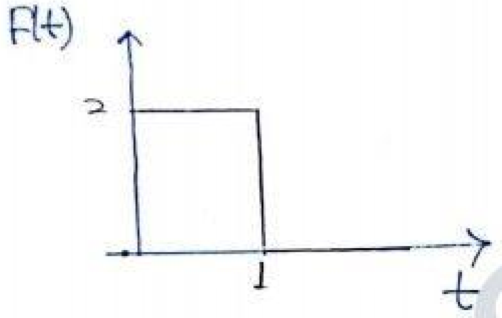
Laplace Transform :-

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

Eg. Find $L(f(t))$ where $f(t)$ is shown in figure



$$L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

$f(t) = 2 \quad t \to 0 \text{ to } 1$

$f(t) = 0 \quad t \to 1 \text{ to } \infty$

$$L(f(t)) = \int_0^1 e^{-st} f(t)\, dt + \int_1^\infty e^{-st} f(t)\, dt$$

$$L(f(t)) = \int_0^1 e^{-st} \cdot 2 \cdot + 0$$

$$L(f(t)) = 2 \cdot \left(\frac{e^{-st}}{-s} \right) \Bigg|_0^1$$

$$L(f(t)) = \frac{2}{s} \left(1 - e^{-s}\right)$$

Unit step function

The unit step function $u(t-a)$ is defined as

$$u(t-a) = \begin{cases} 0 & , \quad t < a \\ 1 & , \quad t > a \end{cases}$$

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

$$L(u(t-a)) = \int_0^a e^{-st} u(t-a)\, dt + \int_a^\infty e^{-st} u(t-a)\, dt$$

$$L(u(t-a)) = 0 + \int_a^\infty e^{-st}\, dt$$

$$L(u(t-a)) = -\frac{1}{s}\left(e^{-\infty} - e^{-as}\right)$$

$$\boxed{L(u(t-a)) = \frac{e^{-as}}{s}}$$

* If $f(t)$ is a periodic function with period T

then $$L(f(t)) = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\, dt$$

eg

The given function is periodic fun of period 2π

$$\text{if } f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

$$\text{then } L(f(t)) = \frac{1}{1-e^{-2\pi s}} \left\{ \int_0^{2\pi} e^{-st} \sin t \, dt + 0 \right\}$$

$$= \boxed{\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left(a \sin bx - b\cos bx\right)}$$

$$L(f(t)) = \frac{1}{1-e^{-2\pi s}} \left\{ \frac{e^{-st}}{s^2+1}\left(-s\sin t - \cos t\right)\Big|_0^{\pi} \right\}$$

$$L(f(t)) = \frac{1}{1-e^{-2\pi s}} \left\{ \frac{e^{-\pi s}}{s^2+1}\left(-\cancel{s}\sin\pi - \cos\pi - \frac{1}{s^2+1}(0-1)\right) \right\}$$

$$L(f(t)) = \frac{1}{1-e^{-2\pi s}} \left\{ \frac{e^{-\pi s}}{s^2+1}(0+1) - \frac{1}{s^2+1}(0-1) \right\}$$

$$L(f(t)) = \frac{1}{1-e^{-2\pi s}} \left(\frac{e^{-\pi s}+1}{s^2+1} \right)$$

* $L(1) = \dfrac{1}{s}$

* $L(t) = \dfrac{1}{s^2}$

* $L(t^n) = \dfrac{n!}{s^{n+1}}$ or $\dfrac{\Gamma(n+1)}{s^{n+1}}$

* $L(e^{at}) = \dfrac{1}{s-a}$

* $L(e^{-at}) = \dfrac{1}{s+a}$

* $L(u(t-a)) = \dfrac{e^{-as}}{s}$

* $L(\sin at) = \dfrac{a}{s^2+a^2}$ * $L(\sinh at) = \dfrac{a}{s^2-a^2}$

* $L(\cos at) = \dfrac{s}{s^2+a^2}$ * $L(\cosh at) = \dfrac{s}{s^2-a^2}$

* $L(\delta(t)) = 1$ (delta function)

