CHAPTER 3

Continuity and Differentiability

- Continuity
- Types of Discontinuity
- Continuity of Special Types of Functions
- Intermediate Value Theorem
- Differentiability

CONTINUITY

In mathematics, a *continuous function* is a function for which, intuitively, small changes in the input result in small changes in the output. Otherwise, a function is said to be *discontinuous*.

A continuous function is a function whose graph can be drawn without lifting the pen from the paper.

For an example, consider the function $h(t)$ which describes the height of a growing flower at time t. This function is continuous. In fact, according to classical physics everything in nature is continuous. By contrast, if $M(t)$ denotes the amount of money in a bank account at time t, then the function jumps whenever money is deposited or withdrawn, so the function $M(t)$ is discontinuous.

Definition of Continuity of a Function

A function $f(x)$ is said to be continuous at $x = a$ if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

i.e., L.H.L. = R.H.L. = value of a function at $x = a$

or $\lim_{x\to a} f(x) = f(a)$.

A function $f(x)$ is said to be discontinuous at $x = a$ if

a. $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist, but are not equal.

b. $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are equal but not equal to $f(a)$.

c. $f(a)$ is not defined.

d. At least one of the limits does not exist.

Note:

It should be noted that continuity of a function is the property of interval and is meaningful at $x = a$ only if the function has a graph in the immediate neighbourhood of $x = a$, not necessarily at $x = a$. Hence, it should not be mislead that continuity of a function is talked only in its domain.

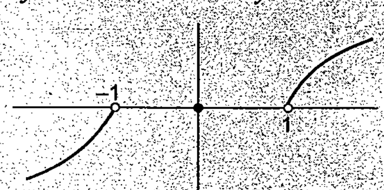


Fig. 3.1

For example, discussing continuity of $f(x) = \frac{1}{x-1}$ at $x = 1$ is meaningful, but continuity of $f(x) = \log_e x$ at $x = -2$ is meaningless. Similarly, if $f(x)$ has a graph as shown in Fig. 3.1, then continuity at $x = 0$ is meaningless.

Also, continuity at $x = a \Rightarrow$ existence of limit at $x = a$, but existence of limit at $x = a$ does not mean continuity at $x = a$.

Directional Continuity

A function may happen to be continuous in only one direction, either from the "left" or from the "right".

A *right-continuous* function is a function which is continuous at all points when approached from the right, that is, $c < x < c + \delta$ [Fig. 3.2(b)].

Similarly, a *left-continuous* function is a function which is continuous at all points when approached from the left, that is, $-\delta < x < c$ [Fig. 3.2(a)].

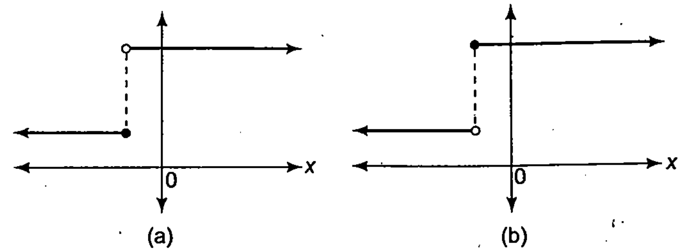


Fig. 3.2

A function is continuous $x = a$ if and only if it is both right-continuous and left-continuous $x = a$.

Continuity in Interval

A function is said to be continuous in the open interval (a, b) if $f(x)$ is continuous at each and every point $\in (a, b)$. For any $c \in (a, b)$, $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$.

A function $f(x)$ is said to be continuous in the closed interval $[a, b]$ if it is continuous at every point in this interval and the continuity at the end points is defined as $f(x)$ is continuous at $x = a$ if $f(a) = \lim_{x\to a^+} f(x) =$ R.H.L. (L.H.L. should not be evaluated) and at $x = b$ if $f(b) = \lim_{x\to b^-} f(x) =$ L.H.L. (R.H.L. should not be evaluated).

Example 3.1 A function $f(x)$ satisfies the following property: $f(x+y) = f(x)f(y)$

Show that the function is continuous for all values of x if it is continuous at $x = 1$.

Sol. As the function is continuous at $x = 1$, we have

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \lim_{h\to 0} f(1-h) = \lim_{h\to 0} f(1+h) = f(1)$$

$$\Rightarrow \lim_{h\to 0} f(1)f(-h) = \lim_{h\to 0} f(1)f(h) = f(1)$$

[Using $f(x+y) = f(x)f(y)$]

$$\Rightarrow \lim_{h\to 0} f(-h) = \lim_{h\to 0} f(h) = 1 \quad (1)$$

Now, consider any arbitrary point $x = a$.

$$\text{L.H.L.} = \lim_{h\to 0} f(a-h)$$

$$= \lim_{h\to 0} f(a)f(-h)$$

$$= f(a)\lim_{h\to 0} f(-h) = f(a)$$

[as $\lim_{h\to 0} f(-h) = 1$, using equation (1)]

$$\text{R.H.L.} = \lim_{h\to 0} f(a+h)$$

$$= \lim_{h\to 0} f(a)f(h)$$

$$= f(a)\lim_{h\to 0} f(h) = f(a)$$

[as $\lim_{h\to 0} f(h) = 1$, using equation (1)]

Hence, at any arbitrary point ($x = a$), L.H.L = R.H.L $= f(a)$.
Therefore, function is continuous for all values of x if it is continuous at 1.

Example 3.2 Let f be a function satisfying $f(x+y) + \sqrt{6-f(y)} = f(x)f(y)$ and $f(h) \to 6$ as $h \to 0$. Discuss the continuity of f.

Sol. R.H.L. $= \lim_{x \to x+} f(x)$

$= \lim_{h \to 0} f(x+h)$

$= \lim_{h \to 0}\left[f(x)f(h) - \sqrt{6-f(h)}\right]$

$= f(x)\lim_{h \to 0} f(h) - \lim_{h \to 0}\sqrt{6-f(h)}$

$= f(x)\,6 - 0 = 6f(x) \neq f(x)$

This shows that if $f(x) \neq 0$, then f is discontinuous at x. If $f(x) = 0$, then $f(x)$ is continuous at x.

Concept Application Exercise 3.1

1. Let $f(x+y) = f(x) + f(y)$ for all x and y. If the function $f(x)$ is continuous at $x = 0$. Show that $f(x)$ is continuous for all x.
2. A function $f(x)$ satisfies the following property: $f(x.y) = f(x)f(y)$. Show that the function $f(x)$ is continuous for all values of x if it is continuous at $x = 1$.
3. If $f(x+y) = f(x)f(y)$ for all $x, y \in R$ and $f(x) = 1 + g(x)\,G(x)$, where $\lim_{x \to 0} g(x) = 0$ and $\lim_{x \to 0} G(x)$ exist, prove that $f(x)$ is continuous at all $x \in R$.

TYPES OF DISCONTINUITY

Removable Discontinuity

Here $\lim_{x \to a} f(x)$ necessarily exists, but is either not equal to $f(a)$ or $f(a)$ is not defined. In this case, it is therefore possible to redefine the function in such a manner that $\lim_{x \to a} f(x) = f(a)$ and thus makes the function continuous.

Consider the functions $g(x) = (\sin x)/x$. Function is not defined at $x = 0$, so the domain is $R - \{0\}$. Since the limit of g at 0 is 1, g can be extended continuously to R by defining its value at 0 to be 1.

Thus redefined function

$$G(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \text{ is continuous at } x = 0.$$

Thus, a point in the domain that can be filled in so that the resulting function is continuous is called a ***removable discontinuity***.

Consider function $f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$

In this example, the function is nicely defined away from the point $x = 1$.

In fact, if $x \neq 1$, the function is

$$f(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$$

However, if we were to consider the point $x = 1$, this definition no longer makes sense since we would have to divide by zero. The function instead tells us that the value of the function is $f(1) = 3$.

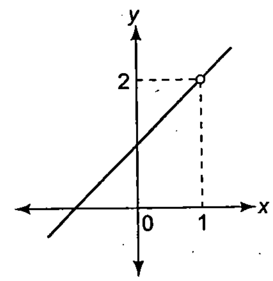


Fig. 3.3

In this example, the graph has a "hole" at the point $x = 1$, which can be filled by redefined $f(x)$ at $x = 1$ as 2 (see Fig. 3.3).

This type of discontinuity is also called ***missing point discontinuity***.

Non-removable Discontinuity

If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $f(x)$ is said to have the first kind of non-removable discontinuity.

Consider the function $f(x) = 1/x$. Function is not defined at $x = 0$. The function f cannot be extended to a continuous function whose domain is R, since no matter what value is assigned at 0, the resulting function will not be continuous. A point in the domain that cannot be filled in so that the resulting function is continuous is called a ***non-removable discontinuity***.

Graphical View of Non-removable Discontinuity

Both the limits are finite and not equal

Consider the function $f(x) = [x]$, greatest integer function. As shown in Fig. 3.4, the graph has jump of discontinuity at all integral values of x.

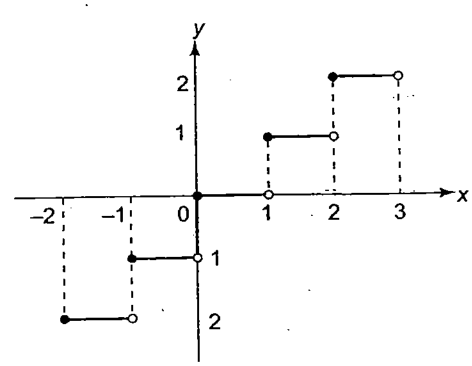


Fig. 3.4

At least one of left and right limit is infinity or vertical asymptote

Consider the function $f(x) = \tan x$

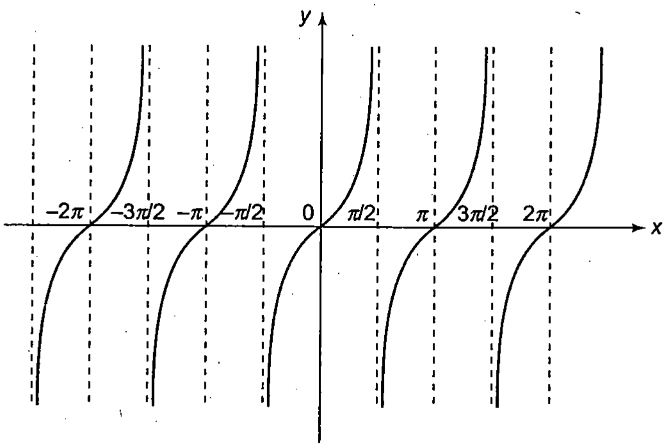


Fig. 3.5

Here the function is not defined at points $\pm\frac{\pi}{2}, \pm 3\frac{\pi}{2}$ and near these points, the function becomes both arbitrarily large and small. Since the function is not defined at these points, it cannot be continuous.

