Chapter 1

Relations and Functions

Relations & Its Types

Relations

Given any two non-empty sets A and B, a relation R from A to B is a subset of the Cartesian product $A \times B$ and is derived by describing a relationship between the first element (say x) and the other element (say y) of the ordered pairs in A & B.

Further, if $(x, y) \in R$, then we say that x is related to y and write this relation as x R y. Hence, $R = \{(x, y); x \in A, y \in B, x\ R\ y\}$.

Note: Ordered pairs means (x, y) and (y, x) are two different pairs.

Example: If R is a relation between two sets $A = \{1,2,3\}$ and $B = \{1,4,9\}$ defined as "square root of ". Here, 1R1, 2R4, 3R9
Then, $R = \{(1, 1), (2, 4), (3, 9)\}$.

Representation of a Relation

1. Roster form: In this form, a relation is represented by the set of all ordered pairs belonging to R. If R is a relation from set $A = \{1,2,3,4\}$ to set $B = \{1,4,9,16,25\}$ such that the second elements are square of the first elements. So, R can be written in roster form as

$R = \{(1,1),(2,4),(3,9),(4,16)\}$

2. Set-builder form: In this form the relation R from set A to B is as

$R = \{(x,y): x\in A, y\in B;$ The rule which asociates x and y$\}$

Example: $R = \{(1,1),(2,4),(3,9),(4,16)\}$ can be written in set-builder form as

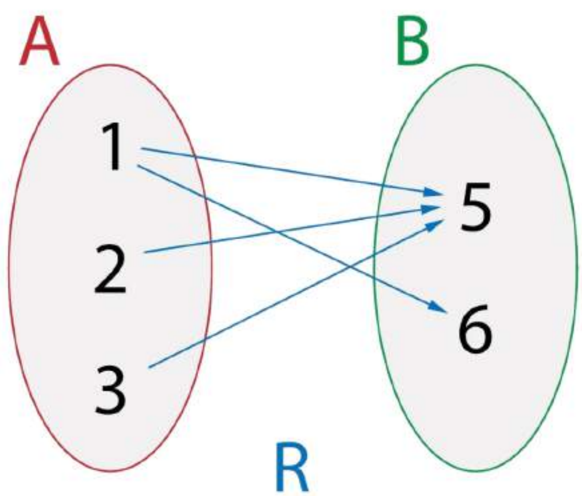
$R = \{(x,y): x\in A, y\in B; y=x^2\}$, where $A = \{1,2,3,4\}$, $B = \{1,4,9,16\}$.

3. Visual Representation (Arrow Diagram)

In this form of representation, we draw arrows from first element to the second element for all ordered pairs belonging to R.

Example: Let R = {(1,5),(1,6),(2,5),(3,5)} from set A = {1,2,3} to set B = {5,6}

It can be represented by the following arrow diagram.



Domain and Range of a Relation

1. Domain of a Relation

Let R be a relation from A to B. The domain of relation R is the set of all those elements a ∈ A such that (a,b) ∈ R for some b ∈ B.

Thus domain of R = {a ∈ A: (a,b) ∈ R for some b ∈ B}

= set of first elements of all the ordered pairs belonging to R

2. Range of a Relation

Let R be a relation from A to B. The range of R is the set of all those elements b ∈ B such that (a,b) ∈ R for some a ∈ A.

Thus range of R = {b ∈ B: (a,b) ∈ R for some a ∈ A}

= set of second elements of all the ordered pairs belonging to R.

3. Co-domain of a Relation

If R be a relation from A to B, then B is called the co-domain of relation R.

Clearly range of a relation $\subseteq$ co-domain.

Total Number of Relations

For two non-empty finite sets A and B. If the number of elements in A is h i.e., $n(A) = h$ & that of B is k i.e. $n(B) = k$, then the number of ordered pair in the Cartesian product will be $n(A \times B) = hk$. The total number of relations is 2^{hk}.

Types of Relations

1. Empty Relation

If no element of set X is related or mapped to any element of X, then the relation R in A is an empty relation, i.e, $R = \Phi$. Think of an example of set A consisting of only 100 hens in a poultry farm. Is there any possibility of finding a relation R of getting any elephant in the farm? No! R is a void or empty relation since there are only 100 hens and no elephant.

2. Universal Relation

A relation R in a set, say A is a universal relation if each element of A is related to every element of A, i.e., $R = A \times A$. Also called Full relation. Suppose A is a set of all - natural numbers and B is a set of all whole numbers. The relation between A and B is universal as every element of A is in set B. Empty relation and Universal relation are sometimes called trivial relation.

3. Identity Relation

In Identity relation, every element of set A is related to itself only. $I = \{(a, a), \in A\}$.

Example: If we throw two dice, we get 36 possible outcomes, (1, 1), (1, 2), … , (6, 6). If we define a relation as R: {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}, it is an identity relation.

4. Inverse Relation

Let R be a relation from set A to set B i.e., $R \subseteq A \times B$. The relation R^{-1} is said to be an Inverse relation if R^{-1} from set B to A is denoted by $R^{-1} = \{(b, a): (a, b) \in R\}$.

Considering the case of throwing of two dice if $R = \{(1, 2), (2, 3)\}$, $R^{-1} = \{(2, 1), (3, 2)\}$. Here, the domain of R is the range of R^{-1} and vice-versa.

5. Reflexive Relation

If every element of set A maps to itself, the relation is Reflexive Relation. For every $a \in A$, $(a, a) \in R$.

Example: Let $A = \{1, 2, 3\}$

Then $R_1 = \{(1, 1), (2, 2), (3, 3)\}$
$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$
$R_3 = \{(1, 1), (2, 2), (1, 2)\}$
Clearly ,R_1, R_2 both are reflexive relation, but R_3 is not reflexive relation because $(3, 3) \notin R_3$

Note: Every identity relation is reflexive relation,but every reflexive relation is not identity relation.

6. Symmetric Relation

A relation R on a set A is said to be symmetric if $(a, b) \in R$ then $(b, a) \in R$, a & $b \in A$.

Example: Let $A = \{1, 2, 3\}$

Then $R_1 = \{(1, 1), (2, 2), (3, 3)\}$
$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$
$R_3 = \{(1, 1), (2, 2), (1, 2)\}$
Clearly, R_1, R_2 both are symmetric relation, but R_3 is not symmetric relation because $(1, 2) \in R_3$, but $(2, 1) \notin R_3$

Note: A relation is called symmetric if $R = R^{-1}$.

7. Transitive Relation

A relation in a set A is transitive if, $(a, b) \in R$, $(b, c) \in R$, then $(a, c) \in R$, $a, b, c \in A$

Example: Let $A = \{1, 2, 3\}$
Then $R_1 = \{(1, 1), (2, 2), (3, 3)\}$
$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$
$R_3 = \{(1, 1), (2, 2), (1, 2), (2, 3)\}$
Clearly ,R_1, R_2 both are transitive relation, but R_3 is not transitive relation because$(1, 2)$ & $(2, 3) \in R_3$, but $(1, 3) \notin R_3$

Note:

Every empty relation defined on a non empty set is always symmetric and transitive but not reflexive.

Every universal set defined on non empty set is always reflexive, symmetric and transitive.

8. Equivalence Relation

A relation is said to be equivalence if and only if it is Reflexive, Symmetric, and Transitive.

Example: If we throw two dices A & B and note down all the possible outcome.

Define a relation R= {(a, b): a ∈ A, b ∈ B}, we find that {(1, 1), (2, 2), ..., (6, 6) ∈ R} (reflexive). If {(a, b) = (1, 2) ∈ R} then, {(b, a) = (2, 1) ∈ R} (symmetry). If {(a, b) = (1, 2) ∈ R} and {(b, c) = (2, 3) ∈ R} then {(a, c) = (1, 3) ∈ R} (transitive)

Solved Examples

Q.1. If A is a set of all triangles and the relation R is defined by "is congruent to" prove that R is an equivalence relation.

Ans.

(i) R is reflexives as every triangle is congruent to itself.

(ii) R is symmetric: if a triangle x is congruent to another triangle y, then the triangle y is congruent to the triangle x.

(iii) If 'a triangle x is congruent to a triangle y and y is congruent to a third triangle z, then x is also congruent to z.

Hence the relation R is transitive. (i), (ii) and (iii) ⇒ that the relation R is an equivalence relation.

Q.2. Three friends A, B, and C live near each other at a distance of 5 km from one another. We define a relation R between the distances of their houses. Is R an equivalence relation?

Ans.

- For an equivalence Relation, R must be reflexive, symmetric and transitive.
- R is not reflexive as A cannot be 5 km away to itself.

- The relation, R is symmetric as the distance between A & B is 5 km which is the same as the distance between B & A.
- R is transitive as the distance between A & B is 5 km, the distance between B & C is 5 km and the distance between A & C is also 5 km. Therefore, this relation is not equivalent.

Equivalence class

An equivalence class is subset of a given set in which each element is related with each other. Moreover equivalence class can be found if the given relation on given set is an equivalence relation.

Example: Let A = {1, 3, 5, 9, 11, 18} and a relation on A given by

R = {(a, b) | a - b is divisible by 4 } find all equivalence classes.

Solution. Clearly R ={(1, 1), (3, 3), (5, 5), (9, 9), (11, 11), (18, 18), (1, 5)(5, 1), (1, 9), (3, 11)(11, 3), (5, 9), (9, 5)}
Now $A_1 = \{1, 5, 9\}$, $A_2 = \{3, 11\}$, $A_3 = \{18\}$
Hence A_1, A_2, A_3 are called equivalence classes of a set A on the given relation R
Moreover we can represent these classes in a very specific way as below:
$A_1 = [1]$ or $[5]$ or $[9]$
$A_2 = [3]$ or $[11]$
$A_3 = [18]$
Clearly, A_1, A_2 , A_3 are disjoint and $A_1 \cup A_2 \cup A_3 = A$ and $A_1 \cap A_2 \cap A_3 = \emptyset$

Note:

- All elements of A_i are related to each other, for all i.
- No element of A_i is related to any element of A_j, $i \neq j$.
- $\cup A_j = X$ and $A_i \cap A_j = \emptyset$, $i \neq j$.

The subsets A_i are called equivalence classes and are called partitions or subdivisions of set A and these are mutually disjoint to one another.

Short Answer Type Questions

Q.1. Show that the number of equivalence relation in the set {1, 2, 3} containing (1, 2) and (2, 1) is two.

Ans. $A = \{1, 2, 3\}$

The maximum possible relation (i.e. universal relation) is
$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$
The smallest equivalence relation R_1 containing $(1, 2)$ and $(2, 1)$ is
$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$
we are left with four pairs (from universal relation) i.e. $(2, 3), (3, 2), (1, 3)$ and $(3, 1)$
If we add $(2, 3)$ to R_1, then for symmetric by we must add $(3, 2)$ and now for transitivity we are forced to add $(1, 3)$ and $(3, 1)$
Thus the only relation bigger than R_1 is universal relation i.e R
$\therefore$ The no. of equivalence relations containing $(1, 2)$ and $(2, 1)$ is two.

Q.2. If $R = \{(x, y) : x^2 + y^2 \leq 4 ; x, y \in z\}$ is a relation on z. Write the domain of R.

Ans. $R = \{(0, 1), (0, -1), (0, 2), (0, -2), (1, 1), (1, -1), (-1, 0), (-1, 1), (-1, -1), (2, 0), (-2, 0)\}$
$\therefore$ Domain of $R = \{0, 1, -1, 2, -2\}$
(i.e the first domain of each ordered pairs)

Q.3. Let $R = \{(x, y) : |x^2 - y^2| < 1\}$ be a relation on set $A = \{1, 2, 3, 4, 5\}$. Write R as a set of ordered pairs.

Ans. $A = \{1, 2, 3, 4, 5\}$
for $|x^2 - y^2| < 1$: x should be equal to y
$\therefore R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$

Q.4. R is a relation in Z defined as $(a, b) \in R \Leftrightarrow a^2 + b^2 = 25$. Find the range.

Ans. We have, $a^2 + b^2 = 25$ and $a, b \in z$
$\therefore R = \{(0, 5), (0, -5), (3, 4), (3, -4), (-3, 4), (-3, -4), (4, 3), (4, -3), (-4, 3), (-4, -3), (5, 0), (-5, 0)\}$
$\therefore$ Range $= \{-5, 5, 4, -4, 4, 3, -3, 0\}$
(i.e. second elements of each order pairs)

Real Valued Functions

FUNCTIONS

A function is a relation that maps each element x of a set A with one and only one element y of another set B. In other words, it is a relation between a set of inputs and a set of outputs in which each input is related with a unique output. A function is a rule that relates an input to exactly one output.

It is a special type of relation. A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B and no two distinct elements of B have the same mapped first element. A and B are the non-empty sets. The whole set A is the domain and the whole set B is codomain.

Definition: A function is a rule (or a set of rules) which relates or associates each and every element of a non empty set A with the unique element of the non empty set B.

REPRESENTATION

A function $f: A \rightarrow B$ is represented as $f(a) = b$ such that for $a \in A$ there is a unique element $b \in B$ such that $(a, b) \in f$.

For any function f, the notation f(a) is read as "f of a" and represents the value of b when a is replaced by the number or expression inside the parenthesis. The element b is the image of a under f and a is the pre-image of b under f.

The set 'A' is called the domain of 'f'.

The set 'B' is called the co-domain of 'f'.

Set of images (outputs) of different elements of the set A is called the range of 'f'. It is obvious that range could be a subset of the co-domain as we may have some elements in the co-domain which are not the images of any element of A (of course, these elements of the co-domain will not be included in the range). Range is also called domain of variation.

If R is the set of real numbers and A and B are subsets of R, then the function f (A) is called a real—valued function or a real function.

Domain of a function 'f' is normally represented as Domain (f). Range is represented as Range (f). Note that some times domain of the function is not explicitly defined. In these cases domain would mean the set of values of 'x' for which f (x) assumes real values that is if $y = f(x)$ then called Domain $(f) = \{x : f(x) \text{ is a real number}\}$.

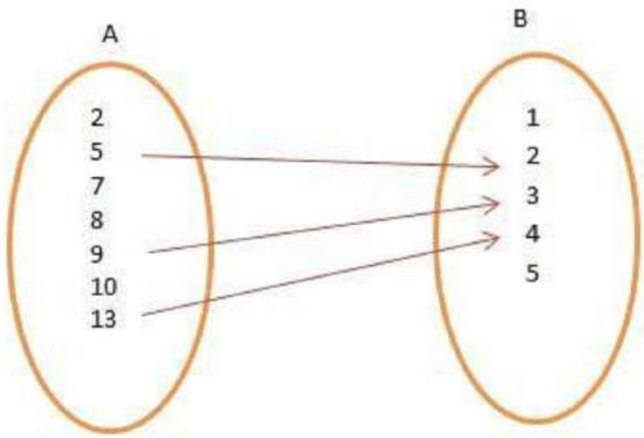
In other words, domain is defined as a set of all those values of x for which the given function is defined.

