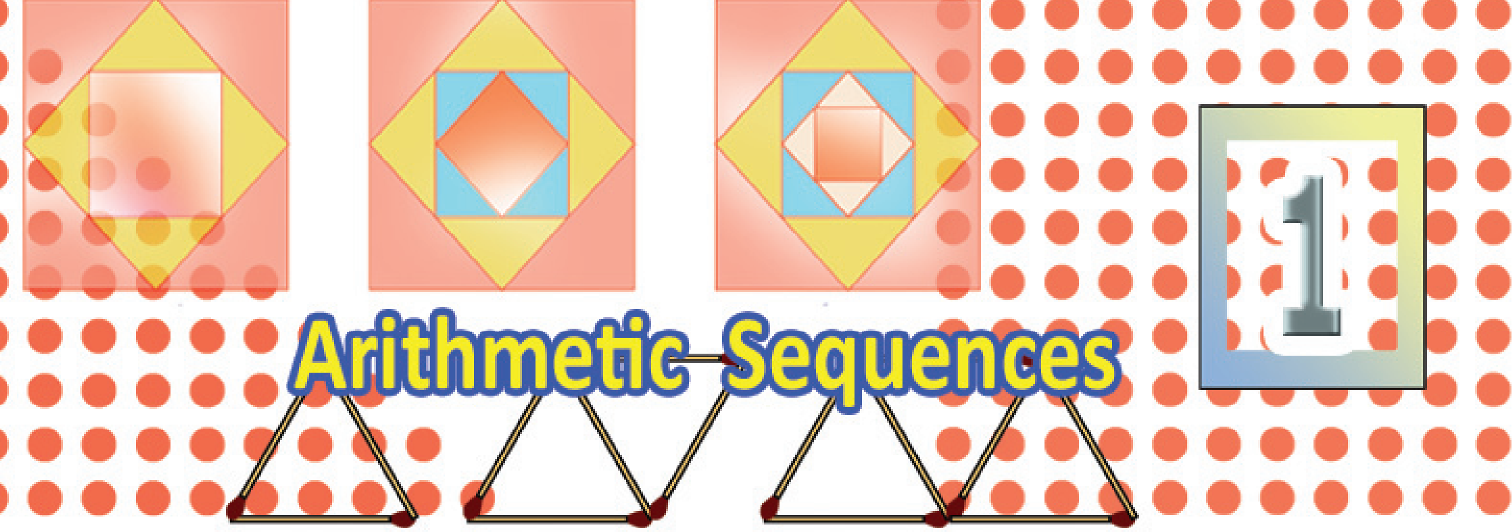
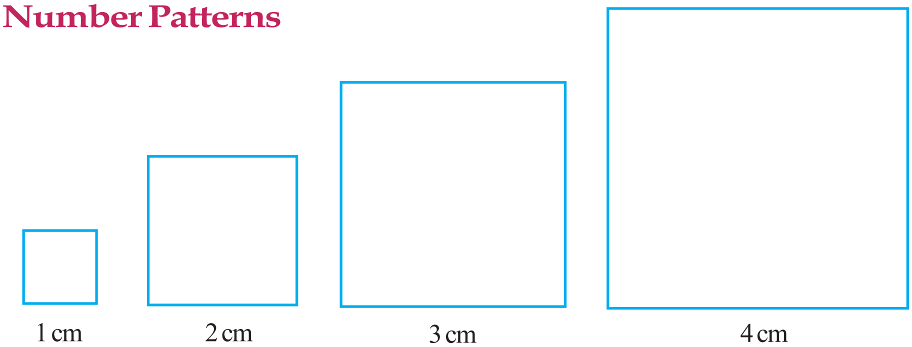


1 Arithmetic Sequences

Number Patterns



See the squares. What are their perimeters?

And areas?

As the lengths of the sides go

1 cm, 2 cm, 3 cm, 4 cm, ...

the perimeters are

4 cm, 8 cm, 12 cm, 16 cm, ...

And the areas

1 sq.cm,   4 sq.cm,   9 sq.cm,   16 sq.cm, ...

Let's look at the numbers alone.

The lengths of the sides are just the natural numbers, written in order;

1, 2, 3, 4, ...

The perimeters form the multiples of 4, in order

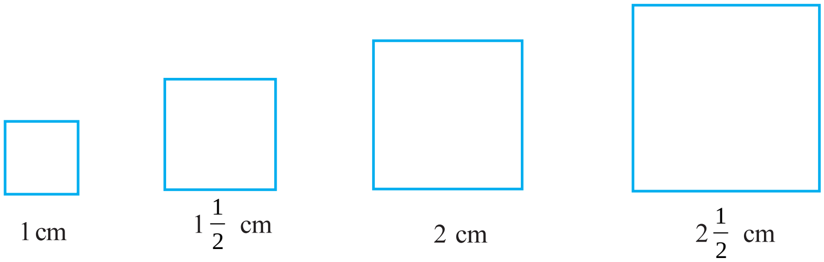
4, 8, 12, 16, ...

And the areas form the perfect squares in order

1, 4, 9, 16, ...

What about their diagonals? Write those numbers also.

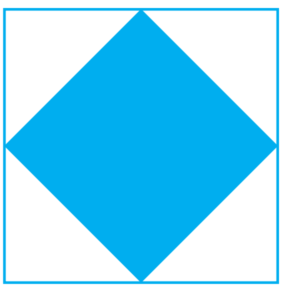
How about increasing the lengths of sides in steps of half a centimetre instead of one centimetre?



Sides	1,	$1\frac{1}{2}$,	2,	$2\frac{1}{2}$,	...
Perimeter	4,	6,	8,	10,	...
Area	1,	$2\frac{1}{4}$,	4,	$6\frac{1}{4}$,	...
Diagonal	$\sqrt{2}$,	$\frac{3}{2}\sqrt{2}$,	$2\sqrt{2}$,	$\frac{5}{2}\sqrt{2}$,	...

A set of numbers written like this, as the first, second, third and so on, is called a *sequence*.

We can make another sequence with squares. Imagine a square of side 1 metre. Joining the midpoints of the sides, we get another square:



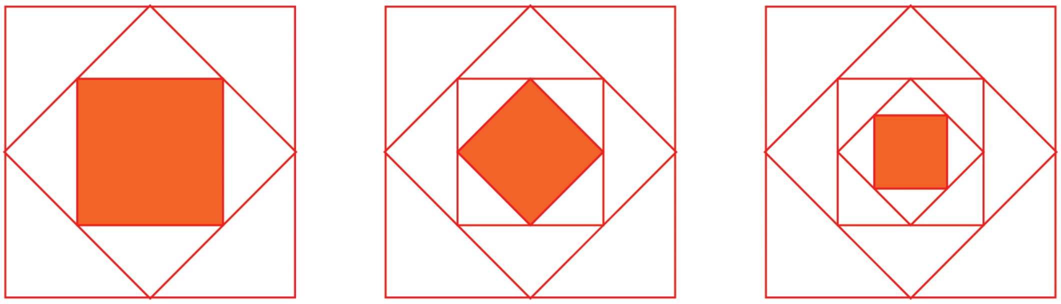
1 metre

What is the area of this smaller square?

Its diagonal is 1 metre; and we know that the area of a square is half the square of its diagonal (The section, **Rhombus** of the lesson **Areas of Quadrilaterals**, in the Class 8 textbook).

So, the area of the small square is half a square metre.

Continuing, the area is halved each time:



1 metre 1 metre 1 metre

What number sequence do we get from this?

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

We get sequences from physics also. The speed of an object falling from a height increases every instant. If the speed at t seconds is taken as v metres per second, the time–speed equation is

$$v = 9.8t$$

If the distance travelled in t seconds is taken as s metres, then the time–distance equation is

$$s = 4.9t^2$$

So, we get two sequences from this:

Time	1,	2,	3,	4,	...
Speed	9.8,	19.6,	29.4,	39.2,	...
Distance	4.9,	19.6,	44.1,	78.4,	...

We can form sequences from peculiarities of pure numbers, instead of numbers as measures. For example, the prime numbers written in order gives the sequence

$$2, 3, 5, 7, 11, 13, \ldots$$

The digits in the decimal form of $\frac{21}{37}$, written in order is the sequence

$$5, 6, 7, 5, 6, 7, 5, 6, 7, \ldots$$

If we take π instead, we get this sequence:

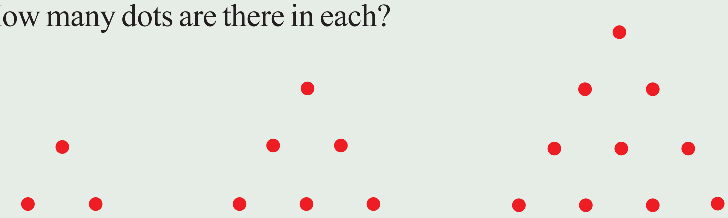
$$3, 1, 4, 1, 5, 9, 2, 6, \ldots$$

The same sequence can be described in different ways. For example, this is the sequence of natural numbers ending in 1:

$$1,\ 11,\ 21,\ 31,\ \ldots$$

We can also say that this is the sequence of natural numbers which leave remainder 1 on division by 10.

(1) Look at these triangles made with dots.
How many dots are there in each?



Compute the number of dots needed to make the next two triangles.

(2) Make the following number sequences, from the sequence of equilateral triangles, squares, regular pentagons and so on, of regular polygons:

Number of sides 3, 4, 5, ...

Sum of interior angles

Sum of exterior angles

One interior angle

One exterior angle

(3) Write down the sequence of natural numbers leaving remainder 1 on division by 3 and the sequence of natural numbers leaving remainder 2 on division by 3.

(4) Write down the sequence of natural numbers ending in 1 or 6 and describe it in two other ways.

(5) One cubic centimetre of iron weighs 7.8 grams. Write as sequences, the volumes of weights of iron cubes of sides 1 centimetre, 2 centimetres and so on.

Algebra of sequences

We have seen that the perimeters of squares of sides 1 centimetre, 2 centimetres, 3 centimetres and so on, form the sequence

$$4,\ 8,\ 12,\ \ldots$$

The numbers forming a sequence are called its *terms*.

Thus 4, 8, 12, ... are the terms of the above sequence. More precisely, 4 is the first term, 8 is the second term, 12 is the third term and so on.

We can write this as given below:

Position	1,	2,	3,	...
Term	4,	8,	12,	...

What is the 5th term? The 20th term?

What is the relation between the positions and the terms?

Each term is four times its position.

Using a bit of algebra, we can put it like this:

The n^{th} term of the sequence is $4n$

Usually the terms in a sequence are written in algebra as $x_1, x_2, x_3, \ldots$ or $y_1, y_2, y_3, \ldots$

So, we can shorten the above sequence rule further

$$x_n = 4n$$

When we take the natural numbers 1, 2, 3, ... as n, we get all the terms of the sequence as

$x_1 = 4$

$x_2 = 8$

$x_3 = 12$

...

We can compute the 100th term directly as

$x_{100} = 400$

We can draw sequences of polygons with a common side using GeoGebra.

Mark the points A, B and type

Sequence [Polygon [A, B, n], n, 3, 10]

in the Input Bar. This command means, like the numbers from 3 to 10 as n and draw the reqular polygon of n sides with AB as a side.

To draw the polygon one by one make an Integer Slider m and change the command to

Sequence [Polygon [A, B, n + 2], n, 1, m]

This means, draw polygons of 3, 4, 5, ... sides as we move the slider to change m through, 1, 2, 3, ...

If instead of n + 2, we type 2n in the above command, what sort of polygons do we get? What if we type 2n + 1 instead?

The sequence of areas is

1, 4, 9, 16, ...

What is the relation between a term and its position here?

Each term is the square of its position.

How about putting this in algebra?

$$x_n = n^2$$

We can also write the lengths of diagonals of these squares as a sequence. How do we put it in algebra?

Let's look at the sequences got on increasing the sides by half centimetre steps.

Side	1,	$1\frac{1}{2}$,	2,	$2\frac{1}{2}$,	...
Perimeter	4,	6,	8,	10,	...
Area	1,	$2\frac{1}{4}$,	4,	$6\frac{1}{4}$,	...
Diagonal	$\sqrt{2}$,	$\frac{3}{2}\sqrt{2}$,	$2\sqrt{2}$,	$\frac{5}{2}\sqrt{2}$,	...

How can we find the algebraic expression for the sequence of the lengths of sides?

First, let's write it like this:

$$1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \ldots$$

There are both natural numbers and fractions in it; and all the fractions have the denominator 2. How about writing the whole numbers also as fractions with denominator 2?

$$\frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \frac{6}{2}, \frac{7}{2}, \ldots$$

The sequence of numerators is

$$2, 3, 4, 5, 6, 7, \ldots$$

What is the algebraic expression for this sequence? Try it!

So, how do we write the sequence of sides in algebra?

Writing the length of a side of the n^{th} square as s_n, we find

$$s_n = \frac{n+1}{2}$$

We can also write it like this

$$s_n = \frac{1}{2}(n+1)$$

What about the sequence of perimeters?

Perimeter is four times the length of a side. So, the algebraic expression for the sequence of perimeters is

$$p_n = 4 \times \frac{1}{2}\ (n + 1) = 2\ (n + 1)$$

For example, the length of a side of the 25th square is

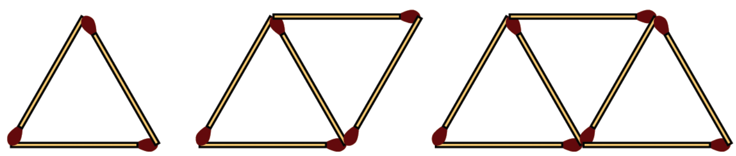
$$s_{25} = \frac{1}{2} \times (25 + 1) = 13$$

and the perimeter of the 50th square is

$$p_{50} = 2 \times (50 + 1) = 102$$

Like this, can't you write the algebraic expressions for the sequence of areas and the sequence of diagonals?

Let's look at another problem.



Three matchsticks to make one triangle, five to make two triangles, seven to make three triangles.

How many matchsticks are needed to make four triangles?

Five triangles?

Three matchsticks to make the first triangle. Thereafter, we need only two more matchsticks to make each triangle. Thus the number of matchsticks form the sequence

3, 5, 7, 9, 11, ...

How many matchsticks do we need to make 10 triangles like this?

3 matchsticks for the first, 2 each for the remaining 9, which makes $9 \times 2 = 18$. Altogether $3 + 18 = 21$.

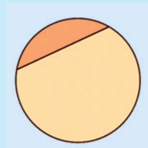
How many matchsticks do we require to make for 100 triangles?

$3 + (99 \times 2) = 201$

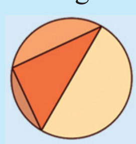
Let's now write this in algebra. How can we write the number of matchsticks we need to make n triangles?

Circle division

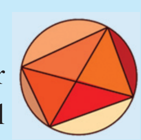
A line joining two points on a circle, divides the circle into two parts.



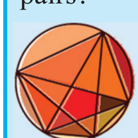
Taking three points on a circle and joining them in pairs, we get four parts



What if we take four points and join all pairs?



And five points?



How many parts do you expect by joining six points like this? Check it by actually drawing such lines.

3 sticks for the first triangle and $2(n-1) = 2n - 2$ sticks for the remaining $n - 1$ triangles; altogether $3 + 2n - 2 = 2n + 1$

That is, the number of sticks we need to make n triangles is

$$x_n = 2n + 1$$

This is the algebraic expression for the sequence 3, 5, 7, ... got by adding 2 again and again to 3. Thus, we can easily compute the number of matchsticks needed to make 500 triangles:

$$x_{500} = (2 \times 500) + 1 = 1001$$

Once we have found out the algebraic expression for a sequence, we can use a computer to write down its terms. For example, the algebraic expression for the weights of iron cubes of sides 1 centimetre, 2 centimetres, 3 centimetres and so on is

$$x_n = 7.8n^3$$

To get a list of the weights of the first hundred cubes, we can use the python language (python 3) and write

```
for n in range (1,101):
    print (7.8*n**3)
```

If we save this piece of code in a file `weights.py` and run the command

```
python 3.2 weights.py > weights.txt
```

We get these numbers written in order in a file `weights.txt`

(1) Write the algebraic expression for each of the sequences below:

i) Sequence of odd numbers

ii) Sequence of natural numbers which leave remainder 1 on division by 3.

iii) The sequence of natural numbers ending in 1.

iv) The sequence of natural numbers ending in 1 or 6.

(2) For the sequence of regular polygons starting with an equilateral triangle, write the algebraic expressions for the sequence of the sums of interior angles, the sums of the exterior angles, the measures of an interior angle, and the measures of an exterior angle.

(3) Look at these pictures:

The second picture is obtained by removing the small triangle formed by joining the midpoints of the first triangle. The third picture is got by removing such a middle triangle from each of the red triangles of the second picture.

i) How many red triangles are there in each picture?

ii) Taking the area of the first triangle as 1, compute the area of a small triangle in each picture.

iii) What is the total area of all the red triangles in each picture?

iv) Write the algebraic expressions for these three sequences obtained by continuing this process.

Arithmetic sequences

When we computed the perimeters of squares with the length of a side 1, 2, 3, 4, ... we got the sequence

4, 8, 12, 16, ...

Here the lengths of sides are increased in steps of 1 and so the perimeter increases in steps of 4. What if we take the side as 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, ...?

Since the lengths of sides increase in steps of $\frac{1}{2}$, the perimeter increases in steps of $4 \times \frac{1}{2} = 2$. The sequence we get is

4, 6, 8, 10, ...