Oscillations (limits oscillate between two finite quantities)

$f(x) = \sin\frac{\pi}{x}$. When $x \to 0$, $\frac{1}{x} \to \pm\infty$ and $\sin(\to \pm\infty)$ can take any value between −1 to 1 or we can say when $x \to 0$, $f(x)$ oscillates between −1 and 1 as shown in Fig. 3.6.

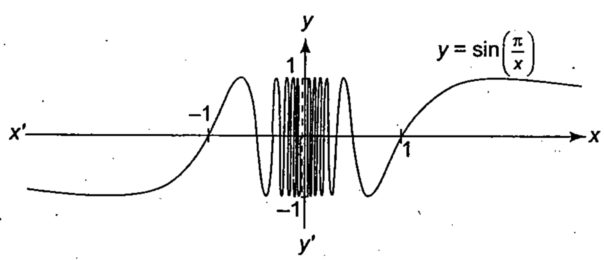


Fig. 3.6

Example 3.3 Find the points of discontinuity of the following functions.

a. $f(x) = \dfrac{1}{2\sin x - 1}$

b. $f(x) = \dfrac{1}{x^2 - 3|x| + 2}$

c. $f(x) = \dfrac{1}{x^4 + x^2 + 1}$

d. $f(x) = \dfrac{1}{1 - e^{\frac{x-1}{x-2}}}$

e. $f(x) = [[x]] - [x - 1]$, where [.] represents the greatest integer function.

Sol. a. $f(x) = \dfrac{1}{2\sin x - 1}$

$f(x)$ is discontinuous when $2\sin x - 1 = 0$

$\Rightarrow \sin x = \frac{1}{2} \Rightarrow x = 2n\pi + \frac{\pi}{6}$ or $x = 2n\pi + \frac{5\pi}{6}, n \in Z$

b. $f(x) = \dfrac{1}{x^2 - 3|x| + 2}$

$f(x)$ is discontinuous when $x^2 - 3|x| + 2 = 0$

$\Rightarrow |x|^2 - 3|x| + 2 = 0$

$\Rightarrow (|x| - 1)(|x| - 2) = 0$

$\Rightarrow |x| = 1, 2$

$\Rightarrow x = \pm 1, \pm 2$

c. $f(x) = \dfrac{1}{x^4 + x^2 + 1} = \dfrac{1}{\left(x^2 + \frac{1}{2}\right)^2 + \frac{3}{4}}$

Now, $x^4 + x^2 + 1 = \left(x^2 + \frac{1}{2}\right)^2 + \frac{3}{4} \geq 1 \ \forall x \in R$

$\Rightarrow f(x)$ is continuous $\forall x \in R$

d. $f(x) = \dfrac{1}{1 - e^{\frac{x-1}{x-2}}}$

$f(x)$ is discontinuous when $x - 2 = 0$. Also

when $1 - e^{\frac{x-1}{x-2}} = 0$

$\Rightarrow x = 2$ and $e^{\frac{x-1}{x-2}} = 1$

$\Rightarrow x = 2$ and $\frac{x-1}{x-2} = 0$

$\Rightarrow x = 2$ and $x = 1$

e. $f(x) = [[x]] - [x - 1] = [x] - ([x] - 1) = 1$

$\Rightarrow f(x)$ is continuous $\forall x \in R$

Example 3.4 Let $f(x) = \left\{\dfrac{\log(1 + x)^{1+x} - x}{x^2}\right\}$, then find the value of $f(0)$ so that the function f is continuous at $x = 0$.

Sol. We must have $f(0) = \lim\limits_{x \to 0} f(x)$

$$= \lim_{x \to 0} \frac{(1 + x)\log(1 + x) - x}{x^2} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\log(1 + x) + 1 - 1}{2x}$$

(Using L'Hopital's rule)

$$= \frac{1}{2} \lim_{x \to 0} \frac{\log(1 + x)}{x} = \frac{1}{2}.$$

Example 3.5 What value must be assigned to k so that the function $f(x) = \begin{cases} \dfrac{x^4 - 256}{x - 4}, & x \neq 4 \\ k, & x = 4 \end{cases}$ is continuous at $x = 4$?

Sol. $f(x)$ is continuous at $x=4$

$$\Rightarrow f(4)=\lim_{x\to 4}\frac{x^4-256}{x-4}$$

$$=\lim_{x\to 4}\frac{x^4-4^4}{x-4}$$

$$=4\times 4^{4-1}=256$$

Example 3.6 A function $f(x)$ is defined as follows

$$f(x)=\begin{cases} ax-b, & x\le 1\\ 3x, & 1<x<2\\ bx^2-a, & x\ge 2\end{cases}$$

Prove that if $f(x)$ is continuous at $x=1$ but discontinuous at $x=2$, then the locus of the point (a, b) is a straight line excluding the point where it cuts the line, $y=3$.

Sol. Given $f(x)$ is continuous at $x=1$

$\therefore f(1)=$ R.H.L.

$$\Rightarrow f(1)=\lim_{x\to 1+}f(x)$$

$$\Rightarrow f(1)=\lim_{h\to 0}f(1+h)$$

$$\Rightarrow a-b=\lim_{h\to 0}3(1+h)$$

$$\Rightarrow a-b=3 \qquad (1)$$

Again, given $f(x)$ is discontinuous at $x=2$

$\therefore$ L.H.L. $\ne f(2)$

$$\Rightarrow \lim_{x\to 2-}f(x)\ne f(2)$$

$$\Rightarrow \lim_{h\to 0}f(2-h)\ne f(2)$$

$$\Rightarrow \lim_{h\to 0}3(2-h)\ne 4b-a$$

$$\Rightarrow 6\ne 4b-a$$

Let $6=4b-a$, then (2)
from equations (1) and (2), we get $b=3$
$\therefore$ locus $y=3$,
which is impossible. $(\because 6\ne 4b-a)$
Hence, the locus of (a, b) is $x-y=3$ excluding the point when it cuts the line, $y=3$.

Example 3.7 Let $f(x)$ be a function defined as

$$f(x)=\begin{cases}\dfrac{x^2-1}{x^2-2|x-1|-1}, & x\ne 1\\[2mm] \dfrac{1}{2}, & x=1\end{cases}$$

Discuss the continuity of the function at $x=1$.

Sol. $$f(1^+)=\lim_{x\to 1^+}\frac{x^2-1}{x^2-2|x-1|-1}$$

$$=\lim_{x\to 1^+}\frac{x^2-1}{x^2-2(x-1)-1}$$

$$=\lim_{x\to 1^+}\frac{(x+1)}{(x+1)-2}=\infty$$

$$f(1^-)=\lim_{x\to 1^-}\frac{x^2-1}{x^2-2|x-1|-1}$$

$$=\lim_{x\to 1^-}\frac{x^2-1}{x^2-2(1-x)-1}$$

$$=\lim_{x\to 1^+}\frac{(x+1)}{(x+1)+2}=\frac{1}{2}$$

Hence $f(x)$ is discontinuous at $x=1$.

Example 3.8 Let $f(x)=\begin{cases}\dfrac{\sin ax^2}{x^2}, x\ne 0\\[2mm] \dfrac{3}{4}+\dfrac{1}{4a}; x=0\end{cases}$. For what values of $a, f(x)$ is continuous at $x=0$.

Sol. $f(x)=\begin{cases}\dfrac{\sin ax^2}{x^2}, & x\ne 0\\[2mm] \dfrac{3}{4}+\dfrac{1}{4a}, & x=0\end{cases}$ is continuous at $x=0$

$$\therefore \lim_{x\to 0}f(x)=f(0)$$

$$\Rightarrow \lim_{x\to 0}\frac{a\sin ax^2}{ax^2}=\frac{3}{4}+\frac{1}{4a}$$

$$\Rightarrow a=\frac{3}{4}+\frac{1}{4a}$$

$$\Rightarrow 4a^2-3a-1=0$$

$$\Rightarrow (4a+1)(a-1)=0$$

$$\Rightarrow a=-1/4, 1$$

Example 3.9 Let $f(x)=\begin{cases}\dfrac{a+3\cos x}{x^2}, & x<0\\[2mm] b\tan\left(\dfrac{\pi}{[x+3]}\right), & x\ge 0\end{cases}$

If $f(x)$ is continuous at $x=0$, then find a and b, where [.] denotes the greatest integer function.

Sol. $f(x)=\begin{cases}\dfrac{a+3\cos x}{x^2}, & x<0\\[2mm] b\tan\left(\dfrac{\pi}{[x+3]}\right), & x\ge 0\end{cases}$

$$f(0^+)=\lim b\tan\left(\frac{\pi}{[0+h+3]}\right)=b\tan\left(\frac{\pi}{3}\right)=\sqrt{3}b$$

$$f(0^-) = \lim_{h\to 0} \frac{a+3\cos(-h)}{(-h)^2}$$

$\Rightarrow a+3=0$ as $f(x)$ is continuous at $x=0$, then $f(0^-)$ must be finite.

$\Rightarrow a=-3$

$$\Rightarrow f(0^-) = \lim_{h\to 0}\frac{-3+3\cosh}{h^2} = \lim_{h\to 0}\frac{-3\cosh}{2} = \frac{-3}{2}$$

Since $f(x)$ is continuous at $x=0$, then

$$\sqrt{3}b = \frac{-3}{2} \Rightarrow b = \frac{-\sqrt{3}}{2}$$

Example 3.10 $f(x)=\begin{cases}\cos^{-1}\{\cot x\}, & x<\frac{\pi}{2}\\ \pi[x]-1, & x\geq\frac{\pi}{2}\end{cases}$

where [·] represents the greatest function and {·} represents the fractional part function. Find the jump of discontinuity.

