Chapter 7

TRANSFORMATION OF COORDINATES

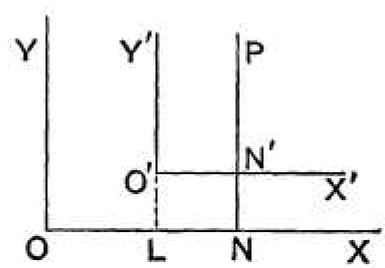
127. It is sometimes found desirable in the discussion of problems to alter the origin and axes of coordinates, either by altering the origin without alteration of the direction of the axes, or by altering the directions of the axes and keeping the origin unchanged, or by altering the origin and also the directions of the axes. The latter case is merely a combination of the first two. Either of these processes is called a transformation of coordinates.

We proceed to establish the fundamental formulæ for such transformation of coordinates.

128. *To alter the origin of coordinates without altering the directions of the axes.*

Let OX and OY be the original axes and let the new axes, parallel to the original, be

$$O'X' \text{ and } O'Y'.$$



Let the coordinates of the new origin O', referred to the original axes be h and k, so that, if $O'L$ be perpendicular to OX, we have

$$OL = h \text{ and } LO' = k.$$

Let P be any point in the plane of the paper, and let its coordinates, referred to the original axes, be x and y, and referred to the new axes let them be x' and y'.

Draw PN perpendicular to OX to meet $O'X'$ in N'.

Then

$$ON = x,\quad NP = y,\quad O'N' = x',\ \text{ and }\ N'P = y'.$$

We therefore have

$$x = ON = OL + O'N' = h + x',$$

and $$y = NP = LO' + N'P = k + y'.$$

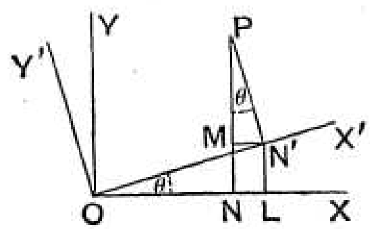
The origin is therefore transferred to the point (h, k) when we substitute for the coordinates x and y the quantities

$$x' + h \ \text{ and } \ y' + k.$$

The above article is true whether the axes be oblique or rectangular.

129. *To change the direction of the axes of coordinates, without changing the origin, both systems of coordinates being rectangular.*

Let OX and OY be the original system of axes and OX' and OY' the new system, and let the angle, XOX', through which the axes are turned be called θ.



Take any point P in the plane of the paper.

Draw PN and PN' perpendicular to OX and OX', and also $N'L$ and $N'M$ perpendicular to OX and PN.

If the coordinates of P, referred to the original axes, be x and y, and, referred to the new axes, be x' and y', we have

$$ON = x,\quad NP = y,\quad ON' = x',\ \text{ and }\ N'P = y'.$$

The angle

$$MPN' = 90° - \angle MN'P = \angle MN'O = \angle XOX' = \theta.$$

We then have

$$x = ON = OL - MN' = ON' \cos\theta - N'P \sin\theta$$
$$= x' \cos\theta - y' \sin\theta \quad \text{.......................} (1),$$

and $$y = NP = LN' + MP = ON' \sin\theta + N'P \cos\theta$$
$$= x' \sin\theta + y' \cos\theta \quad \text{.......................} (2).$$

If therefore in any equation we wish to turn the axes, being rectangular, through an angle θ we must substitute

$$\mathbf{x'}\cos\theta - \mathbf{y'}\sin\theta \text{ and } \mathbf{x'}\sin\theta + \mathbf{y'}\cos\theta$$

for x and y.

When we have both to change the origin, and also the direction of the axes, the transformation is clearly obtained by combining the results of the previous articles.

If the origin is to be transformed to the point (h, k) and the axes to be turned through an angle θ, we have to substitute

$$h + x'\cos\theta - y'\sin\theta \text{ and } k + x'\sin\theta + y'\cos\theta$$

for x and y respectively.

The student, who is acquainted with the theory of projection of straight lines, will see that equations (1) and (2) express the fact that the projections of OP on OX and OY are respectively equal to the sum of the projections of ON' and $N'P$ on the same two lines.