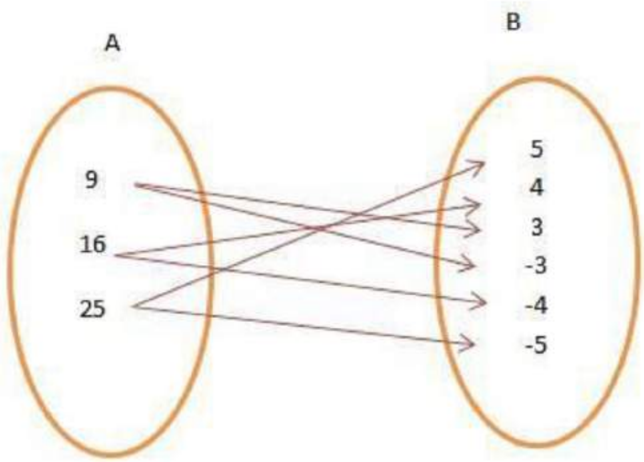
Note: Every function is a relation but every relation need not be a function.

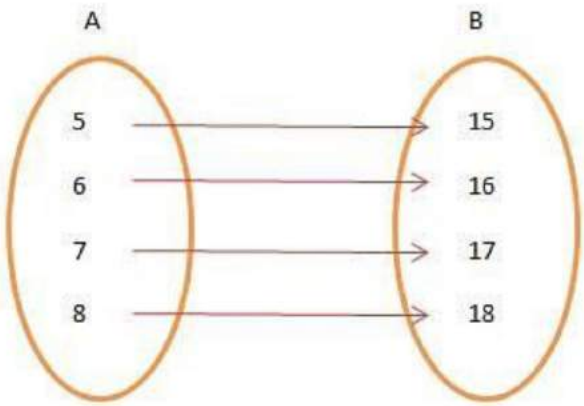
For example: Let A = {1, 2, 3}
$R_1 = \{(1, 1), (2, 3), (3, 3)\}$
$R_2 = \{(1, 1)(1, 2), (3, 2)(2, 1)\}$

Here R_1 is a function but R_2 is not a function because here element 1 has two images as 1 and 2.

Q. Which of the following is a function?







Ans. Figure 3 is an example of function since every element of A is mapped to a unique element of B and no two distinct elements of B have the same pre-image in A.

Q. Find the domain and range of the function

$$\mathbf{f(x) = \sqrt{x-5}}$$

Ans. The function

$$f(x) = \sqrt{x-5}$$

is defined for $x \geq 5$

⇒ The domain is $[5, \infty]$.

Also, for any

$$x = a \geqslant 5, f(a) = \sqrt{a-5} \geqslant 0$$

⇒ The range of the function is $[0, \infty]$.

Q. Find the domain and the range of the function y = f (x), where f (x) is given by

(i) $x^2 - 2x - 3$

(ii) $\sqrt{x^2 - 2x - 3}$

(iii) $f(x) = \begin{cases} x^2 - 1 & 0 \leqslant x < 1 \\ 0, & x = 1 \\ x^2 + 1 & x > 1 \end{cases}$

(iv) tan x

(v) $\tan^{-1} x$

(vi) $\log_{10}$ (x).

(vii) sin x

Ans.

(i) Here $y = (x - 3)(x + 1)$.
The function is defined for all real values of x
$\Rightarrow$ Its domain is R.
Also $x^2 - 2x - 3 - y = 0$ for real x
$\Rightarrow 4 + 4(3 + y) \geq 0 \Rightarrow -4 \leq y < \infty$.
Hence the range of the given function is $[-4, \infty]$.

(ii) Here
$y = \sqrt{(x-3)(x+1)}$

$\Rightarrow (x - 3)(x + 1) \geq 0$ so that $x \geq 3$ or $x \leq -1$.
Hence the domain is $R - (-1, 3)$ or $(-\infty, -1] \cup [3, \infty]$.

Since f(x) is non−negative in the domain, the range of f (x) is the interval $[0, \infty]$.

(iii) The given function is defined for all $x \geq 0$

$\Rightarrow$ The domain is $[0, \infty]$.
Moreover $-1 \leq f(x) < \infty$
$\Rightarrow$ The range is $[-1, \infty]$.

(iv) The function $f(x) = \tan x = \sin x/\cos x$ is not defined when
$\cos x = 0$, or $x = (2n + 1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots\ldots$
Hence domain of tan x is $R - \{(2n + 1)\pi/2, n = 0, \pm 1, \pm 2, \ldots\}$, and its range is R.

(v) The function $f(x) = \tan^{-1} x$, is defined for all real values of x and $-\pi/2 < \tan^{-1} <$ $\pi/2$.
Hence its domain is R and the range is $(-\pi/2, \pi/2)$.

(vi) The function $f(x) = \log_{10} x$ is defined for all $x > 0$. Hence its domain is $(0, \infty)$ and range is R.

(vii) The function $f(x) = \sin x$, (x in radians) is defined for all real values of x
$\Rightarrow$ domain of f (x) is R. Also $-1 \leq \sin x \leq 1$, for all x,
so that the range of f (x) is $[-1, 1]$.

Different types of functions and their graphical representation

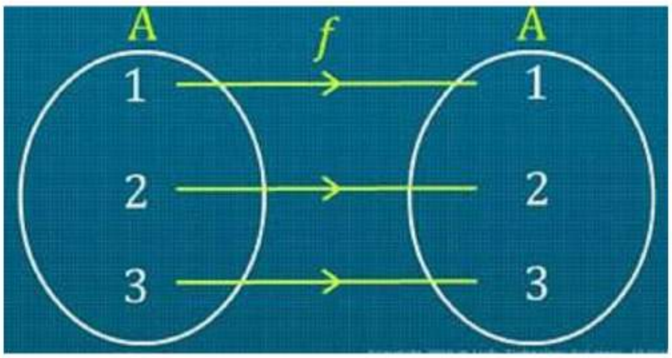
Identity function

A function f:R → R is said to be an identity function if f(x) = x, ∀ x ϵ R denoted by IR.

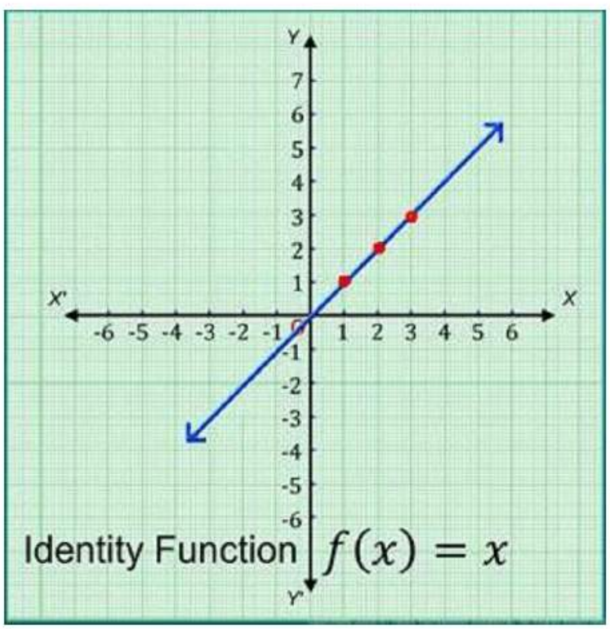
Let A = {1, 2, 3}

The function f: A → A defined by f(x) = x is an identity function.

f = {(1,1), (2,2), (3,3)}.



The graph of an identity function is a straight line passing through the origin.



Each point on this line is equidistant from the coordinate axes.

The straight line makes an angle of 45° with the coordinate axes.

Example:

Let A = {1, 2, 3, 4, 5, 6}

Then Identity function on set A will be defined as
$I_A : A \rightarrow A$, $I_A = x$, $x \in A$
for x = 1 , $I_A(1) = x = 1$
for x = 2 , $I_A(1) = x = 2$
for x = 3 , $I_A(1) = x = 3$
for x = 4 , $I_A(1) = x = 4$
for x = 5 , $I_A(1) = x = 5$
Domain, Range and co-domain will be Set A

Constant function

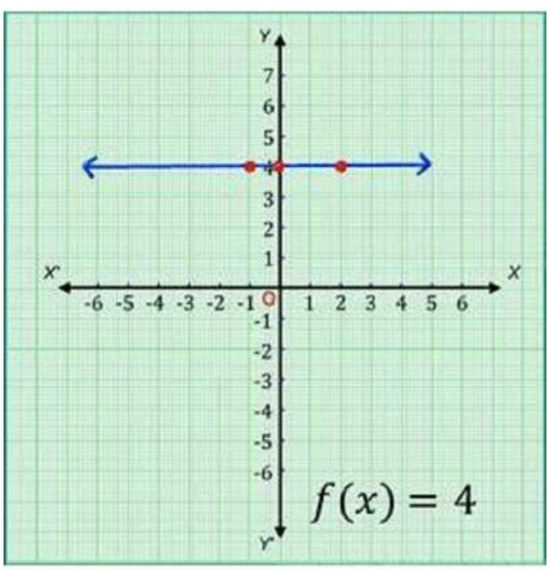
A function f:R → R is said to be a constant function, if f(x) = c , ∀ x ϵ R, where c is a constant.

Let f:R → R be a constant function defined by f(x) = 4 , ∀ x ϵ R.

The ordered pairs satisfying the linear function are: (0, 4), (-1, 4), (2, 4).

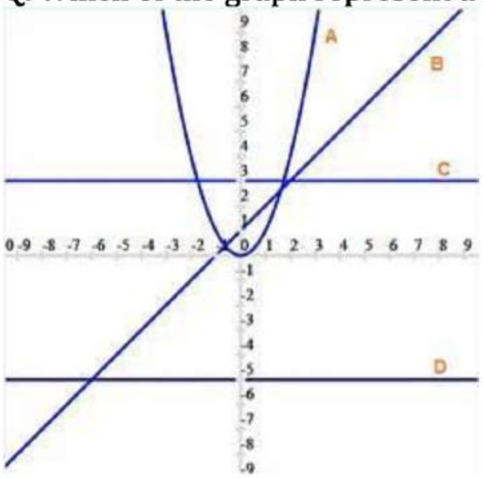
If the range of a function is a singleton set, then it is known as a constant function.

On plotting these points on the Cartesian plane and then joining them, we get the graph of the constant function f of x = 4,∀ x ϵ R as shown.



Also, from the graph, we can conclude that the graph of a constant function, $f(x) = c$, is always a straight line parallel to the X-axis, intersecting the Y-axis at (0, c).

Q. Which of the graph represent a constant function?



Ans.

The graph should be parallel to x-axis for the function to be constant function
So C and D are constant function.

Polynomial function

A polynomial function is defined by $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where n is a non-negative integer and $a_0, a_1, a_2, \ldots, n \in R$. The highest power in the expression is the degree of the polynomial function. Polynomial functions are further classified based on their degrees:
If the degree is zero, the polynomial function is a constant function (explained above).

Examples of polynomial functions

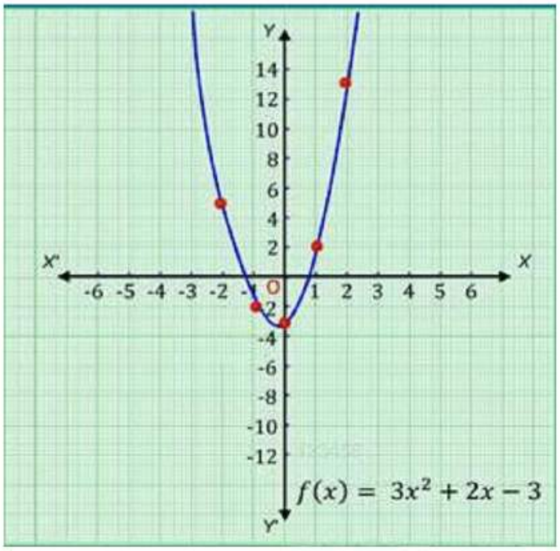
$f(x) = x^2 + 5x + 6$,$\forall x \in R$ (Degree is 2)

and $f(x) = x^3 + 4x + 2$,$\forall x \in R$ (Degree is 3)

Note: In a polynomial function, the powers of the variables should be non-negative integers.For example, $f(x) = \sqrt{x} + 2$ ($\forall x \in R$) is not a polynomial function because the power of x is a rational number.

Consider the polynomial function, $f(x) = 3x^2 +2x -3$,$\forall x \in R$The ordered pairs satisfying the polynomial function are (0, -3), (-1, -2), (1, 2), (2, 13), (-2, 5).

On plotting these points on the Cartesian plane and then joining them, we get the graph of the polynomial function f of x is equal to $3x^2 + 2x - 3$, $\forall x \in R$ as shown.



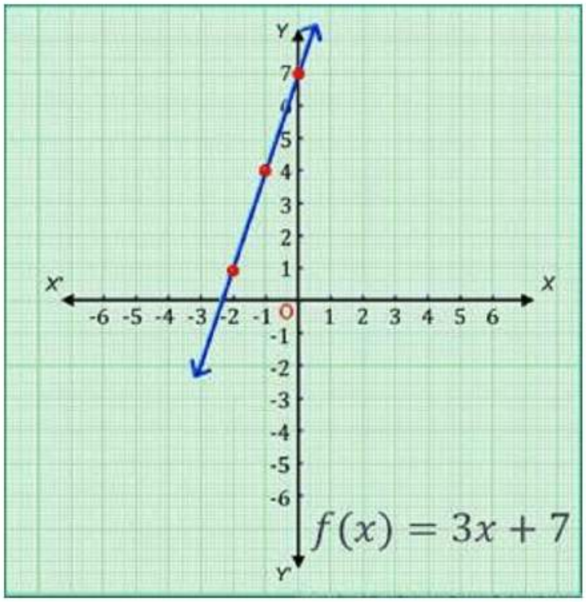
Linear function

A function $f: R \rightarrow R$ is said to be a linear function if $f(x) = ax + b$, where $a \neq 0$, a and b are real constants, and x is a real variable. It is a polynomial function with degree one.

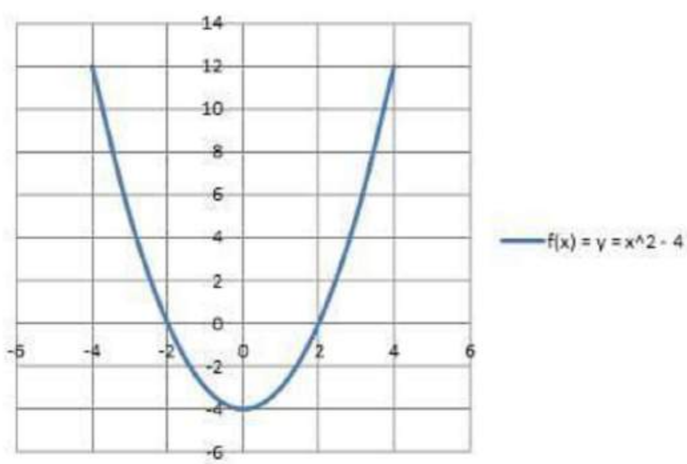
Consider the linear function, $f(x) = 3x + 7$, $\forall x \in R$

The ordered pairs satisfying the linear function are (0, 7), (-1, 4), (-2, 1).