Sol. $f(x)=\begin{cases}\cos^{-1}\{\cot x\}, & x<\frac{\pi}{2}\\ \pi[x]-1, & x\geq\frac{\pi}{2},\end{cases}$

$$\lim_{x\to\pi^-/2} f(x) = \lim_{x\to\pi^-/2}\cos^{-1}\{\cot x\}$$

$$= \cos^{-1}\{0^+\} = \cos^{-1}0 = \frac{\pi}{2}$$

$$\lim_{x\to\pi^+/2} f(x) = \lim_{x\to\pi^+/2}(\pi[x]-1) = \pi-1$$

$\therefore$ jump of discontinuity $= \pi - 1 - \frac{\pi}{2} = \frac{\pi}{2} - 1$

Theorems of Continuity

1. Sum, difference, product, and quotient of two continuous functions are always a continuous function. However, $h(x)=\dfrac{f(x)}{g(x)}$ is continuous at $x=a$ only if $g(a)\neq 0$.
2. If $f(x)$ is continuous and $g(x)$ is discontinuous, then $f(x)+g(x)$ is a discontinuous function. (Prove by contradiction.)
   $f(x)=x$ and $g(x)=[x]$ are the greatest integer functions.
   Here, $f(x)$ is continuous at $x=0$, but $g(x)$ is discontinuous at $x=0$.
   Hence, $F(x) = x + [x]$ is discontinuous at $x = 0$ as $f(0^+)=0$ and $f(0^-)=-1$.
3. If $f(x)$ is continuous and $g(x)$ is discontinuous at $x=a$, then the product function $h(x)=f(x)g(x)$ is not necessarily be discontinuous at $x=a$.
   Consider, $f(x)=x^3$ and $g(x)=\text{sgn}(x)$
   Here $f(x)$ is continuous at $x=0$ and $g(x)$ is discontinuous at $x=0$. But the product function is
   $F(x)=f(x)g(x)=\begin{cases}x^3, & x>0\\ 0, & x=0\\ -x^3, & x<0\end{cases}$, which is continuous at $x=0$.
4. If $f(x)$ and $g(x)$ are discontinuous at the same point, then the sum or product of the functions may be continuous.
   For example, both $f(x)=[x]$ (greatest integer function) and $g(x)=\{x\}$ (fractional part function) are discontinuous at $x=1$, but their sum $f(x)+g(x)=x$ is continuous at $x=1$.
   Also, $f(x)=\begin{cases}-1; & x\leq 0\\ 1; & x>0\end{cases}$ and $g(x)=\begin{cases}1; & x\leq 0\\ -1; & x>0\end{cases}$
   Here both the functions are discontinuous at $x=0$, but their product $f(x)g(x)=-1, \forall x\in R$, is continuous at $x=0$.
5. Every polynomial function is continuous at every point of the real line.
   $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n \ \forall x\in R$
6. Every rational function is continuous at every point where its denominator is different from zero.
7. Logarithmic functions, exponential functions, trigonometric functions, inverse circular functions, and modulus functions are continuous in their domain.

Example 3.11 If $f(x)=\begin{cases}|x+1|; & x\leq 0\\ x; & x>0\end{cases}$ and

$g(x)=\begin{cases}|x|+1; & x\leq 1\\ -|x-2|; & x>1\end{cases}$.

Draw its graph and discuss the continuity of $f(x)+g(x)$.

Sol. Since $f(x)$ is discontinuous at $x=0$ and $g(x)$ is continuous at $x=0$, then $f(x)+g(x)$ is discontinuous at $x=0$.
Since $f(x)$ is continuous at $x=1$ and $g(x)$ is discontinuous at $x=1$, then $f(x)+g(x)$ is discontinuous at $x=1$.

Alternative method

$f(x)=\begin{cases}-x-1, & x<-1\\ x+1, & -1\leq x\leq 0\\ x; & x>0\end{cases}$ and

$g(x)=\begin{cases}-x+1, & x\leq 0\\ x+1, & 0<x\leq 1\\ x-2, & 1<x<2\\ -x+2, & x\geq 2\end{cases}$

Also, $f(x)$ and $g(x)$ are re-written as

$$f(x)=\begin{cases}-x-1, & x<-1\\ x+1, & -1\le x\le 0\\ x, & 0<x\le 1\\ x, & 1<x<2\\ x, & x\ge 2\end{cases} \quad \text{and}$$

$$g(x)=\begin{cases}-x+1, & x<-1\\ -x+1, & -1\le x\le 0\\ x+1, & 0<x\le 1\\ x-2, & 1<x<2\\ -x+2, & x\ge 2\end{cases}$$

$$f(x)+g(x)=\begin{cases}-2x, & x<-1\\ 2, & -1\le x\le 0\\ 2x+1, & 0<x\le 1\\ 2x-2 & 1<x<2\\ 2, & x\ge 2\end{cases}$$

The graph of $f(x)+g(x)$ is shown in Fig. 3.7.

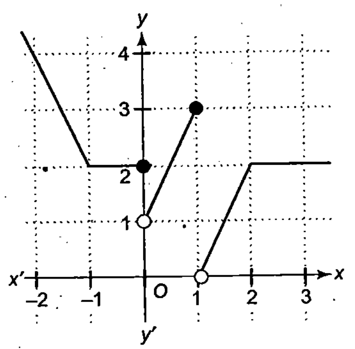


Fig. 3.7

From the graph, $f(x)+g(x)$ is discontinuous at $x=0, 1$.

Concept Application Exercise 3.2

1. Find the value of $f(0)$ so that the function $f(x)=\dfrac{\sqrt{1+x}-\sqrt[3]{1+x}}{x}$ becomes continuous at $x=0$.

2. If the function $f(x)=\dfrac{x^2-(A+2)x+A}{x-2}$ for $x\neq 2$ and $f(2)=2$ is continuous at $x=2$, then find the value of A.

3. If the function $f: R\setminus\{0\}\to R$ given by $f(x)=\dfrac{1}{x}-\dfrac{2}{e^{2x}-1}$ is continuous at $x=0$, then find the value of $f(0)$.

4. Let $f(x)=\dfrac{1-\tan x}{4x-\pi}$, $x\neq\dfrac{\pi}{4}$, $x\in\left[0,\dfrac{\pi}{2}\right]$. If $f(x)$ is continuous in $\left[0,\dfrac{\pi}{4}\right]$, then find the value of $f\left(\dfrac{\pi}{4}\right)$.

5. Discuss the continuity of $f(x)=\begin{cases}\dfrac{x^2}{|x|}, & x\neq 0\\ 0, & x=0\end{cases}$.

6. Let $f(x)=\begin{cases}(1+3x)^{1/x}, & x\neq 0\\ e^3, & x=0\end{cases}$. Discuss the continuity of $f(x)$ at (*a*) $x=0$, (b) $x=1$.

7. Discuss the continuity of $f(x)=\begin{cases}\dfrac{x-1}{e^{\frac{1}{x-1}}+1}, & x\neq 1\\ 0, & x=1\end{cases}$ at $x=1$.

8. Which of the following functions is not continuous $\forall x\in R$?

   a. $\sqrt{2\sin x+3}$ b. $\dfrac{e^x+1}{e^x+3}$

   c. $\left(\dfrac{2^{2x}+1}{2^{3x}+5}\right)^{5/7}$ d. $\sqrt{\text{sgn}\, x+1}$

9. If the function $f(x)=\begin{cases}Ax-B, & x\le 1\\ 3x, & 1<x<2\\ Bx^2-A, & x\ge 2\end{cases}$ be continuous at $x=1$ and discontinuous at $x=2$, find the values of A and B.

10. Discuss the continuity of

$$f(x)=\begin{cases}\dfrac{x^4-5x^2+4}{|(x-1)(x-2)|}, & x\neq 1, 2\\ 6, & x=1\\ 12, & x=2\end{cases}$$

11. Match the following for the type of discontinuity at $x=1$ in column II for the function in column I.

Column I	Column II
a. $f(x)=\dfrac{1}{x-1}$	p. Removable discontinuity
b. $f(x)=\dfrac{x^3-x}{x^2-1}$	q. Non-removable discontinuity
c. $f(x)=\dfrac{\lvert x-1\rvert}{x-1}$	r. Jump of discontinuity
d. $f(x)=\sin\left(\dfrac{1}{x-1}\right)$	s. Discontinuity due to vertical asymptote
	t. Missing point discontinuity
	u. Oscillating discontinuity

CONTINUITY OF SPECIAL TYPES OF FUNCTIONS

Continuity of Functions in which Greatest Integer Function is Involved

$f(x) = [x]$ is discontinuous when x is an integer.

Similarly, $f(x) = [g(x)]$ is discontinuous at all integers when $g(x)$ is an integer, but this is true only when $g(x)$ is monotonic [$g(x)$ is strictly increasing or strictly decreasing].

For example, $f(x) = [\sqrt{x}]$ is discontinuous at all integers when $\sqrt{x}$ is an integer, as $\sqrt{x}$ is strictly increasing (monotonic function).

$f(x) = [x^2], x \geq 0$, is discontinuous at all integers when x^2 is an integer, as x^2 is strictly increasing for $x \geq 0$.

Now consider, $f(x) = [\sin x]$, $x \in [0, 2\pi]$. $g(x) = \sin x$ is not monotonic in $[0, 2\pi]$. For this type of function, points of discontinuity can be determined easily by graphical methods. We can note that at $x = 3\pi/2$, $\sin x$ takes integral value -1, but at $x = 3\pi/2, f(x) = [\sin x]$ is continuous.

Example 3.12 Discuss the continuity of following functions ([·] represents the greatest integer function.)

a. $f(x) = [\log_e x]$

b. $f(x) = [\sin^{-1} x]$

c. $f(x) = \left[\dfrac{2}{1+x^2}\right], x \geq 0$

Sol. **a.** $\log_e x$ function is a monotonically increasing function. Hence $f(x) = [\log_e x]$ is discontinuous, where $\log_e x = k$ or $x = e^k, k \in Z$.

Thus $f(x)$ is discontinuous at $x = \cdots e^{-2}, e^{-1}, e^0, e^1, e^2, \cdots$

b. $\sin^{-1} x$ is a monotonically increasing function.

Hence, $f(x) = [\sin^{-1} x]$ is discontinuous where $\sin^{-1} x$ is an integer.

$\Rightarrow \sin^{-1} x = -1, 0, 1$ or $x = -\sin 1, 0, \sin 1$

c. $\dfrac{2}{1+x^2}, x \geq 0$, is a monotonically decreasing function.

Hence, $f(x) = \left[\dfrac{2}{1+x^2}\right], x \geq 0$ is discontinuous, when $\dfrac{2}{1+x^2}$ is an integer.

$\Rightarrow \dfrac{2}{1+x^2} = 1, 2$

$\Rightarrow x = 1, 0$

Example 3.13 The number of points where $f(x) = [x/3]$, $x \in [0, 30]$ is discontinuous (where [·] represents greatest integer function).

Sol. $f(x) = [x/3]$ is discontinuous when $x/3$ is integer.

For $x \in [0, 30]$, $f(x)$ is discontinuous when $x = 3, 6, 9, \cdots, 27, 30$.

Hence $f(x)$ is discontinuous at exactly 10 values of x.

Example 3.14 Draw the graph and find the points of discontinuity for $f(x) = [2\cos x]$, $x \in [0, 2\pi]$. ([·] represents the greatest integer function.)