Now look at the problem of matchstick triangles. We need 3 sticks for the first angle and added two for each triangle thereafter. Thus we start with 3 and add 2 repeatedly to get the sequence

3, 5, 7, 9, ...

> A sequence got by starting with any number and adding a fixed number repeatedly is called an *arithmetic sequence*.

In the second problem of square, the lengths of sides are

$$1, \quad 1\frac{1}{2}, \quad 2, \quad 2\frac{1}{2}, \ldots$$

This is also an arithmetic sequence: We start with 1 and add $\frac{1}{2}$ repeatedly.

The sums of exterior angles of regular polygons give the sequence

$$360, 360, 360, \ldots$$

This again is an arithmetic sequence where we start with 360 and add 0 again and again.

Let's look at another problem:

An object moves along a straight line at 10 metres/second. Applying a constant force in the opposite direction, the speed is reduced by 2 metres/second every second.

The sequence of speeds is

$$10, 8, 6, \ldots$$

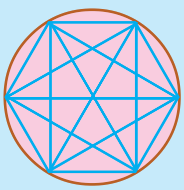
Here, the terms are got by subtracting 2 repeatedly from 10. This is also considered an arithmetic sequence. There are two ways to include such sequence among arithmetic sequences: either change "adding a fixed number repeatedly" to "adding or subtracting a fixed number repeatedly" in the definition of an arithmetic sequence or interpret "subtract 2" as "add – 2" to justify the inclusion.

We can describe arithmetic sequences in another manner. In such a sequence, we add the same number to move from a term in any position to the term immediately after it. So, if we subtract from any term, the term immediately before it, we get this number.

> An arithmetic sequence is a sequence in which we get the same number on subtracting from any term, the term immediately preceding it.

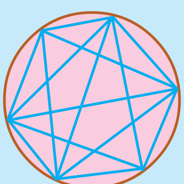
Think before you leap!

We have seen how lines joining points on a circle dissect it (The sidebar, circle division). For 2, 3, 4, 5 points, we get 2, 4, 8, 16 parts. For 6 points? 32 may be our guess. What do we actually get on drawing?



If the points are equally spaced, we get 30 parts; otherwise 31 parts.

Anyhow, the maximum number of parts is 31. We can show that the maximum number of parts, by joining n points is



$$\frac{1}{24}n(n-1)(n-2)(n-3)+\frac{1}{2}n(n-1)+1$$

What is interesting is that for $n = 1, 2, 3, 4, 5$, this expression and 2^{n-1} give the same numbers 1, 2, 4, 8, 16. From $n = 6$ onwards, they give different numbers.

This constant difference got by subtracting from any term the just previous term, is called the *common difference* of an arithmetic sequence.

Very often, we find out whether a given sequence is an arithmetic sequence by checking whether the difference between the terms is constant. For example, consider the multiples of 3:

$$3, 6, 9, \ldots$$

The difference of two consecutive multiples of 3 is 3 itself. So, it is an arithmetic sequence with common difference 3.

Now suppose we add 1 to each of these mutliples. We get the sequence

$$4, 7, 10, \ldots$$

This again is an arithmetic sequence with common difference 3.

Now look at the sequence of the powers of 3:

$$3, 9, 27, \ldots$$

$9 - 3 = 6$ and $27 - 9 = 18$. Here the difference between consecutive terms is not the same.

So, it is not an arithmetic sequence.

Now go through all the sequences we have discussed so far and pick out the arithmetic sequences from them.

(1) Check whether each of the sequences given below is an arithmetic sequence. Give reasons. For the arithmetic sequences, write the common difference also.

i) Sequence of odd numbers

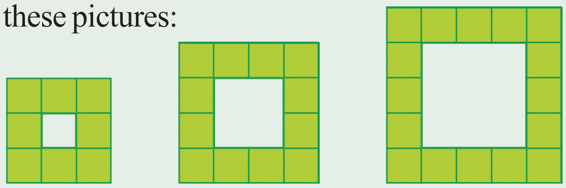
ii) Sequence of even numbers

iii) Sequence of fractions got as half the odd numbers

iv) Sequence of powers of 2

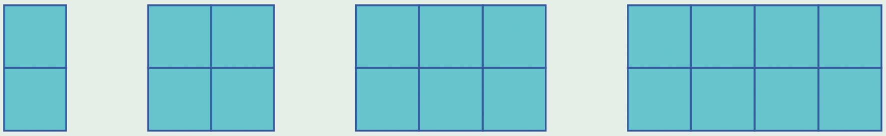
v) Sequence of reciprocals of natural numbers

(2) Look at these pictures:



If the pattern is continued, do the numbers of coloured squares form an arithmetic sequence? Give reasons.

(3) See the pictures below:



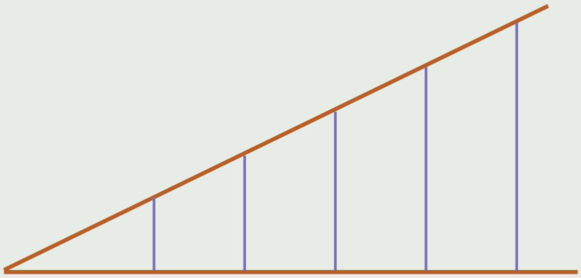
i) How many small squares are there in each rectangle?

ii) How many large squares?

iii) How many squares in all?

Continuing this pattern, is each such sequence of numbers, an arithmetic sequence?

(4) In this picture, the perpendiculars to the bottom line are equally spaced. Prove that, continuing like this, the lengths of perpendiculars form an arithmetic sequence.



(5) The algebraic expression of a sequence is

$$x_n = n^3 - 6n^2 + 13n - 7$$

Is it an arithmetic sequence?

Position and term

Can you make an arithmetic sequence with 1 and 11 as the first and second terms?

Simple, isn't it? To get 11 from 1, we must add 10. Adding 10 again and again, we get the arithmetic sequence

$$1, 11, 21, 31, \ldots$$

Another question then: Can you make an arithmetic sequence with 1 and 11 as the first and third terms?

How do we calculate the common difference of such a sequence?

The common difference added to 1 gives the second term; and we don't know what it is. Adding it once more should give the third term, 11.

That is, 11 is got by adding twice the common difference to 1.

So, twice the common difference is 10, which means the common difference is 5.

Now we can write the sequence as

$$1, 6, 11, 16, 21, \ldots$$

How about an arithmetic sequence with the 3rd term 37 and 7th term 72?

To get the 7th term from the 3rd term, we must add the common difference $7 - 3 = 4$ times.

And the actual number to be added is $73 - 37 = 36$

Thus 4 times the common difference is 36 and so the common difference is $36 \div 4 = 9$

How do we find the first term?

Subtract the common difference twice from the 3rd term; that is $37 - (2 \times 9) = 19$

Now we can write the sequence from the beginning

Sequence and remainder

The even numbers 2, 4, 6,... form an arithmetic sequence. So do the odd numbers 1, 3, 5, ... The common difference is 2 in both. Even numbers are numbers divisible by 2, or numbers leaving the remainder 0 on division by 2. And odd numbers are those leaving remainder 1 on division by 2.

Similarly, we can make three arithmetic sequences, leaving remainders 0, 1, 2 on division by 3. What is the common difference of all these?

How about division by 4? Now let's think in reverse: in an arithmetic sequence of natural numbers, the difference of any two terms is a multiple of the common difference. So they all leave the same remainder on division by the common difference (why?)

$$19, 28, 37, \ldots$$

How do we compute the 25th term of this sequence?

There are several ways:

To the 3rd term, add $(25 - 3) = 22$ times the common difference: $37 + (22 \times 9) = 235$

To the 2nd term, add $(25 - 2) = 23$ times the common difference: $28 + (23 \times 9) = 235$

To the 1st term, add $(25 - 1) = 24$ times the common difference: $19 + (24 \times 9) = 235$

In general, if we know any two terms of an arithmetic sequence and their positions, we can compute the entire sequence.

What is the general principle used?

The difference between any two terms of an arithmetic sequence is the product of the difference of positions and the common difference.

We can put it like this also:

In an arithmetic sequence, term difference is proportional to position difference; and the constant of proportionality is the common difference.

We can use this to check whether a given number is a term of a given arithmetic sequence.

For example, let's take a sequence seen before:

$$19,\quad 28,\quad 37,\ \ldots$$

The difference between any two terms of this is a multiple of 9. On the other hand, what if the difference of a number from a term of this sequence is multiple of 9?

For example, the difference of the number 1000 with the first term 19 of this sequence is $1000 - 19 = 981 = 109 \times 9$, which is multiple of 9. Thus 1000 is got by adding 109 times the common difference 9 to the first term 19. So, it is the 110^{th} term of the sequence.

Sequence rule

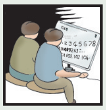
What is the next term of the sequence 3, 5, 7, ...? It's not said to be an arithmetic sequence. so the next term need not be 9. For example, if it is supposed to be the sequence of odd primes, the next term is 11. What is the moral here? Just by writing down the first few terms of a sequence, we cannot predict exactly the terms to come next. To do this either the rule of formation of the sequence or the context in which the sequence arises must be specified.

Consider these sequences:

$$x_n = 2n - 1$$
$$x_n = n^2 - n + 1$$
$$x_n = n^3 - 3n^2 + 4n - 1$$

All of them have 1 and 3 as the first two terms. What about the third terms?

Is every power of 10, a term of the arithmetic sequence 19, 28, 37, ...?



(1) In each of the arithmetic sequences below, some terms are missing and their positions are marked with ◯. Find them.

i) 24, 42, ◯, ◯, ...

ii) ◯, 24, 42 , ◯, ...

iii) ◯, ◯, 24, 42, ...

iv) 24, ◯, 42, ◯, ...

v) ◯, 24, ◯, 42, ...

vi) 24, ◯, ◯, 42, ...

(2) The terms in two positions of some arithmetic sequences are given below. Write the first five terms of each:

i) 3^{rd} term 34
6^{th} term 67

ii) 3^{rd} term 43
6^{th} term 76

iii) 3^{rd} term 2
5^{th} term 3

iv) 4^{th} term 2
7^{th} term 3

v) 2^{nd} term 5
5^{th} term 2

(3) The 5^{th} term of an arithmetic sequence is 38 and the 9^{th} term is 66. What is its 25^{th} term?

(4) Is 101 a term of the arithmetic sequence 13, 24, 35, ...? What about 1001?

(5) How many three-digit numbers are there, which leave a remainder 3 on division by 7?

(6) Fill up the empty cells of the square below such that the numbers in each row and column form arithmetic sequences:

1			4
7			28

What if we use other numbers instead of 1, 4, 28 and 7?

(7) In the table below, some arithmetic sequences are given with two numbers against each. Check whether each belongs to the sequence or not.

Sequence	Numbers	Yes/No
11, 22, 33, ...	123	
	132	
12, 23, 34, ...	100	
	1000	
21, 32, 43, ...	100	
	1000	
$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots$	3	
	4	
$\frac{3}{4}, 1\frac{1}{2}, 2\frac{1}{4}, \ldots$	3	
	4	

Algebra of arithmetic sequences

The simplest arithmetic sequence is that of natural numbers and we have seen many of its peculiarities in classes seven and eight. For example, look at these sums:

$$1 + 2 + 3 = 6 = 3 \times 2$$
$$2 + 3 + 4 = 9 = 3 \times 3$$
$$3 + 4 + 5 = 12 = 3 \times 4$$

The sum of any three consecutive natural numbers is three times the middle number. Why?

To understand this, let's take the middle number as x. Then the first number is $x - 1$ and the third number is $x + 1$. Their sum is

$$(x - 1) + x + (x + 1) = 3x$$

which is thrice the middle number.

Is this true for three consecutive odd numbers?

And even numbers?

Now what if we take three consecutive terms of some arithmetic sequence?

For example, consider the arithmetic sequence

$$2, 7, 12, 17, \ldots$$

Pick three consecutive terms of this, say, 37, 42, 47, ...

Their sum is

$$37 + 42 + 47 = 126$$

Isn't this three times 42?

Is it true for all arithmetic sequences?

Let's use algebra to check.

Let x be the middle one of three consecutive terms of an arithmetic sequence.

To write the terms on either side, we must have the common difference.

Let's take it as y. Then the first term is $x - y$ and the third term is $x + y$. The sum of all three is

$$(x - y) + x + (x + y) = 3x$$

So, this is true for all arithmetic sequences:

> For any arithmetic sequence, the sum of three consecutive terms is thrice the middle one.

We can observe another fact from this. The first and the last terms added to the middle term makes three times the middle one; so the sum of the first and the last should be twice the middle (Three times a number is double the number added to it, isnt' it?)

We can put this in a slightly different way:

> In any three consecutive terms of an arithmetic sequence, the middle one is half the sum of the first and the last.

We can write these in algebra.

If x, y, z are three consecutive terms of an arithmetic sequence, then

- $x + y + z = 3y$
- $y = \frac{1}{2}(x + z)$

What can you say about the sum of five consecutive terms of an arithmetic sequence and the sum of the first and last terms? What if we consider seven consecutive terms? What is the general conclusion?

Now let's look at the algebraic expressions for some arithmetic sequences.

First let's take the arithmetic sequence

$$19, 28, 37, \dots$$

To get the term at any position of this sequence, we must multiply the position difference from the first by the common difference 9, and add to the first term 19. For example, if we take the 15th term, the position difference from the first is 15 – 1 = 14; so to get the 15th term, we must add 14 times the common difference 9 to the first term 19.

$$15^{\text{th}} \text{ term is } 19 + (14 \times 9) = 145$$

What about the 20th term?

In general, for any natural number n, the n^{th} term is

$$19 + (n - 1) \times 9 = 9n + 10$$

Thus the algebraic expression for this sequence is

$$x_n = 9n + 10$$

Now look at the arithmetic sequence,

$$\frac{1}{2}, \frac{3}{4}, 1, \dots$$

Thinking along the same lines as in the first problem, the n^{th} term is

$$\frac{1}{2} + (n - 1) \times \frac{1}{4} = \frac{1}{4}n + \frac{1}{4}$$

That is, the algebraic expression for the sequence is

$$x_n = \frac{1}{4}(n + 1)$$

In the first sequence, we multiply the position n by the common difference 9 and add 10; in the second, we multiply by $\frac{1}{4}$ and add $\frac{1}{4}$.

Is every arithmetic sequence in this form?

Taking the first term of an arithmetic sequence as f and the common difference as d, the n^{th} term is

$$f + (n - 1)d = dn + (f - d)$$

That is, we multiply n by number d and add the number $f - d$.

Thus each term of an arithmetic sequence is got by multiplying the position number by the common difference and adding a fixed number.

That is, the algebraic expression for any arithmetic sequence is of the form.

$$x_n = an + b$$

On the other hand, is any sequence $x_n = an + b$, an arithmetic sequence?

Any two consecutive terms of this sequence are of the form $an + b$ and $a(n+1) + b$; and their difference is

$$a(n + 1) + b - (an + b) = a$$

That is the difference of any two consecutive terms is the same number a; and so it is an arithmetic sequence.

Any arithmetic sequence is of the form

$$x_n = an + b$$

where a and b are fixed numbers; conversely, any sequence of this form is an arithmetic sequence.

We have also seen that in the arithmetic sequence $x_n = an + b$, the common difference is a.

Using this, we can easily find the algebraic expression for a given arithmetic sequence. For example, let's look at the sequence got by starting at $\frac{1}{2}$ and adding $\frac{1}{3}$ repeatedly:

$$\frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \ldots$$

Mark a point A in GeoGebra and type the command

Sequence [Circle [A, n], n, 1, 10]

We get circles centered at A with radii 1 to 10. To change the number of circles, make an Integer slider m and change the command to

Sequence [Circle [A , n], n, 1, m]

Now by changing n to 2n or 2n + 1, we can draw circles with radii even or odd numbers.

Making two more sliders a, b with Min = 0 and changing the command to

Sequence [Circle [A , an + b], n, 1, m]

we can draw circles of radii in different arithmetic sequences by changing a and b.

Draw a regular hexagon and draw a sequence of circles like this, centred at each vertex.

Since the common difference is $\frac{1}{3}$, its algebraic expression is of the form $\frac{1}{3}n + b$. Take $n = 1$, first term is $\frac{1}{3} + b$.

So we get

$$\frac{1}{3} + b = \frac{1}{2}$$

from which we find

$$b = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

So the algebraic expression for the sequence is

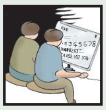
$$\frac{1}{3}n + \frac{1}{6}$$

We can write the algebraic expression for this sequence as a fraction like this;

$$x_n = \frac{2n+1}{6}$$

We can note many things from this. All fractions in this sequence have odd numbers as numerations and 6 as the denominator. The odd numbers don't have 2 as a factor and so don't have 6 as a factor either. Thus no term of this sequence is a natural number.

In other words, this sequence does not contain any natural number.



(1) Write three arithmetic sequences with 30 as the sum of the first five terms.

(2) The first term of an arithmetic sequence is 1 and the sum of the first four terms is 100. Find the first four terms.

(3) Prove that for any four consecutive terms of an arithmetic sequence, the sum of the two terms on the two ends and the sum of the two terms in the middle are the same.

(4) Write four arithmetic sequences with 100 as the sum of the first four terms.

(5) The 8^{th} term of an arithmetic sequence is 12 and its 12^{th} term is 8. What is the algebraic expression for this sequence?

(6) The Bird problem in Class 8 (The lesson, **Equations**) can be slightly changed as follows.

