Dispersion and Absorption of Light (Part - 1)

Q.200. A free electron is located in the field of a monochromatic light wave. The intensity of light is I = 150 W/m², its frequency is $\omega = 3.4.10^{15}$ s^{-1}. Find:
(a) the electron's oscillation amplitude and its velocity amplitude;
(b) the ratio F_m/F_e, where F_m and F_e are the amplitudes of forces with which the magnetic and electric components of the light wave field act on the electron; demonstrate that that ratio is equal to
$\frac{1}{2} v/c,$
where v is the electron's velocity amplitude and c is the velocity of light. Instruction. The action of the magnetic field component can be disregarded in the equation of motion of the electron since the calculations show it to be negligible.

Ans. In a travelling plane electromagnetic wave the intensity is simply the time averaged magnitude of the Pointing vector

$$I = \langle |\vec{E} \times \vec{H}| \rangle = \left\langle \sqrt{\frac{\varepsilon_0}{\mu_0}} E^2 \right\rangle = \langle c\, \varepsilon_0 E^2 \rangle$$

on using

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}, \quad E\sqrt{\varepsilon_0} = H\sqrt{\mu_0}\,.$$

Now time averaged value of E^2 is $E_0^2/2$ so

$$I = \frac{1}{2} c\, \varepsilon_0 E_0^2 \quad \text{or} \quad E_0 = \sqrt{\frac{2I}{c\,\varepsilon_0}}$$

(a) Represent the electric field at any point by $E = E_0 \sin \omega t$. Then for the electron we have the equation.

$$m\ddot{x} = eE_0 \sin \omega t$$

So $$x = -\frac{eE_0}{m\omega^2} \sin \omega t$$

He amplitude of the forced oscillation is

$$\frac{eE_0}{m\omega^2} = \frac{e}{m\omega^2}\sqrt{\frac{2I}{c\varepsilon_0}} = 5{\cdot}1 \times 10^{-16}\ \text{cm}$$

The velocity amplitude is clearly

$$\frac{eE_0}{m\omega} = 5{\cdot}1 \times 10^{-16} \times 3{\cdot}4 \times 10^{15} = 1{\cdot}73\ \text{cm/sec}$$

(b) For the electric force F_e = amplitude of the electric force
- e E_0

For the magnetic force (which we have neglected above), it is

$$(e\,\mathrm{v}B) = (e\,\mathrm{v}\,\mu_0 H)$$

$$= e\,\mathrm{v}E\sqrt{\varepsilon_0\mu_0} = e\,\mathrm{v}\frac{E}{c}$$

writing $\mathrm{v} = -\mathrm{v}_0 \cos\omega t$

where $\mathrm{v}_0 = \frac{eE_0}{m\omega}$

we see that the magnetic force is apart from a sign $\frac{e\,\mathrm{v}_0 E_0}{2c}\sin 2\omega t$

Hence $\frac{F_m}{F_e} =$ Ratio of amplitudes of the two forces

$$= \frac{\mathrm{v}_0}{2c} = 2{\cdot}9 \times 10^{-11}$$

This is negligible and justifies the neglect of magnetic field of the electromagnetic wave in calculating v_o

Q.201. An electromagnetic wave of frequency ω propagates in dilute plasma. The free electron concentration in plasma is equal to n_o. Neglecting the interaction of the wave and plasma ions, find:
(a) the frequency dependence of plasma permittivity;
(b) how the phase velocity of the electromagnetic wave depends on its wavelength λ in plasma.

Ans. (a) It turns out that one can neglect the spatial dependence of the electric field as well as the magnetic field. Thus for a typical electron

$$m\ddot{\vec{r}} = e\vec{E}_0 \sin \omega t$$

so $\vec{r} = -\dfrac{e\vec{E}_0}{m\omega^2} \sin \omega t$ (neglecting any nonsinusoidal part).