Sol. $f(x) = [2\cos x]$

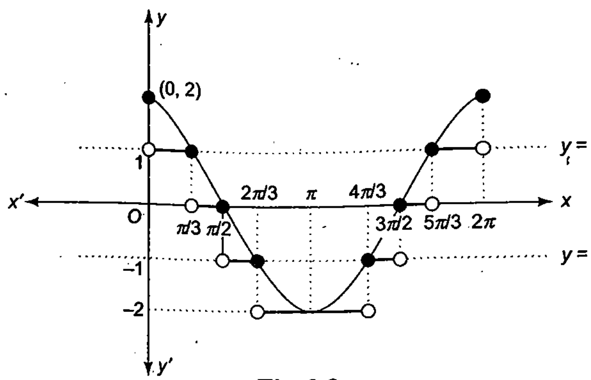


Fig. 3.8

Clearly from the graph given in Fig. 3.8, $f(x)$ is discontinuous at $x = 0, \dfrac{\pi}{3}, \dfrac{\pi}{2}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{3\pi}{2}, \dfrac{5\pi}{3}, 2\pi$.

Example 3.15 Draw the graph and discuss the continuity of $f(x) = [\sin x + \cos x]$, $x \in [0, 2\pi]$, where [·] represents the greatest integer function.

Sol. $f(x) = [\sin x + \cos x] = [g(x)]$ where $g(x) = \sin x + \cos x$

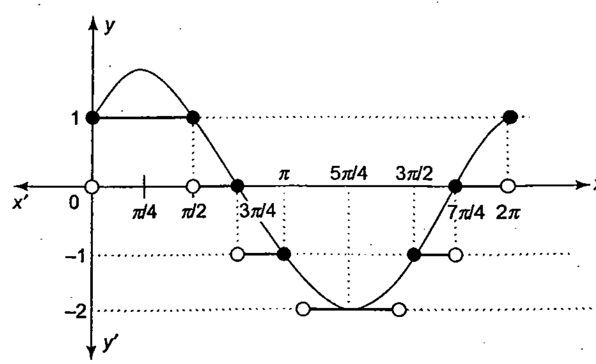


Fig. 3.9

$g(0) = 1,\ g\left(\dfrac{\pi}{4}\right) = \sqrt{2},\ g\left(\dfrac{\pi}{2}\right) = 1$

$g\left(\dfrac{3\pi}{4}\right) = 0,\ g(\pi) = -1,\ g\left(\dfrac{5\pi}{4}\right) = -\sqrt{2}$

$g\left(\dfrac{3\pi}{2}\right) = -1,\ g\left(\dfrac{7\pi}{4}\right) = 0,\ g(2\pi) = 1$

Clearly from the graph given in Fig. 3.9, $f(x)$ is discontinuous at $x = 0, \dfrac{\pi}{2}, \dfrac{3\pi}{4}, \pi, \dfrac{3\pi}{2}, \dfrac{7\pi}{2}, 2\pi$.

Example 3.16 If the function $f(x) = \left[\dfrac{(x-2)^3}{a}\right] \sin(x-2) + a\cos(x-2)$. [.] denotes the greatest integer function which is continuous in [4, 6], then find the values of a.

Sol. $\sin(x-2)$ and $\cos(x-2)$ are continuous for all x.
Since $[x^3]$ is not continuous at integral point.

So $f(x)$ is continuous in [4, 6] if $\left[\frac{(x-2)^3}{a}\right] = 0 \ \forall x \in [4, 6]$.

Now $(x-2)^3 \in [8, 64]$ for $x \in [4, 6]$.

$\Rightarrow a > 64$ for $\left[\frac{(x-2)^3}{a}\right] = 0$

Example 3.17 Discuss the continuity of

$$f(x) = \begin{cases} x\{x\}+1, & 0 \le x < 1 \\ 2-\{x\}, & 1 \le x \le 2 \end{cases}$$

where $\{x\}$ denotes the fractional part function.

Sol. $f(0) = f(0^+) = 1$
$f(2) = 2$ and $f(2^-) = 1$
Hence $f(x)$ is discontinuous at $x = 2$. Also $f(1^+) = 2$
$f(1^-) = 1 + 1 = 2$ and $f(1) = 2$
Hence $f(x)$ is continuous at $x = 1$

Continuity of Functions in which Signum Function is Involved

We know that $f(x) = \text{sgn}(x)$ is discontinuous at $x = 0$.
In general, $f(x) = \text{sgn}(g(x))$ is discontinuous at $x = a$ if $g(a) = 0$.

Example 3.18 Discuss the continuity of

a. $f(x) = \text{sgn}(x^3 - x)$
b. $f(x) = \text{sgn}(2\cos x - 1)$
c. $f(x) = \text{sgn}(x^2 - 2x + 3)$

Sol. **a.** $f(x) = \text{sgn}(x^3 - x)$
Here $x^3 - x = 0 \Rightarrow x = 0, -1, 1$
Hence $f(x)$ is discontinuous at $x = 0, -1, 1$.

b. $f(x) = \text{sgn}(2\cos x - 1)$
Here, $2\cos x - 1 = 0 \Rightarrow \cos x = 1/2 \Rightarrow x = 2n\pi + (\pi/3)$, $n \in Z$, where $f(x)$ is discontinuous.

c. $f(x) = \text{sgn}(x^2 - 2x + 3)$
Here, $x^2 - 2x + 3 > 0$ for all x.
Thus, $f(x) = 1$ for all x, hence continuous for all x.

Example 3.19 If $f(x) = \text{sgn}(2\sin x + a)$ is continuous for all x, then find the possible values of a.

Sol. $f(x) = \text{sgn}(2\sin x + a)$ is continuous for all x.
Then $2\sin x + a \ne 0$ for any real x.
$\Rightarrow \sin x \ne -a/2 \Rightarrow |a/2| > 1 \Rightarrow a < -2$ or $a > 2$

Example 3.20 Discuss the continuity of $f(x) = |x| \,\text{sgn}(x^3 - x)$.

Sol. $\text{sgn}(x^3 - x)$ is discontinuous when $x^3 - x = 0$ or $x = 0, \pm 1$.
But $f(0) = f(0^+) = f(0^-) = 0$.
Hence $f(x)$ is continuous at $x = 0$
Hence $f(x) = |x|\text{sgn}(x^3 - x)$ is discontinuous at $x = \pm 1$ only.

Example 3.21 If $f(x) = \begin{cases} \text{sgn}(x-2) \times [\log_e x], & 1 \le x \le 3 \\ \{x^2\}, & 3 < x \le 3.5 \end{cases}$

where $[\cdot]$ denotes the greatest integer function and $\{.\}$ represents the fractional part function. Find the point where the continuity of $f(x)$ should be checked. Hence, find the points of discontinuity.

Sol. **a.** Continuity should be checked at the endpoints of intervals of each definition, i.e., $x = 1, 3, 3.5$.
b. For $\{x^2\}$, continuity should be checked when $x^2 = 10, 11, 12$ or $x = \sqrt{10}, \sqrt{11}, \sqrt{12}$, $\{x^2\}$ is discontinuous for those values of x where x^2 is an integer (note, here x^2 is monotonic for given domain).
c. For $\text{sgn}(x-2)$, continuity should be checked when $x - 2 = 0$ or $x = 2$.
d. For $[\log_e x]$, continuity should be checked when $\log_e x = 1$ or $x = e (\in [1, 3])$.
Hence, the overall continuity must be checked at $x = 1, 2, e, 3, \sqrt{10}, \sqrt{11}, \sqrt{12}, 3.5$.
Further, $f(1) = 0$ and

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \text{sgn}(x-2) \times [\log_e x] = 0$.
Hence $f(x)$ is continuous at $x = 1$.

$\lim_{x\to2^-} f(x) = \lim_{x\to2^-} \text{sgn}(x-2) \times [\log_e x] = (-1) \times 0 = 0$

$\lim_{x\to2^+} f(x) = \lim_{x\to2^+} \text{sgn}(x-2) \times [\log_e x] = (1) \times 0 = 0$.
Also $f(2) = 0$
Hence, $f(x)$ is continuous at $x = 2$

$\lim_{x\to e^-} f(x) = \lim_{x\to e^-} \text{sgn}(x-2) \times [\log_e x] = (1) \times 0 = 0$

$\lim_{x\to e+} f(x) = \lim_{x\to e^+} \text{sgn}(x-2) \times [\log_e x] = (1) \times (1) = 1$
Hence, $f(x)$ is discontinuous at $x = e$.

$\lim_{x\to3^-} f(x) = \lim_{x\to3^-} \text{sgn}(x-2) \times [\log_e x] = 1$

$\lim_{x\to3^+} f(x) = \lim_{x\to3^+} \{x^2\} = 0$
Hence, $f(x)$ is discontinuous at $x = 3$.
Also $\{x^2\}$ and hence $f(x)$ is discontinuous at $x = \sqrt{10}, \sqrt{11}, \sqrt{12}$.

$\lim_{x\to3.5^-} f(x) = \lim_{x\to3.5^-} \{x^2\} = 0.25 = f(3.5)$
Hence, $f(x)$ is discontinuous at $x = e, 3, \sqrt{10}, \sqrt{11}, \sqrt{12}$.

Continuity of Functions Involving Limit $\lim_{n\to\infty} a^n$

We know that $\lim_{n\to\infty} a^n = \begin{cases} 0, & 0 \le a < 1 \\ 1, & a = 1 \\ \infty, & a > 1 \end{cases}$

Example 3.22 Discuss the continuity of $f(x) = \lim_{n\to\infty} \frac{x^{2n} - 1}{x^{2n} + 1}$.

Sol. $f(x) = \lim_{n\to\infty} \frac{(x^2)^n - 1}{(x^2)^n + 1}$

$$= \lim_{n\to\infty} \frac{1 - \frac{1}{(x^2)^n}}{1 + \frac{1}{(x^2)^n}} = \begin{cases} -1, & 0 \le x^2 < 1 \\ 0, & x^2 = 1 \\ 1, & x^2 > 1 \end{cases} = \begin{cases} 1, & x < -1 \\ 0, & x = -1 \\ -1, & -1 < x < 1 \\ 0, & x = 1 \\ 1, & x > 1 \end{cases}$$

Thus, $f(x)$ is discontinuous at $x = \pm 1$.

Example 3.23 Discuss the continuity of $f(x) = \lim_{n\to\infty} \cos^{2n} x$.

Sol. $f(x) = \lim_{n\to\infty}\left(\cos^2 x\right)^n$

$$= \begin{cases} 0, & 0 \le \cos^2 x < 1 \\ 1, & \cos^2 x = 1 \end{cases} = \begin{cases} 0, & x \ne n\pi,\ n \in I \\ 1, & x = n\pi,\ n \in I \end{cases}$$

Hence, $f(x)$ is discontinuous when $x = n\pi$, $n \in I$.

Example 3.24 Find the values of a if $f(x) = \lim_{n\to\infty} \dfrac{ax^{2n}+2}{x^{2n}+a+1}$ is continuous at $x = 1$.