130. Ex. 1. *Transform to parallel axes through the point* $(-2, 3)$ *the equation*

$$2x^2 + 4xy + 5y^2 - 4x - 22y + 7 = 0.$$

We substitute $x = x' - 2$ and $y = y' + 3$, and the equation becomes

$$2(x'-2)^2 + 4(x'-2)(y'+3) + 5(y'+3)^2 - 4(x'-2) - 22(y'+3) + 7 = 0,$$

i.e.
$$2x'^2 + 4x'y' + 5y'^2 - 22 = 0.$$

Ex. 2. *Transform to axes inclined at* $30°$ *to the original axes the equation*

$$x^2 + 2\sqrt{3}xy - y^2 = 2a^2.$$

For x and y we have to substitute

$$x'\cos 30° - y'\sin 30° \text{ and } x'\sin 30° + y'\cos 30°,$$

i.e.
$$\frac{x'\sqrt{3} - y'}{2} \text{ and } \frac{x' + y'\sqrt{3}}{2}.$$

The equation then becomes

$$(x'\sqrt{3} - y')^2 + 2\sqrt{3}(x'\sqrt{3} - y')(x' + y'\sqrt{3}) - (x' + y'\sqrt{3})^2 = 8a^2,$$

i.e.
$$x'^2 - y'^2 = a^2.$$

EXAMPLES XV

1. Transform to parallel axes through the point $(1, -2)$ the equations

(1) $y^2 - 4x + 4y + 8 = 0$,

and (2) $2x^2 + y^2 - 4x + 4y = 0$.

2. What does the equation

$$(x-a)^2 + (y-b)^2 = c^2$$

become when it is transferred to parallel axes through

(1) the point $(a-c, b)$,

(2) the point $(a, b-c)$?

3. What does the equation

$$(a-b)(x^2+y^2) - 2abx = 0$$

become if the origin be moved to the point $\left(\frac{ab}{a-b}, 0\right)$?

4. Transform to axes inclined at 45° to the original axes the equations

(1) $x^2 - y^2 = a^2$,

(2) $17x^2 - 16xy + 17y^2 = 225$,

and (3) $y^4 + x^4 + 6x^2y^2 = 2$.

5. Transform to axes inclined at an angle α to the original axes the equations

(1) $x^2 + y^2 = r^2$,

and (2) $x^2 + 2xy \tan 2\alpha - y^2 = a^2$.

6. If the axes be turned through an angle $\tan^{-1} 2$, what does the equation $4xy - 3x^2 = a^2$ become?

7. By transforming to parallel axes through a properly chosen point (h, k), prove that the equation

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$$

can be reduced to one containing only terms of the second degree.

8. Find the angle through which the axes may be turned so that the equation $Ax + By + C = 0$

may be reduced to the form $x =$ constant, and determine the value of this constant.

ANSWERS

1. (1) $y'^2 = 4x'$; (2) $2x'^2 + y'^2 = 6$.

2. (1) $x'^2 + y'^2 = 2cx'$; (2) $x'^2 + y'^2 = 2cy'$.

3. $(a-b)^2(x'^2 + y'^2) = a^2b^2$.

4. (1) $2x'y'+a^2=0$; $9x'^2+25y'^2=225$; $x'^4+y'^4=1$.
5. $x'^2+y'^2=r^2$; $x'^2-y'^2=a^2\cos 2a$. **6.** $x'^2-4y'^2=a^2$.
8. $\tan^{-1}\dfrac{B}{A}$; $-C\div\sqrt{A^2+B^2}$.