* $L(\sin^2 t) = L\left(\dfrac{1-\cos 2t}{2}\right) = L\left(\dfrac{1}{2}\right) - L\left(\dfrac{\cos 2t}{2}\right)$

$= \dfrac{1}{2} L(t) - \dfrac{1}{2} L(\cos 2t) = \dfrac{1}{2}\cdot\dfrac{1}{s} - \dfrac{1}{2}\cdot\dfrac{s}{s^2+4}$

* $L(t^{-1/2}) = \dfrac{\Gamma(-1/2+1)}{s^{-1/2+1}} = \dfrac{\Gamma(1/2)}{\sqrt{s}} = \sqrt{\dfrac{\pi}{s}}$

* $L(t^{1/2}) = \dfrac{\Gamma(1/2+1)}{s^{1/2+1}} = \dfrac{\frac{1}{2}\Gamma(1/2)}{s^{3/2}} = \dfrac{\frac{1}{2}\sqrt{\pi}}{(s)^{3/2}}$

* if $L(f(t)) = F(s)$ then $L(e^{at} f(t)) = F(s-a)$

first shifting rule.

eg $L(e^{3t} t^5) = ?$

$$L(t^5) = \frac{5!}{s^{5+1}} = \frac{120}{s^6}.$$

$$L(e^{3t} t^5) = \frac{120}{(s-3)^6}$$

eg $L(e^{2t} \cos 4t) = ?$

$$L(\cos 4t) = \frac{s}{s^2+16}$$

$$L(e^{2t} \cos 4t) = \frac{(s-2)}{(s-2)^2+16}$$

eg. $L(e^{-t} \sin t) = ?$

$$L(\sin t) = \frac{1}{s^2+1}$$

$$L(e^{-t} \sin t) = \frac{1}{(s^2-1)^2+1} = \frac{1}{s^2+2s+2}$$

★ If $L(f(t)) = F(s)$ then

$$L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}(F(s))$$ Multiplication by t^n

eg

$$L(t^2 \sin 2t) = ?$$

$$L(\sin 2t) = \frac{2}{s^2+4}$$

$$L(t^2 \sin 2t) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)$$

$$= \frac{d}{ds} \cdot \frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{d}{ds}\left(2 \cdot \frac{-1}{(s^2+4)^2} \cdot 2s\right)$$

$$L(t^2 \sin 2t) = \frac{d}{ds}\left(\frac{-4s}{(s^2+4)^2}\right)$$

$$= \frac{(s^2+4)^2(-4) - (-4s)(2(s^2+4))(\cdot 2s)}{(s^2+4)^4}$$

$$L(t^2 \sin 2t) = \frac{12s^2-16}{(s^2+4)^3}$$

eg $L(t\cos 3t) = ?$

$$L(\cos 3t) = \frac{s}{s^2+9}$$

$$L(t\cos 3t) = (-1)^1 \frac{d^1}{ds^1}\left(\frac{s}{s^2+9}\right)$$

$$= -\,d\left(\frac{(s^2+9).1 - s(2s)}{(s^2+9)^2}\right)$$

$$= \frac{-(s^2+9-2s^2)}{(s^2+9)^2} = \frac{-(-s^2+9)}{(s^2+9)^2}$$

$$L(t\cos 3t) = \frac{s^2-9}{(s^2+9)^2}$$

eg The value of $\int_0^\infty e^{-4t}\, t \cos 3t = ?$

$$L(t\cos 3t) = \int_0^\infty e^{-st}\, t\cos 3t\, dt = \frac{s^2-9}{(s^2+9)^2}$$

Compare both Put $s = 4$ on both side

$$\int_0^\infty e^{-4t}\, t\cos 3t\, dt = \frac{4^2-9}{(4^2+9)^2} = \frac{7}{625}$$

