Chapter 11

Scattering Theory

Much of our understanding about the structure of matter is extracted from the scattering of particles. Had it not been for scattering, the structure of the microphysical world would have remained inaccessible to humans. It is through scattering experiments that important building blocks of matter, such as the atomic nucleus, the nucleons, and the various quarks, have been discovered.

11.1 Scattering and Cross Section

In a scattering experiment, one observes the collisions between a beam of incident particles and a target material. The total number of collisions over the duration of the experiment is proportional to the total number of incident particles and to the number of target particles per unit area in the path of the beam. In these experiments, one counts the collision products that come out of the target. After scattering, those particles that do not interact with the target continue their motion (undisturbed) in the forward direction, but those that interact with the target get scattered (deflected) at some angle as depicted in Figure 11.1. The number of particles coming out varies from one direction to the other. The number of particles scattered into an element of solid angle $d\Omega$ ($d\Omega = \sin\theta \, d\theta \, d\varphi$) is proportional to a quantity that plays a central role in the physics of scattering: the *differential cross section*. The differential cross section, which is denoted by $d\sigma(\theta, \varphi)/d\Omega$, is defined as the number of particles scattered into an element of solid angle $d\Omega$ in the direction (θ, φ) per unit time and incident flux:

$$\frac{d\sigma(\theta, \varphi)}{d\Omega} = \frac{1}{J_{inc}} \frac{dN(\theta, \varphi)}{d\Omega}, \tag{11.1}$$

where J_{inc} is the incident flux (or incident current density); it is equal to the number of incident particles per area per unit time. We can verify that $d\sigma/d\Omega$ has the dimensions of an area; hence it is appropriate to call it a differential cross section.

The relationship between $d\sigma/d\Omega$ and the total cross section σ is obvious:

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} \frac{d\sigma(\theta, \varphi)}{d\Omega} d\varphi. \tag{11.2}$$

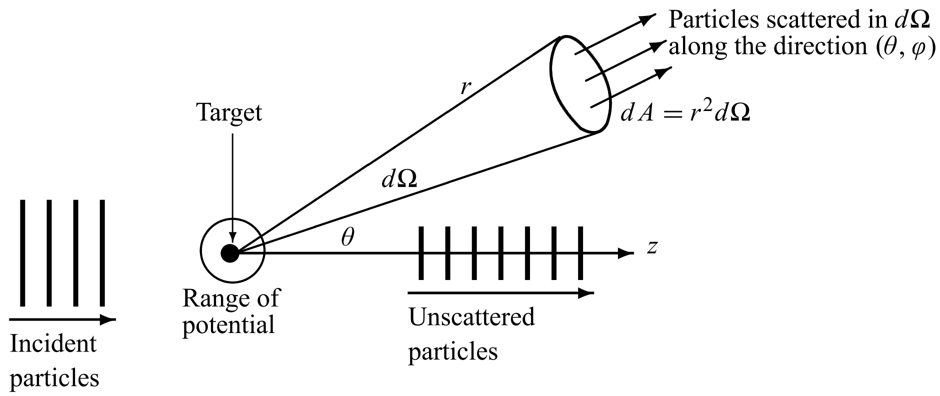

Figure 11.1 Scattering between an incident beam of particles and a fixed target: the scattered particles are detected within a solid angle $d\Omega$ along the direction (θ, φ).

Most scattering experiments are carried out in the laboratory (Lab) frame in which the target is initially at rest while the projectiles are moving. Calculations of the cross sections are generally easier to perform within the center of mass (CM) frame in which the center of mass of the projectiles–target system is at rest (before and after collision). In order to be able to compare the experimental measurements with the theoretical calculations, one has to know how to transform the cross sections from one frame into the other. We should note that the total cross section σ is the same in both frames, since the total number of collisions that take place does not depend on the frame in which the observation is carried out. As for the differential cross sections $d\sigma(\theta, \varphi)/d\Omega$, they are not the same in both frames, since the scattering angles (θ, φ) are frame dependent.

11.1.1 Connecting the Angles in the Lab and CM frames

To find the connection between the Lab and CM cross sections, we need first to find how the scattering angles in one frame are related to their counterparts in the other. Let us consider the scattering of two (structureless, nonrelativistic) particles of masses m_1 and m_2; m_2 represents the target, which is initially at rest, and m_1 the projectile. Figure 11.2 depicts such a scattering in the Lab and CM frames, where θ_1 and θ are the scattering angles of m_1 in the Lab and CM frames, respectively; we are interested in detecting m_1. In what follows we want to find the relation between θ_1 and θ. If $\vec{r}_{1_L}$ and $\vec{r}_{1_C}$ denote the position of m_1 in the Lab and CM frames, respectively, and if $\vec{R}$ denotes the position of the center of mass with respect to the Lab frame, we have $\vec{r}_{1_L} = \vec{r}_{1_C} + \vec{R}$. A time derivative of this relation leads to

$$\vec{V}_{1_L} = \vec{V}_{1_C} + \vec{V}_{CM}, \tag{11.3}$$

where $\vec{V}_{1_L}$ and $\vec{V}_{1_C}$ are the velocities of m_1 in the Lab and CM frames *before* collision and $\vec{V}_{CM}$ is the velocity of the CM with respect to the Lab frame. Similarly, the velocity of m_1 *after* collision is

$$\vec{V'}_{1_L} = \vec{V'}_{1_C} + \vec{V}_{CM}. \tag{11.4}$$

From Figure 11.2a we can infer the x and y components of (11.4):

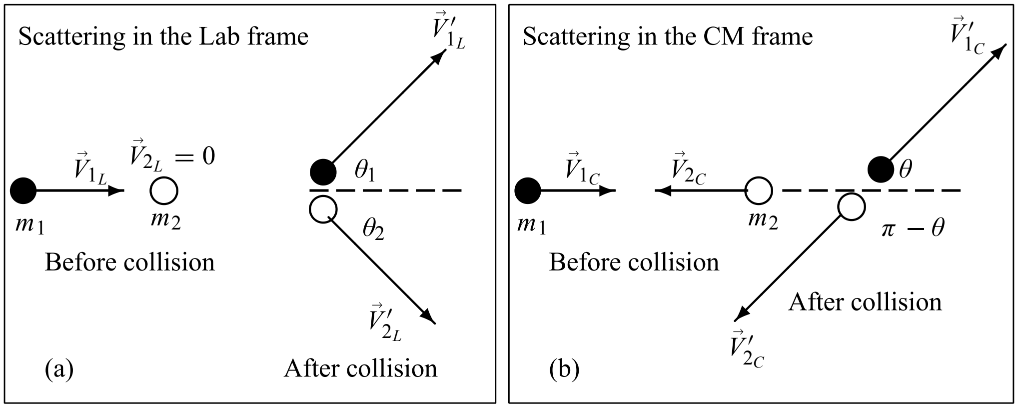

Figure 11.2 Elastic scattering of two structureless particles in the Lab and CM frames: a particle of mass m_1 strikes a particle m_2 initially at rest.

$$V'_{1_L} \cos\theta_1 = V'_{1_C} \cos\theta + V_{CM}, \tag{11.5}$$
$$V'_{1_L} \sin\theta_1 = V'_{1_C} \sin\theta. \tag{11.6}$$

Dividing (11.6) by (11.5), we end up with

$$\tan\theta_1 = \frac{\sin\theta}{\cos\theta + V_{CM}/V'_{1_C}}, \tag{11.7}$$

where V_{CM}/V'_{1_C} can be shown to be equal to m_1/m_2. To see this, since $\vec{V}_{2_L} = 0$, we have

$$\vec{V}_{CM} = \frac{m_1\vec{V}_{1_L} + m_2\vec{V}_{2_L}}{m_1 + m_2} = \frac{m_1}{m_1 + m_2}\vec{V}_{1_L}, \tag{11.8}$$

which when inserted into (11.3) leads to $\vec{V}_{1_L} = \vec{V}_{1_C} + m_1\vec{V}_{1_L}/(m_1 + m_2)$; hence

$$\vec{V}_{1_C} = \left(1 - \frac{m_1}{m_1 + m_2}\right)\vec{V}_{1_L} = \frac{m_2}{m_1 + m_2}\vec{V}_{1_L}. \tag{11.9}$$

On the other hand, since the center of mass is at rest in the CM frame, the total momenta before and after collisions are separately zero:

$$p_C = m_1 V_{1_C} - m_2 V_{2_C} = 0 \implies V_{2_C} = \frac{m_1}{m_2} V_{1_C}, \tag{11.10}$$
$$p'_{C_x} = m_1 V'_{1_C}\cos\theta - m_2 V'_{2_C}\cos\theta = 0 \implies V'_{2_C} = \frac{m_1}{m_2} V'_{1_C}. \tag{11.11}$$

In the case of *elastic* collision, the speeds of the particles in the CM frame are the same before and after collision; to see this, since the kinetic energy is conserved, a substitution of (11.10) and (11.11) into $\frac{1}{2}m_1 V_{1_C}^2 + \frac{1}{2}m_2 V_{2_C}^2 = \frac{1}{2}m_1 {V'_{1_C}}^2 + \frac{1}{2}m_2 {V'_{2_C}}^2$ yields $V'_{1_C} = V_{1_C}$ and $V'_{2_C} = V_{2_C}$. Thus, we can rewrite (11.9) as

$$\vec{V}'_{1_C} = \vec{V}_{1_C} = \frac{m_2}{m_1 + m_2}\vec{V}_{1_L}. \tag{11.12}$$

Dividing (11.8) by (11.12) we obtain

$$\frac{V_{CM}}{V'_{1_C}} = \frac{m_1}{m_2}. \tag{11.13}$$

Finally, a substitution of (11.13) into (11.7) yields

$$\tan\theta_1 = \frac{\sin\theta}{\cos\theta + V_{2C}/V_{1C}} = \frac{\sin\theta}{\cos\theta + m_1/m_2}, \tag{11.14}$$

which, using $\cos\theta_1 = 1/\sqrt{\tan^2\theta_1 + 1}$, becomes

$$\boxed{\cos\theta_1 = \frac{\cos\theta + \frac{m_1}{m_2}}{\sqrt{1 + \frac{m_1^2}{m_2^2} + 2\frac{m_1}{m_2}\cos\theta}}.} \tag{11.15}$$

Remark
By analogy with the foregoing analysis, we can establish a connection between θ_2 and θ. From (11.4) we have $\vec{V}'_{2_L} = \vec{V}'_{2_C} + \vec{V}_{CM}$. The x and y components of this relation are

$$V'_{2_L}\cos\theta_2 = -V'_{2_C}\cos\theta + V_{CM} = (-\cos\theta + 1)V'_{2_C}, \tag{11.16}$$

$$V'_{2_L}\sin\theta_2 = -V'_{2_C}\sin\theta; \tag{11.17}$$

in deriving (11.16), we have used $V_{CM} = V'_{2_C} = V_{2C}$. A combination of (11.16) and (11.17) leads to

$$\boxed{\tan\theta_2 = \frac{\sin\theta}{-\cos\theta + V_{CM}/V'_{2_C}} = \frac{\sin\theta}{1-\cos\theta} = \cot\left(\frac{\theta}{2}\right) \Longrightarrow \theta_2 = \frac{\pi - \theta}{2}.} \tag{11.18}$$

11.1.2 Connecting the Lab and CM Cross Sections

The connection between the differential cross sections in the Lab and CM frames can be obtained from the fact that the number of scattered particles passing through an infinitesimal cross section $d\sigma$ is the same in both frames: $d\sigma(\theta_1, \varphi_1) = d\sigma(\theta, \varphi)$. What differs is the solid angle $d\Omega$, since it is given in the Lab frame by $d\Omega_1 = \sin\theta_1 d\theta_1 d\varphi_1$ and in the CM frame by $d\Omega = \sin\theta\, d\theta d\varphi$. Thus, we have

$$\left(\frac{d\sigma}{d\Omega_1}\right)_{Lab} d\Omega_1 = \left(\frac{d\sigma}{d\Omega}\right)_{CM} d\Omega \Longrightarrow \left(\frac{d\sigma}{d\Omega_1}\right)_{Lab} = \left(\frac{d\sigma}{d\Omega}\right)_{CM} \frac{\sin\theta}{\sin\theta_1}\frac{d\theta}{d\theta_1}\frac{d\varphi}{d\varphi_1}, \tag{11.19}$$

where (θ_1, φ_1) are the scattering angles of particle m_1 in the Lab frame and (θ, φ) are its angles in the CM frame. Since there is cylindrical symmetry around the direction of the incident beam, we have $\varphi = \varphi_1$ and hence

$$\left(\frac{d\sigma}{d\Omega_1}\right)_{Lab} = \left(\frac{d\sigma}{d\Omega}\right)_{CM} \frac{d(\cos\theta)}{d(\cos\theta_1)}. \tag{11.20}$$

From (11.15) we have

$$\frac{d\cos\theta_1}{d\cos\theta} = \frac{1 + \frac{m_1}{m_2}\cos\theta}{\left(1 + \frac{m_1^2}{m_2^2} + 2\frac{m_1}{m_2}\cos\theta\right)^{3/2}}, \tag{11.21}$$

which when substituted into (11.20) leads to

$$\boxed{\left(\frac{d\sigma}{d\Omega_1}\right)_{Lab} = \frac{(1+\frac{m_1^2}{m_2^2}+2\frac{m_1}{m_2}\cos\theta)^{3/2}}{1+\frac{m_1}{m_2}\cos\theta}\left(\frac{d\sigma}{d\Omega}\right)_{CM}.} \tag{11.22}$$

Similarly, we can show that (11.20) and (11.18) yield

$$\boxed{\left(\frac{d\sigma}{d\Omega_2}\right)_{Lab} = 4\cos\theta_2\left(\frac{d\sigma}{d\Omega_2}\right)_{CM} = 4\sin\left(\frac{\theta}{2}\right)\left(\frac{d\sigma}{d\Omega_2}\right)_{CM}.} \tag{11.23}$$

Limiting cases: (a) If $m_2 \gg m_1$, or when $\frac{m_1}{m_2} \to 0$, the Lab and CM results are the same, since (11.15) leads to $\theta_1 = \theta$ and (11.22) to $\left(\frac{d\sigma}{d\Omega_1}\right)_{Lab} = \left(\frac{d\sigma}{d\Omega}\right)_{CM}$. (b) If $m_2 = m_1$ then (11.15) leads to $\tan\theta_1 = \tan(\theta/2)$ or to $\theta_1 = \theta/2$; in this case (11.22) yields $\left(\frac{d\sigma}{d\Omega_1}\right)_{Lab} = 4\left(\frac{d\sigma}{d\Omega}\right)_{CM}\cos(\theta/2)$.

Example 11.1
In an elastic collision between two particles of equal mass, show that the two particles come out at right angles with respect to each other in the Lab frame.