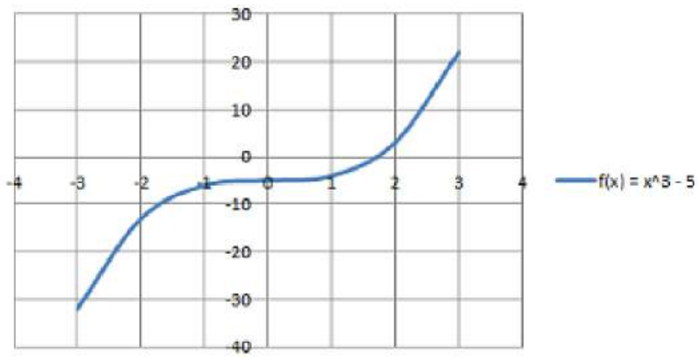
On plotting these points on the Cartesian plane and then joining them, we get the graph of the linear function f of x is equal to $3x + 7$, $\forall x \in R$ as shown in the figure.



Quadratic Function: If the degree of the polynomial function is two, then it is a quadratic function. It is expressed as $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, c are constant & x is a variable. The domain and the range are R. The graphical representation of a quadratic function say, $f(x) = x^2 - 4$ is



Cubic Function: A cubic polynomial function is a polynomial of degree three and can be denoted by $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$ and a, b, c, and d are constant & x is a variable. Graph for $f(x) = y = x^3 - 5$. The domain and the range are R.



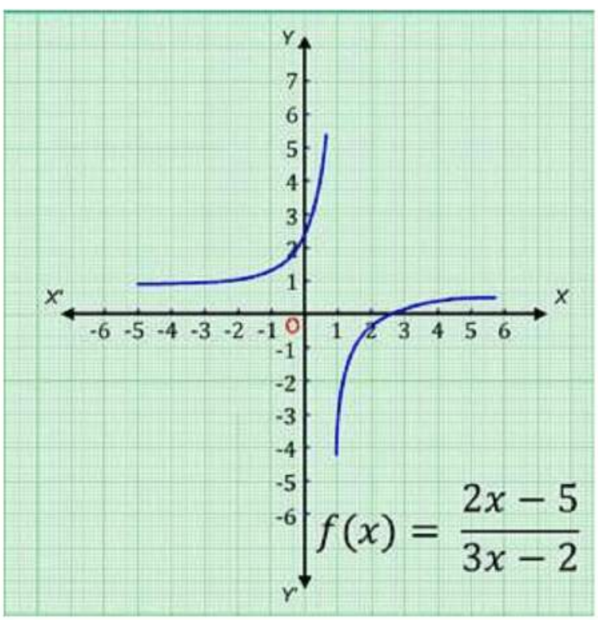
Rational function

If f(x) and g(x) be two polynomial functions, then f(x)/g(x) such that g(x) ≠ 0 and ∀ x ϵ R, is known as a rational function.

Let us consider the function f (x) = 2x -5/3x - 2 (x ≠ ⅔).

The ordered pairs satisfying the polynomial function are: (0, 5), (2, -¼), (1, -3).

On plotting these points on the Cartesian plane and then joining them, we get the graph of the given rational function as shown.



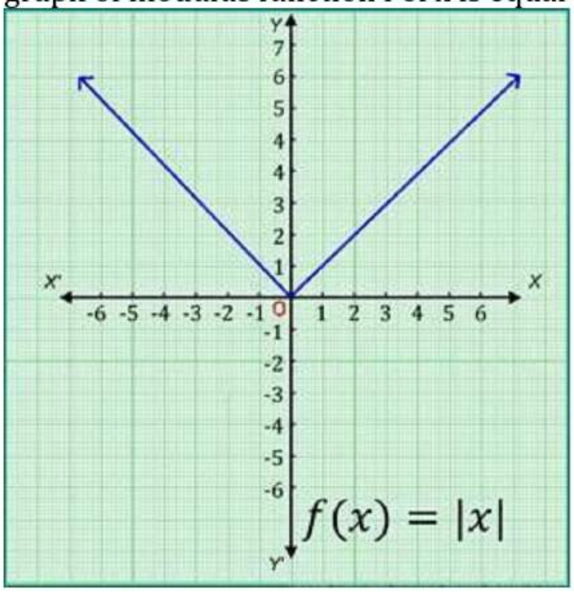
Modulus function

A function f: R → R defined by f(x) = |x| (∀ x ϵ R) is known as a modulus function.

If x is negative, then the value of the function is minus x, and if x is non-negative, then the value of the function is x. i.e. $f(x) = x$ if $x \geq 0 = - x$ if $x < 0$.

The ordered pairs satisfying the polynomial function are (0, 0), (-1, 1), (1, 1), (-3, 3), (3, 3).

On plotting these points on the Cartesian plane and then joining them, we get the graph of modulus function f of x is equal to mod of x.



Greatest integer function

A function f: R → R defined by $f(x) = [x], \forall\ x \in R$ assumes the value of the greatest integer, less than or equal to x.

From the definition of [x], we can see that

[x] = -1 for -1 £ x < 0

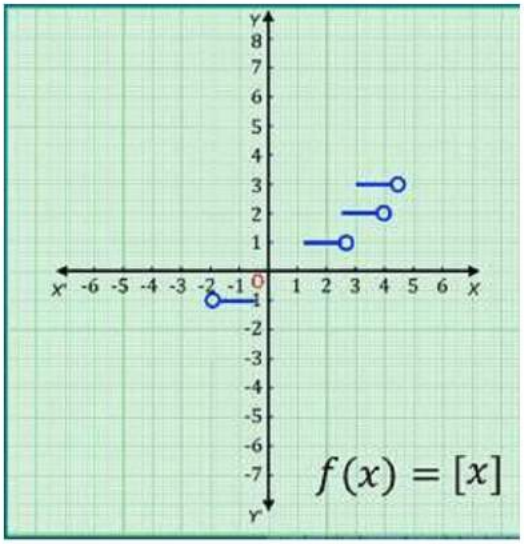
[x] = 0 for 0 £ x < 1

[x] = 1 for 1 £ x < 2

[x] = 2 for 2 £ x < 3, and so on.

⇒ f(2.5) will give the value 2 and f(1.2) will give the value 1, and so on…

Hence, the graph of the greatest integer function is as shown.

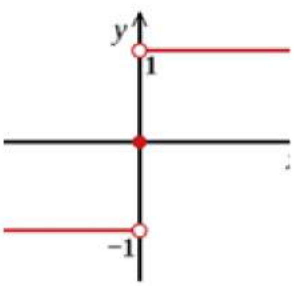


Signum Function

A function f: R→ R defined by

f(x) = { 1, if x > 0; 0, if x = 0; -1, if x < 0

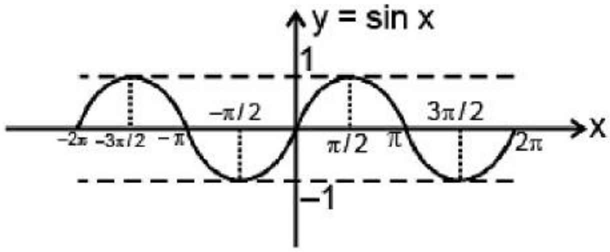
Signum or the sign function extracts the sign of the real number and is also known as step function.



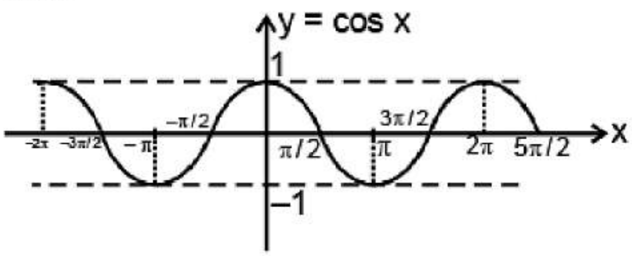
Trigonometric function

(i) Function: f(x) = sin x
Domain: x ∈ R
Range: y ∈ [−1, 1]

Curve:

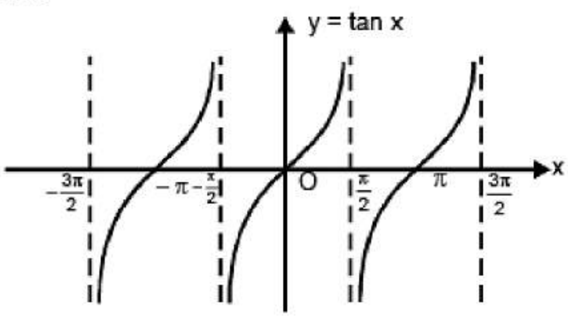


(ii) Function: f(x) = cos x
Domain: x ∈ R
Range: y ∈ [−1, 1]
Curve:



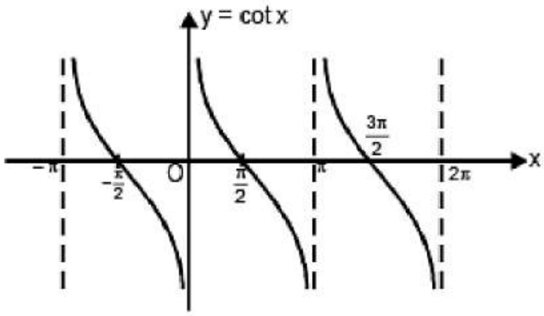
(iii) Function: f(x) = tan x
Domain: x ∈ R
Range: y ∈ R

Curve:

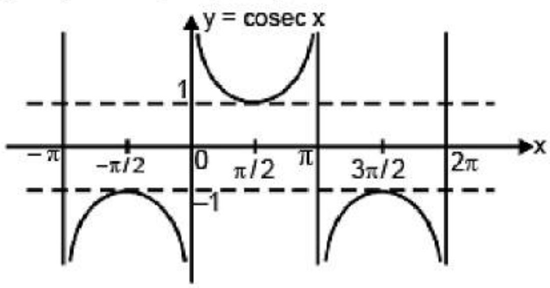


(iv) Function: f(x) = cot x
Domain: x ∈ R – (2n + 1) π / 2, n ∈ I
Range: y ∈ R

Curve:

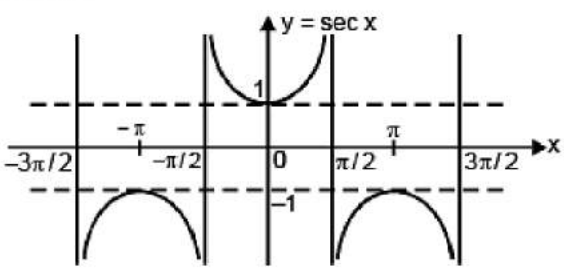


(v) Function: $f(x) = \text{cosec } x$
Domain: $x \in R - n\pi, n \in I$
Range: $y \in (-\infty, -1] \cup [1, \infty)$



Curve:

(vi) Function: $f(x) = \sec x$
Domain: $x \in R - (2n + 1)\,\pi / 2, n \in I$
Range: $y \in (-\infty, -1] \cup [1, \infty)$
Curve:



Algebraic Function

A function is called an algebraic function. If it can be constructed using algebraic operations such as additions, subtractions, multiplication, division taking roots etc. All polynomial functions are algebraic but converse is not true

Example: $f(x) = \sqrt{x^4 + 5x^2} + x + (x^3 + 5)^{3/5}$, $f(x) = x^3 + 3x^2 + x + 5$

Remark : Function which are not algebraic are called as **transcendental function**.
Example:

$$f(x) = \frac{(x^5 + 5x^2)^{3/5}}{x^3} + 3\sqrt{x^2 + 5x + 6} + \ell n\ x \rightarrow \text{transcidental function}$$

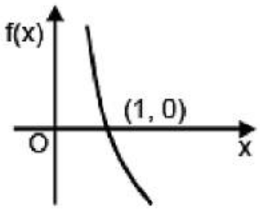
Example:

$$f(x) = \sqrt{x^2 + 7} + e^{\ell n\ x} + \frac{x + 7}{\sqrt{x^2 + 7}} \rightarrow \text{algebraic function.}$$

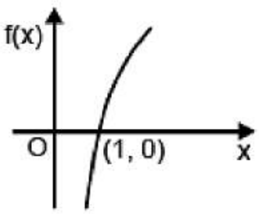
Logarithmic function

$f(x) = \log_a x$, where $x > 0$, $a > 0$, $a \neq 1$
$a \rightarrow$ base, $x \rightarrow$ number or argument of log.

Case – I : $0 < a < 1$
$f(x) = \log_a x$
Domain : $x \in (0, \infty)$
Range : $y \in R$



Case – II : $a > 1$
$f(x) = \ln x$



Exponential function

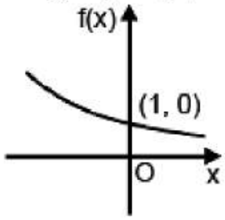
$f(x) = a^x$, where $a > 0$, $a \neq 1$

$a \rightarrow$ Base $x \rightarrow$ Exponent

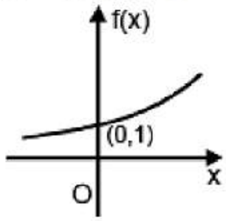
Case – I : $0 < a < 1$; $a = 1 / 2$

$$f(x) = \left(\frac{1}{2}\right)^x$$

Domain : $x \in R$
Range : $y \in (0, \infty)$

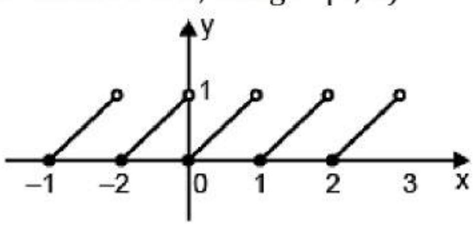


Case – II : $a > 1$



Fractional part function

$y = f(x) = \{x\} = x - [x]$
Domain : $x \in R$; Range : $[0, 1)$



<u>Example:</u>

$2.3 = 2 + 0.3 \rightarrow$ fractional part
$\downarrow$
Integer part

<u>**Properties**</u>:

(i) Fractional part of any integer is zero.
(ii) $\{x + n\} = \{x\}, n \in I$
(iii) $\{x\} + \{-x\} = \begin{cases} 0 ; & x \in I \\ 1 ; & \text{otherwise} \end{cases}$

Examples

Example.1. Find the range of the following functions:

(a) $y = \dfrac{1}{2+\sin 3x+\cos 3x}$

(b) $y = \sin^{-1}\left(\dfrac{x^2+1}{x^2+2}\right)$

Ans. (a) We have,

$y = \dfrac{1}{2+\sin 3x+\cos 3x}$ i.e. $\sin 3x + \cos 3x = \dfrac{1}{y} - 2$

i.e. $\sqrt{2}\ \sin\left(3x+\dfrac{\pi}{4}\right) = \dfrac{1}{y} - 2$

i.e. $\sin\left(3x+\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}\left(\dfrac{1}{y}-2\right)$

since, $\left|\sin\left(3x+\dfrac{\pi}{4}\right)\right| \le 1,$

therefore we have $\left|\dfrac{1}{y}-2\right| \le \sqrt{2}$

i.e. $-\sqrt{2} \le \dfrac{1}{y} - 2 \le \sqrt{2}$

i.e. $2 - \sqrt{2} \le \dfrac{1}{y} \le 2\ \sqrt{2}$

i.e. $\dfrac{1}{2+\sqrt{2}} \le y \le \dfrac{1}{2-\sqrt{2}}$

Hence, the range is $y \in \left[\dfrac{1}{2+\sqrt{2}}, \dfrac{1}{2-\sqrt{2}}\right]$