SOLUTIONS/HINTS

1. (i) $(y'-2)^2-4(x'+1)+4(y'-2)+8=0$, *i.e.* $y'^2=4x$.
(ii) $2(x'+1)^2+(y'-2)^2-4(x'+1)+4(y'-2)=0$,
i.e. $$2x'^2+y'^2=6.$$

2. (1) $(x'+a-c-a)^2+(y'+b-b)^2=c^2$,
i.e. $$x'^2+y'^2=2cx'.$$
(2) $(x'+a-a)^2+(y'+b-c-b)^2=c^2$,
i.e. $$x'^2+y'^2=2cy'.$$

3. $(a-b)\left\{\left(x'+\dfrac{ab}{a-b}\right)^2+y'^2\right\}-2ab\left(x'+\dfrac{ab}{a-b}\right)=0$,
i.e. $$(a-b)^2(x'^2+y'^2)=a^2b^2.$$

4. Substitute $\dfrac{x'-y'}{\sqrt{2}}$ for x, and $\dfrac{x'+y'}{\sqrt{2}}$ for y.

(i) $(x'-y')^2-(x'+y')^2=2a^2$, *i.e.* $2x'y'+a^2=0$.
(ii) $17\{(x'-y')^2+(x'+y')^2\}-16(x'^2-y'^2)=450$,
i.e. $$9x'^2+25y'^2=225.$$
(iii) $(x'-y')^4+(x'+y')^4+2(x'-y')^2(x'+y')^2+4(x'^2-y'^2)^2=8$.
$\therefore\ \{(x'-y')^2+(x'+y')^2\}^2+4(x'^2-y'^2)^2=8$,
i.e. $4(x'^2+y'^2)^2+4(x'^2-y'^2)^2=8$, *i.e.* $x'^4+y'^4=1$.

5. (i) $(x'\cos\alpha-y'\sin\alpha)^2+(x'\sin\alpha+y'\cos\alpha)^2=r^2$;
$$\therefore\ x'^2+y'^2=r^2.$$
(ii) $(x'\cos\alpha-y'\sin\alpha)^2-(x'\sin\alpha+y'\cos\alpha)^2$
$+2(x'\cos\alpha-y'\sin\alpha)(x'\sin\alpha+y'\cos\alpha)\tan 2\alpha=a^2$;
$\therefore\ (x'^2-y'^2)\cos 2\alpha-2x'y'\sin 2\alpha$
$+\{(x'^2-y'^2)\sin 2\alpha+2x'y'\cos 2\alpha\}\tan 2\alpha=a^2$;
$$\therefore\ x'^2-y'^2=a^2\cos 2\alpha.$$

6. $\tan^{-1} 2 = \sin^{-1}\dfrac{2}{\sqrt{5}} = \cos^{-1}\dfrac{1}{\sqrt{5}}$;

$\therefore$ for x substitute $\dfrac{x'-2y'}{\sqrt{5}}$; for y put $\dfrac{2x'+y'}{\sqrt{5}}$; (Art. 129)

$\therefore\ 4(x'-2y')(2x'+y') - 3(x'-2y')^2 = 5a^2$, *i.e.* $x'^2 - 4y'^2 = a^2$.

7. On substituting $x'+h$ for x and $y'+k$ for y, the equation becomes

$$12(x'+h)^2 - 10(x'+h)(y'+k) + 2(y'+k)^2 + 11(x'+h) - 5(y'+k) + 2 = 0,$$

or $12x'^2 - 10x'y' + 2y'^2 + 2x'(12h - 5k + \frac{11}{2}) + 2y'(-5h + 2k - \frac{5}{2}) + 12h^2 - 10hk + 2k^2 + 11h - 5k + 2 = 0.$ ......(i)

There will be no terms of the first degree in x' and y' if h and k be so chosen that

$$12h - 5k + \tfrac{11}{2} = 0, \text{ and } -5h + 2k - \tfrac{5}{2} = 0,$$

i.e. if $$h = -\tfrac{3}{2},\ k = -\tfrac{5}{2}.$$

On substituting these values in (1), the equation reduces to $$12x'^2 - 10x'y' + 2y'^2 = 0.$$