Sol. $f(1^+) = a$ and $f(1^-) = \dfrac{2}{a+1}$

For continuity at $x = 1$, $a = \dfrac{2}{a+1}$

$\Rightarrow a^2 + a = 2 \ \Rightarrow\ a^2 + a - 2 = 0 \ \Rightarrow\ a = -2, a = 1$

Continuity of Functions in which $f(x)$ is Defined Differently for Rational and Irrational Values of x

Example 3.25 Discuss the continuity of the following function:

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}.$$

Sol. For any $x = a$,

L.H.L. $= \lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h) = 0$ or 1

[as $\lim_{h\to 0}(a-h)$ can be rational or irrational]

Similarly, R.H.L. $= \lim_{x\to a+} f(x) = \lim_{h\to 0} f(a+h) = 0$ or 1.

Hence, $f(x)$ oscillates between 0 and 1 as for all values of a.

$\therefore$ L.H.L. and R.H.L. do not exist.

$\Rightarrow f(x)$ is discontinuous at a point $x = a$ for all values of a.

Example 3.26 Find the value of x where

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases} \text{ is continuous.}$$

Sol. $f(x)$ is continuous at some $x = a$, where $x = 1 - x$ or $x = 1/2$.

Hence, $f(x)$ is continuous at $x = 1/2$.

We have $f(1/2) = 1/2$

If $x \to 1/2^+$ then x may be rational or irrational

$\Rightarrow f(1/2^+) = 1/2$ or $1 - 1/2 = 1/2$

If $x \to 1/2^-$ then x may be rational or irrational

$\Rightarrow f(1/2^-) = 1/2$ or $1 - 1/2 = 1/2$

Hence $f(x)$ is continuous at $x = 1/2$.

For some other point, say, $x = 1 \ \Rightarrow\ f(1) = 1$

If $x \to 1^+$ then x may be rational or irrational.

$\Rightarrow f(1^+) = 1$ or $1 - 1 = 0$

Hence, $f(1^+)$ oscillates between 1 and 0, which causes discontinuity at $x = 1$.

Similarly, $f(x)$ oscillates between 0 and 1 for all $x \in R - \{1/2\}$.

Continuity of Composite Functions

$f(x) = f(g(x))$ is discontinuous also at those values of x where $g(x)$ is discontinuous.

For example, $f(x) = \dfrac{1}{1-x}$ is discontinuous at $x = 1$

Now $f(f(x)) = \dfrac{1}{1-\dfrac{1}{1-x}} = \dfrac{x-1}{x}$ is not only discontinuous at $x = 0$ but also at $x = 1$.

Now $f(f(f(x))) = \dfrac{\dfrac{x-1}{x}-1}{\dfrac{x-1}{x}} = x$ seems to be continuous, but it is discontinuous at $x = 1$ and $x = 0$, where $f(x)$ and $f(f(x))$ are discontinuous, respectively.

Example 3.27 If $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = \dfrac{1}{x-2}$, then discuss the continuity of $f(x)$, $g(x)$, and $fog(x)$.

Sol. $f(x) = \dfrac{x+1}{x-1}$

$\therefore$ f is not defined at $x = 1$. $\therefore$ f is discontinuous at $x = 1$.

$g(x) = \dfrac{1}{x-2}$

$g(x)$ is not defined at $x = 2$. $\therefore$ g is discontinuous at $x = 2$.

Now, $fog(x)$ will be discontinuous at

a. $x = 2$ [point of discontinuity of $g(x)$]

b. $g(x) = 1$ [when $g(x)$ = point of discontinuity of $f(x)$]

if $g(x) = 1 \Rightarrow \dfrac{1}{x-2} = 1 \Rightarrow x = 3$

$\therefore$ $fog(x)$ is discontinuous at $x = 2$ and $x = 3$.

Also, $fog(x) = \dfrac{\dfrac{1}{x-2}+1}{\dfrac{1}{x-2}-1}$

Here $fog(2)$ is not defined.

$$\lim_{x\to 2} fog(x) = \lim_{x\to 2} \frac{\dfrac{1}{x-2}+1}{\dfrac{1}{x-2}-1} = \lim_{x\to 2} \frac{1+x-2}{1-x+2} = 1.$$

$\therefore$ $fog(x)$ is discontinuous at $x = 2$ and it has a removable discontinuity at $x = 2$. For continuity at $x = 3$,

$$\lim_{x\to 3^+} fog(x) = \lim_{x\to 3^+} \frac{\dfrac{1}{x-2}+1}{\dfrac{1}{x-2}-1} = -\infty$$

$$\lim_{x\to 3^-} fog(x) = \lim_{x\to 3^-} \frac{\dfrac{1}{x-2}+1}{\dfrac{1}{x-2}-1} = \infty$$

$\therefore$ $fog(x)$ is discontinuous at $x = 3$, and it is a non-removable discontinuity at $x = 3$.

Example 3.28 If $f(x) = \begin{cases} x-2, & x \le 0 \\ 4-x^2, & x > 0 \end{cases}$, then discuss the continuity of $y = f(f(x))$. (Good)

Sol. $f(x)$ is discontinuous at $x = 0$, Hence, $f(f(x))$ may be discontinuous at $x = 0$

$f(f(0^+)) = f(4) = 4 - 16 = -12$

and $f(f(0^-)) = f(-2) = -4$

Hence, $f(x)$ is discontinuous at $x = 0$

$f(f(x))$ is also discontinuous when $f(x) = 0$

$\Rightarrow$ $x - 2 = 0$ when $x \le 0$ or $x^2 - 4 = 0$ when $x > 0$

$\Rightarrow$ at $x = 2$

Also we can see that $f(f(2)) = 0, f(f(2^+)) = f(0^-) = -2$ and $f(f(2^-)) = f(0^+) = 4$

Hence $f(f(x))$ is discontinuous at $x = 0$ and $x = 2$.

Concept Application Exercise 3.3

1. Find the values of x in [1, 3] where the function is $[x^2 + 1]$ ([·] represents the greatest integer function) is discontinuous.
2. Find the number of points of discontinuity for $f(x) = [6\sin x]$, $0 \le x \le \pi$, ([·] represents the greatest integer function).
3. Discuss the continuity of $f(x) = [\tan^{-1}x]$ ([·] represents the greatest integer function).
4. Discuss the continuity of $f(x) = \{\cot^{-1}x\}$ ({·} represents the fractional part function).
5. $f(x) = \lim_{n\to\infty} \dfrac{x^n - \sin x^n}{x^n + \sin x^n}$ for $x > 0, x \ne 1$ and $f(1) = 0$. Discuss the continuity at $x = 1$.
6. If $f(x)$ is a continuous function $\forall x \in R$ and the range of $f(x)$ is $(2, \sqrt{26})$ and $g(x) = \left[\dfrac{f(x)}{c}\right]$ is continuous $\forall x \in R$, then find the least positive integral value of c, where [.] denotes the greatest integer function.
7. Discuss the continuity of $f(x)$ in [0, 2], where $f(x) = \lim_{n\to\infty}\left(\sin\dfrac{\pi x}{2}\right)^{2n}$.
8. Discuss the continuity of $f(x) = \begin{cases} x^2, & x \text{ is rational} \\ -x^2, & x \text{ is irrational.}\end{cases}$
9. If $y = \dfrac{1}{t^2 + t - 2}$, where $t = \dfrac{1}{x-1}$, then find the number of points where $f(x)$ is discontinuous.
10. If $f(x) = \begin{cases} [\sin \pi x], & 0 \le x < 1 \\ \text{sgn}\left(x - \dfrac{5}{4}\right)\times\left\{x - \dfrac{2}{3}\right\}, & 1 \le x \le 2 \end{cases}$ where [·] denotes the greatest integer function and {.} the represents fractional part function. At what points should the continuity checked? Hence, find the points of discontinuity.
11. Find the value of a for which $f(x) = \begin{cases} x^2, & x \in Q \\ x + a, & x \notin Q \end{cases}$ is not continuous at any x.
12. Discuss the continuity of $f(x) = (\log|x|)\text{sgn}(x^2 - 1), x \ne 0$.
13. Find the number of integer lying in the interval (0, 4), where the function $f(x) = \lim_{n\to\infty}\left(\cos\dfrac{x\pi}{2}\right)^{2n}$ is discontinuous.

Properties of Functions Continuous in [a, b]

1. If a function f is continuous on a closed interval $[a, b]$, then it is bounded.
2. A continuous function whose domain is some closed interval must have its range also in the closed interval.
3. If $f(a)$ and $f(b)$ possess opposite signs, then there exists at least one solution of the equation $f(x) = 0$ in the open interval (a, b) provided f is continuous in $[a, b]$.
4. If f is continuous on $[a, b]$, then f^{-1} is also continuous.

Example 3.29 Let f be a continuous function defined onto on [0, 1] with range [0, 1]. Show that there is some c in [0, 1] such that $f(c) = 1 - c$.

Sol. Consider $g(x) = f(x) - 1 + x$

$g(0) = f(0) - 1 \le 0$ [as $f(0) \le 1$]

$g(1) = f(1) \ge 0$ [as $f(1) \ge 0$]

Hence, $g(0)$ and $g(1)$ have values of opposite signs.

Hence, there exists at least one $c \in (0, 1)$ such that $g(c) = 0$.

$\therefore g(c) = f(c) - 1 + c = 0; f(c) = 1 - c$.

Example 3.30 Let f be continuous on the interval [0, 1] to R such that $f(0) = f(1)$. Prove that there exists a point c in $\left[0, \dfrac{1}{2}\right]$ such that $f(c) = f\left(c + \dfrac{1}{2}\right)$.

Sol. Consider a continuous function $g(x) = f\left(x + \dfrac{1}{2}\right) - f(x)$

$\left(g \text{ is continuous } \forall x \in \left[0, \dfrac{1}{2}\right]\right)$

$\Rightarrow g(0) = f\left(\dfrac{1}{2}\right) - f(0) = f\left(\dfrac{1}{2}\right) - f(1)$ [as $f(0) = f(1)$]

and $g\left(\dfrac{1}{2}\right) = f(1) - f\left(\dfrac{1}{2}\right) = -\left[f\left(\dfrac{1}{2}\right) - f(1)\right]$

Since g is continuous and $g(0)$ and $g(1/2)$ have opposite signs, hence the equation $g(x) = 0$ must have at least one root in [0, 1/2].

Hence, for some $c \in \left[0, \dfrac{1}{2}\right]$, $g(c) = 0 \Rightarrow f\left(c + \dfrac{1}{2}\right) = f(c)$.

Example 3.31 Let $f: [0, 1] \to [0, 1]$ be a continuous function. Then prove $f(x) = x$ for at least one $0 \le x \le 1$

Sol.