* If $L(f(t)) = F(s)$ then

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)\,ds \quad \{\text{Division by } t\}$$

eg

$$L\left(\frac{\sin at}{t}\right) = ?$$

$$L(\sin at) = \frac{a}{a^2+s^2}$$

$$L\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{a^2+s^2}\,ds$$

$$= a\frac{1}{a}\tan^{-1}(s/a)\Big|_s^\infty = \tan^{-1}(\infty) - \tan^{-1}(s/a)$$

$$L\left(\frac{\sin at}{t}\right) = \frac{\pi}{2} - \tan^{-1}(s/a) = \cot^{-1}(s/a)$$

eg The value of $\int_0^\infty \frac{\sin at}{t}\,dt = ?$

$$L(f(t)) = L\left(\frac{\sin at}{t}\right) = \int_0^\infty e^{-st}\frac{\sin at}{t}\,dt = \cot^{-1}(s/a)$$

Put $s = 0$ on both side

$$\Rightarrow \int_0^\infty \frac{\sin at}{t} = \cot^{-1}(0) = \pi/2$$

eg The value of $\int_{-\infty}^{\infty} 12\cos(2\pi t)\,\frac{\sin(4\pi t)}{4\pi t}\,dt = ?$

:

$$\int_{-\infty}^{\infty} 12\cos(2\pi t)\,\frac{\sin(4\pi t)}{4\pi t}\,dt \Rightarrow \text{it is even function}$$

$$\Rightarrow\ 2\int_{0}^{\infty} 12\cos(2\pi t)\,\frac{\sin(4\pi t)}{4\pi t}\,dt$$

$$= \frac{12}{4\pi}\int_{0}^{\infty} \frac{2\sin(4\pi t)\cos(2\pi t)\,dt}{t}$$

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

$$= \frac{3}{\pi}\int_{0}^{\infty}\left(\frac{\sin(6\pi t) + \sin(2\pi t)}{t}\right)dt$$

$$= \frac{3}{\pi}\left\{\int_{0}^{\infty}\underbrace{\frac{\sin(6\pi t)}{t}\,dt}_{\cot^{-1}(0)} + \int_{0}^{\infty}\underbrace{\frac{\sin(2\pi t)}{t}\,dt}_{\cot^{-1}(0)}\right\}$$

$$= \frac{3}{\pi}\left\{\frac{\pi}{2} + \frac{\pi}{2}\right\}$$

$$= 3$$

$\underline{*}$ if $L(f(t)) = F(s)$ then

(i) $L(f'(t)) = s\,L(f(t)) - f(0)$

(ii) $L(f''(t)) = s^2 L(f(t)) - s f(0) - f'(0)$

(iii) $L(f'''(t)) = s^3 L(f(t)) - s^2 f(0) - s f'(0) - f''(0)$

<u>Solve</u>:- $y'' + 4y = 12t$, $y(0) = 0$ $y'(0) = 9$

<u>sol^n</u>

$$y'' + 4(y) = 12t \quad (1)$$

take lap. both side

$$L(y'') + L(4y) = L(12t)$$

$$s^2 L(y) - s(y(0)) - y'(0) + 4\,L(y) = 12(L(t))$$

$$s^2 L(y) - 0 - 9 + 4\,L(y) = 12\frac{1}{s^2}$$

$$(s^2+4)\,L(y) = \frac{12}{s^2} + 9$$

$$L(y) = \frac{12 + 9s^2}{s^2(s^2+4)}$$

$$L(y) = \frac{12 + 3s^2 + 6s^2}{s^2(s^2+4)}$$

$$L(y) = \frac{3(4+s^2)}{s^2(4+s^2)} + \frac{6s^2}{s^2(4+s^2)}$$

$$L(y) = \frac{3}{s^2} + \frac{6}{s^2+4}$$

(3×2)