The ions, will be practically unaffected. Then

$$\vec{P} = n_o e\vec{r} = -\frac{n_0 e^2}{m\omega^2}\vec{E}$$

and

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \left(1 - \frac{n_0 e^2}{\varepsilon_0 m \omega^2}\right)\vec{E}$$

Hence the permittivity

$$\varepsilon = 1 - \frac{n_0 e^2}{\varepsilon_0 m \omega^2}.$$

(b) The phase velocity is given by

$$v = \omega/K = \frac{c}{\sqrt{\varepsilon}}$$

So

$$c\,k = \omega\sqrt{1 - \frac{\omega_P^2}{\omega^2}}, \quad \omega_P^2 = \frac{n_0 e^2}{\varepsilon_0 m}$$

$$\omega^2 = c^2 k^2 + \omega_P^2$$

Thus

$$v = c\sqrt{1 + \frac{\omega_P^2}{c^2 k^2}} = c\sqrt{1 + \left(\frac{n_0 e^2}{4\pi^2 m c^2 \varepsilon_0}\right)\lambda^2}$$

Q.202. Find the free electron concentration in ionosphere if its refractive index is equal to n = 0.90 for radio waves of frequency v = 100 MHz

Ans. From the previous problem

$$n^2 = 1 - \frac{n_0 e^2}{\varepsilon_0 m \omega^2}$$

$$= 1 - \frac{n_0 e^2}{4\pi^2 \varepsilon_0 m v^2}$$

Thus $n_0 = (4\pi^2 v^2 m \varepsilon_0/e^2)(1 - n^2) = 2{\cdot}36 \times 10^7 \text{ cm}^{-3}$

Q.203. Assuming electrons of substance to be free when subjected to hard X-rays, determine by what magnitude the refractive index of graphite differs from unity in the case of X-rays whose wavelength in vacuum is equal to λ = 50 pm.

Ans. For hard x- rays, the electrons in graphite will behave as if nearly free and the formula of previous problem can be applied. Thus

$$n^2 = 1 - \frac{n_0 e^2}{\varepsilon_0 m \omega^2}$$

and $$n = 1 - \frac{n_0 e^2}{2 \varepsilon_0 m \omega^2}$$

on taking square root and neglecting higher order terms.

So $$n - 1 = -\frac{n_0 e^2}{2 \varepsilon_0 m \omega^2} = -\frac{n_0 e^2 \lambda^2}{8 \pi^2 \varepsilon_0 m e^2}$$

We calculate n_0 as follows: There are 6 x 6 .023 x 10^{23} electrons in 12 gms of graphite of density 1.6 gm/c.c. Thus

$$n_0 = \frac{6 \times 6{\cdot}023 \times 10^{23}}{(12/1{\cdot}6)} \text{ per c.c}$$

Using the values of other constants and $\lambda = 50 \times 10^{-12}$ metre we get

$$n - 1 = -5{\cdot}4 \times 10^{-7}$$

Q.204. An electron experiences a quasi-elastic force kx and a "fric- tion force" $\gamma\dot{x}$ in the field of electromagnetic radiation. The E-component of the field varies as $E = E_0 \cos \omega t$. Neglecting the action of the magnetic component of the field, find:
(a) the motion equation of the electron;
(b) the mean power absorbed by the electron; the frequency at which that power is maximum and the expression for the maximum mean power.

Ans. (a) The equation of the electron can (under the stated conditions) be written as
$$m\ddot{x} + \gamma \dot{x} + kx = eE_0 \cos \omega t$$

To solve this equation we shall find it convenient to use complex displacements.
Consider the equation

$$m\ddot{z} + \gamma\dot{z} + kz = eE_0 e^{-i\omega t}$$

Its solution is

$$z = \frac{eE_0 e^{-i\omega t}}{-m\omega^2 - i\gamma\omega + k}$$

(we ignore transients.)

$$z = \frac{eE_0 e^{-i\omega t}}{-m\omega^2 - i\gamma\omega + k}$$

Writing $\beta = \frac{\gamma}{2m},\ \omega_0^2 = \frac{k}{m}$

we find $z = \frac{eE_0}{m} e^{-i\omega t} \Big/ (\omega_o^2 - \omega^2 - 2i\beta\omega)$