One bird said:

"We and we again, together with half of us and half of that, and one more is a natural number"

Write the possible number of birds in order. For each of these, write the sum told by the bird also.

Find the algebraic expression for these two sequences.

(7) Prove that the arithmetic sequence with first term $\frac{1}{3}$ and common difference $\frac{1}{6}$ contains all natural numbers.

(8) Prove that the arithmetic sequence with first term $\frac{1}{3}$ and common difference $\frac{2}{3}$ contains all odd numbers, but no even number.

(9) Prove that the squares of all the terms of the arithmetic sequence 4, 7, 10, ... belong to the sequence.

(10) Prove that the arithmetic sequence 5, 8, 11, ... contains no perfect squares.

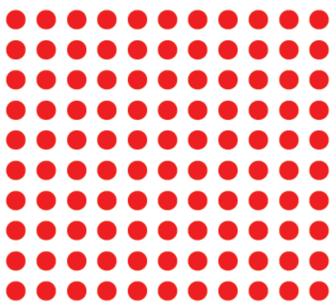
(11) The angles of a pentagon are in arithmetic sequence. Prove that its smallest angle is greater than 36°.

(12) Write the whole numbers in the arithmetic sequence $\frac{11}{8}, \frac{14}{8}, \frac{17}{8}, \dots$

Do they form an arithmetic sequence?

Sums

See this picture:



How many dots are there in it?

We need not count one by one. There are 11 dots in each row and 10 such rows; total $10 \times 11 = 110$.

How many dots are there in this triangle?

We can count row by row and add:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

Is there an easier way?

How about changing this to a rectangle?

Language of law

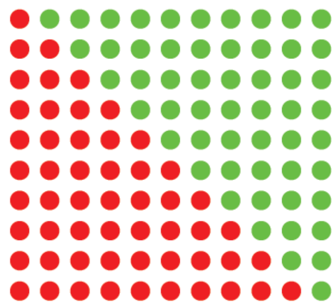
We have noted that to compute the terms of a sequence, the rule or law of formation must be specified. And we have seen how such rules can be algebraically expressed.

But the rule of forming some sequences cannot be written in algebra. For example, consider the sequence 2,3, 5, 7, 11, of primes. No algebraic expression has been found to compute the number in a specified position in this sequence. Again, consider the sequence 3, 1, 4, 1, 5, 9, ... of digits in the decimal form of π. There is no algebraic expression for this sequence also.

In such cases, we can only describe the rule of formation in ordinary language.

For that, make another triangle like this;

Turn it upside down and join with the first:



A math tale

You have heard about the great mathematician Gauss, haven't you? He showed extraordinary talent in mathematics from a very early age. There are several tales about this.

Gauss was 10. The teacher asked his class to add all numbers from 1 to 100 just to keep them quiet. Little Gauss gave the answer very quickly, 5050.

And he explained it like this: 1 and 100 make 101, so do 2 and 99. Thus 50 pairs each of sum 101 give $50 \times 101 = 5050$.

As seen before, this rectangle, $10 \times 11 = 110$ dots.

How many dots are there in each triangle?

Half of 110, which is 55.

We can write in numbers, what we did with pictures. Let's write

$$s = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

Writing in reverse,

$$s = 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$$

How about adding the numbers in the same position?

$$2s = 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11$$

So,

$$s = \frac{1}{2} \times 10 \times 11 = 55$$

We can add numbers from 1 to 100 like this:

$$s = 1 + 2 + 3 + \dots + 98 + 99 + 100$$

$$s = 100 + 99 + 98 + \dots + 3 + 2 + 1$$

Adding numbers in the same place,

$$\begin{aligned} 2s &= \overbrace{101 + 101 + 101 + \ldots + 101 + 101 + 101 + 101}^{100 \text{ times}} \\ &= 100 \times 101 \end{aligned}$$

From this, we get

$$s = \frac{1}{2} \times 100 \times 101 = 5050$$

Can't we find the sum upto any number, instead of 100, like this?

The sum of any number of consecutive natural numbers, starting with one, is half the product of the last number and the next natural number.

In the language of algebra,

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}\ n\ (n + 1)$$

Using this, we can find sums of consecutive terms of other arithmetic sequences also.

For example, let's take the even numbers 2, 4, 6, ..., 100. Even numbers are got by multiplying natural numbers by 2. So, we can write

$$2 + 4 + 6 + \cdots + 100 = 2\ (1 + 2 + 3 + \cdots + 50)$$

We have seen that

$$1 + 2 + 3 + \cdots + 50 = \frac{1}{2} \times 50 \times 51$$

From this, we get

$$2 + 4 + 6 + \cdots + 100 = 2 \times \frac{1}{2} \times 50 \times 51 = 2550$$

In general, the first n even natural numbers are

$$2,\ 4,\ 6,\ \ldots,\ 2n$$

And their sum is

$$2 + 4 + 6 + \cdots + 2n = 2\ (1 + 2 + 3 + \cdots + n) = n\ (n + 1)$$

Can't you find the sum of the first n multiples of 3 in this manner?

How do we find the sum of the first n odd numbers?

First let's write the algebraic expression for the sequence of odd numbers:

$$x_n = 2n - 1$$

Taking $n = 1, 2, 3, \dots$ in this, we get the sequence of odd numbers. So, we can write the sequence like this:

$$x_1 = (2 \times 1) - 1$$
$$x_2 = (2 \times 2) - 1$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$x_n = (2 \times n) - 1$$

How about adding all this from top to bottom?

$$x_1 + x_2 + \dots + x_n = ((2 \times 1) + (2 \times 2) + \dots + (2 \times n)) - \overbrace{(1+1+\dots+1)}^{n \text{ times}}$$
$$= 2\,(1 + 2 + \dots + n) - n$$

Now, if we use

$$1 + 2 + \dots + n = \frac{1}{2}\, n\,(n + 1)$$

we get,

$$x_1 + x_2 + \dots + x_n = 2 \times \frac{1}{2}\, n\,(n + 1) - n = n^2$$

That is, the sum of consecutive odd numbers starting from 1 is the square of the number of odd numbers added.

We have already noted this in Class 7, in the section **Perfect squares** of the lesson **Square and Square root**.

Like this, we can calculate the sum of any arithmetic sequence.

An arithmetic sequence is of the form

$$x_n = an + b$$

To calculate the sum of its first n terms, we put $n = 1, 2, 3, \dots,$ in this and add:

$$x_1 = a + b$$
$$x_2 = 2a + b$$
$$\dots\dots\dots\dots\dots\dots$$
$$x_n = na + b$$

$$x_1 + x_2 + \cdots + x_n = (a + 2a + \cdots + na) + \overbrace{(b + b + \ldots + b)}^{n \text{ times}}$$
$$= a\,(1 + 2 + \cdots + n) + nb$$
$$= a\,\frac{n(n+1)}{2} + nb$$
$$= \frac{1}{2}an\,(n+1) + nb$$

For the arithmetic sequence

$$\boldsymbol{x_n = an + b}$$

the sum of the first n terms is

$$\boldsymbol{x_1 + x_2 + \ldots + x_n = \frac{1}{2}an\,(n+1) + nb}$$

For example, let's see how we find the sum of the first n terms of the arithmetic sequence

$$1, 4, 7, \ldots$$

The algebraic expression of this sequence is

$$x_n = 3n - 2$$

So, the sum of the first 100 terms is

$$\left(\frac{1}{2} \times 3 \times 100 \times 101\right) + \left(100 \times (-2)\right) = 14950$$

In general, the sum of the first n terms of this sequence

$$\frac{1}{2} \times 3 \times n\,(n+1) - 2n = \frac{1}{2}\,(3n^2 - n)$$

We can calculate the sum of an arithmetic sequence in another way. For this, we write the algebraic expression for the sum like this:

$$\frac{1}{2}an\,(n+1) + nb = \frac{1}{2}n\,(a\,(n+1) + 2b)$$
$$= \frac{1}{2}n\,((an+b) + (a+b))$$

Here $an + b$ is the n^{th} term x_n of the sequence and $a + b$ is the first term x_1.

The sum of the first n terms $x_1, x_2, \ldots, x_n$ of an arithmetic

Sum of squares

We have seen the identity

$$(x+1)^2 = x^2 + 2x + 1$$

Like this, we have the identity

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1$$

From this, we see that for any number x,

$$(x+1)^3 - x^3 = 3x^2 + 3x + 1$$

Taking $x = 1, 2, 3, \ldots, n$ in this, and adding all these, we get

$$(n+1)^3 - 1 = 3\,(1^2 + 2^2 + 3^2 + \ldots + n^2) + 3(1 + 2 + 3 + \ldots + n) + n$$

That is,

$$n^3 + 3n^2 + 3n = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) + \frac{3}{2}\,n(n+1) + n$$

So,

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{3}\,\left(n^3 + 3n^2 + 3n - \frac{3}{2}\,n(n+1) - n\right)$$

The expression on the right can be simplified to give

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}\,n(n+1)\,(2n+1)$$

sequence is

$$x_1 + x_2 + \dots + x_n = \frac{1}{2} n\,(x_n + x_1)$$

In ordinary language we can put it like this:

The sum of any number of consecutive terms of an arithmetic sequence is half the product of the number of terms and the sum of the first and last terms.

To compute the sum of the first 100 terms of the arithmetic sequence 1, 3, 5, ... we first find the 100th term as

$$1 + (99 \times 3) = 298$$

Now the sum of the first 100 terms is

$$\frac{1}{2} \times 100 \times (298 + 1) = 14950$$

The sum of the first n terms of any arithmetic sequence has a definite algebraic form. To see it, we write the expression for the sum like this:

$$\frac{1}{2} an\,(n+1) + nb = \frac{1}{2} an^2 + \left(\frac{1}{2}a + b\right)n$$

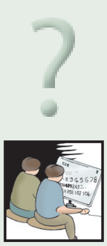
In this, $\frac{1}{2}a$ and $\frac{1}{2}a + b$ are constants associated with the sequence. Thus the sum is the sum of products of n^2 and n with definite numbers.

In other words, the sum of the first n terms of an arithmetic sequence is of the form $pn^2 + qn$.

To find the sum of consecutive terms of a sequence with known algebraic expression, we can use the sum function in python. For example, the command.

```
sum(x**2 for x in range(1,101))
```

gives the sum of the squares of the first hundred natural numbers.



(1) Find the sum of the first 25 terms of each of the arithmetic sequences below.

i) 11, 22, 33, ... ii) 12, 23, 34, ... iii) 21, 32, 43, ...

iv) 19, 28, 37, ... v) 1, 6, 11, ...

(2) What is the difference between the sum of the first 20 terms and the next 20 terms of the arithmetic sequence 6, 10, 14, ...?

(3) Calculate the difference between the sum of the first 20 terms of the arithmetic sequences 6, 10, 14, ... and 15, 19, 23, ...

(4) Find the sum of all three digit numbers, which are multiples of 9.

(5) Find n in the equation $5^2 \times 5^4 \times 5^6 \times ... \times 5^{2n} = (0.008)^{-30}$.

(6) The expressions for the sum to n terms of some arithmetic sequences are given below. Find the expression for the n^{th} term of each:

i) $n^2 + 2n$ ii) $2n^2 + n$ iii) $n^2 - 2n$

iv) $2n^2 - n$ v) $n^2 - n$

(7) Calculate in head, the sums of the following arithmetic sequences.

i) $51 + 52 + 53 + ... + 70$

ii) $1\frac{1}{2} + 2\frac{1}{2} + ... + 12\frac{1}{2}$

iii) $\frac{1}{2} + 1 + 1\frac{1}{2} + 2 + 2\frac{1}{2} + ... + 12\frac{1}{2}$

(8) The sum of the first 10 terms of an arithmetic sequence is 350 and the sum of the first 5 terms is 100. Write the algebraic expression for the sequence.

(9) Prove that the sum of any number of terms of the arithmetic sequence 16, 24, 32, ... starting from the first, added to 9 gives a perfect square.

(10) 4

7 10

13 16 19

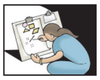
22 25 28 31

....

....

....

Write the next two lines of the pattern above. Calculate the first and last terms of the 20th line.

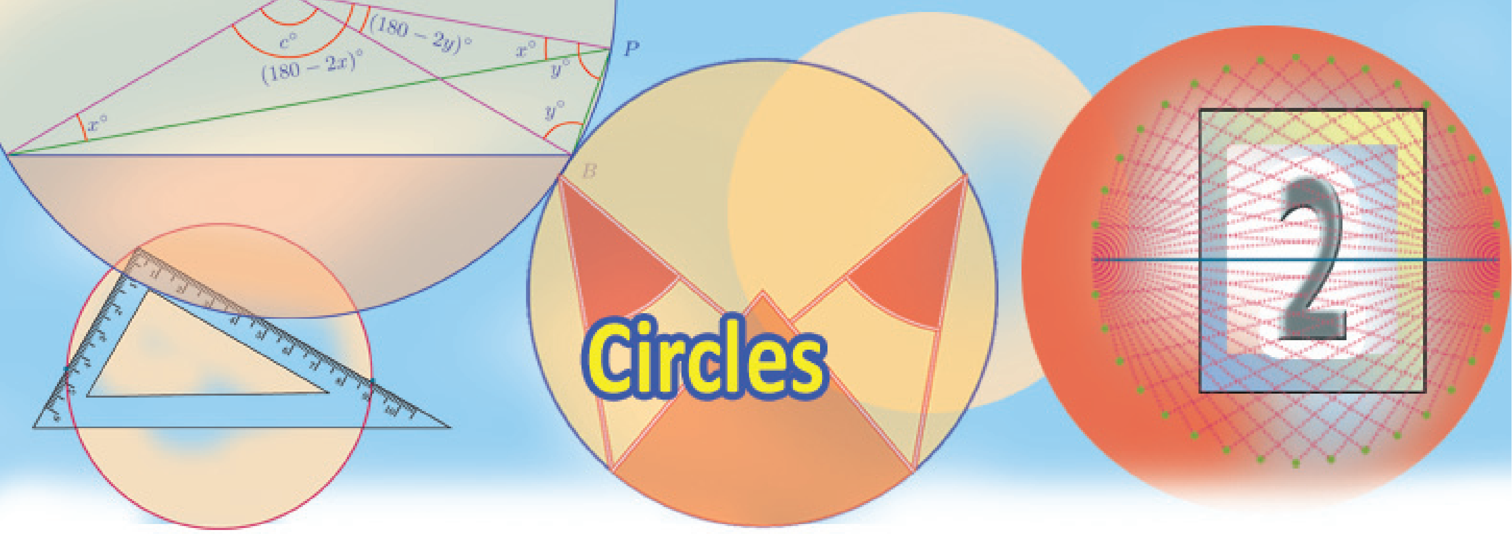


Project

- Find those arithmetic sequences, which contain all powers of their terms.
- Find those arithmetic sequences in which the sum of any number of consecutive terms, starting from the first is a perfect square.

Looking back

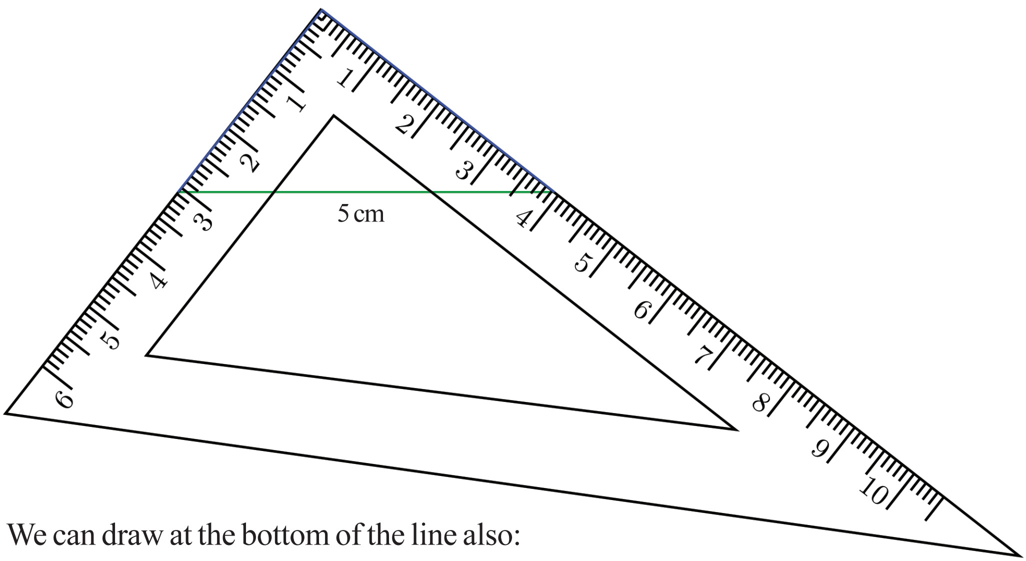
Learning outcomes	On my own	With teacher's help	Must improve
• Writing the rule of formulation of a sequence and writing the algebraic expression for it. • Recognising arithmetic sequence from other sequences. • Formulating the relation between terms, position of terms and common difference in arithmetic sequences • Using various methods to compute the sum of arithmetic sequences.			



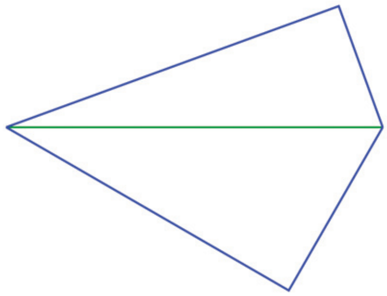
Can you draw a right triangle of hypotenuse 5 centimetres? The perpendicular sides can be of any length.