8. If the axes be turned through an angle θ the equation becomes

$$A(x'\cos\theta - y'\sin\theta) + B(x'\sin\theta + y'\cos\theta) + C = 0,$$

or $$(A\cos\theta + B\sin\theta)x' + (B\cos\theta - A\sin\theta)y' + C = 0.$$

Hence we must have $B\cos\theta - A\sin\theta = 0$; $\therefore \tan\theta = \dfrac{B}{A}$,

and the equation becomes $\dfrac{A^2+B^2}{\sqrt{A^2+B^2}}x' + C = 0$;

$$\therefore \text{ the constant} = -\frac{C}{\sqrt{A^2+B^2}}.$$

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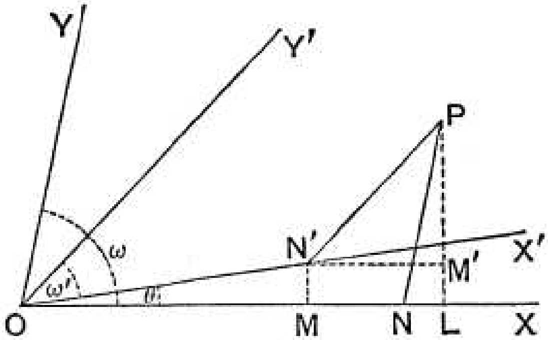
131. The general proposition, which is given in the next article, on the transformation from one set of oblique axes to any other set of oblique axes is of very little importance and is hardly ever required.

***132.** *To change from one set of axes, inclined at an angle ω, to another set, inclined at an angle ω', the origin remaining unaltered.*



Let OX and OY be the original axes, OX' and OY' the new axes, and let the angle XOX' be θ.

Take any point P in the plane of the paper.

Draw PN and PN' parallel to OY and OY' to meet OX and OX' respectively in N and N', PL perpendicular to OX, and $N'M$ and $N'M'$ perpendicular to OL and LP.

Now

$$\angle PNL = \angle YOX = \omega, \text{ and } PN'M' = Y'OX = \omega' + \theta.$$

Hence if

$$ON = x, \quad NP = y, \quad ON' = x', \text{ and } N'P = y',$$

we have $\quad y \sin \omega = NP \sin \omega = LP = MN' + M'P$

$$= ON' \sin \theta + N'P \sin (\omega' + \theta),$$

so that $\quad y \sin \omega = x' \sin \theta + y' \sin (\omega' + \theta) \quad \ldots\ldots\ldots\ldots\ldots(1).$

Also

$$x + y \cos \omega = ON + NL = OL = OM + N'M'$$
$$= x' \cos \theta + y' \cos (\omega' + \theta) \quad \ldots\ldots\ldots\ldots\ldots(2).$$

Multiplying (2) by $\sin \omega$, (1) by $\cos \omega$, and subtracting, we have

$$x \sin \omega = x' \sin (\omega - \theta) + y' \sin (\omega - \omega' - \theta) \ldots\ldots(3).$$

[This equation (3) may also be obtained by drawing a perpendicular from P upon OY and proceeding as for equation (1).]

The equations (1) and (3) give the proper substitutions for the change of axes in the general case.

As in Art. 130 the equations (1) and (2) may be obtained by equating the projections of OP and of ON' and $N'P$ on OX and a straight line perpendicular to OX.

***133.** *Particular cases of the preceding article.*

(1) Suppose we wish to transfer our axes from a rectangular pair to one inclined at an angle ω'. In this case ω is 90°, and the formulæ of the preceding article become

$$x = x' \cos\theta + y' \cos(\omega' + \theta),$$

and $$y = x' \sin\theta + y' \sin(\omega' + \theta).$$

(2) Suppose the transference is to be from oblique axes, inclined at ω, to rectangular axes. In this case ω' is 90°, and our formulæ become

$$x \sin\omega = x' \sin(\omega - \theta) - y' \cos(\omega - \theta),$$

and $$y \sin\omega = x' \sin\theta + y' \cos\theta.$$

These particular formulæ may easily be proved independently, by drawing the corresponding figures.

Ex. *Transform the equation* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *from rectangular axes to axes inclined at an angle* 2α, *the new axis of* x *being inclined at an angle* $-\alpha$ *to the old axes and* $\sin\alpha$ *being equal to* $\frac{b}{\sqrt{a^2+b^2}}$.