Solution
In the special case $m_1 = m_2$, equations (11.14) and (11.18) respectively become

$$\tan\theta_1 = \tan\left(\frac{\theta}{2}\right), \qquad \tan\theta_2 = \cot\left(\frac{\theta}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right). \tag{11.24}$$

These two equations yield

$$\theta_1 = \frac{\theta}{2}, \qquad \theta_2 = \frac{\pi}{2} - \frac{\theta}{2} = \frac{\pi}{2} - \theta_1; \tag{11.25}$$

hence $\theta_1 + \theta_2 = \pi/2$. In these cases, (11.22) and (11.23) yield

$$\left(\frac{d\sigma}{d\Omega_1}\right)_{Lab} = 4\left(\frac{d\sigma}{d\Omega}\right)_{CM}\cos\theta_1 = 4\left(\frac{d\sigma}{d\Omega}\right)_{CM}\cos\left(\frac{\theta}{2}\right), \tag{11.26}$$

$$\left(\frac{d\sigma}{d\Omega_2}\right)_{Lab} = 4\left(\frac{d\sigma}{d\Omega_2}\right)_{CM}\cos\theta_2 = 4\left(\frac{d\sigma}{d\Omega}\right)_{CM}\sin\left(\frac{\theta}{2}\right). \tag{11.27}$$

11.2 Scattering Amplitude of Spinless Particles

The foregoing discussion dealt with definitions of the cross section and how to transform it from the Lab to the CM frame; the conclusions reached apply to classical as well as to quantum mechanics. In this section we deal with the quantum description of scattering. For simplicity, we consider the case of elastic[1] scattering between two spinless, nonrelativistic particles of masses m_1 and m_2. During the scattering process, the particles interact with one another. If the interaction is *time independent*, we can describe the two-particle system with stationary states

$$\Psi(\vec{r}_1, \vec{r}_2, t) = \psi(\vec{r}_1, \vec{r}_2) e^{-iE_T t/\hbar}, \tag{11.28}$$

where E_T is the total energy and $\psi(\vec{r}_1, \vec{r}_2,)$ is a solution of the time-independent Schrödinger equation:

$$\left[-\frac{\hbar^2}{2m_1}\vec{\nabla}_1^2 - \frac{\hbar^2}{2m_2}\vec{\nabla}_2^2 + \hat{V}(\vec{r}_1, \vec{r}_2) \right] \psi(\vec{r}_1, \vec{r}_2) = E_T \psi(\vec{r}_1, \vec{r}_2); \tag{11.29}$$

$\hat{V}(\vec{r}_1, \vec{r}_2)$ is the potential representing the interaction between the two particles.

In the case where the interaction between m_1 and m_2 depends only on their relative distance $r = |\vec{r}_1 - \vec{r}_2|$ (i.e., $\hat{V}(\vec{r}_1, \vec{r}_2) = \hat{V}(r)$), we can, as seen in Chapter 6, reduce the eigenvalue problem (11.29) to two decoupled eigenvalue problems: one for the center of mass (CM), which moves like a free particle of mass $M = m_1 + m_2$ and which is of no concern to us here, and another for a fictitious particle with a reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ which moves in the potential $\hat{V}(r)$:

$$-\frac{\hbar^2}{2\mu}\vec{\nabla}^2 \psi(\vec{r}) + \hat{V}(r)\psi(\vec{r}) = E\psi(\vec{r}). \tag{11.30}$$

The problem of scattering between two particles is thus reduced to solving this equation. We are going to show that the differential cross section in the CM frame can be obtained from an asymptotic form of the solution of (11.30). Its solutions can then be used to calculate the probability per unit solid angle per unit time that the particle μ is scattered into a solid angle $d\Omega$ in the direction $(\theta,\ \varphi)$; this probability is given by the differential cross section $d\sigma/d\Omega$. In quantum mechanics the incident particle is described by means of a wave packet that interacts with the target. The incident wave packet must be spatially large so that spreading during the experiment is not appreciable. It must be large compared to the target's size and yet small compared to the size of the Lab so that it does not overlap simultaneously with the target and detector. After scattering, the wave function consists of an unscattered part propagating in the forward direction and a scattered part that propagates along some direction $(\theta,\ \varphi)$.

We can view (11.30) as representing the scattering of a particle of mass μ from a fixed scattering center that is described by $V(r)$, where r is the distance from the particle μ to the center of $V(r)$. We assume that $V(r)$ has a *finite* range a. Thus the interaction between the particle and the potential occurs only in a limited region of space $r \leq a$, which is called the *range* of $V(r)$, or the scattering region. Outside the range, $r > a$, the potential vanishes, $V(r) = 0$; the eigenvalue problem (11.30) then becomes

$$\left(\nabla^2 + k_0^2\right) \phi_{inc}(\vec{r}) = 0, \tag{11.31}$$

where $k_0^2 = 2\mu E/\hbar^2$. In this case μ behaves as a *free* particle before collision and hence can be described by a *plane* wave

$$\phi_{inc}(\vec{r}) = A e^{i\vec{k}_0 \cdot \vec{r}}, \tag{11.32}$$

where $\vec{k}_0$ is the wave vector associated with the incident particle and A is a normalization factor. Thus, prior to the interaction with the target, the particles of the incident beam are independent of each other; they move like free particles, each with a momentum $\vec{p} = \hbar\vec{k}_0$.

[1] In *elastic* scattering, the internal states and the structure of the colliding particles do not change.

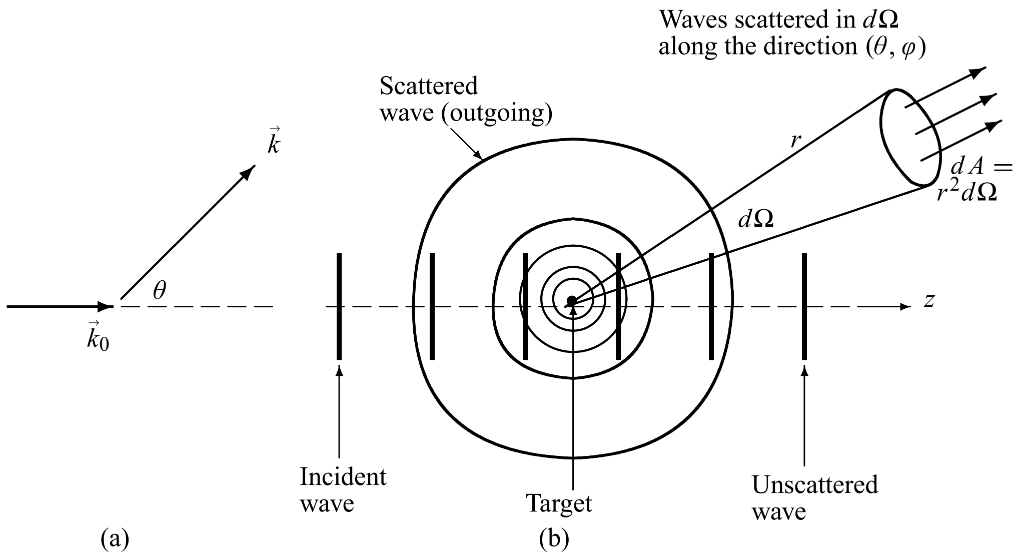

Figure 11.3 (a) Angle between the incident and scattered wave vectors $\vec{k}_0$ and $\vec{k}$. (b) Incident and scattered waves: the incident wave is a plane wave, $\phi_{inc}(\vec{r}) = Ae^{i\vec{k}_0\cdot\vec{r}}$, and the scattered wave, $\phi_{sc}(\vec{r}) = Af(\theta, \varphi)\frac{e^{i\vec{k}\cdot\vec{r}}}{r}$, is an outgoing wave.

When the incident wave (11.32) collides or interacts with the target, an outgoing wave $\phi_{sc}(\vec{r})$ is scattered out. In the case of an *isotropic* scattering, the scattered wave is *spherically symmetric*, having the form $e^{i\vec{k}\cdot\vec{r}}/r$. In general, however, the scattered wave is not spherically symmetric; its amplitude depends on the direction (θ, φ) along which it is detected and hence

$$\phi_{sc}(\vec{r}) = Af(\theta, \varphi)\frac{e^{i\vec{k}\cdot\vec{r}}}{r}, \tag{11.33}$$

where $f(\theta, \varphi)$ is called the *scattering amplitude*, $\vec{k}$ is the wave vector associated with the scattered particle, and θ is the angle between $\vec{k}_0$ and $\vec{k}$ as displayed in Figure 11.3a. After the scattering has taken place (Figure 11.3b), the total wave consists of a superposition of the incident plane wave (11.32) and the scattered wave (11.33):

$$\boxed{\psi(\vec{r}) = \phi_{inc}(\vec{r}) + \phi_{sc}(\vec{r}) \simeq A\left[e^{i\vec{k}_0\cdot\vec{r}} + f(\theta, \varphi)\frac{e^{i\vec{k}\cdot\vec{r}}}{r}\right],} \tag{11.34}$$

where A is a normalization factor; since A has no effect on the cross section, as will be shown in (11.40), we will take it equal to one throughout the rest of the chapter. We now need to determine $f(\theta, \varphi)$ and $d\sigma/d\Omega$. In the following section we are going to show that the differential cross section is given in terms of the scattering amplitude by $d\sigma/d\Omega = |f(\theta, \varphi)|^2$.

11.2.1 Scattering Amplitude and Differential Cross Section

The scattering amplitude $f(\theta, \varphi)$ plays a central role in the theory of scattering, since it determines the differential cross section. To see this, let us first introduce the incident and scattered flux densities:

$$\vec{J}_{inc} = \frac{i\hbar}{2\mu}(\phi_{inc}\vec{\nabla}\phi^*_{inc} - \phi^*_{inc}\vec{\nabla}\phi_{inc}), \tag{11.35}$$

$$\vec{J}_{sc} = \frac{i\hbar}{2\mu}(\phi_{sc}\vec{\nabla}\phi^*_{sc} - \phi^*_{sc}\vec{\nabla}\phi_{sc}). \tag{11.36}$$

Inserting (11.32) into (11.35) and (11.33) into (11.36) and taking the magnitudes of the expressions thus obtained, we end up with

$$J_{inc} = |A|^2\frac{\hbar k_0}{\mu}, \qquad J_{sc} = |A|^2\frac{\hbar k}{\mu r^2}\left| f(\theta, \varphi)\right|^2. \tag{11.37}$$

Now, we may recall that the number $dN(\theta, \varphi)$ of particles scattered into an element of solid angle $d\Omega$ in the direction (θ, φ) and passing through a surface element $dA = r^2 d\Omega$ per unit time is given as follows (see (11.1)):

$$dN(\theta, \varphi) = J_{sc}r^2 d\Omega. \tag{11.38}$$

When combined with (11.37) this relation yields

$$\frac{dN}{d\Omega} = J_{sc}r^2 = |A|^2\frac{\hbar k}{\mu}\left| f(\theta, \varphi)\right|^2. \tag{11.39}$$

Now, inserting (11.39) and $J_{inc} = |A|^2\hbar k_0/\mu$ into (11.1), we end up with

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{J_{inc}}\frac{dN}{d\Omega} = \frac{k}{k_0}\left| f(\theta, \varphi)\right|^2.} \tag{11.40}$$

Since the normalization factor A does not contribute to the differential cross section, we will be taking it equal to one. For elastic scattering k_0 is equal to k; hence (11.40) reduces to

$$\boxed{\frac{d\sigma}{d\Omega} = \left| f(\theta, \varphi)\right|^2.} \tag{11.41}$$

The problem of determining the differential cross section $d\sigma/d\Omega$ therefore reduces to that of obtaining the scattering amplitude $f(\theta, \varphi)$.

11.2.2 Scattering Amplitude

We are going to show here that we can obtain the differential cross section in the CM frame from an asymptotic form of the solution of the Schrödinger equation (11.30). Let us first focus on the determination of $f(\theta, \varphi)$; it can be obtained from the solutions of (11.30), which in turn can be rewritten as

$$(\nabla^2 + k^2)\psi(\vec{r}) = \frac{2\mu}{\hbar^2}V(\vec{r})\psi(\vec{r}). \tag{11.42}$$

The general solution to this equation consists of a sum of two components: a general solution to the homogeneous equation:

$$(\nabla^2 + k_0^2)\psi_{homo}(\vec{r}) = 0, \tag{11.43}$$

and a particular solution to (11.42). First, note that $\psi_{homo}(\vec{r})$ is nothing but the incident plane wave (11.32). As for the particular solution to (11.42), we can express it in terms of *Green's function*. Thus, the general solution of (11.42) is given by

$$\psi(\vec{r}) = \phi_{inc}(\vec{r}) + \frac{2\mu}{\hbar^2}\int G(\vec{r}-\vec{r}\,')V(\vec{r}\,')\psi(\vec{r}\,')\,d^3r', \tag{11.44}$$

where $\phi_{inc}(\vec{r}) = e^{i\vec{k}_0\cdot\vec{r}}$ and $G(\vec{r}-\vec{r}\,')$ is Green's function corresponding to the operator on the left-hand side of (11.43). The function $G(\vec{r}-\vec{r}\,')$ is obtained by solving the point source equation

$$(\nabla^2 + k^2)G(\vec{r}-\vec{r}\,') = \delta(\vec{r}-\vec{r}\,'), \tag{11.45}$$

where $G(\vec{r}-\vec{r}\,')$ and $\delta(\vec{r}-\vec{r}\,')$ are given by their Fourier transforms as follows:

$$G(\vec{r}-\vec{r}\,') = \frac{1}{(2\pi)^3}\int e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')}\tilde{G}(\vec{q})\,d^3q, \tag{11.46}$$

$$\delta(\vec{r}-\vec{r}\,') = \frac{1}{(2\pi)^3}\int e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')}\,d^3q. \tag{11.47}$$

A substitution of (11.46) and (11.47) into (11.45) leads to

$$\left(-\vec{q}^{\,2} + \vec{k}^{\,2}\right)\tilde{G}(\vec{q}) = 1 \qquad \Longrightarrow \qquad \tilde{G}(\vec{q}) = \frac{1}{\vec{k}^{\,2} - \vec{q}^{\,2}}. \tag{11.48}$$

The expression for $G(\vec{r}-\vec{r}\,')$ is obtained by inserting (11.48) into (11.46):

$$G(\vec{r}-\vec{r}\,') = \frac{1}{(2\pi)^3}\int \frac{e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')}}{k^2-q^2}d^3q. \tag{11.49}$$