Draw a line 5 centimetres long. Draw any angle at one end and draw 90° minus this angle at the other end to make a triangle.

We can also use a set square to do this. Place it with the right angle on top and edges passing through the ends of the line. Try it!

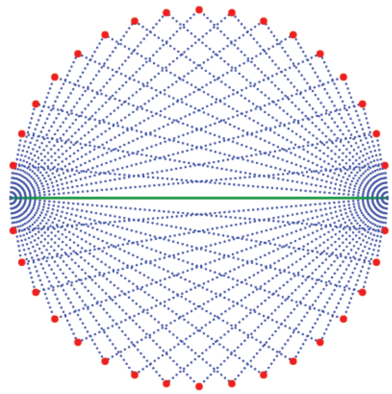


We can draw at the bottom of the line also:



Draw several such triangles and look at the third vertices.

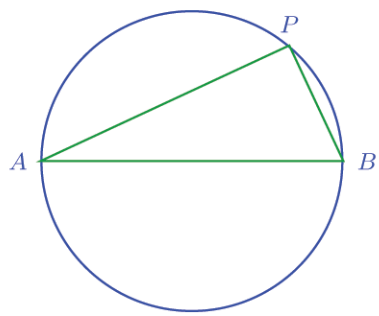
Why do all of them lie on a circle? Let's think about this.



Using GeoGebra, we can actually see how right triangles make a circle. First draw a line 5 centimetres long, using **Line with Given Length**. Make slider a from 0 to 180 with increment 5. Make a 90° angle at one end and (90 – a)° clockwise at the other end, using **Angle with Given Size**. Join the points so got with the ends of the line, using **Line**. Join the intersection of these lines and the ends of the first line to make a triangle. Apply **Trace On** to the top vertex and sides of the triangle and **Animation** on to the slider.

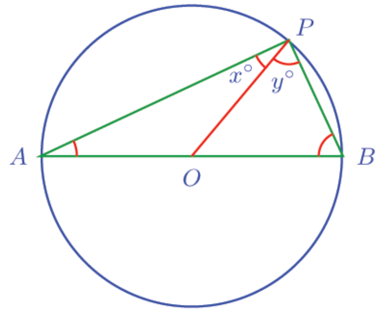
Right angle and circles

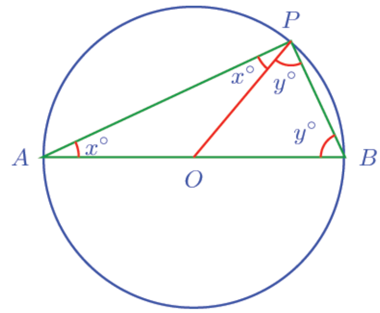
Remember the discussion on the angle got by joining a point on a circle with the ends of a diameter, at the end of the lesson, **Equal Triangles** in Class 8.



How did we show that the angle at P is a right angle?

Join P to the centre of the circle. Now the angle at P is split into two:





The triangles AOP and BOP on the left and right are isosceles (Why?). So the angle at A is x° and the angle at B is y°.

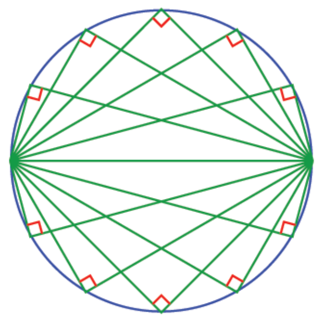
In the large triangle ABP, the sum of the angles is 180°. So,

$$x + y + (x + y) = 180$$

From this, we get

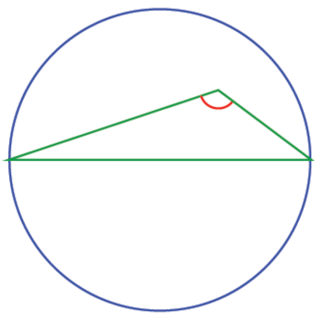
$$x + y = 90$$

If we join the ends of a diameter of a circle to a point on the circle, we get a right angle.

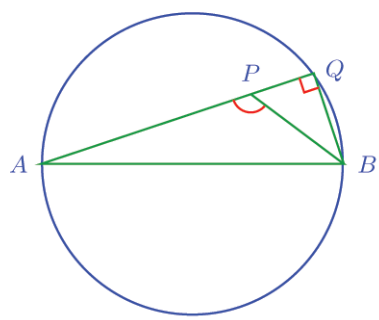


We can shorten this:

Angle in a semicircle is right.



We can think a bit more about this. Joining ends of a diameter to a point on the circle gives a right angle. What if we join the ends to a point inside the circle?

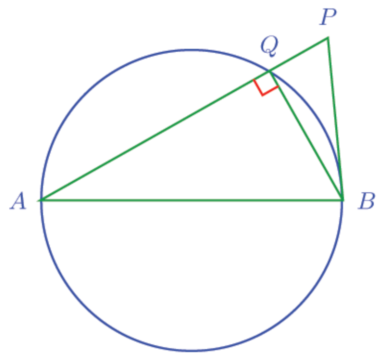


Extend one of the lines to meet the circle. Join this point to the other end of the diameter.

Now APB is the exterior angle at P of ΔAQB. So, it is the sum of the interior angles at Q and B.

Of these, the angle at Q is right. So $\angle APB$ is larger than a right angle.

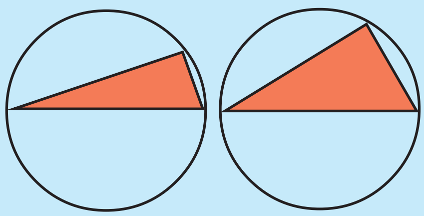
What about a point outside the circle? In this case, APB is an interior angle of ΔPQB; and the right angle AQB is an exterior angle. So we can see that $\angle APB$ is less than right angle.



Now suppose that the angle obtained when ends of a diameter joined to some point is a right angle. This point can't be inside the circle. (For any point inside the circle, such an angle is larger than a right angle). It can't be outside the circle either (for any point outside, such an angle is less than a right angle). So the point must be on the circle.

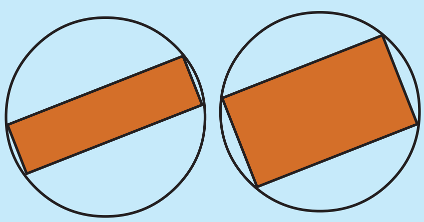
Special square

We can draw several right triangles in a circle, by joining any point on it to the ends of a diameter.



Which point on the circle gives a triangle of maximum area?

Another question: We can draw several rectangles with all four corners on the circle.



What's the speciality of the rectangle of maximum area?

What do we see here?

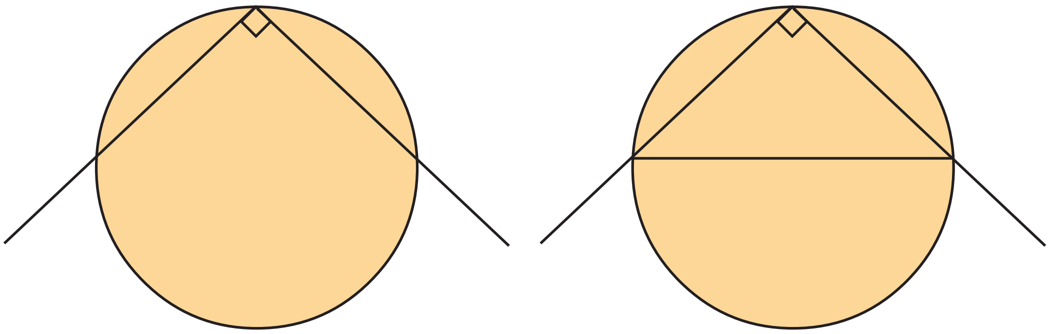
If a pair of lines drawn from the ends of a diameter of a circle are perpendicular to each other, then they meet on the circle.

We can change this slightly so that the circle appears at the end:

All pairs of mutually perpendicular lines, drawn from the ends of a fixed line, meet on the circle with that line as diameter.

Now don't you see why the third corners of right triangles in the first picture form a circle?

Now let's think in reverse. If we draw mutually perpendicular lines from a point on a circle and join the points where they cut the circle, do we get a diameter?



Then we get a right triangle and its circumcircle, and we have seen in Class 9 that the circumcentre of a right triangle is the midpoint of the hypotenuse.

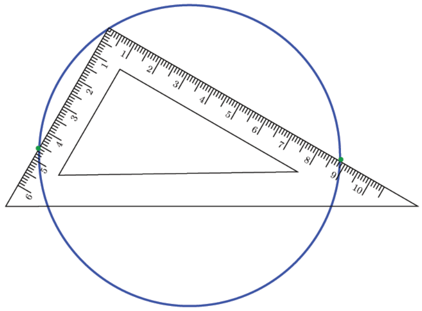
So, the bottom line is a diameter.

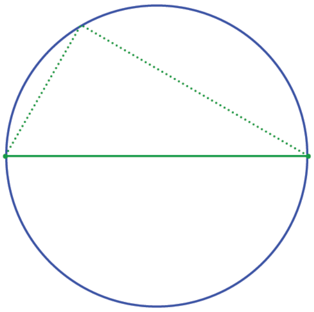
Now let's see some applications of these ideas.

Remember the method to locate the centre of a circle drawn using a bangle or a lid?

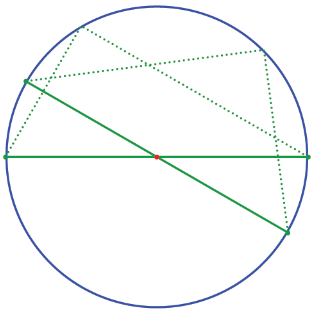
There's another way. Place a set square with its right angle on the circle and mark the points where the perpendicular edges cross the circle.

The line joining them is a diameter.





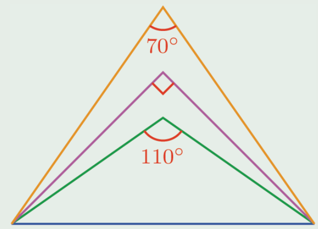
Now change the position of the set square and draw another diameter:



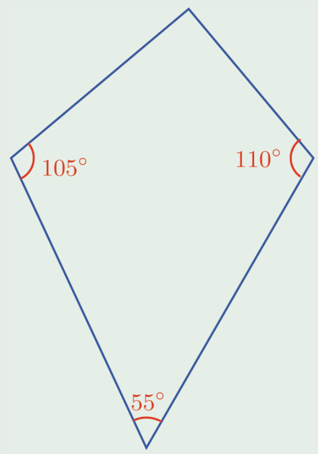
The point where these diameters cross is the centre.

Draw a circle centred at A and mark two points B and C on it. Draw a line from B through C, using **Ray**. Draw a perpendicular to this through B and mark the point D where it meets the circle. Join CD. Is it a diameter? Change the positions of B, C and see.

(1) Suppose we draw circle with the bottom side of the triangles in the picture as diameter. Find out whether the top corner of each triangle is inside the circle, on the circle or outside the circle.

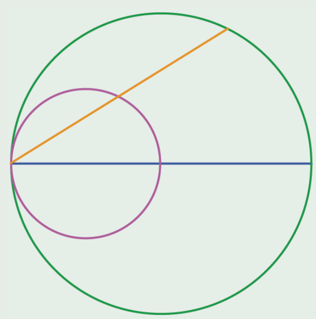


(2) For each diagonal of the quadrilateral shown, check whether the other two corners are inside, on or outside the circle with that diagonal as diameter.



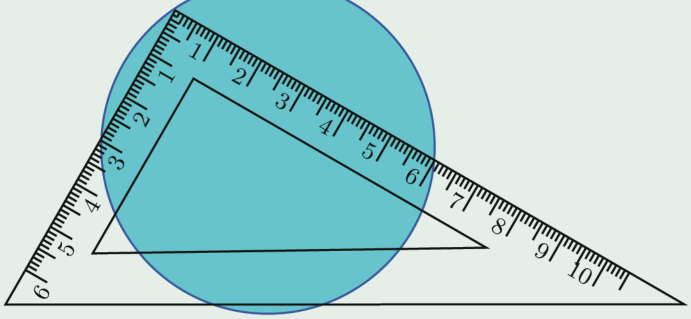
(3) If circles are drawn with each side of a triangle of sides 5 centimetres, 12 centimetres and 13 centimetres, as diameters, then with respect to each circle, where would be the third vertex?

(4) In the picture, a circle is drawn with a line as diameter and a smaller circle with half the line as diameter. Prove that any chord of the larger circle through the point where the circles meet is bisected by the small circle.

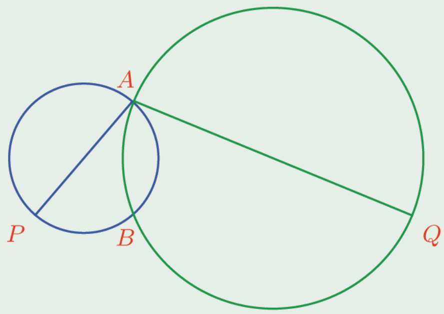


(5) Prove that the two circles drawn on the two equal sides of an isosceles triangle as diameters pass through the mid point of the third side.

(6) Use a calculator to determine upto two decimal places, the perimeter and the area of the circle in the picture.



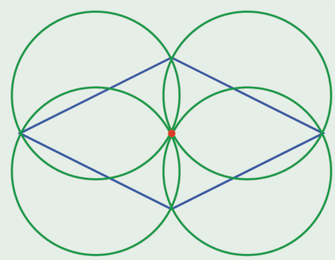
(7) The two circles in the picture cross each other at A and B. The points P and Q are the other ends of the diameters through A.



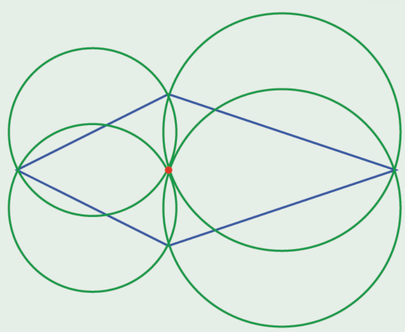
i) Prove that P, B, Q lie on a line.

ii) Prove that PQ is parallel to the line joining the centres of the circles and is twice as long as this line.

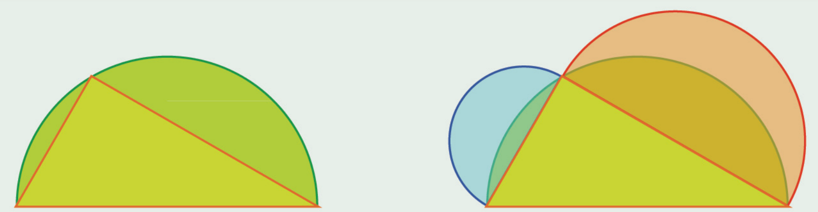
(8) Prove that all four circles drawn with the sides of a rhombus as diameters pass through a common point.



Prove that this is true for any quadrilateral with adjacent sides equal, as in the picture.



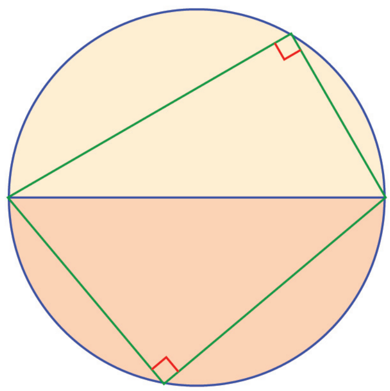
(9) A triangle is drawn by joining a point on a semicircle to the end of the diameter. Then semicircles are drawn with the other two sides as diameter.



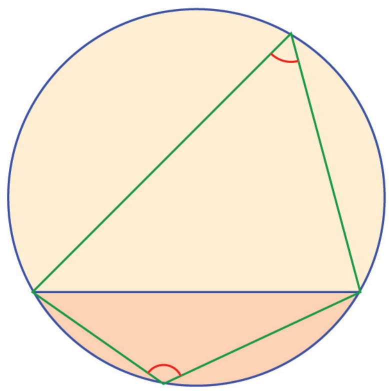
Prove that the sum of the areas of the blue and red crescents in the second picture is equal to the area of the triangle.

Chord, angle and arc

A diameter of a circle divides it into two equal parts, and joining the ends of the diameter to a point in any part gives a right angle.

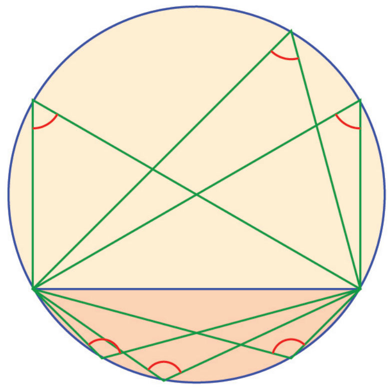


What about a chord, which is not a diameter?

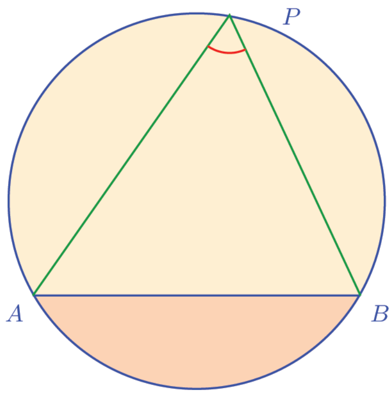


The parts are not equal; nor are the angles right.

But here also, are all the angles on the same side equal?