Now x = Real part of z

$$= \frac{eE_0}{m} \cdot \frac{\cos(\omega t + \varphi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} = a\cos(\omega t + \varphi)$$

where

$$\tan\varphi = \frac{2\beta\omega}{\omega^2 - \omega_0^2}$$

$$\left(\sin\varphi = -\frac{2\beta\omega}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}}\right).$$

$$P = \langle F\dot{x} \rangle = \langle eE_0 \cos\omega t(-\omega a \sin(\omega t + \varphi)) \rangle$$

$$= eE_0 \cdot \frac{eE_0}{m}\frac{1}{2} \cdot \frac{2\beta\omega}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \cdot \omega = \left(\frac{eE_0}{m}\right)^2 \frac{\beta m\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

$$\omega_0 = \omega$$

$$P = \left(\frac{eE_0}{m}\right)^2 \frac{\beta m}{\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 + 4\beta^2}$$

$$P_{max} = \frac{m}{4\beta}\left(\frac{eE_0}{m}\right)^2 \quad \text{for } \omega = \omega_0 .$$

$$P = \langle \gamma \dot{x} \cdot \dot{x} \rangle$$

$$= (\gamma \omega^2 a^2/2) = \frac{\beta m \omega^2 (eE_0/m)^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}.$$

Q.205. In some cases permittivity of substance turns out to be a complex or a negative quantity, and refractive index, respectively, a complex (n' = n + ix) or an imaginary (n' = ix) quantity. Write the equation of a plane wave for both of these cases and find out the physical meaning of such refractive indices.

Ans. Let us write the solutions of the wave equation in the form

where $k = \frac{2\pi}{\lambda}$ and λ. is the wavelength in the medium. If $n' = n + i\chi$, then

$$k = \frac{2\pi}{\lambda_0} n'$$

(λ_0 is the wavelength in vacuum) and the equation becomes

$$A = A_0 e^{\chi' x} \exp(i(\omega t_1 - k' x))$$

where $\chi' = \frac{2\pi}{\lambda_0}\chi$ and $k' = \frac{2\pi}{\lambda_0} n$

$$A = A_0 e^{\chi' x} \cos(\omega t - k' x)$$

This represents a plane wave whose amplitude diminishes as it propagates to the right (provided X' < 0).

when $n' = i\chi$, then similarly

$$A = A_0 e^{\chi' x} \cos \omega t$$

(on putting n = 0 in the above equation).

This represents a standing wave whose amplitude diminishes as one goes to the right (if X' < 0). The wavelength of the wave is infinite (k' - 0).

Waves of the former type are realized inside metals as well as inside dielectrics when there is total reflection, (penetration of wave).

Q.5.206. A sounding of dilute plasma by radio waves of various frequencies reveals that radio waves with wavelengths exceeding λ_o = 0.75 m experience total internal reflection. Find the free electron concentration in that plasma.

Ans.

$$0 = 1 - \frac{n_0 e^2}{\varepsilon_0 m \omega^2} \quad \text{if} \quad \omega = \frac{2\pi c}{\lambda_0}$$

Hence

$$\frac{n_0 e^2 \lambda_0^2}{4\pi^2 \varepsilon_0 m c^2} = 1$$

or

$$n_0 = \frac{4\pi^2 \varepsilon_0 m c^2}{e^2 \lambda_0^2} = 1{\cdot}984 \times 10^9 \text{ per c.c}$$

Q.207. Using the definition of the group velocity u, derive Rayleigh's formula (5.5d). Demonstrate that in the vicinity of λ = λ' the velocity u is equal to the segment v' cut by the tangent of the curve v (λ) at the point λ' (Fig. 5.36).

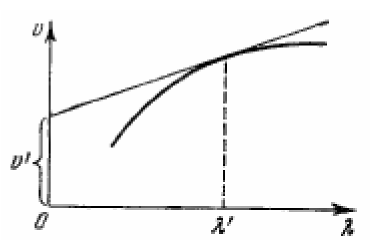


Fig. 5.36.