To integrate over the angles in

$$G(\vec{r}-\vec{r}\,') = \frac{1}{(2\pi)^3}\int_0^\infty \frac{q^2dq}{k^2-q^2}\int_0^\pi e^{iq|\vec{r}-\vec{r}\,'|\cos\theta}\sin\theta\,d\theta\int_0^{2\pi}d\varphi, \tag{11.50}$$

we need simply to make the variables change $x = \cos\theta$:

$$\int_0^\pi e^{iq|\vec{r}-\vec{r}\,'|\cos\theta}\sin\theta\,d\theta = \int_{-1}^1 e^{iq|\vec{r}-\vec{r}\,'|x}dx = \frac{1}{iq|\vec{r}-\vec{r}\,'|}\left(e^{iq|\vec{r}-\vec{r}\,'|} - e^{-iq|\vec{r}-\vec{r}\,'|}\right). \tag{11.51}$$

Hence (11.50) becomes

$$G(\vec{r}-\vec{r}\,') = \frac{1}{4\pi^2 i|\vec{r}-\vec{r}\,'|}\int_0^\infty \frac{q}{k^2-q^2}\left(e^{iq|\vec{r}-\vec{r}\,'|} - e^{-iq|\vec{r}-\vec{r}\,'|}\right)dq, \tag{11.52}$$

or

$$G(\vec{r}-\vec{r}\,') = -\frac{1}{4\pi^2 i|\vec{r}-\vec{r}\,'|}\int_{-\infty}^{+\infty}\frac{qe^{iq|\vec{r}-\vec{r}\,'|}}{q^2-k^2}dq. \tag{11.53}$$

We may evaluate this integral by the method of residues by closing the contour in the upper half of the q-plane: it is equal to $2\pi i$ times the residue of the integrand at the poles. Since there

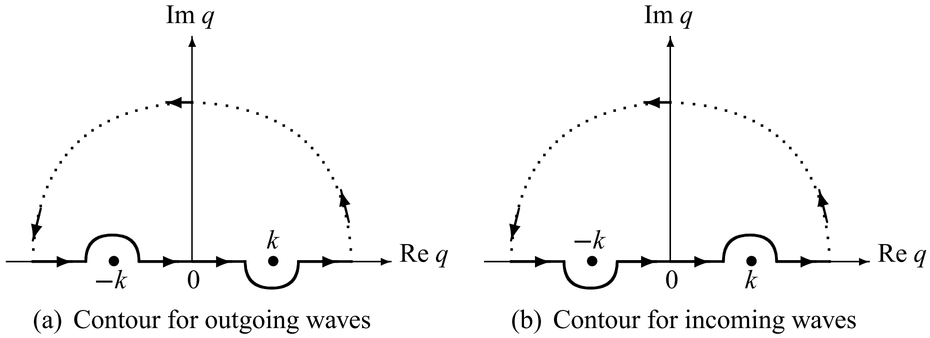

Figure 11.4 Contours corresponding to outgoing and incoming waves.

are two poles, $q = \pm k$, the integral has two possible values. The value corresponding to the pole at $q = k$, which lies inside the contour of integration in Figure 11.4a, is given by

$$G_+(\vec{r} - \vec{r}\,') = -\frac{1}{4\pi} \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r} - \vec{r}\,'|} \tag{11.54}$$

and the value for the pole at $q = -k$ (Figure 11.4b) is

$$G_-(\vec{r} - \vec{r}\,') = -\frac{1}{4\pi} \frac{e^{-ik|\vec{r}-\vec{r}'|}}{|\vec{r} - \vec{r}\,'|}. \tag{11.55}$$

Green's function $G_+(\vec{r} - \vec{r}\,')$ represents an *outgoing spherical wave* emitted from $\vec{r}\,'$ and the function $G_-(\vec{r} - \vec{r}\,')$ corresponds to an *incoming wave* that converges onto $\vec{r}\,'$. Since the scattered waves are outgoing waves, only $G_+(\vec{r} - \vec{r}\,')$ is of interest to us. Inserting (11.54) into (11.44) we obtain the total scattered wave function:

$$\boxed{\psi(\vec{r}) = \phi_{inc}(\vec{r}) - \frac{\mu}{2\pi\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r} - \vec{r}\,'|} V(\vec{r}\,')\psi(\vec{r}\,')\, d^3r'.} \tag{11.56}$$

This is an *integral equation*; it does not yet give the unknown solution $\psi(\vec{r})$ but only contains it in the integrand. All we have done is to rewrite the Schrödinger (differential) equation (11.30) in an integral form (11.56), because the integral form is suitable for use in scattering theory. We are going to show that (11.56) reduces to (11.34) in the asymptotic limit $r \longrightarrow \infty$. But let us first mention that (11.56) can be solved approximately by means of a series of successive or iterative approximations, known as the *Born series*. The zero-order solution is given by $\psi_0(\vec{r}) = \phi_{inc}(\vec{r})$. The first-order solution $\psi_1(\vec{r})$ is obtained by inserting $\psi_0(\vec{r}) = \phi_{inc}(\vec{r})$ into the integral sign of (11.56):

$$\begin{aligned} \psi_1(\vec{r}) &= \phi_{inc}(\vec{r}) - \frac{\mu}{2\pi\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}_1|}}{|\vec{r} - \vec{r}_1|} V(\vec{r}_1)\psi_0(\vec{r}_1)\, d^3r_1 \\ &= \phi_{inc}(\vec{r}) - \frac{\mu}{2\pi\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}_1|}}{|\vec{r} - \vec{r}_1|} V(\vec{r}_1)\phi_{inc}(\vec{r}_1)\, d^3r_1. \end{aligned} \tag{11.57}$$

The second order is obtained by inserting $\psi_1(\vec{r})$ into (11.56):

$$\psi_2(\vec{r}) = \phi_{inc}(\vec{r}) - \frac{\mu}{2\pi\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}_2|}}{|\vec{r} - \vec{r}_2|} V(\vec{r}_2)\psi_1(\vec{r}_2)\, d^3r_2$$

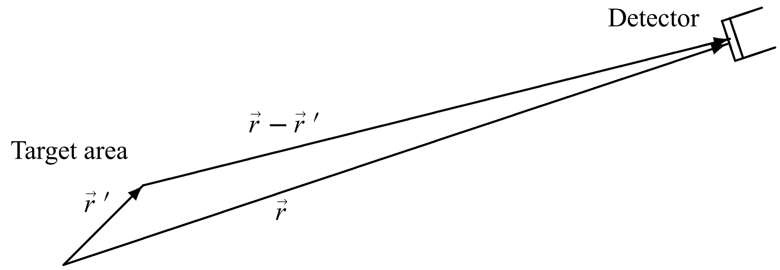

Figure 11.5 The distance r from the target to the detector is too large compared to the size r' of the target: $r \gg r'$.

$$= \phi_{inc}(\vec{r}) - \frac{\mu}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}_2|}}{|\vec{r}-\vec{r}_2|} V(\vec{r}_2)\phi_{inc}(\vec{r}_2)\, d^3r_2$$
$$+ \left(\frac{\mu}{2\pi\hbar^2}\right)^2 \int \frac{e^{ik|\vec{r}-\vec{r}_2|}}{|\vec{r}-\vec{r}_2|} V(\vec{r}_2)\, d^3r_2 \int \frac{e^{ik|\vec{r}_2-\vec{r}_1|}}{|\vec{r}_2-\vec{r}_1|} V(\vec{r}_1)\phi_{inc}(\vec{r}_1)\, d^3r_1. \tag{11.58}$$

Continuing in this way, we can obtain $\psi(\vec{r})$ to the desired order; the nth order approximation for the wave function is a series which can be obtained by analogy with (11.57) and (11.58).

Asymptotic limit of the wave function

We are now going to show that (11.56) reduces to (11.34) for large values of r. In a scattering experiment, since the detector is located at distances (away from the target) that are much larger than the size of the target (Figure 11.5), we have $r \gg r'$, where r represents the distance from the target to the detector and r' the size of the detector. If $r \gg r'$ we may approximate $k|\vec{r} - \vec{r}\,'|$ and $|\vec{r} - \vec{r}\,'|^{-1}$ by

$$k|\vec{r} - \vec{r}\,'| = k\sqrt{\vec{r}^2 - 2\vec{r}\cdot\vec{r}\,' + \vec{r}\,'^2} \simeq kr - k\frac{\vec{r}}{r}\cdot\vec{r}\,' = kr - \vec{k}\cdot\vec{r}\,', \tag{11.59}$$

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \frac{1}{r}\frac{1}{\left|1 - \vec{r}\cdot\vec{r}\,'/\vec{r}^{\,2}\right|} \simeq \frac{1}{r}\left(1 + \frac{\vec{r}\cdot\vec{r}\,'}{r^2}\right) \simeq \frac{1}{r}, \tag{11.60}$$

where $\vec{k} = k\hat{r}$ is the wave vector associated with the scattered particle. From the previous two approximations, we may write the asymptotic form of (11.56) as follows:

$$\psi(\vec{r}) \longrightarrow e^{i\vec{k}_0\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\theta, \varphi) \qquad (r \to \infty), \tag{11.61}$$

where

$$\boxed{f(\theta, \varphi) = -\frac{\mu}{2\pi\hbar^2}\int e^{-i\vec{k}.\vec{r}\,'} V(\vec{r}\,')\psi(\vec{r}\,')\, d^3r' = -\frac{\mu}{2\pi\hbar^2}\langle \phi \mid \hat{V} \mid \psi\rangle,} \tag{11.62}$$

where $\phi(\vec{r})$ is a plane wave, $\phi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$, and $\vec{k}$ is the wave vector of the scattered wave; the integration variable r' extends over the spatial degrees of freedom of the target. The differential cross section is then given by

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta, \varphi)|^2 = \frac{\mu^2}{4\pi^2\hbar^4}\left|\int e^{-i\vec{k}\cdot\vec{r}\,'}\hat{V}(\vec{r}\,')\psi(\vec{r}\,')\,d^3r'\right|^2 = \frac{\mu^2}{4\pi^2\hbar^4}\left|\langle\phi \mid \hat{V} \mid \psi\rangle\right|^2.} \tag{11.63}$$

11.3 The Born Approximation

11.3.1 The First Born Approximation

If the potential $V(\vec{r})$ is weak enough, it will distort only slightly the incident plane wave. The *first Born approximation* consists then of approximating the scattered wave function $\psi(\vec{r})$ by a *plane wave*. This approximation corresponds to the first iteration of (11.56); that is, $\psi(\vec{r})$ is given by (11.57):

$$\psi(\vec{r}) \simeq \phi_{inc}(\vec{r}) - \frac{\mu}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}V(\vec{r}\,')\phi_{inc}(\vec{r}\,')\,d^3r'. \tag{11.64}$$

Thus, using (11.62) and (11.63), we can write the scattering amplitude and the differential cross section in the first Born approximation as follows:

$$f(\theta, \varphi) = -\frac{\mu}{2\pi\hbar^2}\int e^{-i\vec{k}\cdot\vec{r}'}V(\vec{r}\,')\phi_{inc}(\vec{r}\,')\,d^3r' = -\frac{\mu}{2\pi\hbar^2}\int e^{i\vec{q}\cdot\vec{r}\,'}V(\vec{r}\,')\,d^3r', \tag{11.65}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \left|f(\theta, \varphi)\right|^2 = \frac{\mu^2}{4\pi^2\hbar^4}\left|\int e^{i\vec{q}\cdot\vec{r}\,'}V(\vec{r}\,')\,d^3r'\right|^2,} \tag{11.66}$$

where $\vec{q} = \vec{k}_0 - \vec{k}$ and $\hbar\vec{q}$ is the momentum transfer; $\hbar\vec{k}_0$ and $\hbar\vec{k}$ are the linear momenta of the incident and scattered particles, respectively.

In *elastic scattering*, the magnitudes of $\vec{k}_0$ and $\vec{k}$ are equal (Figure 11.6); hence

$$q = \left|\vec{k}_0 - \vec{k}\right| = \sqrt{k_0^2 + k^2 - 2kk_0\cos\theta} = k\sqrt{2(1-\cos^2\theta)} = 2k\sin\left(\frac{\theta}{2}\right). \tag{11.67}$$

If the potential $V(\vec{r}\,')$ is *spherically symmetric*, $V(\vec{r}\,') = V(r')$, and choosing the z-axis along $\vec{q}$ (Figure 11.6), then $\vec{q}\cdot\vec{r}\,' = qr'\cos\theta'$ and therefore

$$\begin{aligned}\int e^{i\vec{q}\cdot\vec{r}\,'}V(\vec{r}\,')\,d^3r' &= \int_0^\infty r'^2 V(r')\,dr'\int_0^\pi e^{iqr'\cos\theta'}\sin\theta' d\theta'\int_0^{2\pi}d\varphi' \\ &= 2\pi\int_0^\infty r'^2 V(\vec{r}\,')\,dr'\int_{-1}^1 e^{iqr'x}dx = \frac{4\pi}{q}\int_0^\infty r'V(r')\sin(qr')\,dr'.\end{aligned} \tag{11.68}$$

Inserting (11.68) into (11.65) and (11.66) we obtain

$$\boxed{f(\theta) = -\frac{2\mu}{\hbar^2 q}\int_0^\infty r'V(r')\sin(qr')\,dr',} \tag{11.69}$$

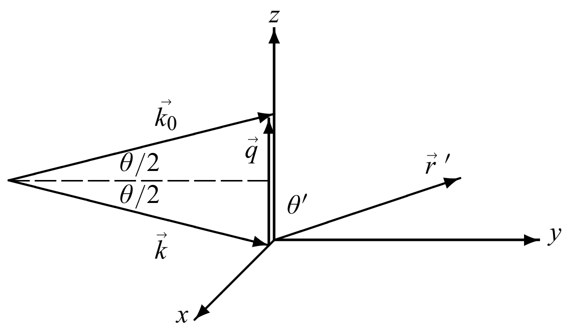

Figure 11.6 Momentum transfer for elastic scattering: $q = |\vec{k}_0 - \vec{k}| = 2k\sin(\theta/2)$, $k_0 = k$.

$$\boxed{\frac{d\sigma}{d\Omega} = \left| f(\theta)\right|^2 = \frac{4\mu^2}{\hbar^4 q^2}\left|\int_0^\infty r' V(r')\sin(qr')\,dr'\right|^2.} \tag{11.70}$$

In summary, we have shown that by solving the Schrödinger equation (11.30) to first-order Born approximation (where the potential $V(\vec{r})$ is weak enough that the scattered wave function is only slightly different from the incident plane wave), the differential cross section is given by equation (11.70) for a spherically symmetric potential.