(b) We have,

$\dfrac{x^2+1}{x^2+2} = 1 - \dfrac{1}{x^2+2}$

Now, we have $2 \le x^2 + 2 < \infty$ i.e. $\frac{1}{2} \ge \frac{1}{x^2+2} > 0$

i.e. $\frac{-1}{2} \le \frac{-1}{x^2+2} < 0$

i.e. $1 - \frac{1}{2} \le 1 - \frac{1}{x^2+2} < 1$

i.e. $\frac{1}{2} \le \frac{x^2+1}{x^2+2} < 1$

i.e. $\sin^{-1}\frac{1}{2} \le \sin^{-1}\left(\frac{x^2+1}{x^2+2}\right) < \sin^{-1} 1$

gives $\frac{\pi}{6} \le y < \frac{\pi}{2}$

Hence, the range is $y \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right)$

Example.2. Find the range of following functions :

(i) $y = \ln(2x - x^2)$
(ii) $y = \sec^{-1}(x^2 + 3x + 1)$

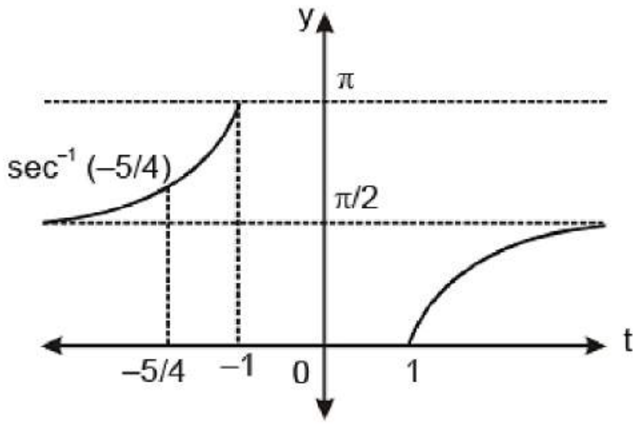
Solution. (i) using maxima–minima, we have $(2x - x^2) \in (-\infty, 1]$
For log to be defined accepted values are $2x - x2 \in (0, 1]$ {i.e. domain $(0, 1]$}
$\ln(2x - x^2) \in (0, 1]$ $\therefore$ range is $(-\infty, 0]$

(ii) $y = \sec^{-1}(x^2 + 3x + 1)$

Let $t = x^2 + 3x + 1$ for $x \in R$ then $t \in \left[-\frac{5}{4}, \infty\right)$

but $y = \sec^{-1}(t)$ $\Rightarrow \ t \in \left[-\frac{5}{4}, -1\right] \cup [1, \infty)$

from graph range is $y \in \left[0, \frac{\pi}{2}\right) \cup \left[\sec^{-1}\left(-\frac{5}{4}\right), \pi\right]$



Example.3. Find the range of $y = \left[\ln(\sin^{-1}\sqrt{x^2+x+1})\right]$

Solution. We have,

$x^2 + x + 1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$ which is a positive quantity whose minimum value is 3 / 4.

Also, for the function $y = \left[\ln(\sin^{-1}\sqrt{x^2+x+1})\right]$ to be defined, we have $x^2 + x + 1 \leq 1$

Thus, we have

$\frac{3}{4} \leq x^2 + x + 1 \leq 1$ i.e. $\frac{\sqrt{3}}{2} \leq \sqrt{x^2+x+1} \leq$ i.e. $\frac{\pi}{3} \leq \sin^{-1}(\sqrt{x^2+x+1}) \leq \frac{\pi}{2}$

[∴ $\sin^{-1}$ x is an increasing function, the inequality sign remains same]

i.e. $\ln\left(\frac{\pi}{3}\right) \leq \ln(\sin^{-1}\sqrt{x^2+x+1} \leq \ln\left(\frac{\pi}{2}\right)$

i.e. $0.046 \leq \ln(\sin^{-1}\sqrt{x^2+x+1})] = 0.$

Hence, the range is $y \in [\ln \pi / 3, \ln \pi / 2]$

Example.4. $f : R \to R,\ f(x) = \frac{3x^2+mx+n}{x^2+1}$ **.If the range of this function is [– 4, 3) then find the value of ($m^2 + n^2$).**

Solution.

$$f(x) = \frac{3(x^2+1)+mx+n-3}{1+x^2}; \; f(x) = 3 + \frac{mx+n-3}{1+x^2}$$

$$y = 3 + \frac{mx+n-3}{1+x^2}$$ for y to lie in $[-4, 3)$ $mx + n - 3 < 0 \; \forall \, x \in R$

this is possible only if $m = 0$ when, $m = 0$ then $y = 3 + \frac{n-3}{1+x^2}$

note that $n - 3 < 0$ (think !) $n < 3$ if $x \to \infty$, $y_{max} \to 3^-$

now y_{min} occurs at $x = 0$ (as $1 + x^2$ is minimum)

$y_{min} = 3 + n - 3 = n \Rightarrow n = -4$ so $m^2 + n^2 = 16$

Example.5. Find the domain and range of f(x) = $\sin\left\{\ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right\}$

Solution. $\sqrt{4-x^2}$ is positive and $x^2 < 4 \Rightarrow -2 < x < 2$

$1 - x$ should also be positive. $\therefore x < 1$

Thus the domain of $\ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)$ is $-2 < x < 1$ sine being defined for all values, the domain of $\sin\left\{\ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right\}$ is the same as the domain of $\ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)$

To study the range. Consider the function $\frac{\sqrt{4-x^2}}{1-x}$

As x varies from -2 to 1, $\frac{\sqrt{4-x^2}}{1-x}$ varies in the open interval $(0, \infty)$ and hence

$\ell n \frac{\sqrt{4-x^2}}{1-x}$ varies from $-\infty$ to $+\infty$. Therefore the range of sin $\left(\ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right)$ is $(-1, +1)$

Example.6. Find the range of the function f(x) = $\sin^{-1} \frac{\sqrt{1+x^4}}{1+5x^{10}}$.

Solution. Consider $g(x) = \frac{\sqrt{1+x^4}}{1+5x^{10}}$. Also g(x) is positive $\forall$ x $\in$ R and g(x) is continuous $\forall$ x $\in$ R and g(0) = 1 and $\lim_{x\to\infty} g(x) = 0$

$\Rightarrow$ g(x) can take all values from (0, 1] $\Rightarrow$ Range of f(x) = sin−1 (g(x)) is (0, π / 2]

Example.7. $f(x) = \cos^{-1} \{\log [\sqrt{[x^3+1]}]\}$, **find the domain and range of f(x) (where [*] denotes the greatest integer function).**

Solution. If $\cos^{-1} x = \theta$, then $-1 \le x \le 1$

$\therefore -1 \le \log [\sqrt{[x^3+1]}] \le 1$

$\Rightarrow e^{-1} \le [\sqrt{[x^3+1]}] \le e$

$0.37 \le [\sqrt{[x^3+1]}] \le 2.7$

$\therefore 1 \le \sqrt{[x^3+1]} < 3 \Rightarrow 1 \le [x^3 + 1] < 9$

$1 \le [x^3] + 1 < 9$ $0 \le [x^3] < 8 \therefore 0 \le x < 2$

$\therefore$ Domain of f(x) = D_r in x $\in$ [0, 2) Range of f(x) When $0 \le x < 2$

Then $1 \le x^3 + 1 < 9 \therefore 1 \le [x^3 + 1] \le 8$

$\Rightarrow 1 \le \sqrt{[x^3+1]} \le 2\sqrt{2} \Rightarrow 1 \le \sqrt{[x^3+1]} \le 2.8$

<u>Case I:</u>

$1 \le \sqrt{[x^3+1]} < 2$ then $[\sqrt{[x^3+1]}] = 1$

$\therefore$ Range in $\cos^{-1}$ {log 1} and $\cos^{-1}$ {log 2}

<u>Case II</u>

$2 \le \sqrt{[x^3+1]} \le 2.8$ then $[\sqrt{[x^3+1]}] = 2$

$\therefore R_f$ is $(\pi / 2, \cos^{-1}(\log 2))$

Example.8. Find the range of the following functions

(i) $f(x) = \log_e (\sin x^{\sin x} + 1)$ where $0 < x < \pi / 2$.
(ii) $f(x) = \log_e (2 \sin x + \tan x - 3x + 1)$ where $\pi / 6 \le x \le \pi /3$

Solution. (i) $0 < x < \pi / 2 \Rightarrow 0 < \sin x < 1$

Range of $\log_e (\sin x^{\sin x} + 1)$ for $0 < x < \pi / 2$ = Range of $\log_e (x^x + 1)$ for $0 < x < 1$

Let $h(x) = x^x + 1 = e^{x\log_e x} + 1$

$h'(x) = e^{x\log_e x} (1 + \log_e x) \Rightarrow h'(x) > 0$ for $x > 1/e$ and $h'(x) < 0$ for $x < 1/e$

$\therefore h(x)$ has a minima at $x = 1/e$

Also $\lim_{x\to 0^+} h(x) = 1 + e^{\lim_{x\to 0}\left(\frac{\ln x}{1/x}\right)} = 1 + e^{\lim_{x\to 0}\left(\frac{1/x}{-1/x^2}\right)}$

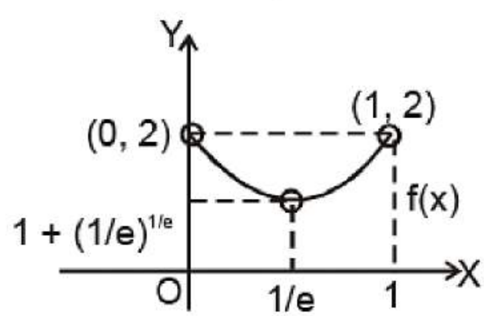
$= 1 + e^0 = 2$ and $\lim_{x\to 1^-} h(x) = 2$

$\therefore 0 < x < 1$

$\Rightarrow 1 + \left(\frac{1}{e}\right)^{\frac{1}{e}} < (x^x + 1) < 2$

$\Rightarrow \log_e \left(1 + \left(\frac{1}{e}\right)^{\frac{1}{e}}\right) < \log_e (x^x + 1) < \log_e 2$

$\therefore$ Range of $f(x) = \left(\log_e\left(1 + e^{\frac{-1}{e}}\right), \log_e 2\right)$



(ii) Let $h(x) = (2\sin x + \tan x - 3x + 1) \Rightarrow h'(x) = (2\cos x + \sec^2 x - 3)$

$$= \frac{2\cos^3 x - 3\cos^2 x + 1}{\cos^2 x}$$

$\therefore h'(x) > 0 \Rightarrow 2\cos^3 x - 3\cos^2 x + 1 > 0$

$$(\cos x - 1)^2\left(\cos x + \frac{1}{2}\right) > 0 \ \forall\ x \in [\pi/6, \pi/3]$$

$\Rightarrow h(x)$ is an increasing function of x

$\Rightarrow h(\pi / 6) \le h(x) \le h(\pi / 3)$

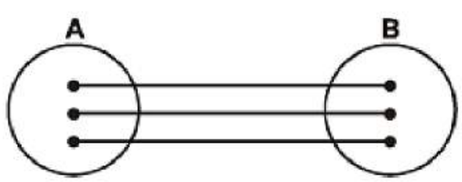
$$\Rightarrow \log_e\left(2 + \frac{1}{\sqrt{3}} - \frac{\pi}{2}\right) \le \log_e h(x) \le \log_e (1 + 2\sqrt{3} - \pi)$$

$$\therefore \text{Range of } f(x) \text{ is } \left[\log_e\left(2 + \frac{1}{\sqrt{3}} - \frac{\pi}{2}\right), \log(1 + 2\sqrt{3} - \pi)\right]$$

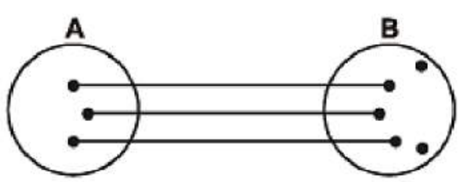
Classification of Functions

1. One-One Function (Injective mapping): A function $f: A \rightarrow B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Diagrammatically an injective mapping can be shown as



OR



Remark:

(i) Any function which is entirely increasing or decreasing in its domain, is one-one.
(ii) If any line parallel to the x-axis cuts the graph of the function at most at one point, then the function is one-one.

Q.1. Determine if the function given below is one to one.

(i) To each state of India assign its Capital

Ans. This is not one to one function because each state of India has not different capital.

(ii) Function = {(2, 4), (3, 6), (-1, -7)}

Ans. The above function is one to one because each value of range has different value of domain.

(iii) $f(x) = |x|$

Ans. Here to check whether the given function is one to one or not, we will consider some values of x (domain) and from the given function find the value of range(y).

1	1
-1	1
-2	2
3	3
-3	3

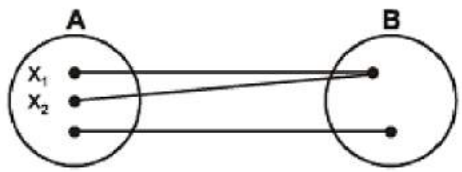
From the above table we can see that an element in the range repeats, then this is not a one to one function.

Q.2. Without using graph prove that the function
$F : R \rightarrow R$ defiend by $f(x) = 4 + 3x$ is one-to-one

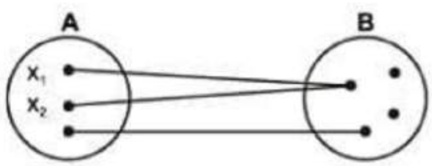
Ans. For a function to be one-one function if
$F(x_1) = F(x_1) \Rightarrow x_1 = x_2 \ \forall \ x_1, x_2 \in$ domain
$\therefore$ Now $f(x_1) = f(x_2)$ gives
$4 + 3x_1 = 4 + 3x_2$ or $x_1 = x_2$
$\therefore$ F is a one-one function.

(2) Many-One function: A function f: A → B is said to be a many one function if two or more elements of A have the same f image in B. Thus f: A → B is many one if for $x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$

Diagrammatically a many one mapping can be shown as



OR



Remark:

(i) A continuous function f(x) which has at least one local maximum or local minimum, is many-one. In other words, if a line parallel to x-axis cuts the graph of the function at least at two points, then f is many-one.

(ii) If a function is one-one, it cannot be many-one and vice versa.

(iii) If f and g both are one-one, then fog and gof would also be one-one (if they exist).

Q.3. Without using graph check the following function f: R → R defined by f (x) = x^2 is one-to-one or not ?