Ans. By definition

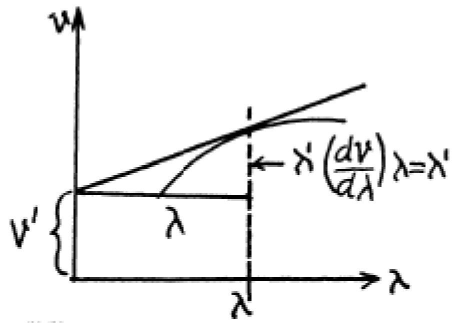
$$u = \frac{d\omega}{dk} = \frac{d}{dk}(\mathrm{v}\,k) \quad \text{as } \omega = \mathrm{v}\,k \quad = \mathrm{v} + k\frac{d\mathrm{v}}{dk}$$

Now $\quad k = \frac{2\pi}{\lambda} \quad$ so $\; dk = -\frac{2\pi}{\lambda^2} d\lambda$

Thus $\quad u = v - \lambda \frac{dv}{d\lambda}$.

Its interpretation is the following :

$\left(\frac{dv}{d\lambda}\right)_{\lambda=\lambda'}$ is the slope of the $v - \lambda$ curve at $\lambda = \lambda'$.



Thus as is obvious from the diagram $v' = v(\lambda') - \lambda' \left(\frac{dv}{d\lambda}\right)_{\lambda=\lambda'}$ is the group velocity for $\lambda = \lambda'$.

Q.208. Find the relation between the group velocity u and phase velocity v for the following dispersion laws:

(a) $v \propto 1/\sqrt{\lambda}$;
(b) $v \propto k$;
(c) $v \propto 1/\omega^2$.

Here λ, k, and ω are the wavelength, wave number, and angular frequency.

Ans.

(a) $v = a/\sqrt{\lambda}$, a = constant

Then $$u = v - \lambda \frac{dv}{d\lambda}$$
$$= \frac{a}{\sqrt{\lambda}} - \lambda\left(-\frac{1}{2} a \lambda^{-3/2}\right) = \frac{3}{2} \cdot \frac{a}{\sqrt{\lambda}} = \frac{3}{2} v.$$

(b) $v = bk = \omega k$, b = constant

so $\quad \omega = bk^2 \quad$ and $\quad u = \frac{d\omega}{dk} = 2bk = 2v$.

(c) $v = \frac{c}{\omega^2}$, c = constant $= \frac{\omega}{k}$.

so $\quad \omega^3 = ck \quad$ or $\quad \omega = c^{1/3} k^{1/3}$

Thus $$u = \frac{d\omega}{dk} = c^{1/3} \frac{1}{3} k^{-2/3} = \frac{1}{3} \frac{\omega}{k} = \frac{1}{3} v$$

Q.209. In a certain medium the relationship between the group and phase velocities of an electromagnetic wave has the form uv = c², where c is the velocity of light in vacuum. Find the dependence of permittivity of that medium on wave frequency, εω.

Ans. We have

$$u\,v = \frac{\omega}{k}\frac{d\omega}{dk} = c^2$$

Integrating we find

$$\omega^2 = A + c^2k^2, \ A \text{ is a constant.}$$

so

$$k = \frac{\sqrt{\omega^2 - A}}{c}$$

and

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{A}{\omega^2}}}$$

writing this as $c/\sqrt{\varepsilon(\omega)}$ we get $\varepsilon(\omega) = 1 - \frac{A}{\omega^2}$

(A can be +ve or negative)

Q.210. The refractive index of carbon dioxide at the wavelengths 509, 534, and 589 nm is equal to 1.647,1.640, and 1.630 respectively. Calculate the phase and group velocities of light in the vicinity of λ = 534 nm.