11.3.2 Validity of the First Born Approximation

The first Born approximation is valid whenever the wave function $\psi(\vec{r})$ is only slightly different from the incident plane wave; that is, whenever the second term in (11.64) is very small compared to the first:

$$\left|\frac{\mu}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}V(r')e^{i\vec{k}_0\cdot\vec{r}\,'}\,d^3r'\right| \ll |\phi_{inc}(\vec{r})|^2. \tag{11.71}$$

Since $\phi_{inc} = e^{i\vec{k}_0\cdot\vec{r}}$ we have

$$\left|\frac{\mu}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}V(r')e^{i\vec{k}_0\cdot\vec{r}\,'}\,d^3r'\right| \ll 1. \tag{11.72}$$

In elastic scattering $k_0 = k$ and assuming that the scattering potential is largest near $r = 0$, we have

$$\left|\frac{\mu}{\hbar^2}\int_0^\infty r' e^{ik\vec{r}\,'}V(r')\,dr'\int_0^\pi e^{ikr'\cos\theta'}\sin\theta' d\theta'\right| \ll 1 \tag{11.73}$$

or

$$\boxed{\frac{\mu}{\hbar^2 k}\left|\int_0^\infty V(r')\left(e^{2ikr'}-1\right)dr'\right| \ll 1.} \tag{11.74}$$

Since the energy of the incident particle is proportional to k (it is purely kinetic, $E_i = \hbar^2k^2/2\mu$), we infer from (11.74) that the Born approximation is valid for large incident energies and weak scattering potentials. That is, when the average interaction energy between the incident particle and the scattering potential is much smaller than the particle's incident kinetic energy, the scattered wave can be considered to be a plane wave.

Example 11.2

(a) Calculate the differential cross section in the first Born approximation for a Coulomb potential $V(r) = Z_1 Z_2 e^2/r$, where $Z_1 e$ and $Z_1 e$ are the charges of the projectile and target particles, respectively.

(b) To have a quantitative idea about the cross section derived in (a), consider the scattering of an alpha particle (i.e., a helium nucleus with $Z_1 = 2$ and $A_1 = 4$) from a gold nucleus ($Z_2 = 79$ and $A_2 = 197$). (i) If the scattering angle of the alpha particle in the Lab frame is $\theta_1 = 60°$, find its scattering angle θ in the CM frame. (ii) If the incident energy of the alpha particle is 8 MeV, find a numerical estimate for the cross section derived in (a).

Solution

In the case of a Coulomb potential, $V(r) = Z_1 Z_2 e^2/r$, equation (11.70) becomes

$$\frac{d\sigma}{d\Omega} = \frac{4Z_1^2 Z_2^2 e^4 \mu^2}{\hbar^4 q^2} \left| \int_0^\infty \sin(qr)\, dr \right|^2, \tag{11.75}$$

where

$$\begin{aligned} \int_0^\infty \sin(qr)\, dr &= \lim_{\lambda \to 0} \int_0^\infty e^{-\lambda r} \sin(qr)\, dr = \frac{1}{2i} \lim_{\lambda \to 0} \left[\int_0^\infty e^{-(\lambda - iq)r}\, dr - \int_0^\infty e^{-(\lambda + iq)r}\, dr \right] \\ &= \frac{1}{2i} \lim_{\lambda \to 0} \left[\frac{1}{\lambda - iq} - \frac{1}{\lambda + iq} \right] = \frac{1}{q}. \end{aligned} \tag{11.76}$$

Now, since $q = 2k \sin(\theta/2)$, an insertion of (11.76) into (11.75) leads to

$$\frac{d\sigma}{d\Omega} = \left(\frac{2Z_1 \mu Z_2 e^2}{\hbar^2 q^2} \right)^2 = \left(\frac{Z_1 Z_2 \mu e^2}{2\hbar^2 k^2} \right)^2 \sin^{-4}\left(\frac{\theta}{2} \right) = \frac{Z_1^2 Z_2^2 e^4}{16E^2} \sin^{-4}\left(\frac{\theta}{2} \right), \tag{11.77}$$

where $E = \hbar^2 k^2/2\mu$ is the kinetic energy of the incident particle. This relation is known as the *Rutherford formula* or the Coulomb cross section.

(b) (i) Since the mass ratio of the alpha particle to the gold nucleus is roughly equal to the ratio of their atomic masses, $m_1/m_2 = A_1/A_2 = \frac{4}{197} = 0.0203$, and since $\theta_1 = 60°$, equation (11.14) yields the value of the scattering angle in the CM frame:

$$\tan 60° = \frac{\sin\theta}{\cos\theta + 0.0203} \qquad \Longrightarrow \qquad \theta = 61°. \tag{11.78}$$

(ii) The numerical estimate of the cross section can be made easier by rewriting (11.77) in terms of the fine structure constant $\alpha = e^2/\hbar c = \frac{1}{137}$ and $\hbar c = 197.33$ MeV fm:

$$\frac{d\sigma}{d\Omega} = \frac{Z_1^2 Z_2^2}{16E^2} \left(\frac{e^2}{\hbar c} \right)^2 (\hbar c)^2 \sin^{-4}\left(\frac{\theta}{2} \right) = \left(\frac{Z_1 Z_2 \alpha}{4} \right)^2 \left(\frac{\hbar c}{E} \right)^2 \sin^{-4}\left(\frac{\theta}{2} \right). \tag{11.79}$$

Since $Z_1 = 2$, $Z_2 = 79$, $\theta = 61°$, $\alpha = \frac{1}{137}$, $\hbar c = 197.33$ MeV fm, and $E = 8$ MeV, we have

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \left(\frac{2 \times 79}{4 \times 137} \right)^2 \left(\frac{197.33 \text{ MeV fm}}{8 \text{ MeV}} \right)^2 \sin^{-4}(30.5°) \\ &= 30.87 \text{ fm}^2 = 0.31 \times 10^{-28} \text{ m}^2 = 0.31 \text{ barn}, \end{aligned} \tag{11.80}$$

where 1 barn $= 10^{-28}$ m^2.

11.4 Partial Wave Analysis

So far we have considered only an approximate calculation of the differential cross section where the interaction between the projectile particle and the scattering potential $V(\vec{r})$ is considered small compared with the energy of the incident particle. In this section we are going to calculate the cross section without placing any limitation on the strength of $V(\vec{r})$.

11.4.1 Partial Wave Analysis for Elastic Scattering

We assume here the potential to be *spherically symmetric*. The angular momentum of the incident particle will therefore be conserved; a particle scattering from a central potential will have the same angular momentum before and after collision. Assuming that the incident plane wave is in the z-direction and hence $\phi_{inc}(\vec{r}) = \exp(ikr\cos\theta)$, we may express it in terms of a superposition of angular momentum eigenstates, each with a definite angular momentum number l (Chapter 6):

$$e^{i\vec{k}\cdot\vec{r}} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\theta). \tag{11.81}$$

We can then examine how each of the partial waves is distorted by $V(r)$ after the particle scatters from the potential. The most general solution of the Schrödinger equation (11.30) is

$$\psi(\vec{r}) = \sum_{lm} C_{lm} R_{kl}(r) Y_{lm}(\theta, \varphi). \tag{11.82}$$

Since $V(r)$ is central, the system is symmetrical (rotationally invariant) about the z-axis. The scattered wave function must not then depend on the azimuthal angle φ; hence $m = 0$. Thus, as $Y_{l0}(\theta, \varphi) \sim P_l(\cos\theta)$, the scattered wave function (11.82) becomes

$$\psi(r, \theta) = \sum_{l=0}^{\infty} a_l R_{kl}(r) P_l(\cos\theta), \tag{11.83}$$

where $R_{kl}(r)$ obeys the following radial equation (Chapter 6):

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2}\right](r R_{kl}(r)) = \frac{2m}{\hbar^2} V(r) \left(r R_{kl}(r)\right). \tag{11.84}$$

Each term of (11.83), which is known as a *partial wave*, is a joint eigenfunction of $\vec{L}^{\,2}$ and $\hat{L}_z$. A substitution of (11.81) into (11.34) with $\varphi = 0$ gives

$$\psi(r, \theta) \simeq \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\theta) + f(\theta)\frac{e^{ikr}}{r}. \tag{11.85}$$

The scattered wave function is given, on the one hand, by (11.83) and, on the other hand, by (11.85).

In almost all scattering experiments, detectors are located at distances from the target that are much larger than the size of the target itself; thus, the measurements taken by detectors pertain to scattered wave functions at large values of r. In what follows we are going to show that, by establishing a connection between the asymptotic forms of (11.83) and (11.85), we can determine the scattering amplitude and hence the differential cross section.

First, since the limit of the Bessel function $j_l(kr)$ for large values of r (Chapter 6) is given by

$$j_l(kr) \longrightarrow \frac{\sin(kr - l\pi/2)}{kr} \qquad (r \longrightarrow \infty), \tag{11.86}$$

the asymptotic form of (11.85) is

$$\psi(r,\theta) \longrightarrow \sum_{l=0}^{\infty} i^l(2l+1)P_l(\cos\theta)\frac{\sin(kr - l\pi/2)}{kr} + f(\theta)\frac{e^{ikr}}{r}, \tag{11.87}$$

and since $\sin(kr - l\pi/2) = [(-i)^l e^{ikr} - i^l e^{-ikr}]/2i$, because $e^{\pm il\pi/2} = \left(e^{\pm i\pi/2}\right)^l = (\pm i)^l$, we can write (11.87) as

$$\psi(r,\theta) \longrightarrow -\frac{e^{-ikr}}{2ikr}\sum_{l=0}^{\infty} i^{2l}(2l+1)P_l(\cos\theta) + \frac{e^{ikr}}{r}\left[f(\theta) + \frac{1}{2ik}\sum_{l=0}^{\infty} i^l(-i)^l(2l+1)P_l(\cos\theta)\right]. \tag{11.88}$$

Second, to find the asymptotic form of (11.83), we need first to determine the asymptotic form of the radial function $R_{kl}(r)$. At large values of r, the scattering potential is effectively zero, for it is short range. In this case (11.84) becomes

$$\left(\frac{d^2}{dr^2} + k^2\right)(rR_{kl}(r)) = 0. \tag{11.89}$$

As seen in Chapter 6, the general solution of this equation is given by a linear combination of the spherical Bessel and Neumann functions

$$R_{kl}(r) = A_l j_l(kr) + B_l n_l(kr), \tag{11.90}$$

where the asymptotic form of the Neumann function is

$$n_l(kr) \longrightarrow -\frac{\cos(kr - l\pi/2)}{kr} \qquad (r \longrightarrow \infty). \tag{11.91}$$

Inserting of (11.86) and (11.91) into (11.90), we obtain the asymptotic form of the radial function:

$$R_{kl}(r) \longrightarrow A_l\frac{\sin(kr - l\pi/2)}{kr} - B_l\frac{\cos(kr - l\pi/2)}{kr} \qquad (r \longrightarrow \infty). \tag{11.92}$$

If $V(r) = 0$ for all r (free particle), the solution of (11.84), $rR_{kl}(r)$, must vanish at $r = 0$; thus $R_{kl}(r)$ must be finite at the origin. Since the Neumann function diverges at $r = 0$, the cosine term in (11.92) does not represent a physically acceptable solution; hence, it needs to be discarded near the origin. By rewriting (11.92) in the form

$$R_{kl}(r) \longrightarrow C_l\frac{\sin(kr - l\pi/2 + \delta_l)}{kr} \qquad (r \longrightarrow \infty), \tag{11.93}$$

we have $A_l = C_l \cos\delta_l$ and $B_l = -C_l \sin\delta_l$, hence $C_l = \sqrt{A_l^2 + B_l^2}$ and

$$\tan\delta_l = -\frac{B_l}{A_l} \qquad \Longrightarrow \qquad \delta_l = -\tan^{-1}\left(\frac{B_l}{A_l}\right). \tag{11.94}$$

We see that, with $\delta_l = 0$, the radial function $R_{kl}(r)$ of (11.93) is finite at $r = 0$, since (11.93) reduces to $j_l(kr)$. So δ_l is a real angle which vanishes for all values of l in the absence of the scattering potential (i.e., $V = 0$); δ_l is called the *phase shift*. It measures, at large values of r, the degree to which $R_{kl}(r)$ differs from $j_l(kr)$ (recall that $j_l(kr)$ is the radial function when there is no scattering). Since this "distortion," or the difference between $R_{kl}(r)$ and $j_l(kr)$, is due to the potential $V(r)$, we would expect the cross section to depend on δ_l. Using (11.93) we can write the asymptotic limit of (11.83) as

$$\psi(r,\theta) \longrightarrow \sum_{l=0}^{\infty} a_l P_l(\cos\theta)\frac{\sin(kr - l\pi/2 + \delta_l)}{kr} \qquad (r \longrightarrow \infty). \tag{11.95}$$

This wave function is known as a *distorted plane wave*, for it differs from a plane wave by having phase shifts δ_l. Since $\sin(kr - l\pi/2 + \delta_l) = [(-i)^l e^{ikr} e^{i\delta_l} - i^l e^{-ikr} e^{-i\delta_l}]/2i$, we can rewrite (11.95) as

$$\psi(r,\theta) \longrightarrow -\frac{e^{-ikr}}{2ikr}\sum_{l=0}^{\infty} a_l i^l e^{-i\delta_l} P_l(\cos\theta) + \frac{e^{ikr}}{2ikr}\sum_{l=0}^{\infty} a_l (-i)^l e^{i\delta_l} P_l(\cos\theta). \tag{11.96}$$

Up to now we have shown that the asymptotic forms of (11.83) and (11.85) are given by (11.96) and (11.88), respectively. Equating the coefficients of e^{-ikr}/r in (11.88) and (11.96), we obtain $(2l+1)i^{2l} = a_l i^l e^{-i\delta_l}$ and hence

$$a_l = (2l+1)i^l e^{i\delta_l}. \tag{11.97}$$

Substituting (11.97) into (11.96) and this time equating the coefficient of e^{ikr}/r in the resulting expression with that of (11.88), we have

$$f(\theta) + \frac{1}{2ik}\sum_{l=0}^{\infty} i^l(-i)^l(2l+1)P_l(\cos\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)i^l(-i)^l e^{2i\delta_l} P_l(\cos\theta), \tag{11.98}$$

which, when combined with $(e^{2i\delta_l} - 1)/2i = e^{i\delta_l}\sin\delta_l$ and $i^l(-i)^l = 1$, leads to

$$\boxed{f(\theta) = \sum_{l=0}^{\infty} f_l(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)(e^{2i\delta_l} - 1) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta),} \tag{11.99}$$

where $f_l(\theta)$ is known as the partial wave amplitude.