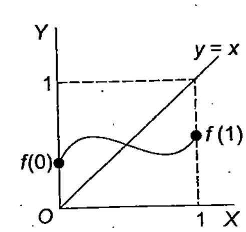


Fig. 3.10

Clearly, $0 \le f(0) \le 1$ and $0 \le f(1) \le 1$. As $f(x)$ is continuous, $f(x)$ attains all the values between $f(0)$ and $f(1)$, and the graph will have no breaks. So, the graph will cut the line $y = x$ at least once, where $0 \le x \le 1$.

So, $f(x) = x$ at that point.

Example 3.32 Suppose f is a continuous map for R to R and $f(f(a)) = a$ for some a. Show that there is some b such that $f(b) = b$.

Sol. If $f(a) = a$ then $b = a$ solves the problem.
So assume $f(a) > a$. Then $g(x) = f(x) - x$ is positive at $x = a$ and is negative at $c = f(a)$ since $g(c) = f(f(a)) - f(a) = a - f(a) < 0$.
Since $g : R \to R$ is continuous, there must be some b, $a < b < c$, such that $g(x) = 0$, i.e., $f(b) = b$.
The same argument works if $f(a) < a$.

INTERMEDIATE VALUE THEOREM

If f is continuous on $[a, b]$ and $f(a) \neq f(b)$, then for any value $c \in (f(a), f(b))$, there is at least one number x_0 in (a, b) for which $f(x_0) = c$.

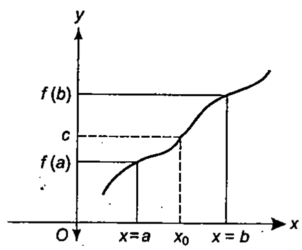


Fig. 3.11 (a)

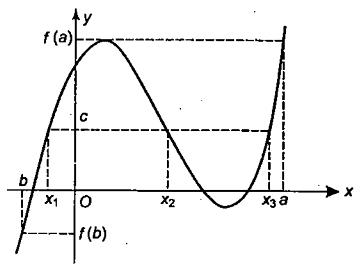


Fig. 3.11 (b)

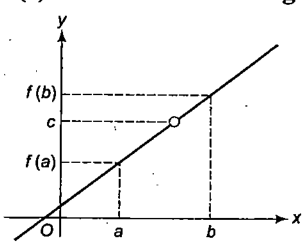


Fig. 3.11 (c)

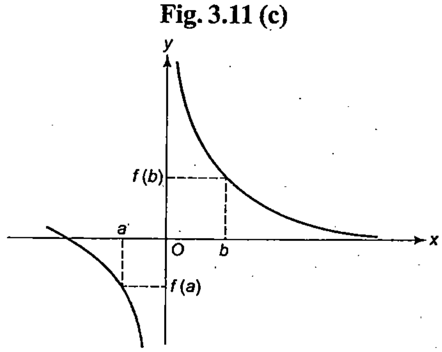


Fig. 3.11 (d)

From Fig. 3.11 (c) and (d), it is clear that continuity in the interval $[a, b]$ is essential for the validity of this theorem.

Example 3.33 Show that the function
$f(x) = (x - a)^2 (x - b)^2 + x$ takes the value $\dfrac{a+b}{2}$ for some value of $x \in [a, b]$.

Sol. $f(a) = a$; $f(b) = b$. Also f is continuous in $[a, b]$ and $\dfrac{a+b}{2} \in [a, b]$.
Hence, using intermediate value theorem, there exists at least one $c \in [a, b]$ such that $f(c) = \dfrac{a+b}{2}$.

Example 3.34 Using intermediate value theorem, prove that there exists a number x such that

$$x^{2005} + \frac{1}{1 + \sin^2 x} = 2005.$$

Sol. Let $f(x) = x^{2005} + (1 + \sin^2 x)^{-2}$
$\therefore f$ is continuous and $f(0) = 1 < 2005$ and $f(2) > 2^{2005}$, which is much greater than 2005. Hence, from the intermediate value theorem, there exists a number c in $(0, 2)$ such that $f(c) = 2005$.

DIFFERENTIABILITY

Existence of Derivative

Right- and Left-Hand Derivatives

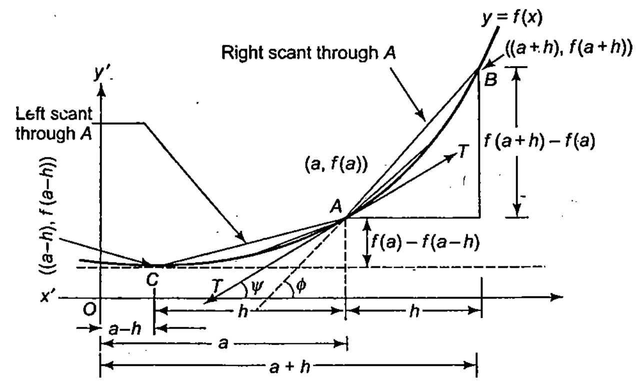


Fig. 3.12

- The right-hand derivative of f at $x = a$, denoted by $f'(a^+)$, is defined by $f'(a^+) = \lim_{h \to 0^+} \dfrac{f(a+h) - f(a)}{h}$ provided the limit exists, and is finite.
When $h \to 0$, the point B moving along the curve tends to A, i.e., $B \to A$, then the chord AB approaches the tangent line AT at the point A and then $\phi \to \psi$

$$\Rightarrow f'(a^+) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \tan \phi = \tan \psi$$

- The left-hand derivative of f at $x = a$, denoted by $f'(a^-)$, is defined by $f'(a^-) = \lim_{h \to 0} \dfrac{f(a-h) - f(a)}{-h}$ provided the limit exists, and is finite.
When $h \to 0$, the point C moving along the curve tends to A, i.e., $C \to A$, then the chord CA approaches the tangent line AT at the point A and then

$$\Rightarrow f'(a^-) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$

- At A, $f(x)$ is differentiable if both $f'(a^+) - f'(a^-)$ exist, equal and both are finite.
In other words, $f(x)$ is differentiable at $x = a$, if a unique tangent can be drawn at this point.

Differentiability and Continuity

If $f(x)$ is differentiable at every point of its domain, then it must be continuous in that domain.

Proof: To prove that f is continuous at a, we have to show that $\lim_{x\to a} f(x) = f(a)$.

We do this by showing that the difference $f(x) - f(a)$ approaches 0.

The given information is that f is differentiable at a, that is,

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \text{ exists.}$$

To connect the given and the unknown, we divide and multiply $f(x)-f(a)$ by $x-a$ (which we can do when $x \neq a$)

$$\Rightarrow f(x)-f(a) = \frac{f(x)-f(a)}{x-a}(x-a)$$

Thus, using the product law, we can write

$$\lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}(x-a)$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \lim_{x\to a}(x-a)$$

$$= f'(a) \times 0 = 0$$

To use what we have just proved, we start with $f(x)$ and add and subtract $f(a)$, we get

$$\lim_{x\to a} f(x) = \lim_{x\to a}[f(a) + (f(x)-f(a))]$$

$$= \lim_{x\to a} f(a) + \lim_{x\to a}[f(x)-f(a)] = f(a) + 0 = f(a)$$

Therefore, f is continuous at a.

Note:
- *The converse of this is false, that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because $\lim_{x\to 0} f(x) = \lim_{x\to 0} |x| = 0 = f(0)$ but non-differentiable as unique tangent cannot be drawn.*
- *If $f(x)$ is differentiable, then its graph must be smooth, i.e., there should be no break or corner.*
  Thus for a function $f(x)$:
  (a) Differentiable $\Rightarrow$ Continuous
  (b) Continuous $\Rightarrow$ May or may not be differentiable
  (c) Not continuous $\Rightarrow$ Not differentiable

How Can a Function Fail to be Differentiable

The function $f(x)$ is said to be non-differentiable at $x = a$ if

a. both $Rf'(a)$ and $Lf'(a)$ exist but are not equal,

b. either or both $Rf'(a)$ and $Lf'(a)$ are not finite, and

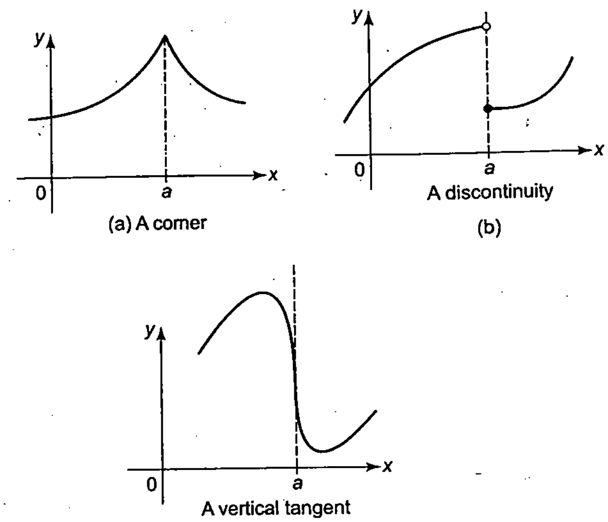
c. either or both $Rf'(a)$ and $Lf'(a)$ do not exist.

The function $y = |x|$ is not differentiable at 0 as its graph changes direction abruptly when $x = 0$. In general, if the graph of a function has a 'corner' or 'kink' in it, then the graph of f has no tangent at this point and f is not differentiable there. (To compute $f'(a)$, we find that the left and right derivatives are different.)

If f is not continuous at a, then f is not differentiable at a. So at any discontinuity (for instance, a jump of discontinuity) f fails to be differentiable.

A third possibility is that the curve has a vertical tangent line when $x = a$, that is, f is continuous at a and $\lim_{x\to a} |f'(x)| = \infty$.

This means that the tangent lines become steeper and steeper as $x \to a$. The following figures illustrate the three possibilities that we have discussed.



(a) A corner

A discontinuity (b)

A vertical tangent (c)

Fig. 3.13

Theorems on Differentiability

1. Addition of differentiable and non-differentiable functions is always non-differentiable.
2. Product of differentiable and non-differentiable functions may be differentiable.

   For example,

   $f(x) = x|x|$ is differentiable at $x = 0$.

   $f(x) = (x-1)|x-1|$ is differentiable at $x = 1$.

   $f(x) = (x-1)\sqrt{|\log x|}$ is differentiable at $x = 1$.

   In general $f(x) = g(x)|g(x)|$ is differentiable at $x = a$ when $g(a) = 0$.

   $f(x) = x|x-1|$ is non-differentiable at $x = 1$.
3. If $g(x)$ is a differentiable function and $f(x) = |g(x)|$ is a non-differentiable function at $x = a$, then $g(a) = 0$.