Ans. The phase velocity of light in the vicinity of $\lambda = 534\,nm = \lambda_0$ is obtained as

$$v(\lambda_0) = \frac{c}{n(\lambda_0)} = \frac{3 \times 10^8}{1.640} = 1.829 \times 10^8\,\text{m/s}$$

To get the group velocity we need to calculate

$\left(\frac{dn}{d\lambda}\right)_{\lambda = \lambda_0}$. We shall use linear

interpolation in the two intervals. Thus

$$\left(\frac{dn}{d\lambda}\right)_{\lambda = 521\cdot5} = -\frac{\cdot007}{25} = -28 \times 10^{-5} \text{ per nm}$$

$$\left(\frac{dn}{d\lambda}\right)_{\lambda = 561\cdot5} = -\frac{\cdot01}{55} = -18\cdot2 \times 10^{-5} \text{ per nm}$$

$(dn/d\lambda)$

$$\left(\frac{dn}{d\lambda}\right)_{\lambda = 534} = \left[-28 + \frac{9\cdot8}{40} \times 12\cdot5\right] \times 10^{-5} \text{ per mm} = -24\cdot9 \times 10^{-5} \text{ per n m}.$$

$$u = \frac{c}{n} - \lambda \frac{d}{d\lambda}\left(\frac{c}{n}\right) = \frac{c}{n}\left[1 + \frac{\lambda}{n}\left(\frac{dn}{d\lambda}\right)\right]$$

At $\lambda = 534$

$$u = \frac{3 \times 10^8}{1\cdot640}\left[1 - \frac{534}{1\cdot640} \times 24\cdot9 \times 10^{-5}\right] \text{m/s} = 1\cdot59 \times 10^8 \text{ m/s}$$

Q.211. A train of plane light waves propagates in the medium where the phase velocity v is a linear function of wavelength: v = a + bλ, where a and b are some positive constants. Demonstrate that in such a medium the shape of an arbitrary train of light waves is restored after the time interval ζ = 1/b.

Ans. We write

$$v = \frac{\omega}{k} = a + b\lambda$$

so

$$\omega = k(a + b\lambda) = 2\pi b + ak.$$

$\left(\text{since } k = \frac{2\pi}{\lambda}\right)$. Suppose a wavetrain at time $t = 0$ has the form

Then at time t it will have the form

$$F(x,t) = \int f(k) e^{ikx - i\omega t} dk$$

$$= \int f(k) e^{ikx - i(2\pi b + ak)t} = \int f(k)\, e^{ik(x - at)}\, e^{-i2\pi bt} dk$$

At $t = \frac{1}{b} = \tau$

$$F(x,\tau) = F(x - a\tau, 0)$$

Q.212. A beam of natural light of intensity I_0 falls on a system of two crossed Nicol prisms between which a tube filled with certain solution is placed in a longitudinal magnetic field of strength H. The length of the tube is l, the coefficient of linear absorption of solution is x, and the Verdet constant is V. Find the intensity of light transmitted through that system.

Ans. On passing through the first (polarizer) Nicol the intensity of light becomes $\frac{1}{2}I_0$ because one of the components has been cut off. On passing through the solution the plane of polarization of the light beam will rotate by

$$\varphi = VlH$$

and its intensity will also decrease by a factor $e^{-\chi l}$. The plane of vibraton of the light wave wll then make an angle $90° - \varphi$ with the principal direction of the analyzer Nicol. Thus by Malus' law the intensity of light coming out of the second Nicol will be

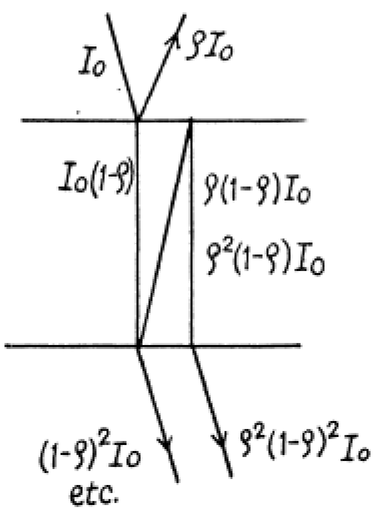
$$\frac{1}{2}I_0 \cdot e^{-\chi l} \cdot \cos^2(90° - \varphi)$$

$$= \frac{1}{2}I_0 e^{-\chi l} \sin^2 \varphi .$$

Dispersion and Absorption of Light (Part - 2)

Q.213. A plane monochromatic light wave of intensity I_0 falls normally on a plane-parallel plate both of whose surfaces have a reflection coefficient p. Taking into account multiple reflections, find the intensity of the transmitted light if
(a) the plate is perfectly transparent, i.e. the absorption is absent;
(b) the coefficient of linear absorption is equal to x, and the plate thickness is d.