From (11.99) we can obtain the differential and the total cross sections

$$\frac{d\sigma}{d\Omega} = \left| f(\theta)\right|^2 = \frac{1}{k^2}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}(2l+1)(2l'+1)e^{i(\delta_l - \delta_{l'})}\sin\delta_l \sin\delta_{l'} P_l(\cos\theta)P_{l'}(\cos\theta), \tag{11.100}$$

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \int_0^{\pi} |f(\theta)|^2 \sin\theta \, d\theta \int_0^{2\pi} d\varphi = 2\pi \int_0^{\pi} |f(\theta)|^2 \sin\theta \, d\theta$$
$$= \frac{2\pi}{k^2} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} (2l+1)(2l'+1) e^{i(\delta_l - \delta_{l'})} \sin\delta_l \sin\delta_{l'} \int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta. \tag{11.101}$$

Using the relation $\int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta = [2/(2l+1)]\delta_{ll'}$, we can reduce (11.101) to

$$\boxed{\sigma = \sum_{l=0}^{\infty} \sigma_l = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l,} \tag{11.102}$$

where σ_l are called the *partial cross sections* corresponding to the scattering of particles in various angular momentum states. The differential cross section (11.100) consists of a superposition of terms with different angular momenta; this gives rise to *interference* patterns between different partial waves corresponding to different values of l. The interference terms go away in the total cross section when the integral over θ is carried out. Note that when $V = 0$ everywhere, all the phase shifts δ_l vanish, and hence the partial and total cross sections, as indicated by (11.100) and (11.102), are zero. Note that, as shown in equations (11.99) and (11.102), $f(\theta)$ and σ are given as infinite series over the angular momentum l. We may recall that, for cases of practical importance with the exception of the Coulomb potential, these series converge after a finite number of terms.

We should note that in the case where we have a scattering between particles that are in their respective s states, $l = 0$, the scattering amplitude (11.99) becomes

$$f_0 = \frac{1}{k} e^{i\delta_0} \sin\delta_0 \qquad (l = 0), \tag{11.103}$$

where we have used $P_0(\cos\theta) = 1$. Since f_0 does not depend on θ, the differential and total cross sections are given by the following simple relations:

$$\frac{d\sigma}{d\Omega} = |f_0|^2 = \frac{1}{k^2} \sin^2\delta_0, \qquad \sigma = 4\pi\,|f_0|^2 = \frac{4\pi}{k^2} \sin^2\delta_0 \qquad (l = 0). \tag{11.104}$$

An important issue here is the fact that the total cross section can be related to the *forward scattering amplitude* $f(0)$. Since $P_l(\cos\theta) = P_l(1) = 1$ when $\theta = 0$, equation (11.99) leads to

$$f(0) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \left(\sin\delta_l \cos\delta_l + i \sin^2\delta_l \right), \tag{11.105}$$

which when combined with (11.102) yields the connection between $f(0)$ and σ:

$$\boxed{\frac{4\pi}{k} \operatorname{Im} f(0) = \sigma = \frac{4\pi}{k} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l.} \tag{11.106}$$

This is known as the *optical theorem* (it is reminiscent of a similar theorem in optics which deals with the scattering of light). The physical origin of this theorem is the conservation of particles (or probability). The beam emerging (after scattering) along the incidence direction ($\theta = 0$)

contains fewer particles than the incident beam, since a number of particles have scattered in various directions. This decrease in the number of particles is measured by the total cross section σ; that is, the number of particles removed from the incident beam along the incidence direction is proportional to σ or, equivalently, to the imaginary part of $f(0)$. We should note that, although (11.106) was derived for elastic scattering, the optical theorem, as will be shown later, is also valid for inelastic scattering.

11.4.2 Partial Wave Analysis for Inelastic Scattering

The scattering amplitude (11.99) can be rewritten as

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1) f_l(k) P_l(\cos\theta), \tag{11.107}$$

where

$$f_l(k) = \frac{1}{k} e^{i\delta_l} \sin\delta_l = \frac{1}{2ik}\left(e^{2i\delta_l} - 1\right) = \frac{1}{2ik}\left(S_l(k) - 1\right), \tag{11.108}$$

with

$$S_l(k) = e^{2i\delta_l}. \tag{11.109}$$

In the case where there is no flux loss, we must have $|S_l(k)| = 1$. However, this requirement is not valid whenever there is *absorption* of the incident beam. In this case of flux loss, $S_l(k)$ is redefined by

$$S_l(k) = \eta_l(k) e^{2i\delta_l}, \tag{11.110}$$

with $0 < \eta_l(k) \leq 1$; hence (11.108) and (11.107) become

$$f_l(k) = \frac{\eta_l e^{2i\delta_l} - 1}{2ik} = \frac{1}{2k}\left[\eta_l \sin 2\delta_l + i(1 - \eta_l \cos 2\delta_l)\right], \tag{11.111}$$

$$\boxed{f(\theta) = \frac{1}{2k}\sum_{l=0}^{\infty}(2l+1)\left[\eta_l \sin 2\delta_l + i(1 - \eta_l \cos 2\delta_l)\right] P_l(\cos\theta).} \tag{11.112}$$

The *total elastic* scattering cross section is given by

$$\boxed{\sigma_{el} = 4\pi \sum_{l=0}^{\infty}(2l+1)|f_l|^2 = \frac{\pi}{k^2}\sum_l (2l+1)(1 + \eta_l^2 - 2\eta_l \cos 2\delta_l).} \tag{11.113}$$

The *total inelastic* scattering cross section, which describes the loss of flux, is given by

$$\boxed{\sigma_{inel} = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\left(1 - \eta_l^2(k)\right).} \tag{11.114}$$

Thus, if $\eta_l(k) = 1$ there is no inelastic scattering, but if $\eta_l = 0$ we have total absorption, although there is still elastic scattering in this partial wave. The sum of (11.113) and (11.114) gives the total cross section:

$$\boxed{\sigma_{tot} = \sigma_{el} + \sigma_{inel} = \frac{2\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\left(1 - \eta_l \cos(2\delta_l)\right).} \tag{11.115}$$

Next, using (11.107) and (11.111), we infer

$$\operatorname{Im} f(0) = \sum_{l=0}^{\infty}(2l+1)\operatorname{Im} f_l = \frac{1}{2k}\sum_{l=0}^{\infty}(2l+1)\left(1-\eta_l\cos(2\delta_l)\right). \tag{11.116}$$

A comparison of (11.115) and (11.116) gives the optical theorem relation, $\operatorname{Im} f(0) = k\sigma_{tot}/4\pi$; hence the optical theorem is also valid for inelastic scattering.

Example 11.3 (High-energy scattering from a black disk)
Discuss the scattering from a black disk at high energies.

Solution
A black disk is totally absorbing (i.e., $\eta_l(k) = 0$). Assuming the values of l do not exceed a maximum value l_{max} ($l \leq l_{max}$) and that k is large (high-energy scattering), we have $l_{max} = ka$ where a is the radius of the disk. Since $\eta_l = 0$, equations (11.113) and (11.114) lead to

$$\sigma_{inel} = \sigma_{el} = \frac{\pi}{k^2}\sum_{l=0}^{ka}(2l+1) = \frac{\pi}{k^2}(ka+1)^2 \simeq \pi a^2; \tag{11.117}$$

hence the total cross section is given by

$$\sigma_{inel} = \sigma_{el} + \sigma_{inel} = 2\pi a^2. \tag{11.118}$$

Classically, the total cross section of a disk is equal to πa^2. The factor 2 in (11.118) is due to purely quantum effects, since in the high-energy limit there are two kinds of scattering: one corresponding to waves that hit the disk, where the cross section is equal to the classical cross section πa^2, and the other to waves that are diffracted. According to Babinet's principle, the cross section for the waves diffracted by a disk is also equal to πa^2.

11.5 Scattering of Identical Particles

First, let us consider the scattering of two identical *bosons* in their center of mass frame (we will consider the scattering of two identical fermions in a moment). *Classically*, the cross section for the scattering of two identical particles whose interaction potential is central is given by

$$\sigma_{cl}(\theta) = \sigma(\theta) + \sigma(\pi-\theta). \tag{11.119}$$

In quantum mechanics there is no way of distinguishing, as indicated in Figure 11.7, between the particle that scatters at an angle θ from the one that scatters at $(\pi-\theta)$. Thus, the scattered wave function must be symmetric:

$$\psi_{sym}(\vec{r}) \longrightarrow e^{i\vec{k}_0\cdot\vec{r}} + e^{-i\vec{k}_0\cdot\vec{r}} + f_{sym}(\theta)\frac{e^{ikr}}{r}, \tag{11.120}$$

and so must also be the scattering amplitude:

$$f_{boson}(\theta) = f(\theta) + f(\pi-\theta). \tag{11.121}$$

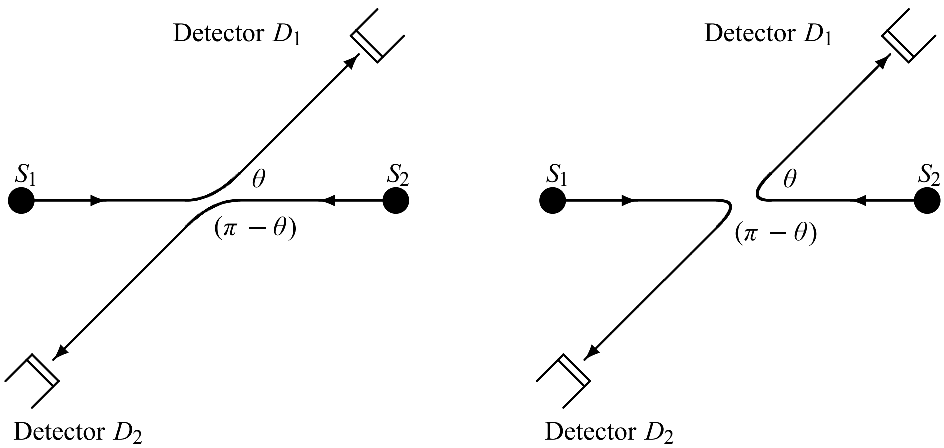

Figure 11.7 When scattering two *identical* particles in the center of mass frame, it is impossible to distinguish between the particle that scatters at angle θ from the one that scatters at $(\pi - \theta)$
.

Therefore, the differential cross section is

$$\begin{aligned}\frac{d\sigma}{d\Omega}_{boson} &= \left| f(\theta) + f(\pi - \theta)\right|^2 = \left| f(\theta)\right|^2 + \left| f(\pi - \theta)\right|^2 + f(\theta)^* f(\pi - \theta) + f(\theta) f^*(\pi - \theta) \\ &= \left| f(\theta)\right|^2 + \left| f(\pi - \theta)\right|^2 + 2\,\mathrm{Re}\left[f^*(\theta) f(\pi - \theta)\right]. \qquad (11.122)\end{aligned}$$

In sharp contrast to its classical counterpart, equation (11.122) contains an interference term $2\,\mathrm{Re}\left[f^*(\theta) f(\pi - \theta)\right]$. Note that when $\theta = \pi/2$, we have $(d\sigma/d\Omega)_{boson} = 4\,|f(\pi/2)|^2$; this is twice as large as the classical expression (which has no interference term): $(d\sigma/d\Omega)_{cl} = 2\,|f(\pi/2)|^2$. If the particles were distinguishable, the differential cross section will be four times smaller, $(d\sigma/d\Omega)_{distinguishable} = |f(\pi/2)|^2$.

Consider now the scattering of two identical spin $\frac{1}{2}$ particles. This is the case, for example, of electron–electron or proton–proton scattering. The wave function of a two spin $\frac{1}{2}$ particle system is known to be either symmetric or antisymmetric. When the spatial wave function is symmetric, that is the two particles are in a spin singlet state, the differential cross section is given by

$$\frac{d\sigma_S}{d\Omega} = |f(\theta) + f(\pi - \theta)|^2, \qquad (11.123)$$

but when the two particles are in a spin triplet state, the spatial wave function is antisymmetric, and hence

$$\frac{d\sigma_A}{d\Omega} = |f(\theta) - f(\pi - \theta)|^2. \qquad (11.124)$$

If the incident particles are unpolarized, the various spin states will be equally likely, so the triplet state will be three times as likely as the singlet:

$$\begin{aligned}\frac{d\sigma}{d\Omega}_{fermion} &= \frac{3}{4}\frac{d\sigma_a}{d\Omega} + \frac{1}{4}\frac{d\sigma_s}{d\Omega} = \frac{3}{4}\,|f(\theta) - f(\pi - \theta)|^2 + \frac{1}{4}\,|f(\theta) + f(\pi - \theta)|^2 \\ &= |f(\theta)|^2 + |f(\pi - \theta)|^2 - \mathrm{Re}\left[f^*(\theta) f(\pi - \theta)\right]. \qquad (11.125)\end{aligned}$$

When $\theta = \pi/2$, we have $(d\sigma/d\Omega)_{fermion} = |f(\pi/2)|^2$; this quantum differential cross section is half the classical expression, $(d\sigma/d\Omega)_{cl} = 2\,|f(\pi/2)|^2$, and four times smaller than the expression corresponding to the scattering of two identical bosons, $(d\sigma/d\Omega)_{boson} = 4\,|f(\pi/2)|^2$.

We should note that, in the case of partial wave analysis for elastic scattering, using the relations $\cos(\pi-\theta) = -\cos\theta$ and $P_l(\cos(\pi-\theta)) = P_l(-\cos\theta) = (-1)^l P_l(\cos\theta)$ and inserting them into (11.99), we can write

$$\begin{aligned} f(\pi-\theta) &= \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos(\pi-\theta)) \\ &= \frac{1}{k}\sum_{l=0}^{\infty}(-1)^l(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta), \end{aligned} \tag{11.126}$$

and hence

$$f(\theta) \pm f(\pi-\theta) = \frac{1}{k}\sum_{l=0}^{\infty}\left[1 \pm (-1)^l\right](2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta). \tag{11.127}$$

Example 11.4

Calculate the differential cross section in the first Born approximation for the scattering between two identical particles having spin 1, mass m, and interacting through a potential $V(r) = V_0 e^{-ar}$.