   For example, $|\sin x|$ is non-differentiable when $\sin x = 0$ or $x = n\pi, n \in Z$.
4. If both $f(x)$ and $g(x)$ are non-differentiable at $x = a$, then $f(x) + g(x)$ may be differentiable at $x = a$.

   For example,

   $$f(x) = \sin|x| - |x| = \begin{cases} -\sin + x, & x < 0 \\ \sin x - x, & x \geq 0 \end{cases}$$

   $$\Rightarrow g'(x) = \begin{cases} -\cos x + 1, & x < 0 \\ \cos x - 1, & x > 0 \end{cases} \Rightarrow f'(0^+) = 0 \text{ and } f'(0^-) = 0$$

Points to Remember

If $y=f(x)$ is differentiable at $x=a$, then it is not necessary that the derivative is continuous at $x=a$.
For example, consider function

$$f(x)=\begin{cases}x^2\sin(1/x), & x\neq 0\\ 0, & x=0\end{cases}$$

For $x\neq 0$,

$$f'(x)=2x\sin(1/x)+x^2\left(-\frac{1}{x^2}\right)\cos\left(\frac{1}{x}\right)$$

$$\Rightarrow f'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}$$

For $x=0$, $f'(x)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$$=\lim_{h\to 0}\frac{h^2\sin\frac{1}{h}-0}{h}=\lim_{h\to 0}h\sin\frac{1}{h}=0$$

Thus $f'(x)=\begin{cases}2x\sin\frac{1}{x}-\cos\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

Now, $f'(x)$ is continuous at $x=0$, if

a. $\lim_{x\to 0}f'(x)$ exists b. $\lim_{x\to 0}f'(x)=f'(0)$

Again, $\lim_{x\to 0}f'(x)=\lim_{x\to 0}\left(2x\sin\frac{1}{x}-\cos\frac{1}{x}\right)$ does not exist.

Since, $\lim_{x\to 0}\cos\frac{1}{x}$ does not exist.
Hence, $f'(x)$ is not continuous at $x=0$.

Differentiability using First Definition of Derivatives

Example 3.35 Discuss the differentiability of

$$f(x)=\begin{cases}\frac{\sin x^2}{x}, & x\neq 0\\ 0, & x=0\end{cases}\quad \text{at } x=0.$$

Sol. For continuity, $\lim_{x\to 0}f(x)=\lim_{h\to 0}\frac{\sin h^2}{h}$

$$=\lim_{h\to 0}h\frac{\sin h^2}{h^2}=0$$

Hence, $f(x)$ is continuous at $x=0$.

Also, $f'(0^+)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{\sin h^2}{h^2}=1$

and $f'(0^-)=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}=\lim_{h\to 0}\frac{\sin h^2}{h^2}=1$

Thus, $f(x)$ is differentiable at $x=0$.

Example 3.36 Discuss the differentiability of

$$f(x)=\begin{cases}x\sin(\ln x^2), & x\neq 0\\ 0, & x=0\end{cases}\quad \text{at } x=0.$$

Sol. For continuity,

$$f(0^+)=\lim_{h\to 0}h\sin(\ln h^2)$$
$$=0\times(\text{any value between }-1\text{ and }1)=0$$
$$f(0^-)=\lim_{h\to 0}(-h)\sin(\ln h^2)$$
$$=0\times(\text{any value between }-1\text{ and }1)=0$$

Hence, $f(x)$ is continuous at $x=0$.
For differentiability,

$$f'(0^+)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$
$$=\lim_{h\to 0}\frac{h\sin(\ln h^2)-0}{h}=\lim_{h\to 0}\sin(\ln h^2)$$

= any value between -1 and 1.
Hence, $f'(0)$ does not take any fixed value.
Hence, $f(x)$ is not differentiable at $x=0$.

Example 3.37 Which of the following function is non-differential at $x=0$?
(i) $f(x)=\cos|x|$
(ii) $f(x)=x|x|$
(iii) $f(x)=|x^3|$

Sol. (i) $f(x)=\cos|x|=\cos x$ which is differentiable at $x=0$

(ii) $f(x)=x|x|=\begin{cases}x^2, & x\geq 0\\ -x^2, & x<0\end{cases}\Rightarrow f'(x)=\begin{cases}2x, & x>0\\ -2x, & x<0\end{cases}$,

$f'(0^+)=f'(0^-)=0$
Hence $f(x)$ is differentiable at $x=0$.

(iii) $f(x)=|x^3|=\begin{cases}x^3, & x\geq 0\\ -x^3, & x<0\end{cases}\Rightarrow f'(x)=\begin{cases}3x^2, & x>0\\ -3x^2, & x<0\end{cases}$,

$f'(0^+)=f'(0^-)=0$
Hence $f(x)$ is differentiable at $x=0$.

Example 3.38 Discuss the differentiability of

$$f(x)=\begin{cases}(x-e)2^{-2^{\left(\frac{1}{(e-x)}\right)}}, & x\neq e\\ 0, & x=e\end{cases}\quad \text{at } x=e.$$

Sol. $f(e^+)=\lim_{h\to 0}(e+h-e)\,2^{-2^{\frac{1}{e-(e+h)}}}$

$$=\lim_{h\to 0}(h)\,2^{-2^{-\frac{1}{h}}}$$

$$=0\times 1=0\ (\text{as for } h\to 0,\ -\frac{1}{h}\to-\infty\Rightarrow 2^{-\frac{1}{h}}\to 0)$$

$$f(e^-)=\lim_{h\to 0}(-h)\,2^{-2^{\frac{1}{h}}}=0\times 0=0$$

Hence, $f(x)$ is continuous $x=e$.

$$f'(e^+)=\lim_{h\to 0}\frac{f(e+h)-f(e)}{h}=\lim_{h\to 0}\frac{h\times 2^{-2^{-\frac{1}{h}}}-0}{h}$$

$$=\lim_{h\to 0}2^{-2^{-\frac{1}{h}}}=1$$

$$f'(e^-)=\lim_{h\to 0}\frac{f(e-h)-f(0)}{-h}=\lim_{h\to 0}\frac{(-h)2^{-2^{\frac{1}{h}}}-0}{-h}$$

$$=\lim_{h\to 0}2^{-2^{\frac{1}{h}}}=0$$

Hence, $f(x)$ is non-differentiable at $x=e$.

Example 3.39 A function $f(x)$ is such that $f\left(x+\frac{\pi}{2}\right)=\frac{\pi}{2}-|x|\ \forall x$. Find $f'\left(\frac{\pi}{2}\right)$, if it exists.

Sol. Given that $f\left(x+\frac{\pi}{2}\right)=\frac{\pi}{2}-|x|$

$$=f'\left(\frac{\pi^+}{2}\right)=\lim_{h\to 0}\frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h}=\frac{\frac{\pi}{2}-|h|-\frac{\pi}{2}}{h}=-1$$

and $$f'\left(\frac{\pi^-}{2}\right)=\lim_{h\to 0}\frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h}=\frac{\frac{\pi}{2}-|-h|-\frac{\pi}{2}}{-h}=1$$

$\Rightarrow f'\left(\frac{\pi}{2}\right)$ does not exist.

Differentiability using Theorems on Differentiability

Example 3.40 Discuss the differentiability of $f(x)=|x|+|x-1|$.

Sol. $f(x)=|x|+|x-1|$
$f(x)$ is continuous everywhere as $|x|$ and $|x-1|$ are continuous for all x.
Also $|x|$ and $|x-1|$ are non-differentiable at $x=0$ and $x=1$, respectively.
Hence, $f(x)$ is non-differentiable at $x=0$ and $x=1$.

Example 3.41 Discuss the differentiability of $f(x)=[x]+|1-x|, x\in(-1,3)$, where [.] represents greatest integer function.

Sol. $[x]$ is non-differentiable at $x=0$, 1, 2 and $|1-x|$ is non-differentiable at $x=1$. Thus, $f(x)$ is definitely non-differentiable at $x=0,2$. Moreover, $[x]$ is discontinuous at $x=1$, whereas $|1-x|$ is continuous at $x=1$. Thus, $f(x)$ is discontinuous and hence non-differentiable at $x=1$.

Example 3.42 Discuss the differentiability of $f(x)=(x^2-1)|x^2-x-2|+\sin(|x|)$.

Sol. $f(x)=(x^2-1)|x^2-x-2|+\sin(|x|)$
$=(x-1)(x+1)|x+1||x-2|+\sin(|x|)$
$(x+1)|x+1|$ is differentiable at $x=-1$
$|x-2|$ is non-differentiable at $x=2$
$\sin(|x|)$ is non-differentiable at $x=0$
Hence $f(x)$ is differentiable at $x=-1$ but not at $x=0$ and $x=2$

Example 3.43 Discuss the differentiability of $f(x)=|x|\sin x+|x-2|\,\text{sgn}(x-2)+|x-3|$

Sol. $|x|\sin x$ is differential at $x=0$, though $|x|$ is non-differentiable at $x=0$, as $\sin 0=0$

$$|x-2|\,\text{sgn}(x-2)=\begin{cases}(2-x)(-1), & x<2\\ 0, & x=2\\ (x-2)(1), & x>2\end{cases}=x-2, x\in R$$

which is differentiable
$|x-3|$ is non-differentiable at $x=3$, hence $f(x)$ is non-differentiable at $x=3$

Differentiability using Graphs

Example 3.44 Discuss the differentiability of
a. $f(x)=\sin|x|$,
b. $f(x)=|\log_e|x||$,
c. $f(x)=\max\{\sec^{-1}x, \text{cosec}^{-1}x\}$,
d. $y=\sin^{-1}(\sin x)$, and
e. $y=\sin^{-1}|\sin x|$.
f. $f(x)=\max\{x^2-3x+2, 2-|x-1|\}$

Sol.
a. $f(x)=\sin|x|$

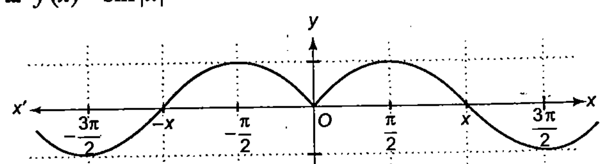


Fig. 3.14

Clearly from the graph, $f(x)$ is non-differentiable at $x=0$.

b. $f(x)=|\log_e|x||$

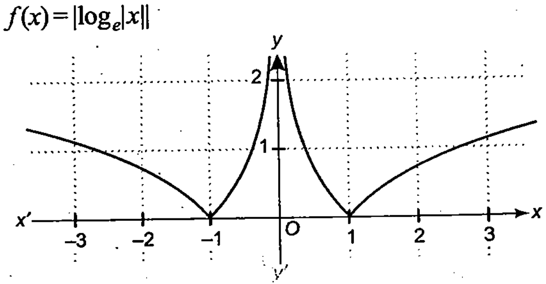


Fig. 3.15

Clearly from the graph, $f(x)$ is non-differentiable at $x=0$, ± 1.