Ans. For the function to be one to one , different elements must have different images
But here if x = 1 then $f(1) = (1)^2 = 1$
And if x = -1 then $f(-1) = (-1)^2 = 1$
Clearly, x = 1 and x = -1 both have same image
So given function is not one to one i.e many to one function
In other words, we can say that a function which is not one to one that can be known as many to one.

Example 16. Show that the function $f(x) = (x^2 - 8x + 18)/(x^2 + 4x + 30)$ is not one-one.

Solution. Test for one-one function

A function is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Now $f(x_1) = f(x_2) \Rightarrow \dfrac{x_1^2 - 8x_1 + 18}{x_1^2 + 4x_1 + 30} = \dfrac{x_2^2 - 8x_2 + 18}{x_2^2 + 4x_2 + 30}$

$\Rightarrow 12x_1^2x_2 - 12x_1x_2^2 + 12x_1^2 - 12x_2^2 - 312x_1 + 312x_2 = 0$

$\Rightarrow (x_1 - x_2)\{12x_1x_2 + 12(x_1 + x_2) - 312\} = 0$

$\Rightarrow \quad x_1 = x_2 \text{ or } x_1 = \dfrac{26 - x_2}{1 + x_2}$

Since $f(x_1) = f(x_2)$ does not imply $x_1 = x_2$ alone, $f(x)$ is not a one-one function.

Example 17. Let f be an injective function such that $f(x)\,f(y) + 2 = f(x) + f(y) + f(xy)\ \forall\ x, y \in R$.
If $f(4) = 65$ and $f(0) \neq 2$, then show that $f(x) - 1 = x^3$ for $x \in R$

Solution. Given that $f(x)\,f(y) + 2 = f(x) + f(y) + f(xy)$ ....(i)
Putting $x = y = 0$ in equation (i), we get $f(0)\,f(0) + 2 = f(0) + f(0) + f(0)$
or $(f(0))^2 + 2 = 3f(0)$ or $(f(0) - 2)\,(f(0) - 1) = 0$ or $f(0) = 1$ $(\because f(0) \neq 2)$ ....(ii)
Again putting $x = y = 1$ in equation (i) and repeating the above steps, we get $(f(1) - 2)\,(f(1) - 1) = 0$
But $f(1) \neq 1$ as $f(x)$ is injective.
$\therefore f(1) = 2$ ...(iii)
Now putting $y = 1/x$ in equation (i), we get

$$f(x)\,f\left(\frac{1}{x}\right) + 2 = f(x) + f\left(\frac{1}{x}\right) + f(1)$$

or $$f(x)\,f\left(\frac{1}{x}\right) + 2 = f(x) + f\left(\frac{1}{x}\right) + 2$$

or $$f(x)\,f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

or $$f(x)\,f\left(\frac{1}{x}\right) - f(x) - f\left(\frac{1}{x}\right) - 1 + 1 = 0$$

or $\quad f(x)\left\{f\left(\frac{1}{x}\right)-1\right\}-1.\left\{f\left(\frac{1}{x}\right)-1\right\}=1$

or $\quad \{f(x)-1\}\left\{f\left(\frac{1}{x}\right)-1\right\}=1 \qquad \ldots(iv)$

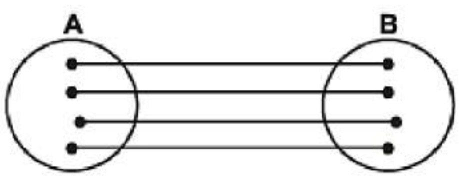
Let $f(x) - 1 = g(x)$

$\Rightarrow \quad f\left(\frac{1}{x}\right)-1=g\left(\frac{1}{x}\right)$

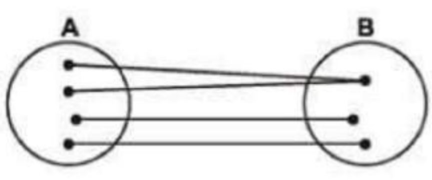
from equation (iv), we get $g(x)\ g(1/x) = 1$ which is only possible when
$g(x) = \pm x^n$
$f(x) = \pm x^n + 1$
or $f(x) = \pm x^n + 1$ or $65 = \pm 4^n + 1$
or $4n = 64 = (4)^3$
$n = 3$
$f(x) = x^3 + 1$ or $f(x) - 1 = x^3$ (neglecting negative sign)

3. Onto-function (Surjective mapping): If the function $f: A \rightarrow B$ is such that each element in B (co-domain) is the f image of at least one element in A, then we say that f is a function of A 'onto' B. Thus $f: A \rightarrow B$ is surjective if $\forall\ b \in B, \exists$ some $a \in$ such that $f(a) = b$.

Diagramatically surjective mapping can be shown as



OR



Note that: if range $\equiv$ co-domain, then $f(x)$ is onto.

Q.4. If f: R → R is defined as f (x)= 3x + 7, x ∈ R, then show that f is an onto function.

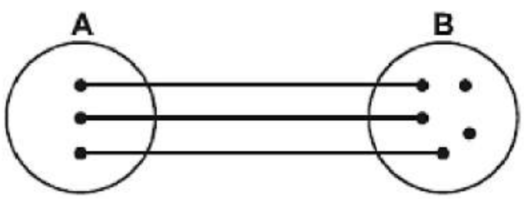
Ans. We have f (x) = 3x + 7, x ∈ R . Let b ∈ R so that f (x) = b ⇒ 3x + 7 = b or x = (b - 7)/3. Since b ∈ R, (b - 7)/3 ∈ R.

Also $f\left(\frac{b-7}{3}\right) = 3\left(\frac{b-7}{3}\right) + 7 = b$

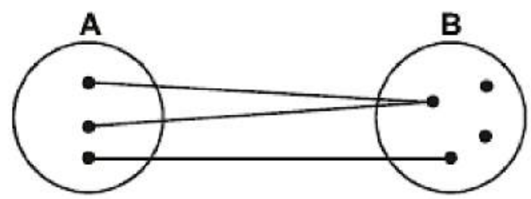
⇒ b is image of (b - 7)/3 where b is arbitrary.
Hence f (x) is an onto function.

4. Into function: If f: A → B is such that there exists at least one element in co-domain which is not the image of any element in the domain, then f(x) is into.

Diagramatically into function can be shown as



OR



Remark:

(i) If a function is onto, it cannot be into and vice versa.

(ii) If f and g are both onto, then gof or fog would be onto (if exists).
Thus a function can be one of these four types:

(a) One-one onto (injective & surjective)

(b) One-one into (injective but not surjective)

(c) Many-one onto (surjective but not injective)

(d) many-one into (neither surjective nor injective)

Remark:

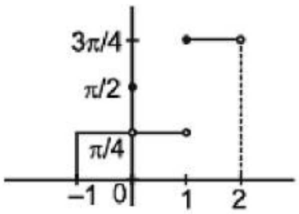
(i) If f is both injective & surjective, then it is called a **Bijective** function. Bijective functions are also named as invertible, non singular or biuniform functions.
(ii) If a set A contains n distinct elements then the number of different functions defined from $A \to A$ is n^n & out of it n ! are one-one.
(iii) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

Q. Prove that
$F : R \to R$ defined by $f(x) = 4x^3 - 5$ is a bijection

Ans. Now $f(x_1) = f(_2)\ \forall\ x_1, x_2$ Domain
$\therefore 4x_1^3 - 5 = 4x_2^3 - 5$
$\Rightarrow x_1^3 = x_2^3$
$\Rightarrow x_1^3 - x_2^3 = 0 \Rightarrow (x_2 - x_1)(x_1^2 + x_1x_2 + x_2^2) = 0$
$\Rightarrow x_1 = x_2$ or
$x_1^2 + x_1x_2 + x_2^2 = 0$ (rejected).
It has no real value of x_1 and x_2.
$\therefore$ F is a one-one function.
Again let $y = (x)$ where $y \in$ codomain, $x \in$ domain.
We have $y = 4x^3 - 5$
or
$$x = \left(\frac{y+5}{4}\right)^{1/3}$$
$\therefore$ For each $y \in$ codomain $\exists\ x \in$ domain such that $f(x) = y$.
Thus F is onto function.
$\therefore$ F is a bijection.

Example 18. A function is defined as , $f: D \to R$ $f(x) = \cot^{-1}(\text{sgn } x) + \sin^{-1}(x - \{x\})$ (where {x} denotes the fractional part function) Find the largest domain and range of the function. State with reasons whether the function is injective or not. Also draw the graph of the function.

Solution. f is many one



D [-1, 2), R = $\left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$

Example 19. Find the linear function(s) which map the interval [0 , 2] onto [1 , 4].

Solution. Let $f(x) = ax + b$
$f(0) = 1$ & $f(2) = 4 \Rightarrow$ $b = 1$ & $a = 3/2$
or $f(0) = 4$ & $f(2) = 1 \Rightarrow$ $b = 4$ & $a = -3/2$
$f(x) = 3x/2 + 1$ or $f(x) = 4 - 3x/2$

Example 20.

(i) Find whether $f(x) = x + \cos x$ is one-one.
(ii) Identify whether the function $f(x) = -x^3 + 3x^2 - 2x + 4$; $R \to R$ is ONTO or INTO
(iii) $f(x) = x^2 - 2x + 3$; $[0, 3] \to A$. Find whether f(x) is injective or not.

Also find the set A, if f(x) is surjective.

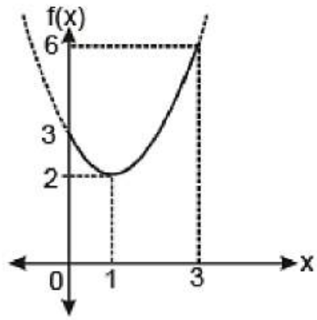
Solution.

(i) The domain of f(x) is R. $f'(x) = 1 - \sin x$.
$f'(x) \geq 0$ for all $x \in$ complete domain and equality holds at discrete points only
$\therefore$ f(x) is strictly increasing on R. Hence f(x) is one-one

(ii) As codomain ≡ range, therefore given function is ONTO

(iii) $f'(x) = 2(x - 1)$; $0 \leq x \leq 3$

$$\therefore f'(x) = \begin{cases} -ve & ; \ 0 \leq x < 1 \\ +ve & ; \ 1 < x < 3 \end{cases}$$



f(x) is a non monotonic continuous function. Hence it is not injective.
For f(x) to be surjective, A should be equal to its range. From graph, range is [2, 6]
$\therefore A \equiv [2, 6]$

Example 21. If f and g be two linear functions from [-1, 1] onto [0, 2] and $\phi : R^+ - \{-1, 1\} \to R$ be defined by

$$\phi(x) = \frac{f(x)}{g(x)}, \text{ then show that } \left|\phi(\phi(x)) + \phi\left(\phi\left(\frac{1}{2}\right)\right)\right| \geq 2.$$

Solution. Let h be a linear function from [-1, 1] onto [0, 2].
Let $h(x) = ax + b$, then $h'(x) = a$
If $a > 0$, then $h(x)$ is an increasing function & $h(-1) = 0$ and $h(1) = 2 \Rightarrow -a + b = 0$ and $a + b = 2 \Rightarrow a = 1$ & $b = 1$.
Hence $h(x) = x + 1$.
If $a < 0$, then $h(x)$ is a decreasing function & $h(-1) = 2$ and $h(1) = 0 \Rightarrow -a + b = 2$ and $a + b = 0 \Rightarrow a = -1$ & $b = 1$.
Hence $h(x) = 1 - x$
Now according to the question $f(x) = 1 + x$ & $g(x) = 1 - x$
or $f(x) = 1 - x$ & $g(x) = 1 + x$

$$\phi(x) = \frac{f(x)}{g(x)} = \frac{1-x}{1+x} \text{ or } \frac{1+x}{1-x}$$

Case-I: When $\phi(x) = \left(\frac{1-x}{1+x}\right)$, $x \neq -1$; $\phi\left(\phi\left(\frac{1}{x}\right)\right) = \frac{1}{x}$

Case-II : When $\phi(x) = \left(\frac{1+x}{1-x}\right), x \neq 1$, $\phi\left(\phi\left(\frac{1}{x}\right)\right) = -x.$

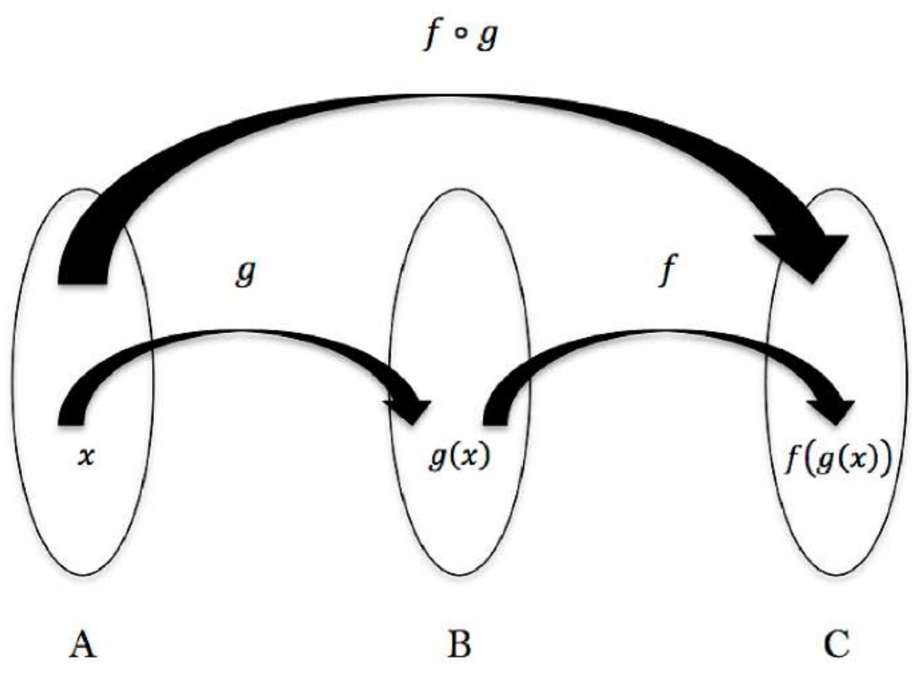
In both cases, $|\emptyset(\emptyset(x)) + \emptyset(\emptyset(1/x))|$

$= \left|x + \frac{1}{x}\right|$ (where $x > 0$) $= \left|\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + 2\right| \geq 2$

Composition of Functions

Composite Functions

Let f: $X \to Y_1$ and g: $Y_2 \to Z$ be two functions and the set $D = \{x \in X: f(x) \in Y_2\}$. If $D \equiv \emptyset$, then the function h defined on D by $h(x) = g\{f(x)\}$ is called composite function of g and f and is denoted by gof. It is also called function of a function.



Remark: Domain of gof is D which is a subset of X (the domain of f). Range of gof is a subset of the range of g. If $D = X$, then $f(X) \subseteq Y_2$.

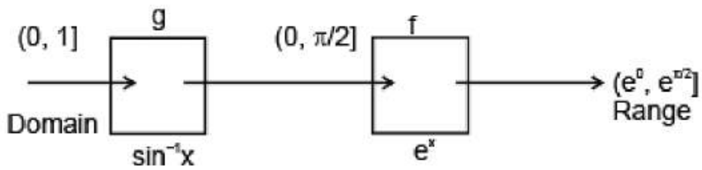
Properties of composite functions

1. The composite of functions is not commutative i.e. gof $\neq$ fog.
2. The composite of functions is associative i.e. if f, g, h are three functions such that fo (goh) & (fog) oh are defined, then fo (goh) = (fog) oh.