Ans. (a) The multiple reflections are shown below. Transmission gives a factor (1 - p) while reflections give factors of p. Thus the transmitted intensity assuming incoherent light is



$$(1-\rho)^2 I_0 + (1-\rho)^2 \rho^2 I_0 + (1-\rho)^2 \rho^4 I_0 + \dots$$
$$= (1-\rho)^2 I_0 (1 + \rho^2 + \rho^4 + \rho^6 + \dots)$$
$$= (1-\rho)^2 I_0 \times \frac{1}{1-\rho^2} = I_0 \frac{1-\rho}{1+\rho}.$$

(b) When there is absorption, we pick up a factor $\sigma = e^{-\chi d}$ in each traversal of the plate. Thus we get

$$(1-\rho)^2 \sigma I_0 + (1-\rho)^2 \sigma^3 \rho^2 I_0 + (1-\rho)^2 \sigma^5 \rho^4 I_0 + \dots$$
$$= (1-\rho)^2 \sigma I_0 (1 + \sigma^2 \rho^2 + \sigma^4 \rho^4 + \dots)$$
$$= I_0 \frac{\sigma (1-\rho)^2}{1 - \sigma^2 \rho^2}$$

Q.214. Two plates, one of thickness d_1 =3 .8 mm and the other of thickness d_2 = 9.0 mm, are manufactured from a certain substance. When placed alternately in the way of monochromatic light, the first transmits ζ_1 = 0.84 fraction of luminous flux and the second, ζ_2 = 0.70. Find the coefficient of linear absorption of that substance. Light falls at right angles to the plates. The secondary reflections are to be neglected.

Ans. We have

$$\tau_1 = e^{-\chi d_1}(1-\rho)^2$$
$$\tau_2 = e^{-\chi d_2}(1-\rho)^2$$

where p is the reflectivity; see previous problem, multiple reflection have been ignored.

Thus $\frac{\tau_1}{\tau_2} = e^{\chi(d_2-d_1)}$

or

$$\chi = \frac{\ln\left(\frac{\tau_1}{\tau_2}\right)}{d_2-d_1} = 0.35 \text{ cm}^{-1}.$$

Q.215. A beam of monochromatic light passes through a pile of N = 5 identical plane-parallel glass plates each of thickness l = = 0.50 cm. The coefficient of reflection at each surface of the plates is p = 0.050. The ratio of the intensity of light transmitted through the pile of plates to the intensity of incident light is ζ = 0.55. Neglecting the secondary reflections of light, find the absorption coefficient of the given glass.

Ans. On each surface we pick up a factor (1 - p) from reflection and a factor $e^{-\chi l}$ due to absorption in each plate.

Thus $$\tau = (1-\rho)^{2N}e^{-\chi N l}$$

Thus $$\chi = \frac{1}{Nl}\ln\frac{(1-\rho)^{2N}}{\tau} = 0.034 \text{ cm}^{-1}.$$

Q.216. A beam of monochromatic light falls normally on the surface of a plane-parallel plate of thickness l. The absorption coefficient of the substance the plate is made of varies linearly along the normal to its surface from x_1 to x_2. The coefficient of reflection at each surface of the plate is equal to p. Neglecting the secondary reflections, find the transmission coefficient of such a plate.