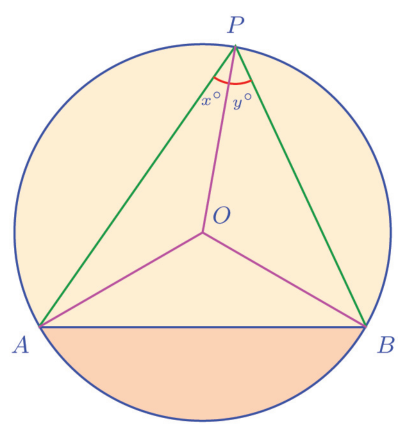


Let's see. First look at an angle at the top.



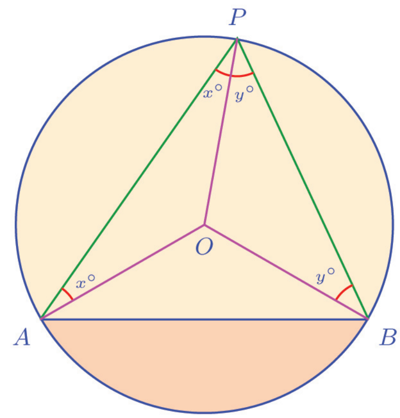
Draw a circle centred on a point A and mark three points B, C, D on it. Mark ∠D. Move D around the circle. What happens to the measure of this angle? Change the positions of B and C. When is ∠D right? When is it acute or obtuse?

As in the case of the diameter, let's join a point P to the centre O of the circle. This line splits the angle at P into two. Let's take the measures of these parts as x° and y°. Here the centre is not on the chord. So, let's join OA and OB also.

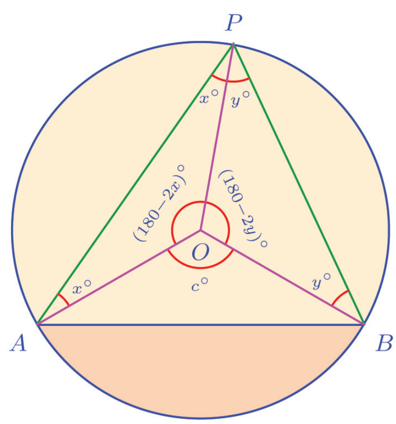


As in the case of a diameter, here also ΔOAP and ΔOBP are isosceles. So, we can write a part of the angles at A and B.

Here unlike the case with the diameter, the two isosceles triangles together do not form a single large triangle. So, the earlier trick of summing up the angles of a triangle won't work.



Instead, let's write all the angles around O:



If we take $\angle AOB = c°$ as in the picture, then

$$(180 - 2x) + (180 - 2y) + c = 360$$

From this, we get

$$2(x + y) = c$$

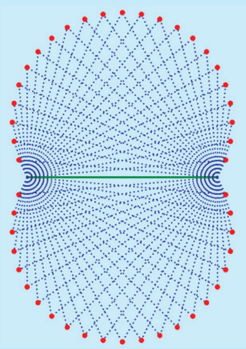
and this gives

$$\angle APB = (x + y)° = \frac{1}{2}c°$$

The point to note here is that once we fix A and B, changing the position of P on the circle changes x and y, but not c.

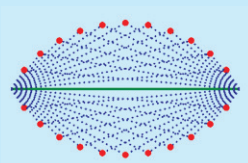
Circle trick

See this picture obtained by drawing angles of the same side above and below a line.



The angle is 60° in this picture.

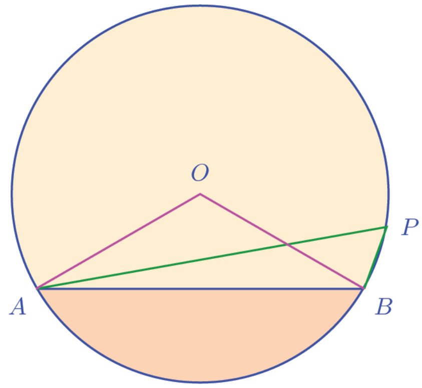
For 120°, we get this picture:



If we take 60° above the line and 120° below the line, we get a full circle. Why is this so?

If we take 30° above, what angle should we take below to get a full circle?

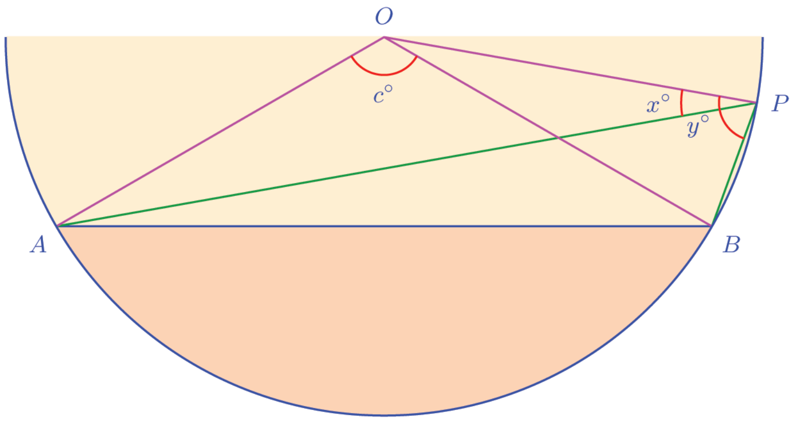
Will we get $\angle APB = \frac{1}{2}c°$, if the position of P is anywhere above AB on the circle?



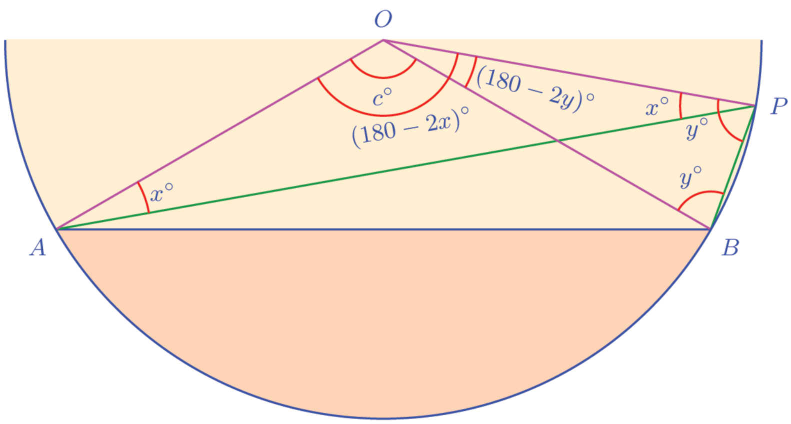
What if its like this?

Let's take $\angle APO = x°$ and $\angle BPO = y°$ as before.

To see the angles clearly, let's zoom in onto the part of the picture we're interested in:



Using the fact that OAP and OBP are isosceles triangles, we can compute some other angles as before:



From the angles at P, we can see that

$$\angle APB = (y - x)°$$

And from the angles at O we see that

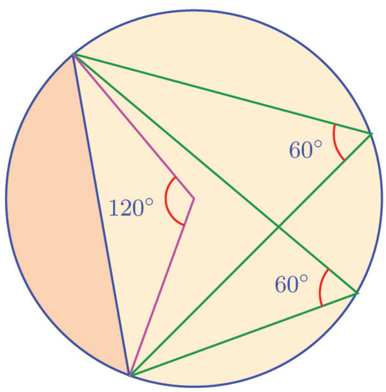
$$c = (180 - 2x) - (180 - 2y) = 2(y - x)$$

So, again we get,

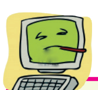
$$\angle APB = \frac{1}{2}c°$$

Thus joining the ends of a non-diametrical chord to any point on the larger part of the circle, we get an angle half the size of the angle we get by joining them to the a centre of the circle.

For example, see this picture:

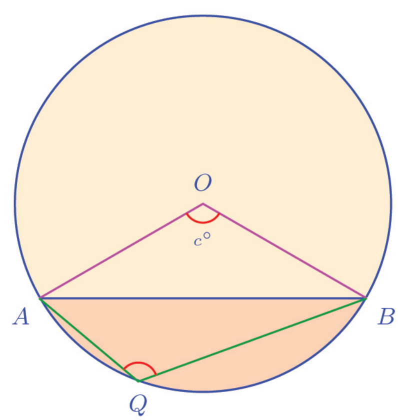


Now let's look at the angles in the smaller part.

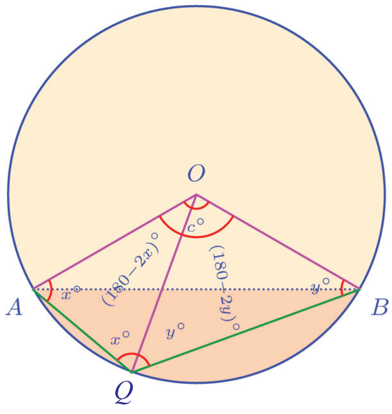


Draw a circle centred at a point A and mark points B, C, D on the circle. Join B and C to A and D. Mark angles BDC and BAC. Is there any relation between these angles?

Change the positions of B, C, D and see.



Here also, we get two isosceles triangles by joining OQ. So, we can compute some angles as before.



From the angles at O, we get,

$$c = (180 - 2x) + (180 - 2y)$$

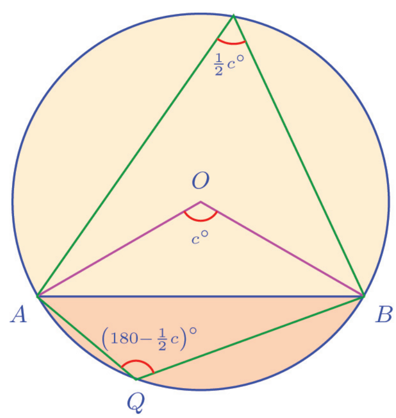
From this we get

$$2(x + y) = 360 - c$$

And this gives

$$\angle AQB = (x + y)^\circ = \left(180 - \frac{1}{2}c\right)^\circ$$

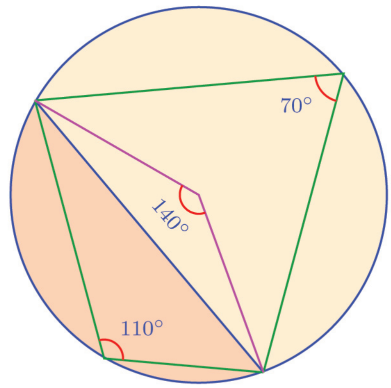
Now let's look at the angles in the two parts of the circle and the angle at the centre together:

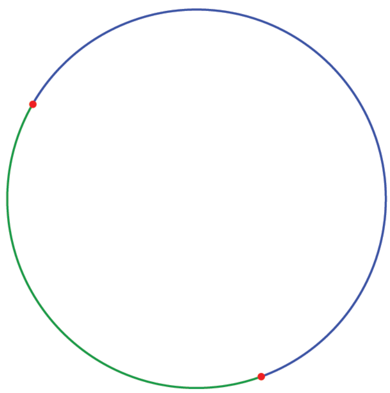


Any chord which is not a diameter splits the circle into unequal parts. The angle got by joining any point on the larger part to the ends of the chord is half the angle got by joining the centre of the circle to these ends. The angle got by joining any point on the smaller part to the ends of the chord is half the angle at the centre subtracted from 180°.

In short, if we know the angle which a chord makes at the centre, then the angle it makes on either side can be computed.

For example, a chord making 140° at the centre makes $\frac{1}{2} \times 140° = 70°$ on the larger part and $180° - \left(\frac{1}{2} \times 140°\right) = 110°$ on the smaller part.



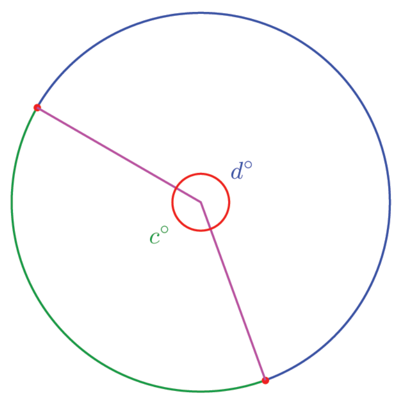


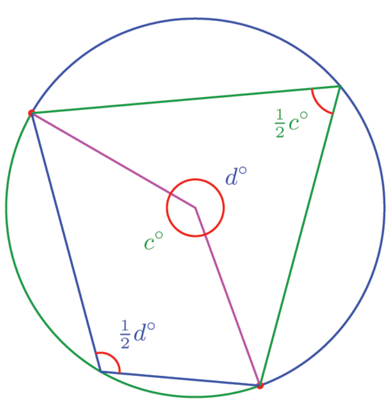
We've talked about central angles of an arc in Class 9. We can put the above results in terms of this idea.

Any two points on a circle divides it into two arcs.

Each of these two arcs can be called the *alternate arc* or *complementary arc* of the other. If we take their central angles as $c°$ and $d°$, then

$$c + d = 360$$





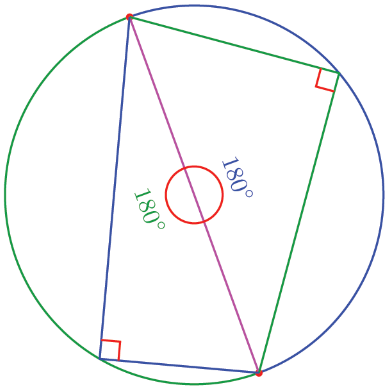
Now let's look at the angles which the first two points make at a point on each arc. As seen above, the angle on the larger arc is $\frac{1}{2}c°$.

And the angle on the smaller arc is

$$\left(180 - \frac{1}{2}c\right)° = \frac{1}{2}(360 - c)° = \frac{1}{2}d°$$

Suppose we start with the end points of a diameter?

The picture is like this.



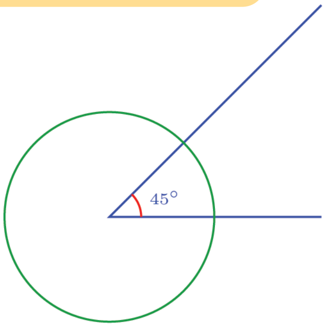
So we can say this about any arc of a circle.

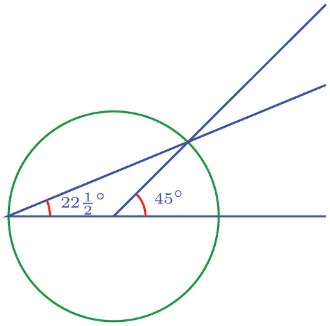
The angle made by any arc of a circle on the alternate arc is half the angle made at the centre.

From this, we also see that all angles made by an arc on the alternate arc are equal; moreover, if we recall the fact that $\frac{1}{2}d° = \left(180 - \frac{1}{2}c\right)°$ as shown in the first picture, we can see that sum of the angles on alternate arcs is 180°. Pairs of angles of sum 180° are usually called *supplementary angles*. Thus we have the following:

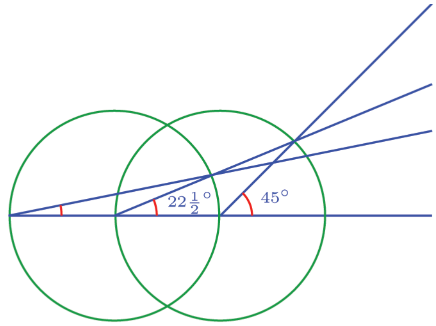
All angles made by an arc on the alternate arc are equal, and a pair of angles on alternate arcs are supplementary.

We can use this principle to halve angles. See this picture.





The corner of the angle is the centre of the circle. Now extend one side of the angle to meet the circle and join this point to the point where the other side cuts the circle. This gives us half the angle.

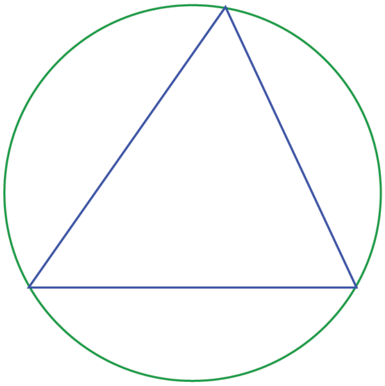


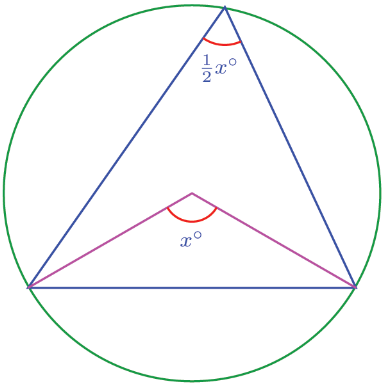
What if we do this again?

How much is the third angle?

We can also use this to draw the triangle with specified angles and circumradius. For example, let's see how we can draw a triangle of angles 30°, 70°, 80° and of circumradius 2.5 centimetres.

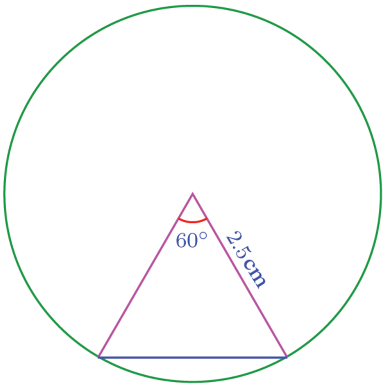
The three sides of the triangle are chords of the circum circle.