Here $\theta = -\alpha$ and $\omega' = 2\alpha$, so that the formulæ of transformation (1) become

$$x = (x' + y') \cos\alpha \text{ and } y = (y' - x') \sin\alpha.$$

Since $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$, we have $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$, and hence the given equation becomes

$$\frac{(x'+y')^2}{a^2+b^2} - \frac{(y'-x')^2}{a^2+b^2} = 1,$$

i.e. $$x'y' = \tfrac{1}{4}(a^2+b^2).$$

***134.** *The degree of an equation is unchanged by any transformation of coordinates.*

For the most general form of transformation is found by combining together Arts. 128 and 132. Hence the most general formulæ of transformation are

$$x = h + x' \frac{\sin(\omega - \theta)}{\sin\omega} + y' \frac{\sin(\omega - \omega' - \theta)}{\sin\omega},$$

and $$y = k + x' \frac{\sin\theta}{\sin\omega} + y' \frac{\sin(\omega' + \theta)}{\sin\omega}.$$

For x and y we have therefore to substitute expressions in x' and y' of the first degree, so that by this substitution the degree of the equation cannot be raised.

Neither can, by this substitution, the degree be lowered. For, if it could, then, by transforming back again, the degree would be raised and this we have just shewn to be impossible.

***135.** *If by any change of axes, without change of origin, the quantity $ax^2+2hxy+by^2$ become*

$$a'x'^2+2h'x'y'+b'y'^2,$$

the axes in each case being rectangular, to prove that

$$a+b=a'+b', \textit{ and } ab-h^2=a'b'-h'^2.$$

By Art. 129, the new axis of x being inclined at an angle θ to the old axis, we have to substitute

$$x'\cos\theta-y'\sin\theta \text{ and } x'\sin\theta+y'\cos\theta$$

for x and y respectively.

Hence $ax^2+2hxy+by^2$

$$=a(x'\cos\theta-y'\sin\theta)^2+2h(x'\cos\theta-y'\sin\theta)(x'\sin\theta+y'\cos\theta)$$
$$+b(x'\sin\theta+y'\cos\theta)^2$$
$$=x'^2[a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta]$$
$$+2x'y'[-a\cos\theta\sin\theta+h(\cos^2\theta-\sin^2\theta)+b\cos\theta\sin\theta]$$
$$+y'^2[a\sin^2\theta-2h\cos\theta\sin\theta+b\cos^2\theta].$$

We then have

$$a'=a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta$$
$$=\tfrac{1}{2}[(a+b)+(a-b)\cos 2\theta+2h\sin 2\theta]\ldots\ldots\ldots\ldots\ldots(1),$$
$$b'=a\sin^2\theta-2h\cos\theta\sin\theta+b\cos^2\theta$$
$$=\tfrac{1}{2}[(a+b)-(a-b)\cos 2\theta-2h\sin 2\theta]\ldots\ldots\ldots\ldots\ldots(2),$$

and
$$h'=-a\cos\theta\sin\theta+h(\cos^2\theta-\sin^2\theta)+b\cos\theta\sin\theta$$
$$=\tfrac{1}{2}[2h\cos 2\theta-(a-b)\sin 2\theta]\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

By adding (1) and (2), we have $a'+b'=a+b$.

Also, by multiplying them, we have

$$4a'b'=(a+b)^2-\{(a-b)\cos 2\theta+2h\sin 2\theta\}^2.$$

Hence $4a'b'-4h'^2$

$$=(a+b)^2-[\{2h\sin 2\theta+(a-b)\cos 2\theta\}^2+\{2h\cos 2\theta-(a-b)\sin 2\theta\}^2]$$
$$=(a+b)^2-[(a-b)^2+4h^2]=4ab-4h^2,$$

so that
$$a'b'-h'^2=ab-h^2.$$

136. *To find the angle through which the axes must be turned so that the expression $ax^2+2hxy+by^2$ may become an expression in which there is no term involving $x'y'$.*

Assuming the work of the previous article the coefficient of $x'y'$ vanishes if h' be zero, or, from equation (3), if

$$2h \cos 2\theta = (a-b) \sin 2\theta,$$

i.e. if

$$\tan 2\theta = \frac{2h}{a-b}.$$

The required angle is therefore

$$\tfrac{1}{2} \tan^{-1}\left(\frac{2h}{a-b}\right).$$

***137.** The proposition of Art. 135 is a particular case, when the axes are rectangular, of the following more general proposition.