Example 1. Let $f(x) = e^x$; $R^+ \rightarrow R$ and $g(x) = \sin^{-1} x$; $[-1, 1] \rightarrow [-\pi/2, \pi/2]$. Find domain and range of fog (x)

Solution. Domain of f(x): $(0, \infty)$, Range of g(x): $[-\pi/2, \pi/2]$

The values in range of g(x) which are accepted by f(x) are $(0, \pi/2]$
$\Rightarrow 0 < g(x) \leq \pi/2 \quad 0 < \sin^{-1} x \leq \pi/2, 0 < x \leq 1$
Hence domain of fog(x) is $x \in (0, 1]$



Therefore Domain : (0, 1], Range : $(1/e^{\pi/2}]$

Example 2. Let $f(x) = (x-1)/(x+1)$, $f^2(x) = f\{f(x)\}$, $f^3(x) = f\{f^2(x)\}$,.....fk + 1 (x) = f{fk(x)}. for k = 1, 2, 3,...., Find f1998 (x).

Solution.

$$f(x) = \frac{x-1}{x+1}, \quad f^2(x) = f\{f(x)\} = \frac{f-1}{f+1} = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} = \frac{-1}{x},$$

$$f^3(x) = f\{f(x)\} = \frac{f^2(x)-1}{f^2(x)+1} = \frac{\frac{-1}{x}-1}{\frac{-1}{x}+1} = \frac{x+1}{x-1},$$

$$f^4 = f\{f^3(x)\} = \frac{f^3(x)-1}{f^3(x)+1} = \frac{\frac{x+1}{x-1}-1}{\frac{x+1}{x-1}+1} = x,$$

$$f^5(x) = f\{f^4(x)\} = \frac{x+1}{x-1} = f(x)$$

Thus, we can see that fk(x) repeats itself at intervals of k = 4.
Hence, we have f1998(x) = f2(x) = -1/x, $[\therefore 1998 = 499 \times 4 + 2]$

Example 3. **Let g : R → R be given by g(x) = 3 + 4x. If $g^n(x)$ = gogo....og(x), show that $f^n(x) = (4^n - 1) + 4^n x$ if $g^{-n}(x)$ denotes the inverse of $g^n(x)$.**

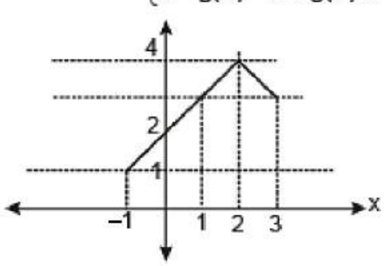
Solution. Since g(x) = 3 + 4x
$g^2(x)$ = (gog) (x) = g {g (x)} = g (3 + 4x) = 3 + 4 (3 + 4x) or $g^2(x) = 15 + 4^2x = (4^2 - 1) + 4^2x$
Now $g^3(x)$ = (gogog) x = g {g^2 (x)} = g (15 + 42 x) = 3 + 4 (15 + 42 x) = 63 + 43 x = (43 –1) + 43x
Similarly we get $g^n(x) = (4^n - 1) + 4^n x$
Now let $g^n(x) = y \Rightarrow x = g^{-n}(y)$ ....(1)
$\therefore y = (4^n - 1) + 4^n x$ or $x = (y + 1 - 4^n)4^{-n}$ ...(2)
From (1) and (2) we get $g^{-n}(y) = (y + 1 - 4^n)\,4^{-n}$.
Hence $g^{-n}(x) = (x + 1 - 4^n)\,4^{-n}$

Example 4. **If f(x) = If f(x) = | |x – 3| – 2 | ; 0 ≤ x ≤ 4 and g(x) = 4 – |2 – x| ; –1 ≤ x ≤ 3 then find fog(x).**

Solution.

$$g(x) = \begin{cases} 2 + x & -1 \le x < 2 \\ 6 - x & 2 \le x \le 3 \end{cases}$$

$$\therefore fog(x) = \begin{cases} 1 - g(x) & 0 \le g(x) < 1 \\ g(x) - 1 & 1 \le g(x) < 3 \\ 5 - g(x) & 3 \le g(x) \le 4 \end{cases}$$



$$\text{or } f(-x) = \begin{cases} x \tan x, & -\frac{\pi}{2} \le x < 0 \\ \frac{\pi}{2}[-x], & -2 \le x < -\frac{\pi}{2} \end{cases}$$

$$= \begin{cases} 2 + x - 1 & -1 \le x < 1 \\ 5 - (2 + x) & 1 \le x < 2 \\ 5 - (6 - x) & 2 \le x \le 3 \end{cases}$$

$$= \begin{cases} x + 1 & -1 \le x < 1 \\ 3 - x & 1 \le x < 2 \\ x - 1 & 2 \le x \le 3 \end{cases}$$

Example 5. Prove that $f(n) = 1 - n$ is the only integer valued function defined on integers such that

(i) $f(f(n)) = n$ for all $n \varepsilon Z$ and
(ii) $f(f(n + 2) + 2) = n$ for all $n \varepsilon Z$ and
(iii) $f(0) = 1$.

Solution. The function $f(n) = 1 - n$ clearly satisfies conditions (i), (ii) and (iii). Conversely, suppose a function
$f: Z \to Z$ satisfies (i), (ii) and (iii). Applying f to (ii) we get, $f(f(f(n + 2) + 2))) = f(n)$ and this gives because of (i), $f(n + 2) + 2 = f(n)$, ...(1)
for all $n \varepsilon Z$. Now using (1) it is easy to prove by induction on n that for all $n \varepsilon Z$,

$$f(n) = \begin{cases} f(0) - n \text{ if n is even} \\ f(1) + 1 - n \text{ if n is odd} \end{cases}$$

Also by (iii), $f(0) = 1$. Hence by (i), $f(1) = 0$. Hence $f(n) = 1 - n$ for all $n \varepsilon Z$.

General Definition

- **Identity function:** A function $f ; A \to$ defined by $f(x) = x \ \forall \ x \in A$ is called the identity of A & denoted by I_A.
  Ex: $f : R^+ \to R^+ ; f(x) = e^{\ell nx}$ and $f : R \to R ; f(x) = \ell n \ e^x$
  Every Identity function is a bijection.
- **Constant function:** A function $f: A \to B$ is said to be constant function. If every element of set A has the same functional image in set B i.e. $f: A \to B ; f(x) = c \ \forall \ x \ \varepsilon \ A \ \& \ c \ \varepsilon \ B$ is called constant function.
- **Homogeneous function:** A function is said to be homogeneous w.r.t. any set of variables when each of its term is of the same degree w.r.t. those variables.
- **Bounded Function:** A function $y = f(x)$ is said to be bounded if it can be express is the form of
  $a \leq f(x) \leq b$ where a and b are finite quantities.
  Example: $-1 \leq \sin x \leq 1 ; 0 \leq \{x\} < 1 ; -1 \leq \text{sgn}(x) \leq 1$ but e^x is not bounded.
  Any function having singleton range like constant function.
- **Implicit function & Explicit function:** If y has been expressed entirely in terms of `x' then it is called an explicit function.
  If x & y are written together in the form of an equation then it is known as implicit equation corresponding to each implicit equation there can be one, two or more explicit function satisfying it
  Example:
  $y = x^3 + 4x^2 + 5x \to$ Explicit function
  $x + y = 1 \to$ Implicit equation
  $y = 1 - x \to$ Explicit function

- **Even & Odd Functions**
  Function must be defined in symmetric interval [-x, x]
  If f (-x) = f (x) for all x in the domain of `f' then f is said to be an even function.
  e.g. f (x) = cos x; g (x) = $x^2 + 3$.
  If f (-x) = -f (x) for all x in the domain of `f' then f is said to be an odd function.
  e.g. f (x) = sin x ; g (x) = $x^3 + x$.

Remark

- f (x) - f (-x) = 0 ⇒ f (x) is even & f (x) + f (-x) = 0 ⇒ f (x) is odd.
- A function may be neither even nor odd.
- Inverse of an even function is not defined.
- Every even function is symmetric about the y-axis & every odd function is symmetric about the origin.
- A function (whose domain is symmetric about origin) can be expressed as a sum of an even & an odd function. e.g.

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$$

- The only function which is defined on the entire number line & is even and odd at the same time is f(x) = 0.
- If f and g both are even or both are odd then the function f.g will be even but if any one of them is odd and other even then f.g will be odd.

Example 6. Which of the following functions is odd ?

(a) sgn x + x^{2000}
(b) | x | - tan x
(c) x^3 cot x
(d) cosec x^{55}

Solution.

Let's name the function of the parts (A), (B), (C) and (D) as f(x), g(x), h(x) & f(x) respectively. Now

(a) $f(-x) = \text{sgn}(-x) + (-x)^{2000} = -\text{sgn } x + x^{2000} \neq f(x)$ & $\neq -f(x)$ ∴ f is neither even nor odd.
(b) $g(-x) = |-x| - \tan(-x) = |x| + \tan x$ ∴ g is neither even nor odd.
(c) $h(-x) = (-x)^3 \cot(-x) = -x^3(-\cot x) = x^3 \cot x = h(x)$ ∴ h is an even function
(d) $f(-x) = \text{cosec}(-x)^{55} = \text{cosec}(-x^{55}) = -\text{cosec } x^{55} = -f(x)$ ∴ f is an odd function.

Alternatively

(a) $f(x) = \text{sgn}(x) + x^{2000} = O + E =$ neither E nor O
(b) $g(x) = E - O =$ Neither E nor O
(c) $h(x) = O \times O = E$ (D) $f(-x) = O \text{ o } O = O$
(d) is the correct option

Example 7.
$\mathbf{f(x)} = (\tan x^5)\ e^{x^3 \text{sgn} x^7}$ **is**

(a) An even function
(b) An odd function
(c) Neither even nor odd function
(d) None of these

Solution.

$$f(x) = (\tan\ (x^5))\ e^{x^3\ \text{sgn}\ (x^7)}$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \ \downarrow$$

$$O \quad (O) \quad e^{O\ \times\ O\ (O)}$$

$$= O \times e^{O \times O} = O \times e^{E} = O \times E = O$$

Example 8. Let f: [-2, 2] → R be a function if f(x) =

$$\begin{cases} x\tan x, & 0 < x \le \frac{\pi}{2} \\ \frac{\pi}{2}[x], & \frac{\pi}{2} x \le 2 \end{cases}$$ Define f for $x \in [-2, 0]$

so that

(i) f is an odd function

(ii) f is an even function (where [*] denotes the greatest integer function)

Solution.

$$\text{Since } f(x) = \begin{cases} x\tan x, & 0 < x \le \frac{\pi}{2} \\ \frac{\pi}{2}[x], & \frac{\pi}{2} x \le 2 \end{cases}$$

$$\therefore f(-x) = \begin{cases} (-x)\tan(-x), & 0 < x \le \frac{\pi}{2} \\ \frac{\pi}{2}[-x], & \frac{\pi}{2} < -x \le 2 \end{cases}$$

$$\text{or } f(-x) = \begin{cases} x\tan x, & -\frac{\pi}{2} \le x < 0 \\ \frac{\pi}{2}[-x], & -2 \le x < -\frac{\pi}{2} \end{cases}$$

(i) If f is an odd function then f(x) = –f (–x)

$$= \begin{cases} -x\tan x, & -\frac{\pi}{2} \le x < 0 \\ -\frac{\pi}{2}[-x], & -2 \le x < -\frac{\pi}{2} \end{cases}$$

(ii) If f is an even function

$$\therefore \quad f(x) = f(-x) = \begin{cases} x\tan x, & -\frac{\pi}{2} \le x < 0 \\ \frac{\pi}{2}[-x], & -2 \le x < -\frac{\pi}{2} \end{cases}$$

Example 9. Let $f(x) = e^x + \sin x$ be defined on the interval [-4, 0]. Find the odd and even extension of f(x) in the interval [-4, 4].

Solution.

Odd Extension: Let g_0 be the odd extension of f(x), then

$$g_0(x) = \begin{cases} f(x) & ; \; x \in [-4,0] \\ -f(-x) & ; \; x \in [0,4] \end{cases} = \begin{cases} e^x + \sin x & ; \; x \in [-4,0] \\ -e^{-x} + \sin x & ; \; x \in [0,4] \end{cases}$$

Even Extension: Let g_e be the odd extension of f(x), then

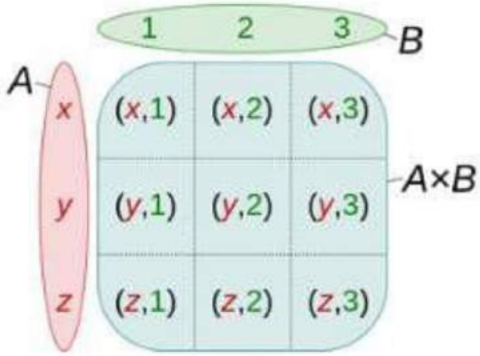
$$g_e(x) = \begin{cases} f(x) & ; \; x \in [-4,0] \\ f(-x) & ; \; x \in [0,4] \end{cases} = \begin{cases} e^x + \sin x & ; \; x \in [-4,0] \\ e^{-x} - \sin x & ; \; x \in [0,4] \end{cases}$$

Cartesian Product of Two Sets

Cartesian Product of Sets

Before getting familiar with this term, let us understand what does Cartesian mean. Remember the terms used when plotting a graph paper like axes (x-axis, y-axis), origin etc. For example, (2, 3) depicts that the value on the x-plane (axis) is 2 and that for y is 3 which is not the same as (3, 2).

The way of representation is fixed that the value of the x coordinate will come first and then that for y (ordered way). Cartesian product means the product of the elements say x and y in an ordered way.



Cartesian Product of Sets

The Cartesian products of sets mean the product of two non-empty sets in an ordered way. Or, in other words, the collection of all ordered pairs obtained by the product of two non-empty sets. An ordered pair means that two elements are taken from each set.

For two non-empty sets (say A & B), the first element of the pair is from one set A and the second element is taken from the second set B. The collection of all such pairs gives us a Cartesian product.

The Cartesian product of two non-empty sets A and B is denoted by A × B. Also, known as the cross-product or the product set of A and B. The ordered pairs (a, b) is such that $a \in A$ and $b \in B$. So, $A \times B = \{(a,b): a \in A, b \in B\}$. For example, Consider two non-empty sets $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$

Cartesian product $A\times B = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_1), (a_2, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_2), (a_3, b_3)\}$.
It is interesting to know that (a_1, b_1) will be different from (b_1, a_1). If either of the two sets is a null set, i.e., either $A = \Phi$ or $B = \Phi$, then, $A \times B = \Phi$ i.e., A × B will also be a null set

Number of Ordered Pairs

For two non-empty sets, A and B. If the number of elements of A is h i.e., $n(A) = h$ & that of B is k i.e., $n(B) = k$, then the number of ordered pairs in Cartesian product will be $n(A \times B) = n(A) \times n(B) = hk$.