Ans. Apart from the factor (1 - p) on each end face of the plate, we shall get a factor due to absorptions. This factor can be calculated by assuming the plate to consist of a large number of very thin slab within each of which the absorption coefficient can be assumed to be constant Thus we shall get a product like

$$\ldots\ldots e^{-\chi(x)dx}\, e^{-\chi(x+dx)dx}\, e^{-\chi(x+2dx)dx} \ldots\ldots$$

This product is nothing but

$$e^{-\int_0^l \chi(x)dx}$$

Now $\chi(0) = \chi_1$, $\chi(l) = \chi_2$ and variation

with x is linear so $\chi(x) = \chi_1 + \frac{x}{l}(\chi_2 - \chi_1)$

Thus the factor becomes

$$e^{-\int_0^l \left[\chi_1 + \frac{x}{l}(\chi_2 - \chi_1)\right]dx} = e^{-\frac{1}{2}(\chi_1 + \chi_2)l}$$

Q.217. A beam of light of intensity l_0 falls normally on a transparent plane-parallel plate of thickness l. The beam contains all the wavelengths in the interval from X_1 t o X_2 of equal spectral intensity. Find the intensity of the transmitted beam if in this wavelength interval the absorption coefficient is a linear function of X, with extreme values x_1 and x_2. The coefficient of reflection at each surface is equal to p. The secondary reflections are to be neglected.

Ans. The spectral density of the incident beam (i.e. intensity of the components whose wave length lies in the interval

λ & $\lambda + d\lambda$) is

$$\frac{I_0}{\lambda_2 - \lambda_1} d\lambda, \; \lambda_1 \le \lambda \le \lambda_2$$

The absorption factor for this component is

$$e^{-\left[\chi_1 + \frac{\lambda + \lambda_1}{\lambda_2 - \lambda_1}(\chi_2 - \chi_1)\right]l}$$

And the transmission factor due to reflection at the surface is $(1 - p)^2$. Thus the intensity of the transmitted beam is

$$(1-\rho)^2 \frac{I_0}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} d\lambda\, e^{-l\left[\chi_1 + \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1}(\chi_2 - \chi_1)\right]}$$

$$= (1-\rho)^2 \frac{I_0}{\lambda_2 - \lambda_1} e^{-\chi_1 l} \left(\frac{1 - e^{-(\chi_2-\chi_1)l}}{(\chi_2-\chi_1)'me} \right) \chi(\lambda_2 - \lambda_1) = (1-\rho)^2 I_0 \frac{e^{-\chi_1 l} - e^{-\chi_2 l}}{(\chi_2 - \chi_1) l}$$

Q.218. A light filter is a plate of thickness d whose absorption coefficient depends on wavelength λ as

$$\varkappa(\lambda) = \alpha(1 - \lambda/\lambda_0)^2 \text{ cm}^{-1},$$

Where α and λ_0 are constants. Find the passband Δλ of this light filter, that is the band at whose edges the attenuation of light is times that at the wavelength λ_0. The coefficient of reflection from the surfaces of the light filter is assumed to be the same at all wavelengths.

Ans. At the wavelength λ_0 the absorption coefficient vanishes and loss in transmission is entirely due to reflection. This factor is the same at all wavelengths and therefore cancels out in calculating the pass band and we need not worry about it now

$$T_0 = (\text{transmissivity at } \lambda = \lambda_0) = (1-\rho)^2$$

$$T = \text{transmissivity at } \lambda = (1-\rho)^2 e^{-\chi(\lambda) d}$$

The edges of the passband are $\lambda_0 \pm \frac{\Delta\lambda}{2}$ and at the edge

$$\frac{T}{T_0} = e^{-\alpha d \left(\frac{\Delta\lambda}{2\lambda_0}\right)^2} = \eta$$

Thus
$$\frac{\Delta\lambda}{2\lambda_0} = \sqrt{\left(\ln\frac{1}{\eta}\right) / \alpha d}$$

or
$$\Delta\lambda = 2\lambda_0 \sqrt{\frac{1}{\alpha d}\left(\ln\frac{1}{\eta}\right)}$$

Q.219. A point source of monochromatic light emitting a luminous flux Φ is positioned at the centre of a spherical layer of substance. The inside radius of the layer is a, the outside one is b. The coefficient of linear absorption of the substance is equal to x, the reflection coefficient of the surfaces is equal to p. Neglecting the secondary reflections, find the intensity of light that passes through that layer.