If by any change of axes, without change of origin, the quantity $ax^2 + 2hxy + by^2$ *becomes* $a'x^2 + 2h'xy + b'y^2$, *then*

$$\frac{a + b - 2h \cos \omega}{\sin^2 \omega} = \frac{a' + b' - 2h' \cos \omega'}{\sin^2 \omega'},$$

and

$$\frac{ab - h^2}{\sin^2 \omega} = \frac{a'b' - h'^2}{\sin^2 \omega'},$$

ω *and* ω' *being the angles between the original and final pairs of axes.*

Let the coordinates of any point P, referred to the original axes, be x and y and, referred to the final axes, let them be x' and y'.

By Art. 20 the square of the distance between P and the origin is $x^2 + 2xy \cos \omega + y^2$, referred to the original axes, and $x'^2 + 2x'y' \cos \omega' + y'^2$, referred to the final axes.

We therefore always have

$$x^2 + 2xy \cos \omega + y^2 = x'^2 + 2x'y' \cos \omega' + y'^2 \ldots\ldots(1).$$

Also, by supposition, we have

$$ax^2 + 2hxy + by^2 = a'x'^2 + 2h'x'y' + b'y'^2 \ldots\ldots(2).$$

Multiplying (1) by λ and adding it to (2), we therefore have

$$x^2 (a + \lambda) + 2xy (h + \lambda \cos \omega) + y^2 (b + \lambda)$$
$$= x'^2 (a' + \lambda) + 2x'y' (h' + \lambda \cos \omega') + y'^2 (b' + \lambda) \ldots(3).$$

If then any value of λ makes the left-hand side of (3) a perfect square, the same value must make the right-hand side also a perfect square.

But the values of λ which make the left-hand a perfect square are given by the condition

$$(h + \lambda \cos \omega)^2 = (a + \lambda)(b + \lambda),$$

i.e. by

$$\lambda^2(1-\cos^2\omega)+\lambda(a+b-2h\cos\omega)+ab-h^2=0,$$

i.e. by $$\lambda^2+\lambda\frac{a+b-2h\cos\omega}{\sin^2\omega}+\frac{ab-h^2}{\sin^2\omega}=0\ \ldots\ldots\ldots\ (4).$$

In a similar manner the values of λ which make the right-hand side of (3) a perfect square are given by the equation

$$\lambda^2+\lambda\frac{a'+b'-2h'\cos\omega'}{\sin^2\omega'}+\frac{a'b'-h'^2}{\sin^2\omega'}=0\ \ldots\ldots\ (5).$$

Since the values of λ given by equation (4) are the same as the values of λ given by (5), the two equations (4) and (5) must be the same.

Hence we have

$$\frac{\mathbf{a+b-2h\cos\omega}}{\mathbf{\sin^2\omega}}=\frac{\mathbf{a'+b'-2h'\cos\omega'}}{\mathbf{\sin^2\omega'}},$$

and $$\frac{\mathbf{ab-h^2}}{\mathbf{\sin^2\omega}}=\frac{\mathbf{a'b'-h'^2}}{\mathbf{\sin^2\omega'}}.$$

EXAMPLES XVI

1. The equation to a straight line referred to axes inclined at 30° to one another is $y=2x+1$. Find its equation referred to axes inclined at 45°, the origin and axis of x being unchanged.

2. Transform the equation $2x^2+3\sqrt{3}xy+3y^2=2$ from axes inclined at 30° to rectangular axes, the axis of x remaining unchanged.

3. Transform the equation $x^2+xy+y^2=8$ from axes inclined at 60° to axes bisecting the angles between the original axes.

4. Transform the equation $y^2+4y\cot\alpha-4x=0$ from rectangular axes to oblique axes meeting at an angle α, the axis of x being kept the same.