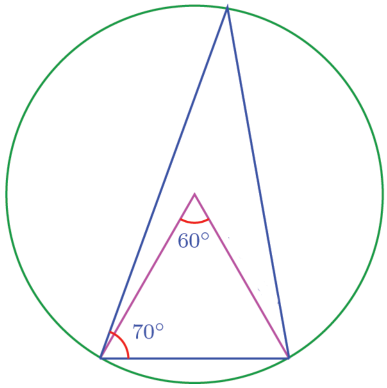




So, the angle opposite each side is half the angle it makes at the centre.

So to draw the triangle we require, we first draw a circle of radius 2.5 centimetres and draw a 60° angle at the centre. Joining its ends, we get the side opposite to 30° angle in our triangle.



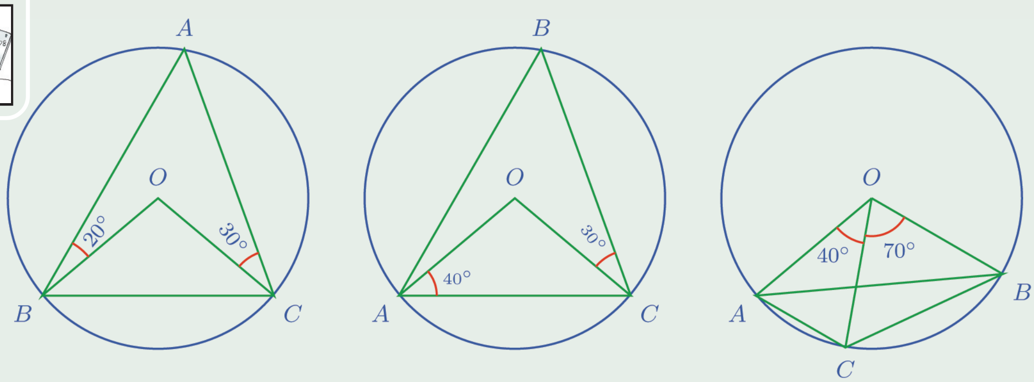


Now draw an angle of 70° at one of its ends and extend the other side to meet the circle. Joining this point to the other end of the line completes the triangle.

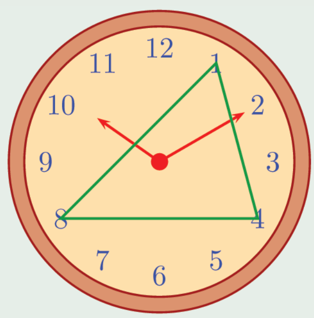
Aren't the other two angles of this triangle 30° and 80°? (Why?)

We note another thing here. We can draw several triangles with the same three angles. Fixing the circum radius also, we determine the triangle completely.

(1) In all the pictures given below, O is the centre of the circle and A, B, C are points on it. Calculate all angles of ΔABC and ΔOBC in each.



(2) The numbers 1, 4, 8 on a clock's face are joined to make a triangle.



Calculate the angles of this triangle.

How many equilateral triangles can we make by joining numbers on the clock's face?

(3) In each problem below, draw a circle and a chord to divide it into two parts such that the parts are as specified;

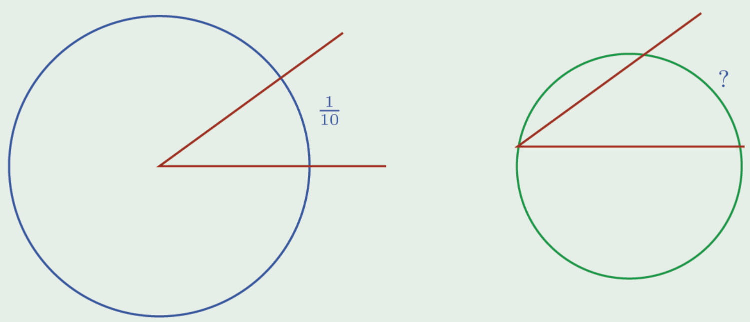
i) All angles on one part 80°.

ii) All angles on one part 110°.

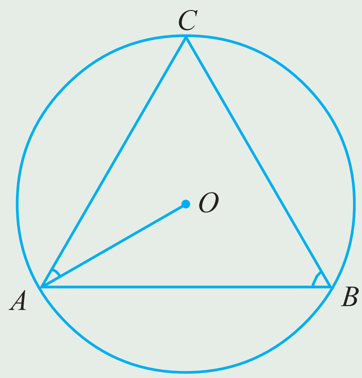
iii) All angles on one part half of all angles on the other.

iv) All angles on one part, one and a half times the angles on the other.

(4) A rod bent into an angle is placed with its corner at the centre of a circle and it is found that $\frac{1}{10}$ of the circle lies within it. If it is placed with its corner on another circle, what part of the circle would be within it?

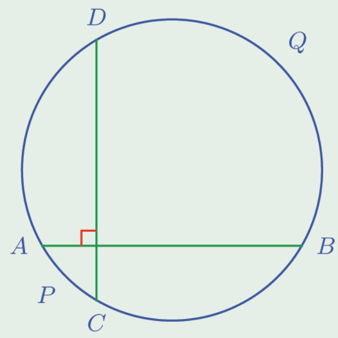


(5) In the picture, O is the centre of the circle and A, B, C, are points on it. Prove that $\angle OAC + \angle ABC = 90°$.

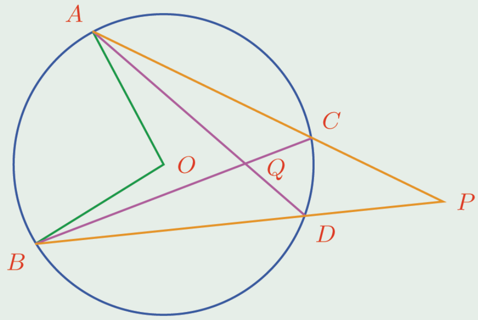


(6) Draw a triangle of circumradius 3 centimetres and two of the angles $32\frac{1}{2}°$ and $37\frac{1}{2}°$.

(7) In the picture, AB and CD are mutually perpendicular chords of the circle. Prove that the arcs APC and BQD joined together would make half the circle.

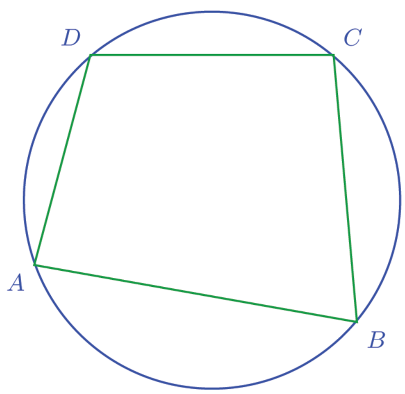


(8) In the picture, A, B, C, D are points on a circle centred at O. The lines AC and BD are extended to meet at P. The line AD and BC intersect at Q. Prove that the angle which the small arc AB makes at O is the sum of the angles it makes at P and Q.



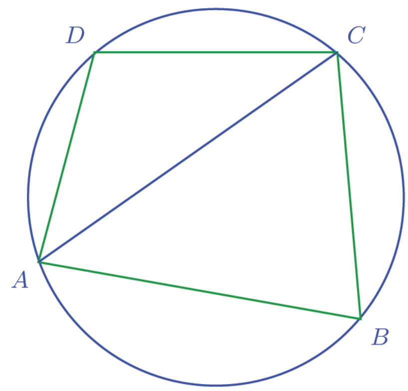
Circle and quadrilateral

Look at this picture:



Is there a relation between the angles at A, B, C, D?

Let's join AC.



Using **Polygon**, draw a quadrilateral joining four points on a circle. Click on **Angle** to measure all angles. Do you see any relation between the angles? Change the positions of the points and see.

Now the angles at B and D are the angles on two parts into which the chord AC divides the circle, so they are supplementary.

Similarly, drawing BD, we see the angles at A and C are supplementary.

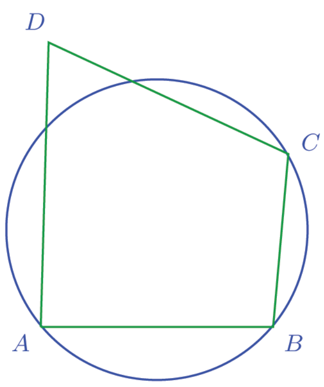
So what can we say in general?

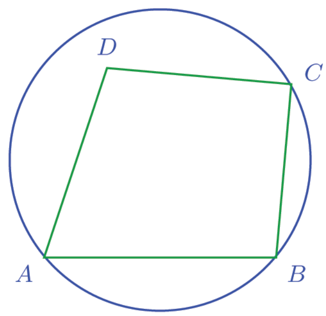
If all four vertices of a quadrilateral are on a circle, then its opposite angles are supplementary.

Is the converse true? That is, if the opposite angles of a quadrilateral are supplementary, can we draw a circle through all four vertices?

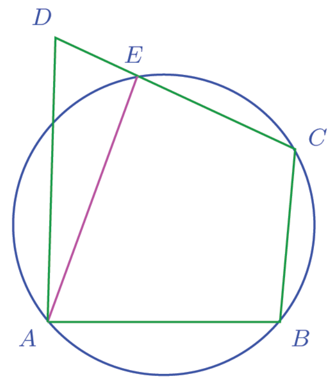
To answer this question, let's see how we can check practically whether the four vertices of a given quadrilateral are on a circle.

We can draw a circle through three of the vertices. (Remember how in Class 9, we drew a circle through any three points not on a line). Now, if the fourth vertex is also on the circle, we are done. But this vertex may be outside the circle or inside it.





Look at the first picture. Joining A and the point where CD cuts the circle, we get a quadrilateral with all four vertices on the circle:



So,

(1) $$\angle B + \angle AEC = 180°$$

Now as in the discussion in the section **Circle and right angle**, we see that

$$\angle AEC = \angle EAD + \angle D$$

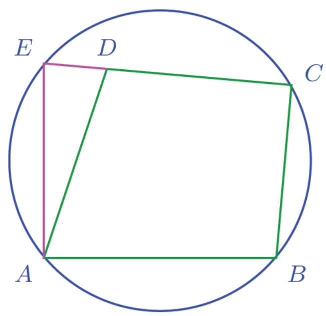
and so

(2) $$\angle D < \angle AEC$$

Here, thinking a bit about the meanings of the relations marked (1) and (2), we see that

$$\angle B + \angle D < 180°$$

Next in the second picture, let's extend CD to meet the circle at E and join AE:



Here we see that

(3) $$\angle B + \angle E = 180°$$

and from ΔEAD

$$\angle ADC = \angle E + \angle EAD$$

so that

(4) $$\angle ADC > \angle E$$

From the relations (3) and (4) we get

$$\angle B + \angle ADC > 180°$$

Mark three points A, B, C on a circle and a point D outside the circle. Use **Polygon** to draw the quadrilateral A, B, C, D and use **Angle** to measure all angles. When D is on the circle, the sum of ∠B and ∠D is 180°. What happens when D is outside? What happens as D is moved away from the centre or towards the centre? What if D is moved inside the circle?

What do we see here?

If one vertex of a quadrilateral is outside the circle drawn through the other three veritces, then the sum of the angles at this vertex and the opposite vertex is less than 180°, if it is inside the circle, the sum is more than 180°.

(We have already seen that if the vertex is on the circle, then the sum of the angles is 180°)

Now suppose in quadrilateral A, B, C, D, we have $\angle B + \angle D = 180°$.

Draw the circle through A, B, C.

Can D be outside the circle? If it is outside, we must have the sum of $\angle B$ and $\angle D$ less than 180° so it is not outside the circle.

Is it inside thet circle? If it is inside, the sum of $\angle B$ and $\angle D$ must be greater than 180°, so it is not inside the circle.

Since it is neither outside nor inside the circle, D must be on the circle.

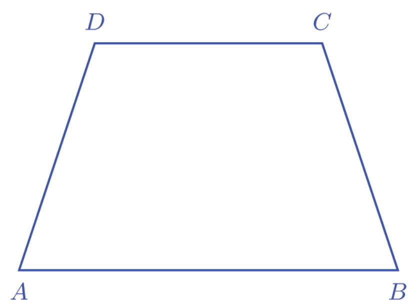
That is,

If the opposite angles of a quadrilateral are supplementary, we can draw a circle passing thorugh all four of its vertices.

Instead of saying, a quadrilateral for which a circle can be drawn through all four vertices, we call it a *cyclic quadrilateral*. As seen now, cyclic quadrilaterals are those quadrilaterals with opposite angles supplementary.

All rectangles are cyclic quadrilaterals, right? So are isosceles trapeziums.

Look at this figure:



ABCD is an isosceles trapezium. So

$$\angle A = \angle B$$

Also, *AB* and *CD* are parallel, so that

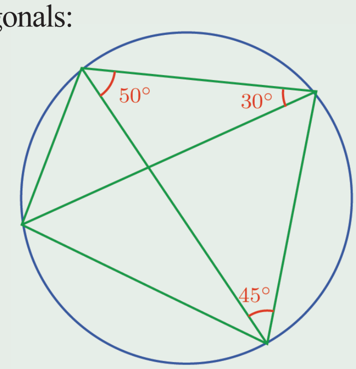
$$\angle A + \angle D = 180^\circ$$

From these two equations, we get

$$\angle B + \angle D = 180^\circ$$

So *ABCD* is a cyclic quadrilateral.

(1) Calculate the angles of the quadrilateral in the picture and also the angles between their diagonals:

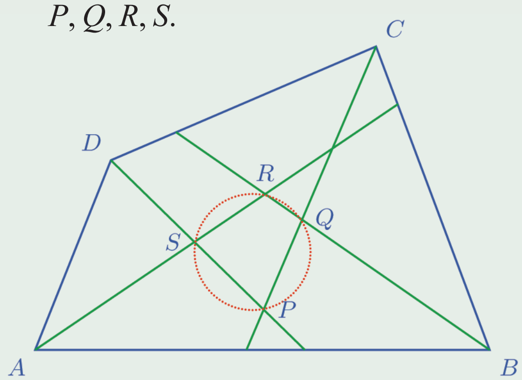


(2) Prove that any exterior angle of a cyclic quadrilateral is equal to the interior angle at the opposite vertex.

(3) Prove that a parallelogram which is not a rectangle is not cyclic.

(4) Prove that a non-isosceles trapezium is not cyolic.

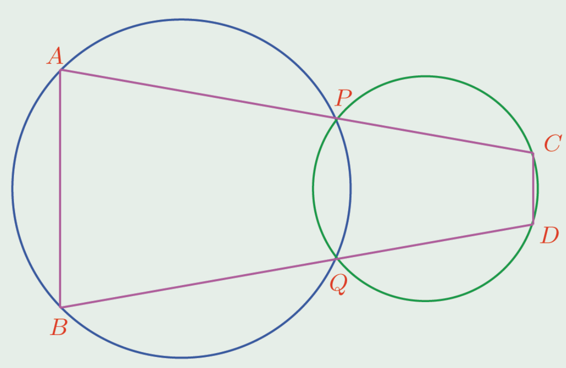
(5) In the picture, bisectors of adjacent angles of the quadrilateral *ABCD* intersect at *P*, *Q*, *R*, *S*.



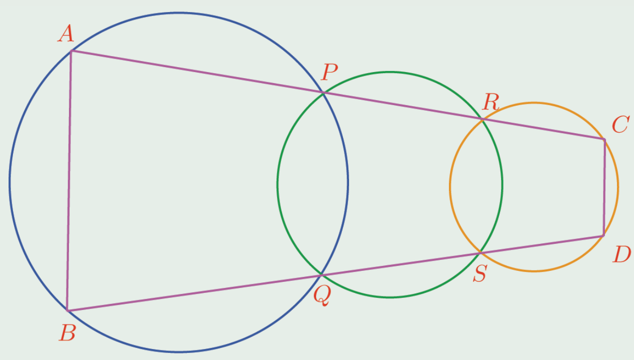
Prove that *PQRS* is a cyclic quadrilateral.

Draw a quadrilateral and the bisectors of its angles in GeoGebra. Mark the points of intersection of bisectors of adjacent angles and draw the quadrilateral joining these points. Check whether it is cyclic. For this use **Circle Through 3 Points** to draw the circle through three vertices and see whether it passes through the fourth. Change the positions of the vertices of the first quadrilateral to make it a parallelogram, rectangle, square, isosceles trapezium and see what happens to the inner quadrilateral (use **Grid** for this).

(6) i) The two circles below intersect at P, Q and lines through these points meet the circles at A, B, C, D. Prove that if AC and BD are of equal length, then $ABCD$ is a cyclic quadrilateral.

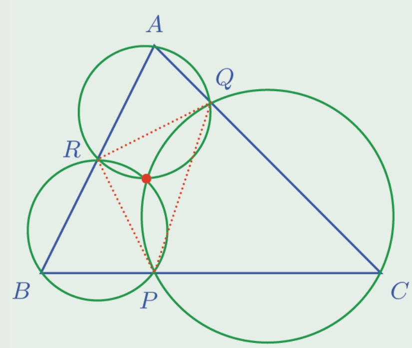


ii) In the picture, the circles on the left and right intersect the middle circle at P, Q, R, S; the lines joining them meet the left and right circles at A, B, C, D. Prove that $ABCD$ is a cyclic quadrilateral.



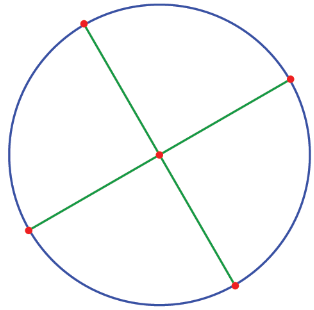
(7) In the picture, points P, Q, R are marked on the sides BC, CA, AB of ΔABC and the circumcircles, of ΔAQR, ΔBRP, ΔCPQ are drawn.