Solution

As seen in Chapter 7, the spin states of two identical particles with spin $s_1 = s_2 = 1$ consist of a total of nine states: a quintuplet $|\,2,\ m\rangle$ (i.e., $|\,2,\ \pm 2\rangle$, $|2,\ \pm 1\rangle$, $|\,2,\ 0\rangle$) and a singlet $|0,\ 0\rangle$, which are symmetric, and a triplet $|1,\ m\rangle$ (i.e., $|1,\ \pm 1\rangle$, $|1,\ 0\rangle$), which are antisymmetric under particle permutation. That is, while the six spin states corresponding to $S = 2$ and $S = 0$ are symmetric, the three $S = 1$ states are antisymmetric. Thus, if the scattering particles are unpolarized, the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{5}{9}\frac{d\sigma_S}{d\Omega} + \frac{1}{9}\frac{d\sigma_S}{d\Omega} + \frac{3}{9}\frac{d\sigma_A}{d\Omega} = \frac{2}{3}\frac{d\sigma_S}{d\Omega} + \frac{1}{3}\frac{d\sigma_A}{d\Omega}, \tag{11.128}$$

where

$$\frac{d\sigma_S}{d\Omega} = \left|\,f(\theta) + f(\pi-\theta)\right|^2, \qquad \frac{d\sigma_A}{d\Omega} = \left|\,f(\theta) - f(\pi-\theta)\right|^2. \tag{11.129}$$

The scattering amplitude is given in the Born approximation by (11.69):

$$\begin{aligned} f(\theta) &= -\frac{2V_0\mu}{\hbar^2 q}\int_0^\infty r e^{-ar}\sin(qr)\,dr = -\frac{V_0\mu}{i\hbar^2 q}\int_0^\infty r e^{-(a-iq)r}\,dr + \frac{V_0\mu}{i\hbar^2 q}\int_0^\infty r e^{-(a+iq)r}\,dr \\ &= \frac{V_0\mu}{\hbar^2 q}\frac{\partial}{\partial q}\int_0^\infty e^{-(a-iq)r}\,dr + \frac{V_0\mu}{\hbar^2 q}\frac{\partial}{\partial q}\int_0^\infty e^{-(a+iq)r}\,dr \\ &= \frac{V_0\mu}{\hbar^2 q}\frac{\partial}{\partial q}\left(\frac{1}{a-iq}\right) + \frac{V_0\mu}{\hbar^2 q}\frac{\partial}{\partial q}\left(\frac{1}{a+iq}\right) = \frac{V_0\mu}{\hbar^2 q}\left[\frac{i}{(a-iq)^2} + \frac{-i}{(a+iq)^2}\right] \\ &= -\frac{4V_0\mu a}{\hbar^2}\frac{1}{(a^2+q^2)^2} = -\frac{4V_0\mu a}{\hbar^2}\frac{1}{\left(a^2 + 4k^2\sin^2(\theta/2)\right)^2}, \end{aligned} \tag{11.130}$$

where we have used $q = 2k\sin(\theta/2)$, with $\mu = m/2$. Since $\sin[(\pi - \theta)/2] = \cos(\theta/2)$, we have

$$\frac{d\sigma_S}{d\Omega} = \frac{16V_0^2\mu^2 a^2}{\hbar^4}\left[\frac{1}{\left(a^2+4k^2\sin^2(\theta/2)\right)^2} + \frac{1}{\left(a^2+4k^2\cos^2(\theta/2)\right)^2}\right]^2, \quad (11.131)$$

$$\frac{d\sigma_A}{d\Omega} = \frac{16V_0^2\mu^2 a^2}{\hbar^4}\left[\frac{1}{\left(a^2+4k^2\sin^2(\theta/2)\right)^2} - \frac{1}{\left(a^2+4k^2\cos^2(\theta/2)\right)^2}\right]^2. \quad (11.132)$$

11.6 Solved Problems

Problem 11.1

(a) Calculate the differential cross section in the Born approximation for the potential $V(r) = V_0 e^{-r/R}/r$, known as the Yukawa potential.

(b) Calculate the total cross section.

(c) Find the relation between V_0 and R so that the Born approximation is valid.

Solution

(a) Inserting $V(r) = V_0 e^{-r/R}/r$ into (11.70), we obtain

$$\frac{d\sigma}{d\Omega} = \frac{4\mu^2 V_0^2}{\hbar^4 q^2}\left|\int_0^\infty e^{-r/R}\sin(qr)\,dr\right|^2, \quad (11.133)$$

where

$$\int_0^\infty e^{-r/R}\sin(qr)\,dr = \frac{1}{2i}\int_0^\infty e^{-(1/R-iq)r}\,dr - \frac{1}{2i}\int_0^\infty e^{-(1/R+iq)r}\,dr$$
$$= \frac{1}{2i}\left[\frac{1}{1/R - iq} - \frac{1}{1/R+iq}\right] = \frac{q}{1/R^2+q^2}; \quad (11.134)$$

hence

$$\frac{d\sigma}{d\Omega} = \frac{4\mu^2 V_0^2}{\hbar^4}\frac{1}{(1/R^2+q)^2} = \frac{4\mu^2 V_0^2}{\hbar^4}\frac{1}{\left[1/R^2+4k^2\sin^2(\theta/2)\right]^2}. \quad (11.135)$$

Note that a connection can be established between this relation and the differential cross section for a Coulomb potential $V(r) = Z_1 Z_2 e^2/r$. For this, we need only to insert $V_0 = -Z_1 Z_2 e^2$ into (11.135) and then take the limit $R \to \infty$; this leads to (11.77):

$$\left(\frac{d\sigma}{d\Omega}\right)_{Rutherford} = \lim_{R\to\infty}\left(\frac{d\sigma}{d\Omega}\right)_{Yukawa}. \quad (11.136)$$

(b) The total cross section can be obtained at once from (11.135):

$$\sigma = \int \frac{d\sigma}{d\Omega}\sin\theta\,d\theta d\varphi = 2\pi\int_0^\pi \frac{d\sigma}{d\Omega}\sin\theta\,d\theta = 2\pi\frac{4\mu^2 V_0^2 R^4}{\hbar^4}\int_0^\pi \frac{\sin\theta\,d\theta}{\left(1+4k^2R^2\sin^2(\theta/2)\right)^2}. \quad (11.137)$$

The change of variable $x = 2kR\sin(\theta/2)$ leads to $\sin\theta\, d\theta = x\,dx/(k^2R^2)$; hence

$$\begin{aligned} \sigma &= \frac{8\pi\mu^2V_0^2R^4}{\hbar^4}\frac{1}{k^2R^2}\int_0^{2kR}\frac{x dx}{\left(1+x^2\right)^2} = \frac{16\pi\mu^2V_0^2R^4}{\hbar^4}\frac{1}{1+4k^2R^2} \\ &= \frac{16\pi\mu^2V_0^2R^4}{\hbar^4}\frac{1}{1+8\mu ER^2/\hbar^2}, \end{aligned} \tag{11.138}$$

where we have used $k^2 = 2\mu E/\hbar^2$; E is the energy of the scattered particle.

(c) The validity condition of the Born approximation is

$$\frac{\mu V_0}{\hbar^2k^2}\left|\int_0^\infty \frac{e^{-ar}}{r}(e^{2ikr}-1)\,dr\right| \ll 1, \tag{11.139}$$

where $a = 1/R$. To evaluate the integral

$$I = \int_0^\infty \frac{e^{-ar}}{r}(e^{2ikr}-1)\,dr \tag{11.140}$$

let us differentiate it with respect to the parameter a:

$$\frac{\partial I}{\partial a} = -\int_0^\infty e^{-ar}(e^{2ikr}-1)\,dr = -\frac{1}{a-2ik}+\frac{1}{a}. \tag{11.141}$$

Now, integrating over the parameter a such that $I(a=+\infty)=0$, we obtain

$$I = \ln a - \ln(a-2ik) = -\ln\left(1-2i\frac{k}{a}\right) = -\frac{1}{2}\ln(1+\frac{4k^2}{a^2}) + i\tan^{-1}\left(\frac{2k}{a}\right). \tag{11.142}$$

Thus, the validity condition (11.139) becomes

$$\frac{\mu V_0}{\hbar^2k^2}\left\{\frac{1}{4}\left[\ln(1+4k^2R^2)\right]^2 + \left(\tan^{-1}(2kR)\right)^2\right\}^{1/2} \ll 1. \tag{11.143}$$

Problem 11.2

Find the differential and total cross sections for the scattering of slow (small velocity) particles from a spherical delta potential $V(r) = V_0\delta(r-a)$ (you may use a partial wave analysis). Discuss what happens if there is no scattering potential.

Solution

In the case where the incident particles have small velocities, only the s-waves, $l = 0$, contribute to the scattering. The differential and total cross sections are given for $l = 0$ by (11.104):

$$\frac{d\sigma}{d\Omega} = |f_0|^2 = \frac{1}{k^2}\sin^2\delta_0, \qquad \sigma = 4\pi\,|f_0|^2 = \frac{4\pi}{k^2}\sin^2\delta_0 \qquad (l=0). \tag{11.144}$$

We need now to find the phase shift δ_0. For this, we need to consider the Schrödinger equation for the radial function:

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} + \left[V_0\delta(r-a) + \frac{l(l+1)\hbar^2}{2mr^2}\right]u(r) = Eu(r), \tag{11.145}$$

where $u(r) = rR(r)$. In the case of s states and $r \neq a$, this equation yields

$$\frac{d^2u(r)}{dr^2} = -k^2u(r), \tag{11.146}$$

where $k^2 = 2mE/\hbar^2$. The acceptable solutions of this equation must vanish at $r = 0$ and be finite at $r \to \infty$:

$$u(r) = \begin{cases} u_1(r) = A\sin(kr), & 0 < r < a, \\ u_2(r) = B\sin(kr + \delta_0), & r > a. \end{cases} \tag{11.147}$$

The continuity of $u(r)$ at $r = a$, $u_2(a) = u_1(a)$, leads to

$$B\sin(ka + \delta_0) = A\sin(ka). \tag{11.148}$$

On the other hand, integrating (11.145) (with $l = 0$) from $r = a - \varepsilon$ to $r = a + \varepsilon$, we obtain

$$-\frac{\hbar^2}{2m}\int_{a-\varepsilon}^{a+\varepsilon}\frac{d^2u(r)}{dr^2}\,dr + V_0\int_{a-\varepsilon}^{a+\varepsilon}\delta(r-a)u(r)\,dr = E\int_{a-\varepsilon}^{a+\varepsilon}u(r)\,dr, \tag{11.149}$$

and taking the limit $\varepsilon \to 0$, we end up with

$$\left.\frac{du_2(r)}{dr}\right|_{r=a} - \left.\frac{du_1(r)}{dr}\right|_{r=a} - \frac{2mV_0}{\hbar^2}u_2(a) = 0. \tag{11.150}$$

An insertion of $u_1(r)$ and $u_2(r)$ as given by (11.147) into (11.150) leads to

$$B\left[k\cos(ka + \delta_0) - \frac{2mV_0}{\hbar^2}\sin(ka + \delta_0)\right] = Ak\cos(ka). \tag{11.151}$$

Dividing (11.151) by (11.148), we obtain

$$k\cot(ka + \delta_0) - \frac{2mV_0}{\hbar^2} = k\cot(ka) \quad\Longrightarrow\quad \tan(ka + \delta_0) = \left[\frac{1}{\tan(ka)} + \frac{2mV_0}{k\hbar^2}\right]^{-1}. \tag{11.152}$$

This equation shows that, when there is no scattering potential, $V_0 = 0$, the phase shift is zero, since $\tan(ka + \delta_0) = \tan(ka)$. In this case, equations (11.103) and (11.104) imply that the scattering amplitude and the cross sections all vanish.

If the incident particles have small velocities, $ka \ll 1$, we have $\tan(ka) \simeq ka$ and $\tan(ka + \delta_0) \simeq \tan(\delta_0)$. In this case, equation (11.152) yields

$$\tan\delta_0 \simeq \frac{ka}{1 + 2mV_0a/\hbar^2} \quad\Longrightarrow\quad \sin^2\delta_0 \simeq \frac{k^2a^2}{k^2a^2 + \left(1 + 2mV_0a/\hbar^2\right)^2}. \tag{11.153}$$

Inserting this relation into (11.144), we obtain

$$\frac{d\sigma}{d\Omega_0} \simeq \frac{a^2}{k^2a^2 + \left(1 + 2mV_0a/\hbar^2\right)^2}, \qquad \sigma_0 \simeq \frac{4\pi a^2}{k^2a^2 + \left(1 + 2mV_0a/\hbar^2\right)^2}. \tag{11.154}$$

Problem 11.3
Consider the scattering of a particle of mass m from a hard sphere potential: $V(r) = \infty$ for $r < a$ and $V(r) = 0$ for $r > a$.

(a) Calculate the total cross section in the low-energy limit. Find a numerical estimate for the cross section for the case of scattering 5 keV protons from a hard sphere of radius $a = 6$ fm.

(b) Calculate the total cross section in the high-energy limit. Find a numerical estimate for the cross section for the case of 700 MeV protons with $a = 6$ fm.

Solution

(a) As the scattering is dominated at low energies by s-waves, $l = 0$, the radial Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} = Eu(r) \qquad (r > a), \tag{11.155}$$

where $u(r) = rR(r)$. The solutions of this equation are

$$u(r) = \begin{cases} u_1(r) = 0, & r < a, \\ u_2(r) = A\sin(kr + \delta_0), & r > a, \end{cases} \tag{11.156}$$

where $k^2 = 2mE/\hbar^2$. The continuity of $u(r)$ at $r = a$ leads to

$$\sin(ka + \delta_0) = 0 \quad \Longrightarrow \quad \tan\delta_0 = -\tan(ka) \quad \Longrightarrow \quad \sin^2\delta_0 = \sin^2(ka), \tag{11.157}$$

since $\sin^2\alpha = 1/(1 + \cot^2\alpha)$. The lowest value of the phase shift is $\delta_0 = -ka$; it is negative, as it should be for a repulsive potential. An insertion of $\sin^2\delta_0 = \sin^2(ka)$ into (11.104) yields

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi}{k^2}\sin^2(ka). \tag{11.158}$$

For low energies, $ka \ll 1$, we have $\sin(ka) \simeq ka$ and hence $\sigma_0 \simeq 4\pi a^2$, which is four times the classical value πa^2.

To obtain a numerical estimate of (11.158), we need first to calculate k^2. For this, we need simply to use the relation $E = \hbar^2k^2/(2m_p) = 5$ keV, since the proton moves as a free particle before scattering. Using $m_pc^2 = 938.27$ MeV and $\hbar c = 197.33$ MeV fm, we have

$$k^2 = \frac{2m_pE}{\hbar^2} = \frac{2(m_pc^2)E}{(\hbar c)^2} = \frac{2(939.57\,\text{MeV})(5 \times 10^{-3}\,\text{MeV})}{(197.33\,\text{MeV fm})^2} = 0.24 \times 10^{-3}\,\text{fm}^{-2}. \tag{11.159}$$

Thus $k = 0.0155\,\text{fm}^{-1}$; the wave shift is given by $\delta_0 = -ka = -0.093\,\text{rad} = -5.33°$. Inserting these values into (11.158), we obtain

$$\sigma = \frac{4\pi}{0.24 \times 10^{-3}\,\text{fm}^{-2}}\sin^2(5.33) = 449.89\,\text{fm}^2 = 4.5\,\text{barn}. \tag{11.160}$$

(b) In the high-energy limit, $ka \gg 1$, the number of partial waves contributing to the scattering is large. Assuming that $l_{max} \simeq ka$, we may rewrite (11.102) as

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{l_{max}}(2l + 1)\sin^2\delta_l. \tag{11.161}$$

Since so many values of l contribute in this relation, we may replace $\sin^2\delta_l$ by its average value, $\frac{1}{2}$; hence

$$\sigma \simeq \frac{4\pi}{k^2}\frac{1}{2}\sum_{l=0}^{l_{max}}(2l+1) = \frac{2\pi}{k^2}(l_{max}+1)^2, \tag{11.162}$$

where we have used $\sum_{l=0}^{n}(2l+1) = (n+1)^2$. Since $l_{max} \gg 1$ we have

$$\sigma \simeq \frac{2\pi}{k^2}l_{max}^2 = \frac{2\pi}{k^2}(ka)^2 = 2\pi a^2. \tag{11.163}$$

Since $a = 6\,\text{fm}$, we have $\sigma \simeq 2\pi(6\,\text{fm})^2 = 226.1\,\text{fm}^2 = 2.26\,\text{barn}$. This is almost half the value obtained in (11.160).