5. If x and y be the coordinates of a point referred to a system of oblique axes, and x' and y' be its coordinates referred to another system of oblique axes with the same origin, and if the formulæ of transformation be

$$x=mx'+ny' \text{ and } y=m'x'+n'y',$$

prove that $$\frac{m^2+m'^2-1}{n^2+n'^2-1}=\frac{mm'}{nn'}.$$

ANSWERS

1. $2x' - \sqrt{6}y' + 1 = 0.$ 2. $x'^2 + \sqrt{3}x'y' = 1.$ 3. $x'^2 + y'^2 = 8.$
4. $y'^2 = 4x' \operatorname{cosec}^2 \alpha.$

SOLUTIONS/HINTS

1. In the formulae of Art. 132, put $\omega = 30°$, $\omega' = 45°$, $\theta = 0$.

For x and y we must substitute

$$\frac{x' \cdot \frac{1}{2} - y' \cdot \dfrac{\sqrt{3}-1}{2\sqrt{2}}}{\frac{1}{2}} \text{ and } \frac{y' \dfrac{1}{\sqrt{2}}}{\frac{1}{2}}.$$

$\therefore$ the equation becomes $\sqrt{2}y' = 2x' - y'(\sqrt{6} - \sqrt{2}) + 1,$

or
$$2x' - \sqrt{6}y' + 1 = 0.$$

2. In the formulae of Art. 135, put $\omega = 30°$, $\omega' = 90°$, $\theta = 0$.

For x and y we must substitute $x' - y'\sqrt{3}$ and $2y'$; and the equation becomes

$$2(x' - y'\sqrt{3})^2 + 6\sqrt{3}y'(x' - y'\sqrt{3}) + 12y'^2 = 2,$$

i.e.
$$x'^2 + \sqrt{3}\,.\,x'y' = 1.$$

3. In the formulae of Art. 132, put

$$\omega = 60°, \ \omega' = 90°, \ \theta = 30°.$$

For x and y we must substitute

$$\frac{\frac{1}{2}x' - y' \dfrac{\sqrt{3}}{2}}{\dfrac{\sqrt{3}}{2}} \text{ and } \frac{\frac{1}{2}x' + y' \dfrac{\sqrt{3}}{2}}{\dfrac{\sqrt{3}}{2}}.$$

The equation then becomes

$$\left(\frac{x'}{\sqrt{3}} - y'\right)^2 + \left(\frac{x'}{\sqrt{3}} + y'\right)^2 + \left(\frac{x'}{\sqrt{3}} + y'\right)\left(\frac{x'}{\sqrt{3}} - y'\right) = 8,$$

i.e.
$$x'^2 + y'^2 = 8.$$

4. In the formulae of Art. 132, put $\omega = 90°$, $\omega' = \alpha$, $\theta = 0$.

For x and y we must substitute $x' + y' \cos \alpha$ and $y' \sin \alpha$; and the equation becomes

$$y'^2 \sin^2 \alpha + 4y' \sin \alpha \,.\, \cot \alpha - 4(x' + y' \cos \alpha) = 0,$$

or
$$y'^2 = 4x' \operatorname{cosec}^2 \alpha.$$

5. Let (x, y) and (x', y') be the coordinates of any point referred to each of the two sets of axes. Then the square of the distance of this point from the origin must be the same in each case.

$$\therefore\ x'^2 + y'^2 + 2x'y' \cos \omega' = x^2 + y^2 + 2xy \cos \omega$$

$$= (mx' + ny')^2 + (m'x' + n'y')^2 + 2(mx' + ny')(m'x' + n'y') \cos \omega.$$

Equating coefficients of x'^2 and y'^2, we have

$$1 = m^2 + m'^2 + 2mm' \cos \omega, \text{ and } 1 = n^2 + n'^2 + 2nn' \cos \omega.$$

Eliminating $\cos \omega$, we have $\dfrac{m^2 + m'^2 - 1}{mm'} = \dfrac{n^2 + n'^2 - 1}{nn'}$.