Prove that they pass through a common point.



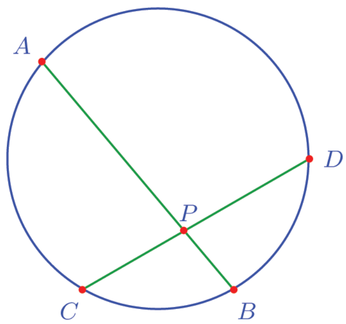
Two chords

Any two diameters of a circle intersect at the centre, and the length of the four pieces are equal.

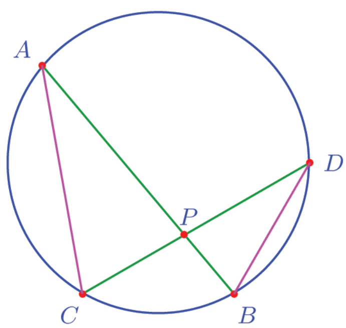


What happens when two chords which are not diameters intersect?

See this picture:



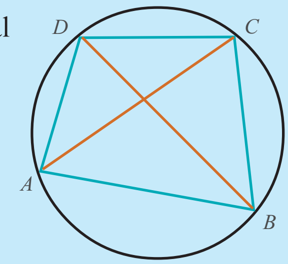
The pieces are not equal; yet there are some relations between them. To see this, we first join *AC* and *BD*.



The angles which the small arc *BC* makes at the points *A* and *D* on the alternate arc are equal. So are the angles which the small arc *AD* makes at *B* and *C*.

Ptolemy's Theorem

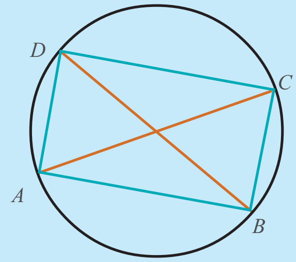
We can prove that the sum of the products of the opposite sides of a cyclic quadrilateral is equal to the product of the diagonals. That is if *ABCD* is a cyclic quadrilateral, then



$$(AB \times CD) + (AD \times BC) = AC \times BD$$

Conversely, if this is true in a quadrilateral, then it is cyclic. This is known as Ptolemey's Theorem.

Now a rectangle is a cyclic quadrilateral and its opposite sides are equal and so are the diagonals. Thus in a rectangle *ABCD* Ptolemey's Theorem becomes



$$AB^2 + BC^2 = AC^2$$

This is Pythagoras Theorem!

Thus ΔPAC and ΔPDB have the same angles. So their sides are in the same ratio. So,

$$\frac{PA}{PC} = \frac{PD}{PB}$$

We write it in multiplicative form like this:

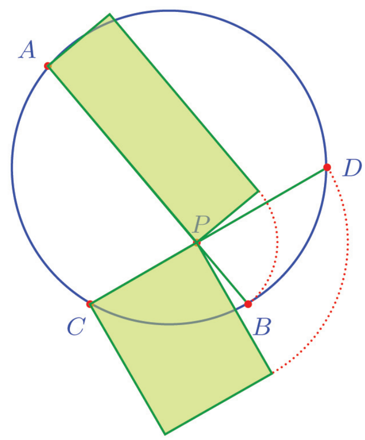
$$PA \times PB = PC \times PD$$

In this, PA, PB are parts of the chord AB and PC, PD are parts of the chord CD. So we can put it like this:

If two chords of a circle intersect within the circle, then the products of the parts of the two chords are equal.

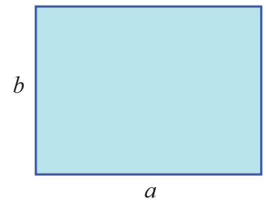
We can interpret the product of two lengths as an area. So this relation can be put in geometric language as below:

If the chords of a circle intersect within a circle, then the rectangles formed by the parts of the same chord have equal area.

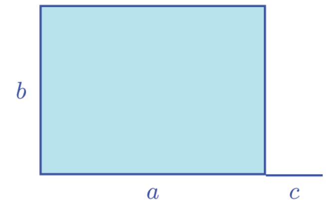


We can solve some problems on areas using this.

For example, see this rectangle:



We want to increase one side a little and draw a rectangle, without changing the area.

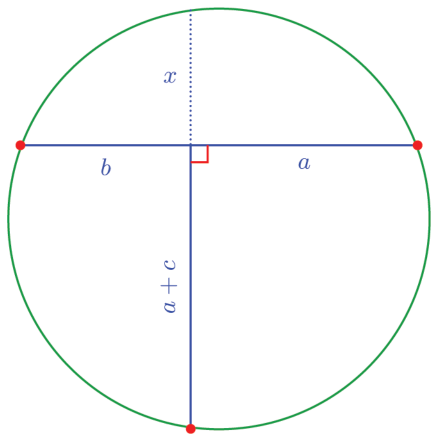


In algebraic language, the problem is this:

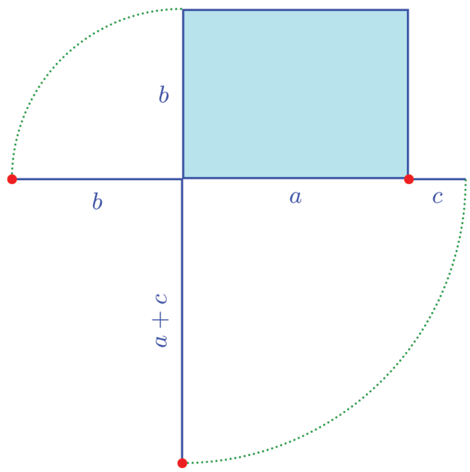
$$\text{Find } x \text{ so that } (a + c)\,x = ab$$

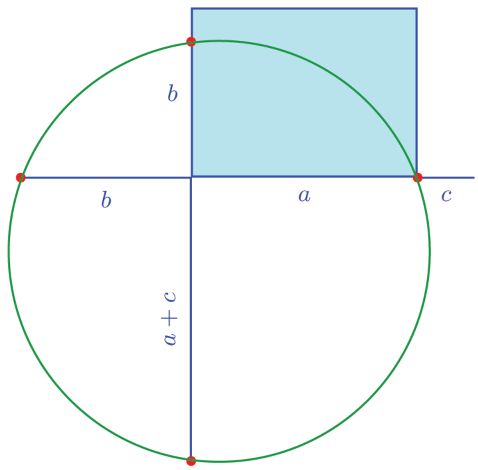
How about using our result on chords?

We have to draw a picture like this:



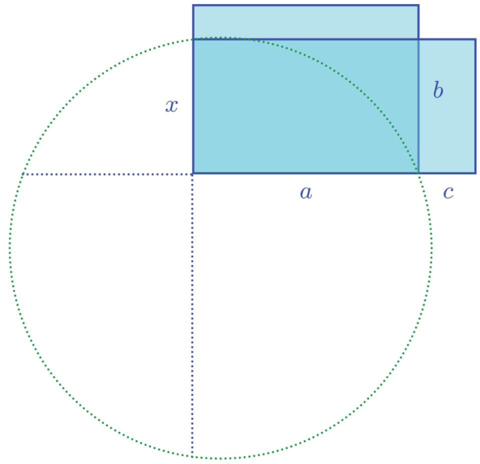
So, first we extend the bottom side of the rectangle to the left by b and the left side downwards by $a + c$.





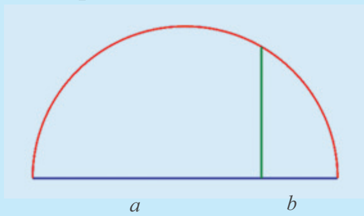
Now draw the circle through the three red dots in the picture. (You know how to draw the circumcircle of a triangle, don't you?)

The part of the left side of the rectangle cut by this circle is the other side of the required rectangle.



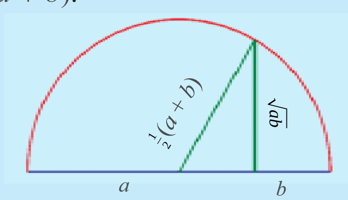
Geometry, algebra and numbers

See this picture:



What is the height of the perpendicular? If we write it as x, then we get $ab = x^2$ or $x = \sqrt{ab}$.

What is the radius of this semicircle? Since diameter is $a + b$, radius is $\frac{1}{2}(a + b)$.

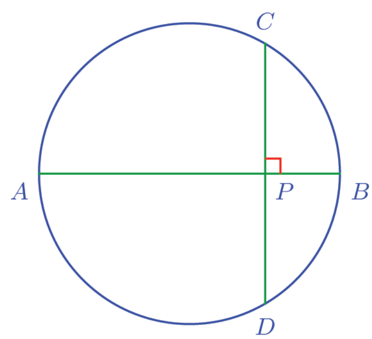


In the picture, radius is longer than the perpendicular. Are they equal in some case?

So, what do we get? For any pair a, b of different numbers

$$\frac{1}{2}(a+b) > \sqrt{ab}$$

A special case of intersecting chords is also of interest. In the picture below, AB is a diameter of the circle and CD is a chord perpendicular to it.



The perpendicular from the centre bisects a chord and so $PC = PD$. So our earlier result in this case is

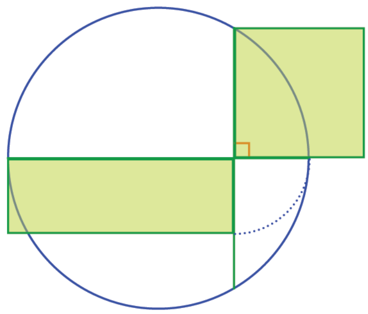
$$PA \times PB = PC^2$$

That is,

The product of the parts into which a diameter of a circle is cut by a perpendicular chord, is equal to the square of half the chord.

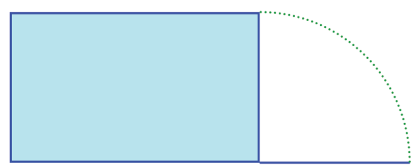
In geometric language, this can be put like this:

The area of the rectangle formed of parts into which a diameter of a circle is cut by a perpendicular chord is equal to the area of the square formed by half the chord.

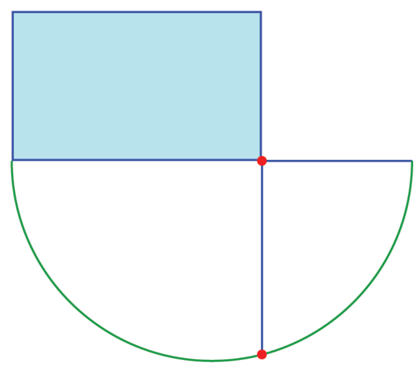


We can use this to change a rectangle into a square of the same area. For example see this rectangle.

To draw a square of the same area, first extend the width by the height.

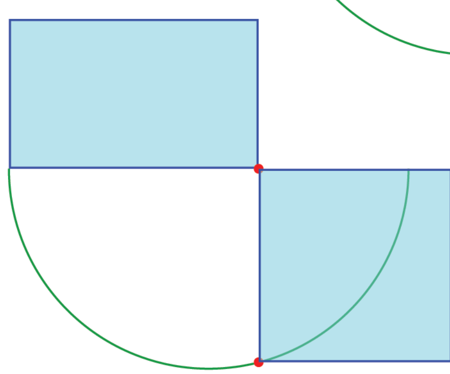


Now draw a circle below with this line as diameter. Extend the right side of the rectangle to meet this semi circle.



This line is the side of the required square (why?)

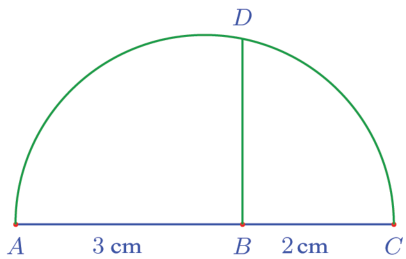
We can also use this to draw a square of specified area.



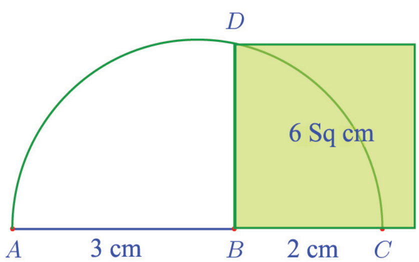
For example, let's see how we can draw a square of area 6 square centimetres.

Since $6 = 3 \times 2$ we need only draw a square of area equal to that of a rectangle of side 3 centimetres and 2 centimetres. We need not draw the rectangle, only a line like this:

Now draw a semicircle with AC as diameter and draw the perpendicular to AC at B. Let it meet the semicircle at D.

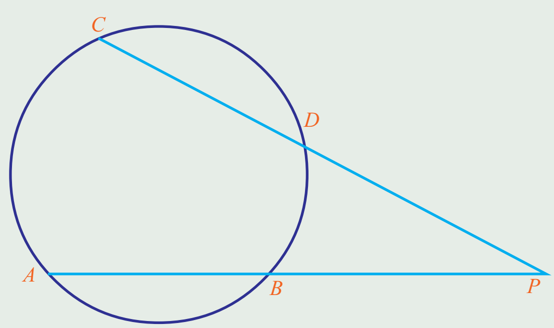


As seen in the general result, $BD^2 = AB \times BC = 6$, so that the square of side BD has an area 6 square centimetres.



Here the length of BD is $\sqrt{6}$ centimetres. Thus we can use this method to draw some lines of irrational lengths.

(1) In the picture, chords AB and CD of the circle are extended to meet at P.



i) Prove that the angles of ΔAPC and ΔPBD, formed by joining AC and BD, are the same.

ii) Prove that $PA \times PB = PC \times PD$.

iii) Prove that if $PB = PD$, then $ABDC$ is an isosceles trapezium.

(2) Draw a rectangle of width 5 centimetres and height 3 centimetres.

i) Draw a rectangle of the same area with width 6 centimetres.

ii) Draw a square of the same area.

(3) Draw a square of area 15 square centimetres.

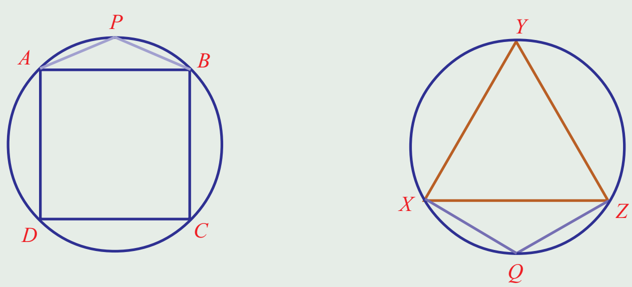
(4) Draw a square of area 5 square centimetres in three different ways.

(5) Draw a triangle of sides 4, 5, 6 centimetres and draw a square of the same area

(6) Draw an equilateral triangle of height 3 centimetres.

(7) Draw an isosceles right triangle of hypotenuse 4 centimetres.

(8) In the picture below, $ABCD$ is a square with vertices on a circle and XYZ is such an equilateral triangle. P and Q are points on the circles:



i) How much is $\angle APB$?

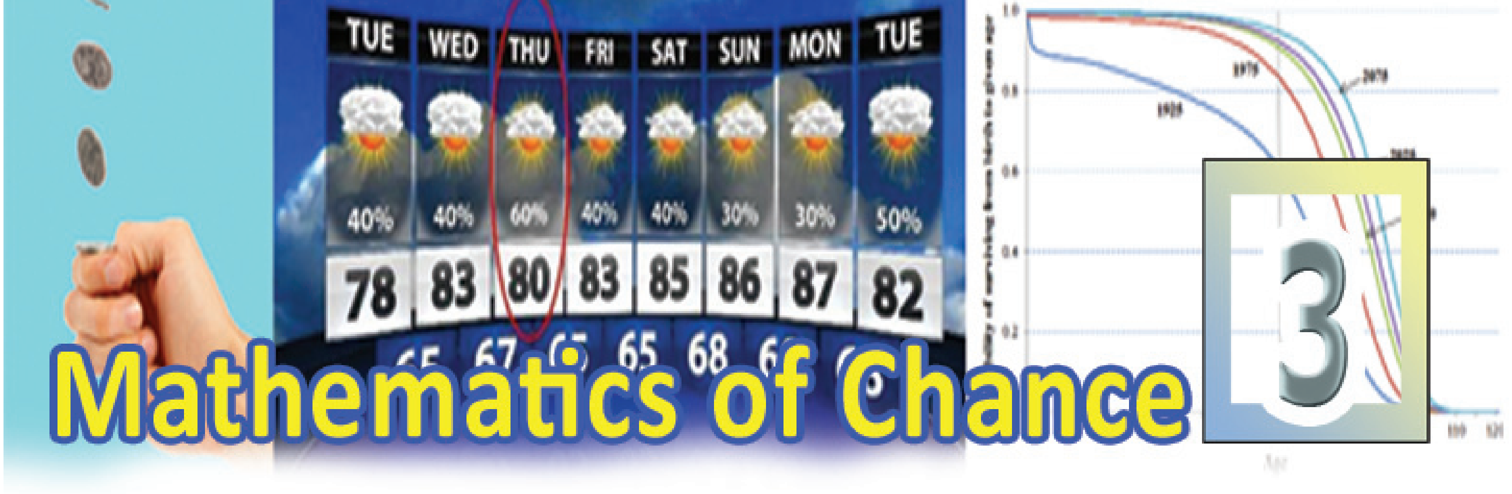
ii) How much is $\angle XQZ$?

Project

- What is the relation between the angles of a polygon with vertices on a circle and the number of sides even?