In conclusion, the cross section from a hard sphere potential is (a) four times the classical value, πa^2, for low-energy scattering and (b) twice the classical value for high-energy scattering.

Problem 11.4

Calculate the total cross section for the low-energy scattering of a particle of mass m from an attractive square well potential $V(r) = -V_0$ for $r < a$ and $V(r) = 0$ for $r > a$, with $V_0 > 0$.

Solution

Since the scattering is dominated at low energies by the s partial waves, $l = 0$, the Schrödinger equation for the radial function is given by

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} - V_0 u(r) = Eu(r) \qquad (r < a), \tag{11.164}$$

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} = Eu(r) \qquad (r > a), \tag{11.165}$$

where $u(r) = rR(r)$. The solutions of these equations for positive energy states are

$$u(r) = \begin{cases} u_1(r) = A\sin(k_1 r), & r < a, \\ u_2(r) = B\sin(k_2 r + \delta_0), & r > a, \end{cases} \tag{11.166}$$

where $k_1^2 = 2m(E+V_0)/\hbar^2$ and $k_2^2 = 2mE/\hbar^2$. The continuity of $u(r)$ and its first derivative, $u'(r) = du(r)/dr$, at $r = a$ yield

$$\left.\frac{u_2(r)}{u_2'(r)}\right|_{r=a} = \left.\frac{u_1(r)}{u_1'(r)}\right|_{r=a} \quad\Longrightarrow\quad \frac{1}{k_2}\tan(k_2 a + \delta_0) = \frac{1}{k_1}\tan(k_1 a), \tag{11.167}$$

which yields

$$\delta_0 = -k_2 a + \tan^{-1}\left[\frac{k_2}{k_1}\tan(k_1 a)\right]. \tag{11.168}$$

Since

$$\tan(k_2 a + \delta_0) = \frac{\sin(k_2 a)\cos\delta_0 + \cos(k_2 a)\sin\delta_0}{\cos(k_2 a)\cos\delta_0 - \sin(k_2 a)\sin\delta_0} = \frac{\tan(k_2 a) + \tan\delta_0}{1 - \tan(k_2 a)\tan\delta_0}, \tag{11.169}$$

we can reduce Eq. (11.167) to

$$\tan\delta_0 = \frac{k_2 \tan(k_1 a) - k_1 \tan(k_2 a)}{k_1 + k_2 \tan(k_1 a)\tan(k_2 a)}. \tag{11.170}$$

Using the relation $\sin^2\delta_0 = 1/(1 + 1/\tan^2\delta_0)$, we can write

$$\sin^2\delta_0 = \left[1 + \left(\frac{k_1 + k_2 \tan(k_1 a)\tan(k_2 a)}{k_2 \tan(k_1 a) - k_1 \tan(k_2 a)}\right)^2\right]^{-1}, \tag{11.171}$$

which, when inserted into (11.104), leads to

$$\sigma_0 = \frac{4\pi}{k_1^2}\sin^2\delta_0 = \frac{4\pi}{k_1^2}\left[1 + \left(\frac{k_1 + k_2 \tan(k_1 a)\tan(k_2 a)}{k_2 \tan(k_1 a) - k_1 \tan(k_2 a)}\right)^2\right]^{-1}. \tag{11.172}$$

If $k_2 a \ll 1$ then (11.170) becomes $\tan\delta_0 \simeq \frac{\tan(k_1 a) - k_1 a}{k_1/k_2 + k_2 a \tan(k_1 a)}$, since $\tan(k_2 a) \simeq k_2 a$. Thus, if $k_2 a \ll 1$ and if E (the scattering energy) is such that $\tan(k_1 a) \simeq k_1 a$, we have $\tan\delta_0 = 0$; hence there will be no s-wave scattering and the cross section vanishes. Note that if the square well potential is extended to a hard sphere potential, i.e., $E \to 0$ and $V_0 \to \infty$, equation (11.168) yields the phase shift of scattering from a hard sphere $\delta_0 = -ka$, since $(k_2/k_1)\tan(k_1 a) \to 0$.

Problem 11.5

Find the differential and total cross sections in the first Born approximation for the elastic scattering of a particle of mass m, which is initially traveling along the z-axis, from a nonspherical, double-delta potential $V(\vec{r}) = V_0\delta(\vec{r} - a\vec{k}) + V_0\delta(\vec{r} + a\vec{k})$, where $\vec{k}$ is the unit vector along the z-axis.

Solution

Since $V(\vec{r})$ is not spherically symmetric, the differential cross section can be obtained from (11.66):

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int V_0\left[\delta(\vec{r} - a\vec{k}) + \delta(\vec{r} + a\vec{k})\right] e^{i\vec{q}\cdot\vec{r}}\, d^3r\right|^2 = \frac{m^2 V_0}{4\pi^2\hbar^4}\,|I|^2. \tag{11.173}$$

Since $\delta(\vec{r} \pm a\vec{k}) = \delta(x)\delta(y)\delta(z \pm a)$ we can write the integral I as

$$\begin{aligned} I &= \int \delta(x) e^{ixq_x}\,dx \int \delta(y) e^{iyq_y}\,dy \int \left[\delta(z - a) + \delta(z + a)\right] e^{izq_z}\,dz \\ &= e^{iaq_z} + e^{-iaq_z} = 2\cos(aq_z). \end{aligned} \tag{11.174}$$

The calculation of q_z is somewhat different from that shown in (11.67). Since the incident particle is initially traveling along the z-axis, and since it scatters elastically from the potential $V(\vec{r})$, the magnitudes of its momenta before and after collision are equal. So, as shown in Figure 11.8, we have $q_z = q\sin(\theta/2) = 2k\sin^2(\theta/2)$, since $q = |\vec{k}_0 - \vec{k}| = 2k\sin(\theta/2)$. Thus, inserting $I = 2\cos(aq_z) = 2\cos\left[2ak\sin^2(\theta/2)\right]$ into (11.173), we obtain

$$\frac{d\sigma}{d\Omega} = \frac{m^2 V_0}{\pi^2\hbar^4}\cos^2\left(2ak\sin^2\frac{\theta}{2}\right). \tag{11.175}$$

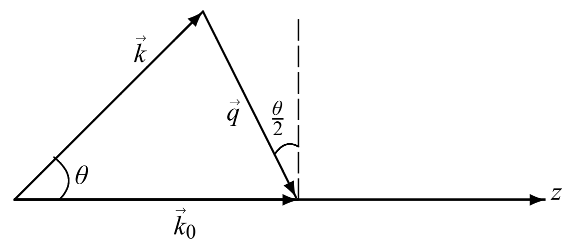

Figure 11.8 Particle traveling initially along the z-axis (taken here horizontally) scatters at an angle θ, with $q = |\vec{k}_0 - \vec{k}| = 2k\sin(\theta/2)$, since $k_0 = k$ and $q_z = q\sin(\theta/2)$.

The total cross section can be obtained at once from (11.175):

$$\begin{aligned}\sigma &= \int \frac{d\sigma}{d\Omega}\sin\theta\, d\theta d\varphi = 2\pi\int_0^\pi \frac{d\sigma}{d\Omega}\sin\theta\, d\theta \\ &= 2\pi\frac{m^2V_0}{\pi^2\hbar^4}\int_0^\pi \sin\theta\cos^2\left(2ak\sin^2\frac{\theta}{2}\right)d\theta,\end{aligned} \tag{11.176}$$

which, when using the change of variable $x = 2ak\sin^2(\theta/2)$ with $dx = 2ak\sin(\theta/2)\cos(\theta/2)\, d\theta$, leads to

$$\begin{aligned}\sigma &= \frac{2m^2V_0}{\pi\hbar^4}\int_0^\pi 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\cos^2\left(2ak\sin^2\frac{\theta}{2}\right)d\theta \\ &= \frac{2m^2V_0}{\pi ak\hbar^4}\int_0^1 \cos^2(x)\, dx \\ &= \frac{m^2V_0}{\pi ak\hbar^4}\int_0^1 [1+\cos(2x)]\, dx \\ &= \frac{m^2V_0}{\pi ak\hbar^4}.\end{aligned} \tag{11.177}$$

Problem 11.6
Consider the elastic scattering of 50 MeV neutrons from a nucleus. The phase shifts measured in this experiment are $\delta_0 = 95°$, $\delta_1 = 72°$, $\delta_2 = 60°$, $\delta_3 = 35°$, $\delta_4 = 18°$, $\delta_5 = 5°$; all other phase shifts are negligible (i.e., $\delta_l \simeq 0$ for $l \geq 6$).

(a) Find the total cross section.

(b) Estimate the radius of the nucleus.

Solution

(a) As $\delta_l \simeq 0$ for $l \geq 6$, equation (11.102) yields

$$\begin{aligned}\sigma &= \frac{4\pi}{k^2}\sum_{l=0}^{6}(2l+1)\sin^2\delta_l \\ &= \frac{4\pi}{k^2}\left(\sin^2\delta_0 + 3\sin^2\delta_1 + 5\sin^2\delta_2 + 7\sin^2\delta_3 + 9\sin^2\delta_4 + 11\sin^2\delta_5\right) = \frac{4\pi}{k^2}\times 10.702.\end{aligned} \tag{11.178}$$

To calculate k^2, we need simply to use the relation $E = \hbar^2 k^2/(2m_n) = 50\,\text{MeV}$, since the neutrons move as free particles before scattering. Using $m_n c^2 = 939.57\,\text{MeV}$ and $\hbar c = 197.33\,\text{MeV fm}$, we have

$$k^2 = \frac{2m_n E}{\hbar^2} = \frac{2(m_n c^2)E}{(\hbar c)^2} = \frac{2(939.57\,\text{MeV})(50\,\text{MeV})}{(197.33\,\text{MeV fm})^2} = 2.41\,\text{fm}^{-2}. \tag{11.179}$$

An insertion of (11.179) into (11.178) leads to

$$\sigma = \frac{4\pi}{2.41\,\text{fm}^{-2}} \times 10.702 = 55.78\,\text{fm}^2 = 0.558\,\text{barn}. \tag{11.180}$$

(b) At large values of l, when the neutron is at its closest approach to the nucleus, it feels mainly the effect of the centrifugal potential $l(l+1)\hbar^2/(2m_n r^2)$; the effect of the nuclear potential is negligible. We may thus use the approximations $E \simeq l(l+1)\hbar^2/(2m_n r_c^2) \simeq 42\hbar^2/(2m_n r_c^2)$ where we have taken $l \simeq 6$, since $\delta_l \simeq 0$ for $l \geq 6$. A crude value of the radius of the nucleus is then given by

$$r_c \simeq \sqrt{\frac{21\hbar^2}{m_n E}} = \sqrt{\frac{21(\hbar c)^2}{(m_n c^2)E}} = \sqrt{\frac{21(197.33\,\text{MeV fm})^2}{(939.57\,\text{MeV})(50\,\text{MeV})}} = 4.17\,\text{fm}. \tag{11.181}$$

Problem 11.7

Consider the elastic scattering of an electron from a hydrogen atom in its ground state. If the atom is assumed to remain in its ground state after scattering, calculate the differential cross section in the case where the effects resulting from the identical nature of the electrons (a) are ignored and (b) are taken into account (in part (b), discuss the three cases when the electrons are in (i) a spin singlet state, (ii) a spin triplet state, or (iii) an unpolarized state).

Solution

(a) By analogy with (11.63) we may write the differential cross section for this process as

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left| -\frac{\mu}{2\pi\hbar^2}\langle \Psi_f \mid \hat{V} \mid \Psi_i \rangle \right|^2, \tag{11.182}$$

where $\mu \simeq m_e/2$, since this problem can be viewed as the scattering of a particle whose reduced mass is half that of the electron. Assuming the atom to be very massive and that it remains in its ground state after scattering, the initial and final states of the system (incident electron plus the atom) are given by $\Psi_i(\vec{r}, \vec{k}_0, \vec{r}\,') = e^{i\vec{k}_0\cdot\vec{r}}\psi_0(\vec{r}\,')$ and $\Psi_f(\vec{r}, \vec{k}, \vec{r}\,') = e^{i\vec{k}\cdot\vec{r}}\psi_0(\vec{r}\,')$, where $e^{i\vec{k}_0\cdot\vec{r}\,'}$ and $e^{i\vec{k}\cdot\vec{r}\,'}$ are the states of the incident electron before and after scattering, and $\psi_0(\vec{r}\,') = (\pi a_0^3)^{-1/2} e^{-r'/a_0}$ is the atom's wave function. We have assumed here that the nucleus is located at the origin and that the position vectors of the incident electron and the atom's electron are given by $\vec{r}$ and $\vec{r}\,'$, respectively. Since the incident electron experiences an attractive Coulomb interaction $-e^2/r$ with the nucleus and a repulsive interaction $e^2/|\vec{r} - \vec{r}\,'|$ with the hydrogen's electron, we have

$$f(\theta) = -\frac{\mu}{2\pi\hbar^2}\int d^3 r e^{i\vec{q}\cdot\vec{r}} \int d^3 r' \psi_0^*(\vec{r}\,') \left[-\frac{e^2}{r} + \frac{e^2}{|\vec{r} - \vec{r}\,'|} \right] \psi_0(\vec{r}\,'), \tag{11.183}$$

with $q = |\vec{k}_0 - \vec{k}| = 2k\sin(\theta/2)$, since $k = k_0$ (elastic scattering). Using $\int_0^\infty \sin(qr)\,dr = 1/q$ (see (11.76)), and since $\int_0^\pi e^{iqr\cos\theta}\sin\theta\,d\theta = \int_{-1}^1 e^{iqrx}dx = (2/qr)\sin(qr)$, we obtain the following relation:

$$\int d^3r\frac{e^{i\vec{q}\cdot\vec{r}}}{r} = \int_0^\infty r\,dr\int_0^\pi e^{iqr\cos\theta}\sin\theta\,d\theta\int_0^{2\pi}d\varphi = \frac{4\pi}{q}\int_0^\infty dr\sin(qr) = \frac{4\pi}{q^2}, \tag{11.184}$$

which, when inserted into (11.183) and since $\int d^3r'\psi_0^*(\vec{r}\,')\psi_0(\vec{r}\,') = 1$, leads to

$$f(\theta) = \frac{\mu}{2\pi\hbar^2}\left[\frac{4\pi e^2}{q^2} - \int d^3r e^{i\vec{q}\cdot\vec{r}}\int d^3r'\psi_0^*(\vec{r}\,')\frac{e^2}{|\vec{r}-\vec{r}\,'|}\psi_0(\vec{r}\,')\right]. \tag{11.185}$$