Solved Example

Q.1. Let P & Q be two sets such that n(P) = 4 and n(Q) = 2. If in the Cartesian product we have (m, 1), (n, -1), (x, 1), (y, -1). Find P and Q, where m, n, x, and y are all distinct.

Solution. P = Set of first elements = {m, n, x, y} and Q = Set of second elements = {1, -1}

Periodic & Inverse Function

Periodic Function

A function f(x) is called periodic if there exists a positive number T ($T > 0$) called the period of the function such that $f(x + T) = f(x)$, for all values of x and $x + T$ within the domain of f(x). The least positive period is called the principal or fundamental period of f.

Example: The function sin x & cos x both are periodic over 2π & tan x is periodic over π

Remark:

(a) A constant function is always periodic, with no fundamental period.
(b) If f(x) has a period p, then $1/f(x)$ and $\sqrt{f(x)}$ also has a period p.
(c) If f(x) has a period T then $f(ax + b)$ has a period T/a ($a > 0$).
(d) If f(x) has a period T_1 & g(x) also has a period T_2 then period of $f(x) \pm g(x)$ or $f(x)/g(x)$ s L.C.M of T_1 & T_2 provided their L.C.M. exists. However that L.C.M. (if exists) need not to be fundamental period. If L.C.M. does not exists then $f(x) \pm g(x)$ or f(x). g(x) or $f(x)/g(x)$ is non-periodic e.g. |sin x| has the period p, |cos x| also has the period π |sin x| + |cos x| also has a period p. But the fundamental period of |sin x| + |cos x| is $\pi/2$.

(e) If g is a function such that gof is defined on the domain of f and f is periodic with T, then gof is also periodic with T as one of its periods. Further if # g is one-one, then T is the period of gof # g is also periodic with T' as the period and the range of f is a subset of [0, T'], then T is the period of gof

(f) Inverse of a periodic function does not exist.

Example 1. Find period of the following functions

(i) $f(x) = \sin\frac{x}{2} + \cos\frac{x}{3}$
(ii) $f(x) = \{x\} + \sin x$

(iii) $f(x) = \cos x \cdot \cos 3x$

(iv) $f(x) =$ $\sin \frac{3x}{2} - \cos \frac{x}{3} - \tan \frac{2x}{3}$.

Solution.

(i) Period of sin x/2 is 4p while period of cos x/3 is 6p. Hence period of sin x/2 + cos x/3 is 12 π {L.C.M. or 4 & 6 is 12}

(ii) Period of sin x = 2π ; Period of {x} = 1; but L.C.M. of 2π & 1 is not possible ∴ it is a periodic

(iii) f(x) = cos x . cos 3x ; Period of f(x) is L.C.M. of (2π, 2π/3) = 2π but 2π may or may not be the fundamental period. The fundamental period can be 2π/n where n ∈ Np may or may not be the fundamental period. The fundamental period can be f(π + x) = (–cos x) (– cos 3x) = f(x)

(iv) Period of f(x) is L.C.M. of $\frac{2\pi}{3/2}, \frac{2\pi}{1/3}, \frac{\pi}{3/2}$

$= \text{L.C.M. of } \frac{4\pi}{3}, 6\pi, \frac{2\pi}{3} = 12\pi$

Example 2. If f(x) = sin x + cos ax is a periodic function, show that a is a rational number.

Solution. Given f(x) = sin x + cos ax

Period of sin x = 2π/1 and period of cos ax 2π/a

Hence period of f(x) = L.C.M.

or

$\left\{\frac{2\pi}{1}, \frac{2\pi}{a}\right\} = \frac{\text{L.C.M. of } \{2\pi, 2\pi\}}{\text{H.C.F. of } \{1, a\}} = \frac{2\pi}{k}$

where k = H.C.F. of 1 and a

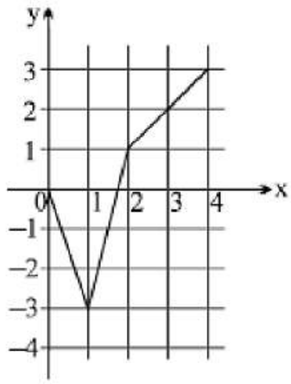
1/k = integer = q (say), (≠0) and k a/k = integer = p (say)

$\frac{a/k}{1/k} = \frac{p}{q}$

a = p/q

a = rational number

Example 3. Given below is a partial graph of an even periodic function *f* whose period is 8. If [*] denotes greatest integer function then find the value of the expression.



f (-3) + 2 | f (-1) | + [f(7/8)] + f (0) + arc cos (f(-2)) + f (-7) + f (20)

Solution.

f (-3) = f (3) = 2 [f(x) is an even function, ∴ f(-x) = f (x)]
again f (-1) = f (1) = - 3
∴ 2 | f (-1) | = 2 | f (1) | = 2 | -3 | = 6
from the graph, -3 < f(7/8) < -3 ∴ [f(7/8)] = -3
f (0) = 0 (obviously from the graph)
cos -1 (f(-2)) = $\cos^{-1}$ (f (2)) = $\cos^{-1}(1)$ = 0
f (-7) = f (-7 + 8) = f (1) = - 3 [f (x) has period 8]
f (20) = f (4 + 16) = f (4) = 3 [f (nT + X) = f (x)]
sum = 2 + 6 - 3 + 0 + 0 - 3 + 3 = 5

Example 4. If the periodic function f(x) satisfies the equation f(x + 1) + f(x - 1) = $\sqrt{3}$ f(x) ∀ x ∈ R then find the period of f(x)

Solution.

We have f(x + 1) + f(x – 1) = $\sqrt{3}$ f(x) ∀ x ∈ R
Replacing x by x – 1 and x + 1 in (1) then f(x) + f(x – 2) = $\sqrt{3}$ f(x – 1) ...(2)
Adding (2) and (3), we get 2f(x) + f(x – 2) + f(x + 2) = $\sqrt{3}$ (f (x – 1) + f(x + 1)) ...(3)
2f (x) + f(x – 2) + f(x + 2) = $\sqrt{3}$. $\sqrt{3}$ f(x) [From (1)]
f(x + 2) + f(x – 2) = f(x) ...(4)
Replacing x by x + 2 in equation (4) then f (x + 4) + f (x) = f (x + 2) ...(5)
Adding equations (4) and (5), we get f(x + 4) + f (x - 2) = 0 ...(6)
Again replacing x by x + 6 in (6) then f (x + 10) + f (x + 4) = 0 ...(7)
Subtracting (6) from (7), we get f (x + 10) - f (x - 2) = 0 ...(8)
Replacing x by x + 2 in (8) then f (x + 12) - f(x) = 0 or f (x + 12) = f(x)
Hence f(x) is periodic function with period 12.

Inverse Of A Function

Let $f : A \to B$ be a one-one & onto function, then their exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x, x \in A$ & $y \in B$. Then g is said to be inverse of f.
Thus $g = f^{-1} : B \to A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

Properties of inverse function :

(i) The inverse of a bijection is unique, and it is also a bijection.
(ii) If $f : A \to B$ is a bijection & $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A & I_B are identity functions on the sets A & B respectively.
(iii) The graphs of f & g are the mirror images of each other in the line $y = x$.
(iv) Normally points of intersection of f and f^{-1} lie on the straight line $y = x$. However it must be noted that $f(x)$ and $f^{-1}(x)$ may intersect otherwise also.
(v) In general fog(x) and gof(x) are not equal. But if either f and g are inverse of each other or atleast one of f, g is an identity function, then gof = fog.
(vi) If f & g are two bijections $f : A \to B$, $g : B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} o g^{-1}$.

Example 5. Find the inverse of the function $f(x) = \ln(x^2 + 3x + 1)$; $x \,\varepsilon\, [1, 3]$ and assuming it to be an onto function.

Solution.
Given $f(x) = \ln(x^2 + 3x + 1)$

$$f'(x) = \frac{2x+3}{(x^2+3x+1)} > 0 \;\forall\; x \in [1, 3]$$

which is a strictly increasing function. Thus f(x) is injective, given that f(x) is onto. Hence the given function f(x) is invertible. Now let $y = f(x) = \ln(x^2 + 3x + 1)$ then $x = f^{-1}(y)$ ...(1)
and $y = \ln(x^2 + 3x + 1) \Rightarrow e^y = x^2 + 3x + 1 \Rightarrow x^2 + 3x + 1 - e^y = 0$

$$x = \frac{-3 \pm \sqrt{9 - 4.1.(1-e^y)}}{2} = \frac{-3 \pm \sqrt{(5+4e^y)}}{2}$$

$$= \frac{-3 + \sqrt{(5+4e^y)}}{2} \quad (\because x \in [1, 3]) \text{ ...(2)}$$

From (1) and (2), we get

$$f^{-1}(y) = \frac{-3+\sqrt{(5+4e^{y})}}{2}$$

Hence $$f^{-1}(x) = \frac{-3+\sqrt{(5+4e^{x})}}{2}$$

Functions Formulas

THINGS TO REMEMBER :

1. GENERAL DEFINITION : If to every value (Considered as real unless other–wise stated) of a variable x, which belongs to some collection (Set) E, there corresponds one and only one finite value of the quantity y, then y is said to be a function (Single valued) of x or a dependent variable defined on the set E ; x is the argument or independent variable .

If to every value of x belonging to some set E there corresponds one or several values of the variable y, then y is called a multiple valued function of x defined on E. Conventionally the word "FUNCTION" is used only as the meaning of a single valued function, if not otherwise stated. Pictorially : $\xrightarrow[\text{input}]{x} \boxed{f} \xrightarrow[\text{output}]{f(x)=y}$, y is called the image of x & x is the pre-image of y under f. Every function from A $\rightarrow$ B satisfies the following conditions.

(i) $f \subset A \times B$

(ii) $\forall\ a \in A \Rightarrow (a, f(a)) \in f$ and

(iii) $(a, b) \in f\ \&\ (a, c) \in f \Rightarrow b = c$

2. DOMAIN, CO-DOMAIN & RANGE OF A FUNCTION : Let $f : A \rightarrow B$, then the set A is known as the domain of f & the set B is known as co-domain of f . The set of all f images of elements of A is known as the range of f . Thus

Domain of $f = \{a \mid a \in A, (a, f(a)) \in f\}$

Range of $f = \{f(a) \mid a \in A, f(a) \in B\}$

It should be noted that range is a subset of co–domain . If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives the range.

3. IMPORTANT TYPES OF FUNCTIONS :

(i) POLYNOMIAL FUNCTION : If a function f is defined by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a^{n-1} x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, ..., a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n

NOTE : (a) A polynomial of degree one with no constant term is called an odd linear function . i.e. $f(x) = ax,\ a \neq 0$

(b) There are two polynomial functions, satisfying the relation; $f(x).f(1/x) = f(x) + f(1/x)$. They are : (i) $f(x) = x^n + 1$ & (ii) $f(x) = 1 - x^n$, where n is a positive integer.

(ii) ALGEBRAIC FUNCTION : y is an algebraic function of x, if it is a function that satisfies an algebraice quation of the form $P_0(x)\, y^n + P_1(x)\, y^{n-1} + ... + P_{n-1}(x)\, y + P_n(x) = 0$ Where n is a positive integer and $P_0(x)$, $P_1(x)$ .... are Polynomials in x.

e.g. $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

Note that all polynomial functions are Algebraic but not the converse. A function that is not algebraic is called TRANSCEDENTAL FUNCTION .

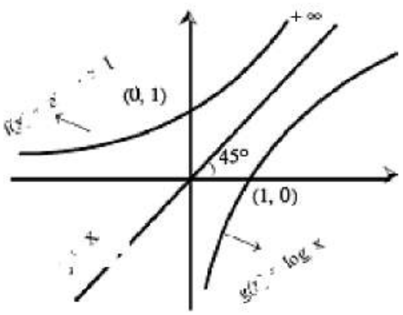
(iii) FRACTIONAL RATIONAL FUNCTION : A rational function is a function of the form. $y = f(x) = \frac{g(x)}{h(x)}$, where g (x) & h (x) are polynomials & $h(x) \neq 0$.

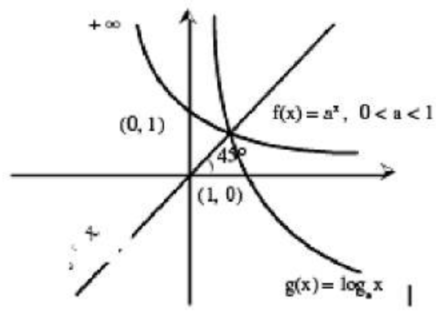
(iv) ABSOLUTE VALUE FUNCTION : A function $y = f(x) = |x|$ is called the absolute value function or Modulus function. It is defined as :

$$y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

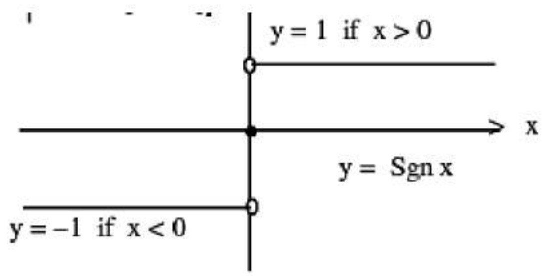
(V) EXPONENTIAL FUNCTION : A function $f(x) = a^x = e^{x/n\,a}$ $(a > 0,\ a \neq 1, x \in R)$ is called anexponential function. The inverse of the exponential function is called the logarithmic function . i.e. $g(x) = \log_a x$.

Note that f(x) & g(x) are inverse of each other & their graphs are as shown.





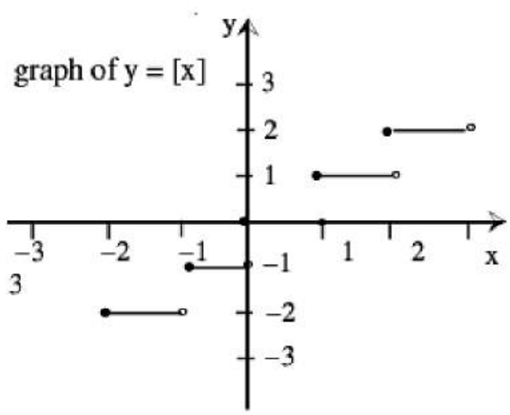
(vi) SIGNUM FUNCTION : A function y= f (x) = Sgn (x) is defined as follows :



$$y = f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

It is also written as $\text{Sgn } x = |x|/x$; $x \neq 0$; $f(0) = 0$

(vii) GREATES INTEGER OR STEP UP FUNCTION : The function $y = f(x) = [x]$ is called the greatest integer function where [x] denotes the greatest integer less than or equal to x. Note that for :



$-1 \le x < 0$; $[x] = -1$

$0 \le x < 1$; $[x] = 0$

$1 \le x < 2$; $[x] = 1$

$2 \le x < 3$; $[x] = 2$ and so on.