Looking back

Learning outcomes	On my own	With teacher's help	Must improve
• Understanding the relation between the angles which an arc of a circle makes at the centre and at various points on the circle. • Realising that of the two parts into which a chord divides a circle, angles on the same part are equal and angles on different parts are supplementary. • Recognising quadrilaterals which have circum circles. • Understanding the relation between the parts into which chords of a circle cut each other. • Converting a rectangle into another rectangle or square without changing the area.			



3 Mathematics of Chance

Possibilities as numbers

There are ten beads in a box, nine black and only one white. If we pick a bead (without looking)...

It is most likely to be black; can be white though.

There are eight black beads and two white in another box. How about picking one from this?

Again it is more likely to be black.

Five black and five white in the third. What if we pick a bead from this? Could be black or white; can't say anything more?

We can say all these in a different way. From the first and the second box, the probability of getting a black bead is more. From the third box, the probabilities are the same.

Let's have a game with beads. Five black and five white beads in one box. Six black and four white in another. One has to choose a box and pick a bead. If it is black, he wins. Which box is the better choice?

The second box contains more blacks. So don't we have a greater probability of getting a black from it?

Suppose we take a black bead from the second box and put it in the first.

So this is how the boxes are now:

First box	:	6 black 5 white
Second	:	5 black 4 white

Now to win the game, which box would you choose?

The first box contains more black beads. Is the probability of getting a black from it also more?

Let's think in terms of totals. The first box has 11 beads in all, of which 6 are black. That is, $\frac{6}{11}$ of the total are black.

What about the second? $\frac{5}{9}$ of the total are black.

Which is larger, $\frac{6}{11}$ or $\frac{5}{9}$?

$\frac{5}{9}$, isn't it?

Thus, the second box has a larger black part. So isn't it still the better choice?

In other words, the probability of getting a black bead from the second box is larger. We can go even further and say that the probability of getting a black bead from the first box is $\frac{6}{11}$ and the probability of getting a black bead from the second box is $\frac{5}{9}$.

What about the probabilities of getting a white bead? $\frac{5}{11}$ from the first and $\frac{4}{9}$ from the second. Which is larger? So if it is white for a win, which box is the better choice?

Let's write down the various probabilities in a table:

		First box		Second box	
		Black	White	Black	White
First	Number	5	5	6	4
	Probability	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{5}$
Later	Number	6	5	5	4
	Probability	$\frac{6}{11}$	$\frac{5}{11}$	$\frac{5}{9}$	$\frac{4}{9}$

And another question. We have seen that at the beginning and after transferring a bead, the probability of getting a black bead from the second box is larger.

Is this probability less or more after the transfer?

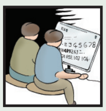
Let's look at another problem.

Numbers from 1 to 25 are written on paper slips and put in a box. One slip is taken from it. What is the probability that it is an even number?

Of all the 25 numbers, 13 are odd and 12 even, right? So the probability of getting an even number is $\frac{12}{25}$.

What about the probability of getting an odd number?

What is the probability of getting a multiple of three? A multiple of six?



(1) A box contains 6 black and 4 white balls. If a ball is taken from it, what is the probability of it being black? And the probability of it being white?

(2) There are 3 red balls and 7 green balls in a bag, 8 red and 7 green balls in another.

i) What is the probability of getting a red ball from the first bag?

ii) From the second bag?

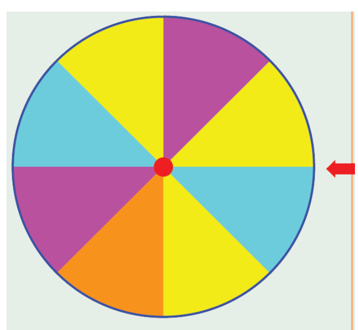
iii) If all the balls are put in a single bag, what is the probability of getting a red ball from it?

(3) One is asked to say a two-digit number. What is the probability of it being a perfect square?

(4) Numbers from 1 to 50 are written on slips of paper and put in a box. A slip is to be drawn from it; but before doing so, one must make a guess about the number: either prime number or a multiple of five. Which is the better guess? Why?

(5) A bag contains 3 red beads and 7 green beads. Another contains one red and one green more. The probability of getting a red from which bag is more?

Geometrical probability



A multicoloured disc spins on a board.

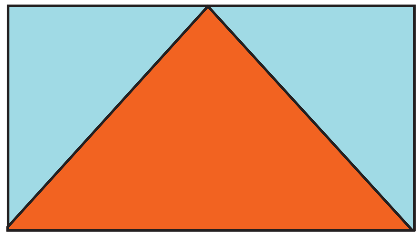
What is the probability of getting yellow against the arrow when it stops?

Any of the eight sectors can came against the arrow. Of these, three are yellow.

So, the probability of yellow is $\frac{3}{8}$.

Can you calculate the probabilities of other colours?

Let's try another problem. A cardboard rectangle is cut out and the midpoint of one side is joined to the ends of the opposite sides to make a triangle. If you shut your eyes and put a dot in this rectangle, what is the probability that it would be within the red triangle?



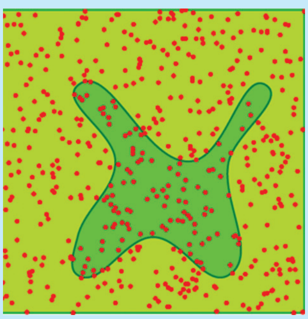
The triangle and rectangle have the same base and height, haven't they?

So the triangle is half the rectangle.

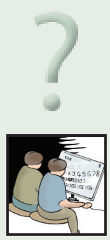
That is, the area of the triangle is $\frac{1}{2}$ of the area of the rectangle. So the probability of the dot falling within the triangle is also $\frac{1}{2}$.

Area and probability

Probability can be used to find areas of complex regions approximately. Draw the region within a specific square and mark points within the square, without any order or pattern.

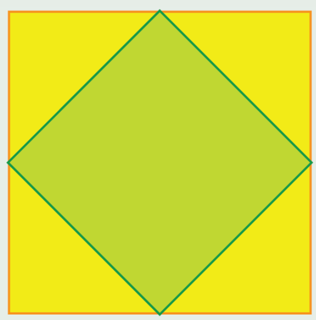


The number of dots within our region divided by the total number of dots would be approximately equal to the area of the region divided by the area of the square. And it becomes better as we increase the number of dots. The generation of dots and the calculations can be done quickly using a computer. It is called the Monte carlo method.

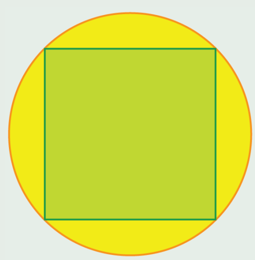


In each picture below, the explanation of the green part is given. Calculate in each, the probability of a dot put without looking to be within the green part.

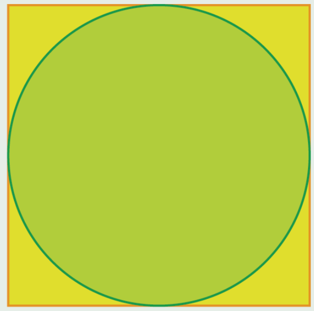
(1) A square got by joining the mid points of a bigger square.



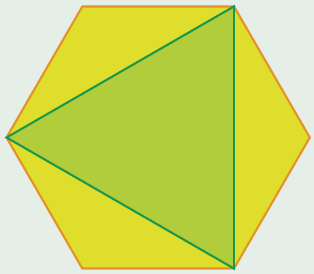
(2) A square with all vertices on a circle.



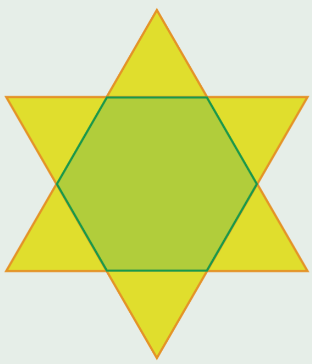
(3) Circle exactly fitting inside a square.



(4) A triangle got by joining alternate vertices of a regular hexagon.

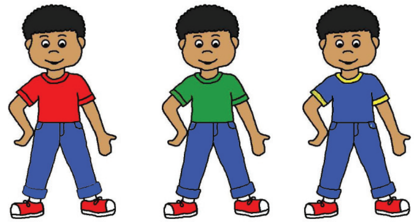


(5) A regular hexagon formed by two overlapping equilateral triangles.

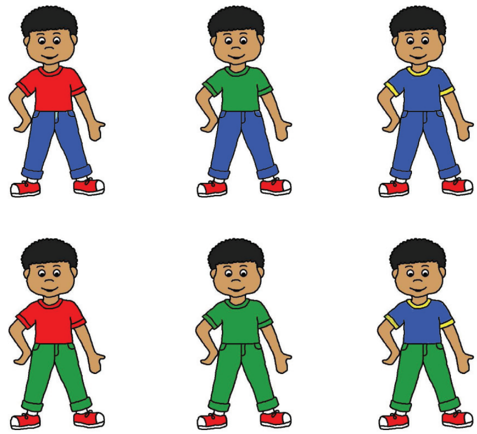


Pairs

Looking for a clean dress, Johny found a pair of blue pants and three shirts, red, green and blue. "In how many ways can I dress?", thought Johny.



Searching again he found a pair of green pants also. Now this can be worn with each of the three shirts, giving three more ways to dress, Johny calculated.



Thus Johny can dress in six different ways.

In how many of these are the colours of shirt and pants the same?

So, what is the probability of Johny wearing shirt and pants of the same colour?

$$\frac{2}{6} = \frac{1}{3}, \text{ right?}$$

Let's look at another problem. A box contains four paper slips numbered 1, 2, 3, 4 and another contains two slips numbered 1,2. One slip is picked from each. What are the possible pairs?

With 1 from the first box, we can combine 1 or 2 from the second box to form two pairs. Let's write them as (1, 1) and (1, 2).

How about writing down all such pairs, combine each number from the first box with the two possibilities from the second?

(1, 1) , (1, 2)

(2, 1) , (2, 2)

(3, 1) , (3, 2)

(4, 1) , (4, 2)

8 pairs in all.

A problem

Galileo has written about a problem posed by a gambler friend. This friend calculated six different ways of getting the sum 9 and sum 10 from three dice:

	9	10
1.	1 + 2 + 6	1 + 3 + 6
2.	1 + 3 + 5	1 + 4 + 5
3.	1 + 4 + 4	2 + 2 + 6
4.	2 + 2 + 5	2 + 3 + 5
5.	2 + 3 + 4	2 + 4 + 4
6.	3 + 3 + 3	3 + 3+ 4

But his experience showed that 10 occurs more. Galileo's solution was like this. The gambler had taken 1, 2, 6 to mean 1 from any one die, 2 from another and 3 from yet another. Instead of this, write (1, 2, 6) to mean 1 from the first die, 6 from the second and 2 from the third. Thus (1, 6, 2), (1, 6, 2), (2, 1, 6), (2, 6, 1). (6, 1, 2), (6, 2, 1) gives six different ways for the sum 9. Expanding all possible triples like this, Galileo computes the total number of ways of getting 9 as 25 and getting 10 as 27. (Try it!)

In how many of these are both numbers odd?

Only in (1, 1) and (3, 1), right? So if we take one slip from each, the probability of both being odd is $\frac{2}{8} = \frac{1}{4}$.

Can you compute the probability of both being even? The probability of one odd and the other even? And both being the same number?

(1) Rajani has three necklaces and three pairs of earrings, of green, blue and red stones. In what all different ways can she wear them? What is the probability of her wearing the necklace and earrings of the same colour? Of different colours?

(2) A box contains four slips numbered 1, 2, 3, 4 and another box contains two slips numbered 1, 2. If one slip is taken from each, what is the probability of the sum of numbers being odd? What is the probability of the sum being even?

(3) A box contains four slips numbered 1, 2, 3, 4 and another contains three slips numbered 1, 2, 3. If one slip is taken from each, what is the probability of the product being odd? The probability of the product being even?

(4) From all two-digit numbers with either digit 1, 2, or 3 one number is chosen.

i) What is the probability of both digits being the same?

ii) What is the probability of the sum of the digits being 4?

(5) A game for two players. First, each has to decide whether he wants odd number or even number. Then both raises some fingers of one hand. If the sum is odd, the one who chose odd at the beginning wins; if it is even, the one who chose even wins. In this game, which is the better choice at the beginning, odd or even?

More pairs

Two boxes again, one containing ten slips numbered 1 to 10 and the other, five slips from 1 to 5. One slip is taken from each box, as usual. What is the probability of both being odd?

The method of solution is simple. Calculate the number of all possible pairs of numbers and then find how many of them have both odd, as required. The second number divided by the first gives the probability.

Easier said than done. It is not very interesting to list all pairs and count.

Probability and frequency

Usually if a coin is tossed a number of times, head and tails would come up more or less equally. But sometimes because of some defect in manufacture or something else, heads may come up more often. How do we find it out?

When the coin is repeatedly tossed, if the number of times each face comes up differs considerably from half, such a doubt is reasonable. So the method is to toss again and again and record the number times each face comes up. For example, look at this table.

Toss	Head	Tail
10	6	4
100	58	42
1000	576	424
10000	5865	4135

Here we can see that it is better to take the probability of a head as 0.6 and probability of a tail as 0.4 instead of 0.5 for both. Methods of making such computations precise are part of the branch of mathematics called Probability Theory.

Let's think. The number from the first box can be any one of the ten numbers; and the number from the second, any one of the five. So how many pairs are there with first number 1? How many with first number 2?

In short, after fixing the first number, we can make 5 different pairs, by changing the second. The first number can be fixed in 10 ways. Thus we can imagine all the pairs arranged like this:

5 pairs.

10 pairs.

(1, 1)	(1, 2)	…	(1, 5)
(2, 1)	(2, 2)	…	(2, 5)
…	…	…	…
…	…	…	…
(10, 1)	(10, 2)	…	(10, 5)

Of these 50 pairs, how many have both odd numbers? For that, the first number must be one of the 5 numbers 1, 3, 5, 7, 9; and the second number one of the 3 numbers 1, 3, 5. Taking each of the first 5 numbers with each of the second 3 numbers, how many different pairs can we form?

5×3, isn't it? (Imagine them as rows and columns if you want). So the probability of getting odd numbers from both these boxes is $\frac{15}{50} = \frac{3}{10}$. Similarly, can you calculate the probability of getting even numbers from both and odd from one and even from the other?

One more problem: there are 50 mangoes in a basket, 20 of which are unripe. Another basket contains 40 mangoes, with 15 unripe. If we take one mango from each basket, what is the probability of both being ripe?

In how many different ways can we choose a pair of mangoes, one from each basket? (If you want, imagine the mangoes from each basket laid on a line; you can also imagine them to be numbered).

So there are $50 \times 40 = 2000$ ways of taking a pair of mangoes. How many of these have both ripe?

The first basket has $50 - 20 = 30$ ripe ones and the second, $40 - 15 = 25$.

Each ripe mango from the first basket paired with a ripe mango from the second gives $30 \times 25 = 750$ pairs. So the probability of both being ripe is $\frac{750}{2000} = \frac{3}{8}$.

Similarly, can't you compute the probability of both being unripe?

What is the probability of getting at least one ripe mango?

At least one ripe means one ripe and the other unripe, or both ripe. In these, one ripe and the other unripe can occur in two ways: the first ripe and the second unripe or the other way round. Thus the total number of pairs with only one ripe is

$$\begin{aligned}(30 \times 15) + (20 \times 25) &= 450 + 500 \\ &= 950\end{aligned}$$

We have already seen that the number of pairs with both ripe is 750. Taking all these together, the number of pairs with at least one ripe is

$$950 + 750 = 1700$$

Measure of uncertainty

Have you noted the time of sunrise and sunset for each day given in a calendar? These can be computed since the Earth and Sun move according to definite mathematical laws. Similarly summer months and rainy months can be computed. But it may not be possible to predict a sudden shower in summer season. The large number of factors influencing rainfall and the complexities of their inter relations make such predictions difficult.

But analysing the context mathematically, probabilities can be calculated. That is why daily weather forecasts are often given as probabilities. Unexpected changes in the circumstances sometimes make them wrong.

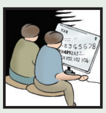
If we look at it rationally, we can see that such probabilistic predictions are more reliable than supposedly certain predictions with no scientific basis.

So, the probability of getting at least one ripe mango is $\frac{1700}{2000} = \frac{17}{20}$.

This can be found in a different way. To have at least one ripe, both cannot be unripe. Of all the 2000 possible pairs, both are unripe in $20 \times 15 = 300$ pairs.

So, in the remaining $2000 - 300 = 1700$ pairs, at least one must be ripe.

Thus the probability of getting at least one ripe mango is $\frac{1700}{2000} = \frac{17}{20}$.



(1) In class 10A, there are 20 boys and 20 girls. In 10B, there are 15 boys and 25 girls. One student is to be selected from each class.

i) What is the probability of both being girls?

ii) What is the probability of both being boys?

iii) What is the probability of one boy and one girl?

iv) What is the probability of atleast one boy?

(2) One is asked to say a two-digit number.

i) What is the probability of both digits being the same?

ii) What is the probability of the first digit being larger?

iii) What is the probability of the first digit being smaller?

(3) Two dice with faces numbered from 1 to 6 are rolled together. What are the possible sums? Which of these sums has the maximum probability?

Looking back

Learning outcomes	On my own	With teacher's help	Must improve
• Explaining probability as a number. • Explaining the method to calculate probability in different contexts. • Justifying the need to analyze probability using numbers in practical situations.			