$$y = L^{-1}\left(\frac{3}{s^2}\right) + L^{-1}\left(\frac{6}{s^2+4}\right)$$

$$y = 3t + 3\sin 2t$$

Solve $\quad y'' + 2y' + 10y = 6\,\delta(t)$

$y(0) = 0 \qquad y'(0) = 0$

take laplace

$$L(y'') + 2L(y') + 10L(y) = 6L(\delta(t))$$

$\cancel{s^2 L(y) - s\,y(0) - y'(0)}$

$$s^2 L(y) - s\,y(0) - y'(0) + 2\{sL(y) - y(0)\} + 10L(y) = 6\times 1$$

$$s^2 L(y) + 2s\,L(y) + 10(L(y)) = 6$$

$$L(y) = \frac{6}{s^2+2s+10}$$

take L^{-1} of both side.

$$y = L^{-1}\left(\frac{6}{s^2+2s+10}\right)$$

$$= L^{-1}\left(\frac{6}{s^2+2s+1+9}\right)$$

$$= L^{-1}\left(\frac{3.2}{(s+1)^2+(3)^2}\right) = 2e^{-t}\sin 3t$$

$$y = 2.L^{-1}\left(\frac{3}{(s+1)^2+(3)^2}\right)$$

$$y = 2e^{-t}\sin 3t$$

Intial Value theorem:-

If $L(f(t)) = F(s)$ then

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s.F(s)$$

Final Value theorem.

If $L(f(t)) = F(s)$ then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s.F(s)$$

Question :-

$$L\left(f(t)\right) = F(s) = \frac{5s^2+23s+6}{s(s^2+2s+2)}$$

then $\underset{t\to 0}{Lt}\ f(t) = ?$

Sol^n

$$\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ s.F(s)$$

$$= \underset{s\to\infty}{Lt}\ \frac{\cancel{s}(5s^2+23s+6)}{\cancel{s}(s^2+2s+2)}$$

$$= \underset{s\to\infty}{Lt}\ \frac{\cancel{s^2}\left(5+\frac{23}{s}+\frac{6}{s^2}\right)}{\cancel{s^2}\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}$$

$$= 5/1 = 5 \quad \underline{Ans}$$

Question $L(t) = F(s) = \frac{2}{s(1+s)}$ then $\underset{t\to\infty}{Lt}\ f(t) = ?$

Sol^n <u>F.V.T</u>

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ s.F(s)$$

$$= \underset{s\to 0}{Lt}\ \frac{s.2}{s(1+s)} = \frac{2}{1+0} = 2$$

Question $L^{-1}\left(\ln\left(\frac{s-2}{s-4}\right)\right)$

Sol^n for nonalgebric funtion use

$$L\left(t\,f(t)\right) = (-1)\frac{d}{ds}F(s)$$

$$F(s) = \ln\left(\frac{s-2}{s-4}\right)$$

$$F(s) = \ln(s-2) - \ln(s-4)$$

$$\frac{d}{ds}F(s) = \frac{1}{s-2} - \frac{1}{s-4}$$

$$(-)\frac{d}{ds}F(s) = \frac{1}{s-4} - \frac{1}{s-2}$$

$$L\left(t\,f(t)\right) = \frac{1}{s-4} - \frac{1}{s-2}$$

$$t\,f(t) = L^{-1}\left(\frac{1}{s-4}\right) - L^{-1}\left(\frac{1}{s-2}\right)$$

$$t\,f(t) = e^{4t} - e^{2t}$$

$$f(t) = \frac{e^{4t} - e^{2t}}{t}$$

Question $L^{-1}\left(\cot^{-1} s\right) = ?$

$$F(s) = \cot^{-1} s$$

$$\frac{d}{ds} F(s) = \frac{-1}{1+s^2}$$

$$(-1)\frac{d}{ds} F(s) = \frac{1}{s^2+1}$$

$$L\left(t(f(t)\right) = \frac{1}{s^2+1}$$

take L^{-1} both side

$$t\, f(t) = \sin t$$

$$f(t) = \frac{\sin t}{t}$$