Properties of greatest integer function :

(a) $[x] \le x < [x] + 1$ and $x - 1 < [x] \le x$, $0 \le x - [x] < 1$

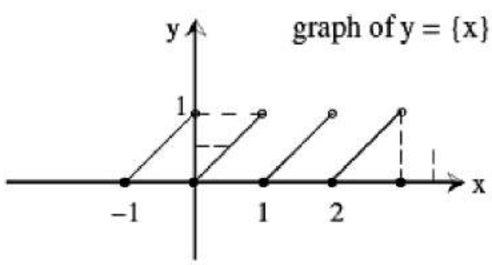
(b) $[x + m] = [x] + m$ if m is an integer .

(c) $[x] + [y] \le [x + y] \le [x] + [y] + 1$

(d) $[x] + [-x] = 0$ if x is an integer $= -1$ otherwise.

(viii) FRACTIONAL PART FUNCTION : It is defined as : $g(x) = \{x\} = x - [x]$.

e.g. the fractional part of the no. 2.1 is $2.1 - 2 = 0.1$ and the fractional part of -3.7 is 0.3. The period of this function is 1 and graph of this function is as shown.



4. DOMAINS AND RANGES OF COMMON FUNCTION :

	Function $(y = f(x))$	Domain (i.e. values taken by x)	Range (i.e. values taken by f (x))
A.	**Algebraic Functions**		
	(i) x^n , $(n \in N)$	R = (set of real numbers)	R , if n is odd $R^+ \cup \{0\}$, if n is even
	(ii) $\frac{1}{x^n}$, $(n \in N)$	$R - \{0\}$	$R - \{0\}$, if n is odd R^+ , if n is even
	(iii) $x^{1/n}$, $(n \in N)$	R , if n is odd $R^+ \cup \{0\}$, if n is even	R , if n is odd $R^+ \cup \{0\}$, if n is even
	Function $(y = f(x))$	**Domain** (i.e. values taken by x)	**Range** (i.e. values taken by f (x))
	(iv) $\frac{1}{x^{1/n}}$, $(n \in N)$	$R - \{0\}$, if n is odd R^+ , if n is even	$R - \{0\}$, if n is odd R^+ , if n is even

B. Trigonometric Functions

(i)	sin x	R	[–1, + 1]
(ii)	cos x	R	[–1, + 1]
(iii)	tan x	$R - (2k+1)\frac{\pi}{2}, k \in I$	R
(iv)	sec x	$R - (2k+1)\frac{\pi}{2}, k \in I$	$(-\infty, -1] \cup [1, \infty)$
(v)	cosec x	$R - k\pi, k \in I$	$(-\infty, -1] \cup [1, \infty)$
(vi)	cot x	$R - k\pi, k \in I$	R

C. Inverse Circular Functions (Refer after Inverse is taught)

(i)	$\sin^{-1} x$	[–1, + 1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(ii)	$\cos^{-1} x$	[–1, + 1]	$[0, \pi]$
(iii)	$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(iv)	$\text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
(v)	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
(vi)	$\cot^{-1} x$	R	$(0, \pi)$

D. Exponential Functions

(i)	e^x	R	R^+
(ii)	$e^{1/x}$	$R - \{0\}$	$R^+ - \{1\}$
(iii)	$a^x, a > 0$	R	R^+
(iv)	$a^{1/x}, a > 0$	$R - \{0\}$	$R^+ - \{1\}$

E. Logarithmic Functions

(i)	$\log_a x, (a > 0)(a \neq 1)$	R^+	R
(ii)	$\log_x a = \frac{1}{\log_a x}\ (a > 0)(a \neq 1)$	$R^+ - \{1\}$	$R - \{0\}$

F. Integral Part Functions Functions

(i)	$[x]$	R	I
(ii)	$\frac{1}{[x]}$	$R - [0, 1)$	$\left\{\frac{1}{n}, n \in I - \{0\}\right\}$

G.	**Fractional Part Functions**		
	(i) $\{x\}$	R	[0, 1)
	(ii) $\frac{1}{\{x\}}$	R – I	$(1, \infty)$
H.	**Modulus Functions**		
	Function ($y = f(x)$)	**Domain** (i.e. values taken by x)	**Range** (i.e. values taken by f(x))
	(i) $\|x\|$	R	$R^+ \cup \{0\}$
	(ii) $\frac{1}{\|x\|}$	R – { 0 }	R^+

I. Signum Function

$\text{sgn}(x) = \frac{\|x\|}{x}, x \neq 0$; $= 0, x = 0$	R	$\{-1, 0, 1\}$

J. Constant Function

say $f(x) = c$	R	$\{c\}$

5. EQUAL OR IDENTICAL FUNCTION :

Two functions f & g are said to be equal if :

(i) The domain of f = the domain of g.

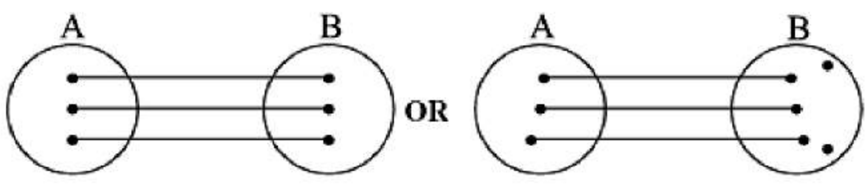
(ii) The range of f = the range of g and

(iii) f(x) = g(x), for every x belonging to their common domain.

eg. $f(x) = \frac{1}{x}$ & $g(x) = \frac{x}{x^2}$ are identical functions.

6. CLASSIFICATION OF FUNCTIONS : One - One Function (Injective mapping) : A function $f : A \to B$ is said to be a one–one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ & $f(x_1)$, $f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Diagramatically an injective mapping can be shown as



Note :

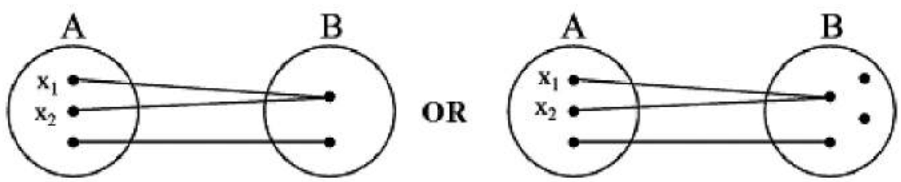
(i) Any function which is entirely increasing or decreasing in whole domain, then f(x) is one−one.

(ii) If any line parallel to x−axis cuts the graph of the function at most at one point, then the function is one−one.

Many–one function :

A function $f : A \rightarrow B$ is said to be a many one function if two or more elements of A have the same f image in B. Thus $f : A \rightarrow B$ is many one if for ; $x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$

Diagramatically a many one mapping can be shown as



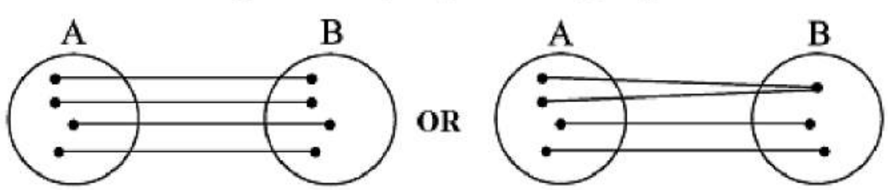
Note :

(i) Any continuous function which has at least one local maximum or local minimum, then f(x) is many−one. In other words, if a line parallel to x−axis cuts the graph of the function at least at two points, then f is many−one .

(ii) If a function is one−one, it cannot be many−one and vice versa.

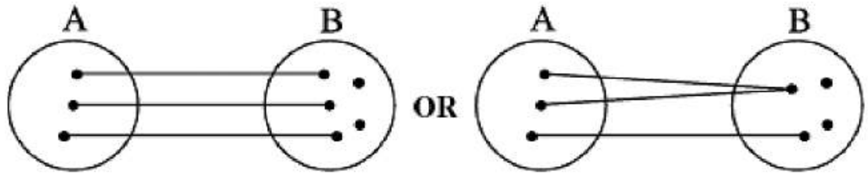
Onto function (Surjective mapping) : If the function $f : A \rightarrow B$ is such that each element in B (co−domain) is the f image of at least one element in A, then we say that f is a function of A 'onto' B . Thus $f : A \rightarrow B$ is surjective if $\forall\ b \in B$, $\exists$ some $a \in A$ such that $f(a) = b$.

Diagramatically surjective mapping can be shown as



Note that : if range = co−domain, then f(x) is onto. Into function : If f : $A \rightarrow B$ is such that there exists at least one element in co−domain which is not the image of any element in domain, then f(x) is into.

Diagramatically into function can be shown as



Note that : If a function is onto, it cannot be into and vice versa . A polynomial of degree even will always be into. Thus a function can be one of these four types :

(a) one−one onto (injective & surjective)

(b) one−one into (injective but not surjective)

(c) many−one onto (surjective but not injective)

(d) many−one into (neither surjective nor injective)

Note : (i) If f is both injective & surjective, then it is called a Bijective mapping.

The bijective functions are also named as invertible, non singular or biuni form functions.

(ii) If a set A contains n distinct elements then the number of different functions defined from $A \rightarrow A$ is n^n & out of it n ! are one one.

Identity function : The function $f: A \rightarrow A$ defined by $f(x) = x \ \forall \ x \in A$ is called the identity of A and is denoted by I_A. It is easy to observe that identity function is a bijection.

Constant function : A function $f: A \rightarrow B$ is said to be a constant function if every element of A has the same f image in B . Thus $f: A \rightarrow B$; $f(x) = c$, $\forall \ x \in A$, $c \in B$ is a constant function. Note that the range of a constant function is a singleton and a constant function may be one-one or many-one, onto or into.

7. ALGEBRAIC OPERATIONS ON FUNCTIONS :

If f & g are real valued functions of x with domain set A, B respectively, then both f & g are defined in $A \cap B$. Now we define $f + g$, $f - g$, $(f . g)$ & (f/g) as follows :

(i) $(f \pm g)(x) = f(x) \pm g(x)$
(ii) $(f . g)(x) = f(x) . g(x)$
domain in each case is $A \cap B$

(iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain is $\{x \mid x \in A \cap B \ s.t \ g(x) \neq 0\}$.

8. COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTIONS :

Let $f: A \rightarrow B$ & $g: B \rightarrow C$ be two functions. Then the function $g \text{ of} : A \rightarrow C$ defined by $(g \text{ of})(x) = g(f(x)) \ \forall \ x \in A$ is called the composite of the two functions f & g .

Diagramatically $\xrightarrow{x} \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x))$. Thus the image of every $x \in A$ under the function g of is the g−image of the f−image of x.

Note that g of is defined only if $\forall \ x \in A$, f(x) is an element of the domain of g so that we can take its g-image. Hence for the product g of of two functions f & g, the range of f must be a subset of the domain of g.

PROPERTIES OF COMPOSITE FUNCTIONS :

(i) The composite of functions is not commutative i.e. $gof \neq fog$.

(ii) The composite of functions is associative i.e. if f, g, h are three functions such that fo (goh) & (fog) oh are defined, then fo (goh) = (fog) oh .

(iii) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

9. HOMOGENEOUS FUNCTIONS :

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For example $5x^2 + 3y^2 - xy$ is homogeneous in x & y . Symbolically if, $f(tx, ty) = t^n . f(x, y)$ then $f(x, y)$ is homogeneous function of degree n.

10. BOUNDED FUNCTION :

A function is said to be bounded if $|f(x)| \le M$, where M is a finite quantity.

11. IMPLICIT & EXPLICIT FUNCTION : A function defined by an equation not solved for the dependent variable is called an IMPLICIT FUNCTION . For eg. the equation $x^3 + y^3 = 1$ defines y as an implicit function. If y has been expressed in terms of x alone then it is called an EXPLICIT FUNCTION.

12. INVERSE OF A FUNCTION : Let $f : A \to B$ be a one−one & onto function, then their exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x, \ \forall \ x \in A \ \& \ y \in B$. Then g is said to be inverse of f . Thus $g = f^{-1} : B \to A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

PROPERTIES OF INVERSE FUNCTION:

(i) The inverse of a bijection is unique.

(ii) If $f : A \to B$ is a bijection & $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A & I_B are identity functions on the sets A & B respectively.

Note that the graphs of f & g are the mirror images of each other in the line $y = x$. As shown in the figure given below a point (x', y') corresponding to $y = x^2$ $(x > 0)$ changes to (y', x') corresponding to $y = +\sqrt{x}$, the changed form of $x = \sqrt{y}$.

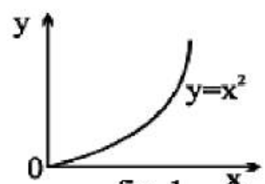


fig. 1

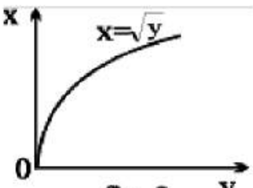


fig. 2

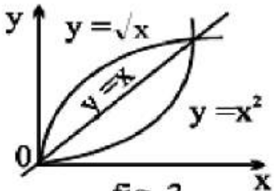


fig. 3

(iii) The inverse of a bijection is also a bijection.

(iv) If f & g two bijections $f: A \to B$, $g: B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} o\, g^{-1}$

13. ODD & EVEN FUNCTIONS : If $f(-x) = f(x)$ for all x in the domain of 'f' then f is said to be an even function. e.g. $f(x) = \cos x$; $g(x) = x^2 + 3$. If $f(-x) = -f(x)$ for all x in the domain of 'f' then f is said to be an odd function. e.g. $f(x) = \sin x$; $g(x) = x^3 + x$.

NOTE :

(a) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even & $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd.

(b) A function may neither be odd nor even.

(c) Inverse of an even function is not defined

(d) Every even function is symmetric about the y−axis & every odd function is symmetric about the origin.

(e) Every function can be expressed as the sum of an even & an odd function.

$$\text{e.g. } f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$$

(f) only function which is defined on the entire number line & is even and odd at the same time is f(x)= 0.

(g) If f and g both are even or both are odd then the function f.g will be even but if any one of them is odd then f.g will be odd.

14. PERIODIC FUNCTION : A function f(x) is called periodic if there exists a positive number T (T > 0) called the period of the function such that f (x + T) = f(x), for all values of x within the domain of x e.g. The function sin x & cos x both are periodic over 2π & tan x is periodic over π

NOTE : (a) f (T) = f (0) = f (−T) , where 'T' is the period.

(b) Inverse of a periodic function does not exist.

(c) Every constant function is always periodic, with no fundamental period.

(d) If f (x) has a period T & g (x) also has a period T then it does not mean that f (x) + g (x) must have a period T . e.g. f(x) = |sinx| + |cosx|.

(e) If f(x) has a period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.

(f) if f(x) has a period T then f(ax + b) has a period T/a (a > 0).

15. GENERAL : If x, y are independent variables, then:

(i) f(xy) = f(x) + f(y) ⇒ f(x) = k ln x or f(x) = 0.

(ii) f(xy) = f(x) . f(y) ⇒ f(x) = x^n, n ∈ R

(iii) f(x + y) = f(x) . f(y) ⇒ f(x) = a^{kx}.