Ans. We have to derive the law of decrease of intensity in ah absorbing medium taking in to account the natural geometrical fall-off (inverse sequare law) as well as absorption.

Consider a thin spherical shell of thickness dx and internal radius x. Let

$I(x)$ and $I(x+dx)$ be the intensities at the inner and outer surfaces of this shell.

Then $$4\pi x^2 I(x)\, e^{-\chi dx} = 4\pi (x+dx)^2 I(x+dx)$$

Except for the factor $e^{-\chi dx}$ this is the usual equation. We rewrite this as

$$x^2 I(x) = I(x+dx)(x+dx)^2(1+\chi dx)$$

$$= \left(I + \frac{dI}{dx}dx\right)(x^2 + 2x\,dx)(1+\chi dx)$$

or $$x^2 \frac{dI}{dx} + \chi x^2 I + 2xI = 0$$

Hence $$\frac{d}{dx}(x^2 I) + \chi(x^2 I) = 0$$

so $$x^2 I = C e^{-\chi x}$$

Where C is a constant of integration.
In our case we apply this equation for $a \le x \le b$
For $x \le a$ the usual inverse square law gives

$$I(a) = \frac{\Phi}{4\pi a^2}$$

Hence $$C = \frac{\Phi}{4\pi} e^{\chi a}$$

And $$I(b) = \frac{\Phi}{4\pi b^2} e^{-\chi(b-a)}$$

This does not take into account reflections. When we do that we get

$$I(b) = \frac{\Phi}{4\pi b^2}(1-\rho)^2 e^{-\chi(b-a)}$$

Q.220. How many times will the intensity of a narrow X-ray beam of wavelength 20 pm decrease after passing through a lead plate of thickness d = 1.0 mm if the mass absorption coefficient for the given radiation wavelength is equal to μ/p = 3.6 cm^2/g?

Ans. The transmission factor is $e^{-\mu d}$ and so the intensity will decrease

$$e^{\mu d}$$

$$= e^{3\cdot 6 \times 11\cdot 3 \times 0\cdot 1} = 58\cdot 4 \text{ timestimes}$$

(we have used $\mu = (\mu/\rho) \times \rho$ and used the known value of density of lead).

Q.221. A. narrow beam of X-ray radiation of wavelength 62 pm penetrates an aluminium screen 2.6 cm thick. How thick must a lead screen be to attenuate the beam just as much? The mass absorption coefficients of aluminium and lead for this radiation are equal to 3.48 and 72.0 cm^2/g respectively.

Ans.

We require $\mu_{Pb}\, d_{Pb} = \mu_{Al}\, d_{Al}$

or
$$\left(\frac{\mu_{Pb}}{\rho_{Pb}}\right)\rho_{Pb}\, d_{Pb} = \left(\frac{\mu_{Al}}{\rho_{Al}}\right)\rho_{Al}\, d_{Al}$$
$$72{\cdot}0 \times 11{\cdot}3 \times d_{Pb} = 3{\cdot}48 \times 2{\cdot}7 \times 2{\cdot}6$$
$$d_{Pb} = 0{\cdot}3 \text{ mm}$$

Q.222. Find the thickness of aluminium layer which reduces by half the intensity of a narrow monochromatic X-ray beam if the corresponding mass absorption coefficient is $\mu/p = 0.32$ cm^2/g.

Ans.

$$\frac{1}{2} = e^{-\mu d}$$

Or
$$d = \frac{\ln 2}{\mu} = \frac{\ln 2}{\left(\frac{\mu}{\rho}\right)\rho} = 0{\cdot}80 \text{ cm}$$

Q.223. How many 50%-absorption layers are there in the plate reducing the intensity of a narrow X-ray beam η = 50 times?

Ans. We require N plates where

$$\left(\frac{1}{2}\right)^N = \frac{1}{50} \quad \text{So } N = \frac{\ln 50}{\ln 2} = 5{\cdot}6$$