By analogy with (11.184), we have $\int d^3r e^{i\vec{q}\cdot|\vec{r}-\vec{r}\,'|}/|\vec{r}-\vec{r}\,'| = 4\pi/q^2$; hence we can reduce the integral in (11.185) to

$$\begin{aligned}\int d^3r\,e^{i\vec{q}\cdot\vec{r}}\int d^3r'\,\psi_0^*(\vec{r}\,')\frac{e^2}{|\vec{r}-\vec{r}\,'|}\psi_0(\vec{r}\,') &= e^2\int d^3r'\,\psi_0^*(\vec{r}\,')e^{i\vec{q}\cdot\vec{r}\,'}\psi_0(\vec{r}\,')\int d^3r\frac{e^{i\vec{q}\cdot|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\\ &= \frac{4\pi e^2}{q^2}\int d^3r'\psi_0^*(\vec{r}\,')e^{i\vec{q}\cdot\vec{r}\,'}\psi_0(\vec{r}\,').\end{aligned} \tag{11.186}$$

The remaining integral of (11.186) can, in turn, be written as

$$\begin{aligned}\int d^3r'\,\psi_0^*(\vec{r}\,')e^{i\vec{q}\cdot\vec{r}\,'}\psi_0(\vec{r}\,') &= \frac{1}{\pi a_0^3}\int_0^\infty r'^2e^{-2r'/a_0}\,dr'\int_0^\pi e^{iqr'\cos\theta'}\sin\theta' d\theta'\int_0^{2\pi}d\varphi'\\ &= \frac{4}{qa_0^3}\int_0^\infty r'e^{-2r'/a_0}\sin(qr')\,dr' = \left(1+\frac{a_0^2q^2}{4}\right)^{-2},\end{aligned} \tag{11.187}$$

where we have used the expression for $\int_0^\infty re^{-ar}\sin(qr)\,dr$ calculated in (11.130). Inserting (11.187) into (11.186), and the resulting expression into (11.185), we obtain

$$f(\theta) = \frac{2\mu e^2}{\hbar^2q^2}\left[1-\left(1+\frac{a_0^2q^2}{4}\right)^{-2}\right] = \frac{\mu e^2}{2k^2\hbar^2\sin^2(\theta/2)}\left[1-\left(1+a_0^2k^2\sin^2\frac{\theta}{2}\right)^{-2}\right]. \tag{11.188}$$

We can thus reduce (11.182) to

$$\frac{d\sigma}{d\Omega} = \frac{4\mu^2e^4}{\hbar^4q^4}\left[1-\left(1+\frac{a_0^2q^2}{4}\right)^{-2}\right]^2 = \frac{\mu^2e^4}{4k^4\hbar^4\sin^4\frac{\theta}{2}}\left[1-\left(1+a_0^2k^2\sin^2(\theta/2)\right)^{-2}\right]^2, \tag{11.189}$$

with $q = 2k\sin(\theta/2)$.

(b) (i) If the electrons are in their spin singlet state (antisymmetric), the spatial wave function must be symmetric; hence the differential cross section is

$$\frac{d\sigma_S}{d\Omega} = \left|\,f(\theta)+f(\pi-\theta)\right|^2, \tag{11.190}$$

where $f(\theta)$ is given by (11.188) and

$$f(\pi-\theta)=\frac{2\mu e^2}{\hbar^2 q^2}\left[1-\left(1+\frac{a_0^2 q^2}{4}\right)^{-2}\right]=\frac{\mu e^2}{2k^2\hbar^2\cos^2\frac{\theta}{2}}\left[1-\left(1+a_0^2k^2\cos^2(\theta/2)\right)^{-2}\right], \tag{11.191}$$

since $\sin(\pi-\theta/2)=\cos(\theta/2)$.

(ii) If, however, the electrons are in their spin triplet state, the spatial wave function must be antisymmetric; hence

$$\frac{d\sigma_A}{d\Omega}=\left|\,f(\theta)-f(\pi-\theta)\right|^2. \tag{11.192}$$

(iii) Finally, if the electrons are unpolarized, the differential cross section must be a mixture of (11.191) and (11.192):

$$\frac{d\sigma}{d\Omega}=\frac{1}{4}\frac{d\sigma_S}{d\Omega}+\frac{3}{4}\frac{d\sigma_A}{d\Omega}=\frac{1}{4}\left|f(\theta)+f(\pi-\theta)\right|^2+\frac{3}{4}\left|f(\theta)-f(\pi-\theta)\right|^2. \tag{11.193}$$

Problem 11.8

In an experiment, 650 MeV π^0 pions are scattered from a heavy, totally absorbing nucleus of radius 1.4 fm.

(a) Estimate the total elastic and total inelastic cross sections.

(b) Calculate the scattering amplitude and check the validity of the optical theorem.

(c) Using the scattering amplitude found in (b), calculate and plot the differential cross section for elastic scattering. Calculate the total elastic cross section and verify that it agrees with the expression found in (a).

Solution

(a) In the case of a totally absorbing nucleus, $\eta_l(k)=0$, the total elastic and inelastic cross sections, which are given by (11.113) and (11.114), become equal:

$$\sigma_{el}=\frac{\pi}{k^2}\sum_{l=0}^{l_{max}}(2l+1)=\sigma_{inel}. \tag{11.194}$$

This experiment can be viewed as a scattering of high-energy pions, $E=650\,\text{MeV}$, from a black "disk" of radius $a=1.4$ fm; thus, the number of partial waves involved in this scattering can be obtained from $l_{max}\simeq ka$, where $k=\sqrt{2m_{\pi^0}E/\hbar^2}$. Since the rest mass energy of a π^0 pion is $m_{\pi^0}c^2\simeq 135$ MeV and since $\hbar c=197.33$ MeV fm, we have

$$k\simeq\sqrt{\frac{2m_{\pi^0}E}{\hbar^2}}=\sqrt{\frac{2(m_{\pi^0}c^2)E}{(\hbar c)^2}}=\sqrt{\frac{2(135\,\text{MeV})(650\,\text{MeV})}{(197.33\,\text{MeV fm})^2}}=2.12\,\text{fm}^{-1}; \tag{11.195}$$

hence $l_{max}=ka\simeq(2.12\,\text{fm}^{-1})(1.4\,\text{fm})=2.97\simeq 3$. We can thus reduce (11.194) to

$$\sigma_{el}=\sigma_{inel}=\frac{\pi}{k^2}\sum_{l=0}^{3}(2l+1)=\frac{16\pi}{k^2}\simeq\frac{16\pi}{(2.12\,\text{fm}^{-1})^2}=40.1\,\text{fm}^2=0.40\,\text{barn}. \tag{11.196}$$

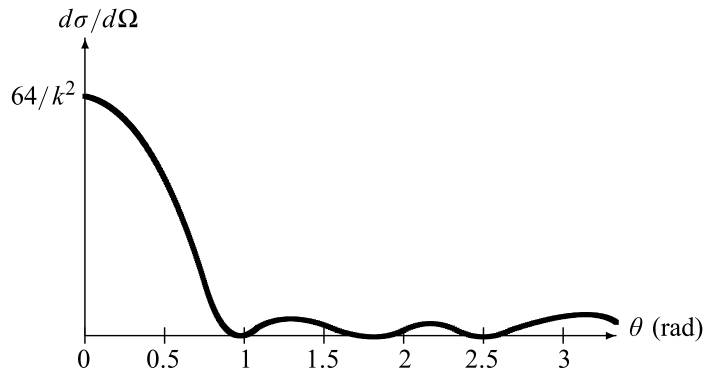

Figure 11.9 Plot of $\frac{d\sigma}{d\Omega} = \frac{1}{4k^2}\left[1 + 3\cos\theta + \frac{5}{2}(3\cos^2\theta - 1) + \frac{7}{2}(5\cos^3\theta - 3\cos\theta)\right]^2$.

The total cross section

$$\sigma_{tot} = \sigma_{el} + \sigma_{inel} = \frac{32\pi}{k^2} = 0.80\,\text{barn}. \tag{11.197}$$

(b) The scattering amplitude can be obtained from (11.112) with $\eta_l(k) = 0$:

$$\begin{aligned} f(\theta) &= \frac{i}{2k}\sum_{l=0}^{3}(2l+1)P_l(\cos\theta) \\ &= \frac{i}{2k}\left[1 + 3\cos\theta + \frac{5}{2}(3\cos^2\theta - 1) + \frac{7}{2}(5\cos^3\theta - 3\cos\theta)\right], \end{aligned} \tag{11.198}$$

where we have used the following Legendre polynomials: $P_0(u) = 1$, $P_1(u) = u$, $P_2(u) = \frac{1}{2}(3u^2 - 1)$, $P_3(u) = \frac{1}{2}(5u^3 - 3u)$. The forward scattering amplitude ($\theta = 0$) is

$$f(0) = \frac{i}{2k}\left[1 + 3 + \frac{5}{2}(3-1) + \frac{7}{2}(5-3)\right] = \frac{8i}{k}. \tag{11.199}$$

Combining (11.197) and (11.199), we get the optical theorem: $\text{Im} f(0) = (k/4\pi)\sigma_{tot} = 8/k$.

(c) From (11.198) the differential elastic cross section is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{1}{4k^2}\left[1 + 3\cos\theta + \frac{5}{2}(3\cos^2\theta - 1) + \frac{7}{2}(5\cos^3\theta - 3\cos\theta)\right]^2. \tag{11.200}$$

As shown in Figure 11.9, the differential cross section displays an interference pattern due to the superposition of incoming and outgoing waves. The total elastic cross section is given by $\sigma_{el} = \int_0^\pi |f(\theta)|^2 \sin\theta\, d\theta \int_0^{2\pi} d\varphi$ which, combined with (11.200), leads to

$$\sigma_{el} = \frac{2\pi}{4k^2}\int_0^\pi \left[1 + 3\cos\theta + \frac{5}{2}(3\cos^2\theta - 1) + \frac{7}{2}(5\cos^3\theta - 3\cos\theta)\right]^2 \sin\theta\, d\theta = \frac{16\pi}{k^2}. \tag{11.201}$$

This is the same expression we obtained in (11.196). Unlike the differential cross section, the total cross section displays no interference pattern because its final expression does not depend on any angle, since the angles were integrated over.

11.7 Exercises

Exercise 11.1
Consider the scattering of a 5 MeV alpha particle (i.e., a helium nucleus with $Z_1 = 2$ and $A_1 = 4$) from an aluminum nucleus ($Z_2 = 13$ and $A_2 = 27$). If the scattering angle of the alpha particle in the Lab frame is $\theta_1 = 30^\circ$,

(a) find its scattering angle θ in the CM frame and

(b) give a numerical estimate of the Rutherford cross section.

Exercise 11.2

(a) Find the differential and total cross sections for the classical collision of two hard spheres of radius r and R, where R is the radius of the larger sphere; the larger sphere is considered to be stationary.

(b) From the results of (a) find the differential and total cross sections for the scattering of pointlike particles from a hard stationary sphere of radius R. *Hint:* You may use the classical relation $d\sigma/d\Omega = -[b(\theta)/\sin\theta]db/d\theta$, where $b(\theta)$ is the impact parameter.

Exercise 11.3
Consider the scattering from the potential $V(r) = V_0 e^{-r^2/a^2}$. Find

(a) the differential cross section in the first Born approximation and

(b) the total cross section.

Exercise 11.4
Calculate the differential cross section in the first Born approximation for the scattering of a particle by an attractive square well potential: $V(r) = -V_0$ for $r < a$ and $V(r) = 0$ for $r > a$, with $V_0 > 0$.

Exercise 11.5
Consider the elastic scattering from the delta potential $V(r) = V_0\delta(r - a)$.

(a) Calculate the differential cross section in the first Born approximation.

(b) Find an expression between V_0, a, μ, and k so the Born approximation is valid.

Exercise 11.6
Consider the elastic scattering from the potential $V(r) = V_0 e^{-r/a}$, where V_0 and a are constant.

(a) Calculate the differential cross section in the first Born approximation.

(b) Find an expression between V_0, a, μ, and k so the Born approximation is valid.

(c) Find the total cross section using the Born approximation.

Exercise 11.7
Find the differential cross section in the first Born approximation for the elastic scattering of a particle of mass m, which is initially traveling along the z-axis, from a nonspherical, double-delta potential:

$$V(\vec{r}) = V_0\delta(\vec{r} - a\vec{k}) - V_0\delta(\vec{r} + a\vec{k}),$$

where $\vec{k}$ is the unit vector along the z-axis.

Exercise 11.8
Find the differential cross section in the first Born approximation for neutron–neutron scattering in the case where the potential is approximated by $V(r) = V_0 e^{-r/a}$.

Exercise 11.9
Consider the elastic scattering of a particle of mass m and initial momentum $\hbar k$ off a delta potential $V(\vec{r}) = V_0\delta(x)\delta(y)\delta(z-a)$, where V_0 is a constant.

(a) What is the physical dimensions of the constant V_0?

(b) Calculate the differential cross sections in the first Born approximation.

(c) Repeat (b) for the case where the potential is now given by

$$V(\vec{r}) = V_0\delta(x)\left[\delta(y-b)\delta(z) + \delta(y)\delta(z-a)\right].$$

Exercise 11.10
Consider the S-wave ($l = 0$) scattering of a particle of mass m from a repulsive spherical potential $V(r) = V_0$ for $r < a$ and $V(r) = 0$ for $r > a$, with $V_0 > 0$.

(a) Calculate $S-$wave ($l = 0$) phase shift and the total cross section.

(b) Show that in the limit $V_0 \to \infty$, the phase shift is given by $\delta_0 = -ka$. Find the total cross section.

Exercise 11.11
Consider the S-wave neutron–neutron scattering where the interaction potential is approximated by $V(r) = V_0\vec{S}_1 \cdot \vec{S}_2 e^{-r/a}$, where $\vec{S}_1$ and $\vec{S}_2$ are the spin vector operators of the two neutrons, and $V_0 > 0$. Find the differential cross section in the first Born approximation.

Exercise 11.12
Consider the S-partial wave scattering ($l = 0$) between two identical spin $1/2$ particles where the interaction potential is given approximately by

$$\hat{V}(r) = V_0\vec{S}_1 \cdot \vec{S}_2\delta(r-a),$$

where $\vec{S}_1$ and $\vec{S}_2$ are the spin vector operators of the two particles, and $V_0 > 0$. Assuming that the incident and target particles are unpolarized, find the differential and total cross sections.

Exercise 11.13
Consider the elastic scattering of 170 MeV neutrons from a nucleus or radius $a = 1.05$ fm. Consider the hypothetical case where the phase shifts measured in this experiment are given by $\delta_l = \frac{180^\circ}{l+2}$.

(a) Estimate the maximum angular momentum l_{max}.

(b) Find the total cross section.