c. $f(x)=\max\{\sec^{-1}x, \text{cosec}^{-1}x\}$

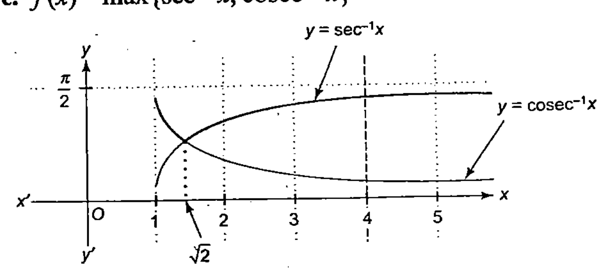


Fig. 3.16

Clearly from the graph, $f(x)$ is non-differentiable at $x=\sqrt{2}$.

d. $y=\sin^{-1}(\sin x)$

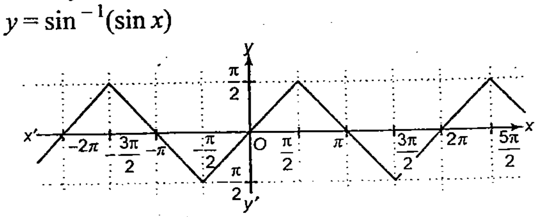


Fig. 3.17

Clearly from the graph, $f(x)$ is non-differentiable at $x = (2n+1)\frac{\pi}{2},\ n \in Z$.

e. $y = \sin^{-1}|\sin x|$

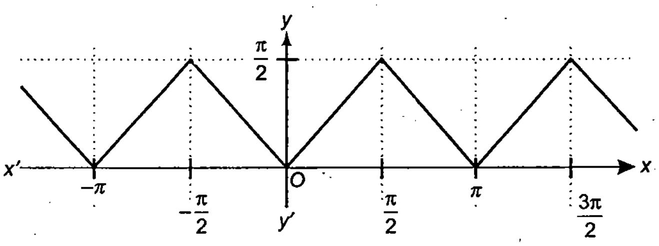


Fig. 3.18

Clearly from the graph, $f(x)$ is non-differentiable at $x = \frac{n\pi}{2},\ n \in Z$.

f. From the graph $f(x)$ is non-differentiable

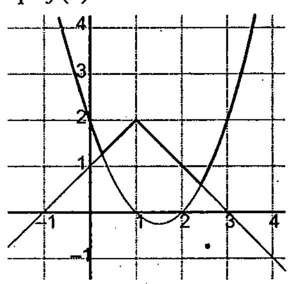


Fig. 3.19

(i) at $x = 1$,
(ii) where $x^2 - 3x + 2 = 2 - (1 - x)$, when $x < 1$
(iii) where $x^2 - 3x + 2 = 2 - (x - 1)$, where $x > 1$

Hence $f(x)$ is discontinuous at $x = 1$, and $x = 2 - \sqrt{3}$ and $x = 1 + \sqrt{2}$

Example 3.45 Discuss the differentiability of $f(x) = \text{maximum}\ \{2 \sin x, 1 - \cos x\}\ \forall x \in (0, \pi)$.

Sol. $f(x) = \max\{2 \sin x, 1 - \cos x\}$ can be plotted as shown in Fig. 3.19.

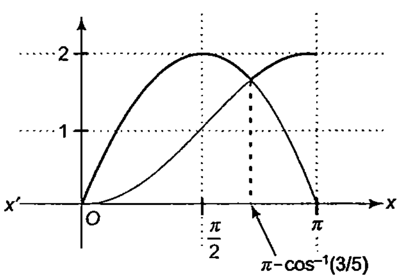


Fig. 3.20

Thus, $f(x) = \text{maximum}\ \{2 \sin x, 1 - \cos x\}$ is not differentiable, when $2 \sin x = 1 - \cos x$

$\Rightarrow 4 \sin^2 x = (1 - \cos x)^2$

$\Rightarrow 4(1 + \cos x) = (1 - \cos x)$

$\Rightarrow 4 + 4 \cos x = 1 - \cos x$

$\Rightarrow \cos x = -3/5$

$\Rightarrow x = \cos^{-1}(-3/5)$

$\Rightarrow$ $f(x)$ is not differentiable at $x = \pi - \cos^{-1}(3/5)$, $\forall x \in (0, \pi)$.

Example 3.46 Discuss the differentiability of $f(x) = e^{-|x|}$.

Sol. We have $f(x) = \begin{cases} e^{-x}, & x \ge 0 \\ e^{x}, & x < 0 \end{cases}$

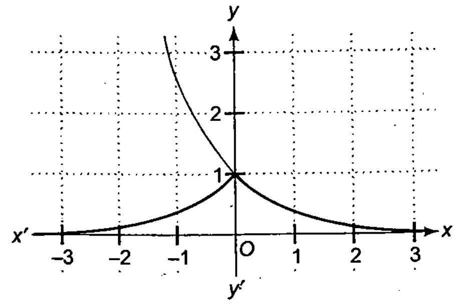


Fig. 3.21

Clearly from the graph, $f(x)$ is non-differentiable at $x = 0$.

Example 3.47 If $f(x) = \max\ \{x^2 + 2ax + 1, b\}$ has two points of non-differentiability, then prove that $a^2 > 1 - b$.

Sol. $f(x) = \max\ \{x^2 + 2ax + 1, b\}$ has two points of non-differentiability if

$y = x^2 + 2ax + 1$ and $y = b$ intersect at two points

or $x^2 + 2ax + 1 = b$ has real and distinct roots

or $x^2 + 2ax + 1 - b = 0$ has real and distinct roots

$\Rightarrow 4a^2 - 4(1 - b) > 0 \quad \Rightarrow \quad a^2 > 1 - b$

Example 3.48 Test the continuity and differentiability of the function $f(x) = \left|\left(x + \frac{1}{2}\right)[x]\right|$ by drawing the graph of the function when $-2 \le x \le 2$, where [.] represents greatest integer function.

Sol. Here, $f(x) = \left|\left(x + \frac{1}{2}\right)[x]\right|,\ -2 \le x \le 2$

$$\Rightarrow f(x) = \begin{cases} \left|\left(x + \frac{1}{2}\right)(-2)\right|, & -2 \le x < -1 \\ \left|\left(x + \frac{1}{2}\right)(-1)\right|, & -1 \le x < 0 \\ \left|\left(x + \frac{1}{2}\right)(0)\right|, & 0 \le x < 1 \\ \left|\left(x + \frac{1}{2}\right)(1)\right|, & 1 \le x < 2 \\ \left|\frac{3}{2} \times 2\right|, & x = 2 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} -(2x+1), & -2 \le x < -1 \\ -\left(x + \frac{1}{2}\right), & -1 \le x < -1/2 \\ (x + 1/2), & -\frac{1}{2} \le x < 0 \\ 0, & 0 \le x < 1 \\ x + \frac{1}{2}, & 1 \le x < 2 \\ 3, & x = 2 \end{cases}$$

which could be plotted as

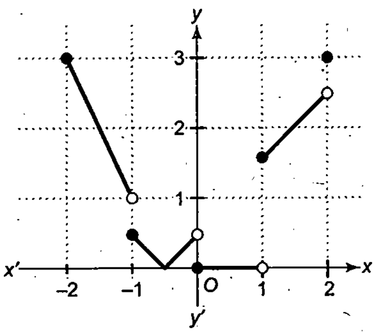


Fig. 3.22

Fig. 3.21 clearly shows that $f(x)$ is not continuous at $x = \{-1, 0, 1, 2\}$ as at these points the graph is broken. $f(x)$ is not differentiable at $x = \left\{-1, \frac{-1}{2}, 0, 1, 2\right\}$ as at $x = \{-1, 0, 1, 2\}$ the graph is broken and at $x = -1/2$ there is a sharp edge.

Differentiability by Differentiation

Example 3.49 If $f(x) = \begin{cases} x, & x \le 1 \\ x^2 + bx + c, & x > 1 \end{cases}$, then find the values of b and c if $f(x)$ is differentiable at $x = 1$.

Sol. $f(x) = \begin{cases} x, & x \le 1 \\ x^2 + bx + c, & x > 1 \end{cases} \Rightarrow f'(x) = \begin{cases} 1, & x < 1 \\ 2x + b, & x > 1 \end{cases}$

$f(x)$ is differentiable at $x = 1$

Then, it must be continuous at $x = 1$

for which $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$

$\Rightarrow 1 + b + c = 1$

$\Rightarrow b + c = 0$ (1)

Also $f'(1^+) = f'(1^-)$

$\Rightarrow \lim_{x \to 1^+} f'(x) = \lim_{x \to 1^-} f'(x)$

$\Rightarrow 2 + b = 1 \Rightarrow b = -1$

$\Rightarrow c = 1$ [from equation (1)]

Example 3.50 Find the values of a and b if

$$f(x) = \begin{cases} a + \sin^{-1}(x + b), & x \ge 1 \\ x, & x < 1 \end{cases} \text{ is differentiable at } x = 1$$

Sol. $f(x) = \begin{cases} a + \sin^{-1}(x + b), & x \ge 1 \\ x, & x < 1 \end{cases}$

$$\Rightarrow f'(x) = \begin{cases} \dfrac{1}{\sqrt{1 - (x + b)^2}}, & x > 1 \\ 1, & x < 1 \end{cases}$$

For $f(x)$ to be continuous at $x = 1$,

$f(1^+) = f(1^-) \Rightarrow a + \sin^{-1}(1 + b) = 1$ (1)

Also $f'(1^+) = f'(1^-) \Rightarrow \dfrac{1}{\sqrt{1 - (1 + b)^2}} = 1 \Rightarrow b = -1.$

$\Rightarrow$ From equation (1), $a = 1$

Example 3.51 The function $f(x) = \begin{cases} ax(x - 1) + b, & x < 1 \\ x - 1, & 1 \le x \le 3. \\ px^2 + qx + 2, & x > 3 \end{cases}$

Find the values of the constants a, b, p and q so that all the following conditions are satisfied.

a. $f(x)$ is continuous for all x.
b. $f'(1)$ does not exist.
c. $f'(x)$ is continuous at $x = 3$.

Sol. $f(x)$ is continuous $\forall x \in R$.

Hence, it must be continuous at $x = 1, 3$.

$f(1^-) = \lim_{x \to 1} ax(x - 1) + b = b$

$f(1^+) = \lim_{x \to 1}(x - 1) = 0$

Now $f(1^-) = f(1^+)$ (for continuity at $x = 1$)

$\Rightarrow b = 0$

$f(3^-) = \lim_{x \to 3}(x - 1) = 2$

$f(3^+) = \lim_{x \to 3}(px^2 + qx + 2) = 9p + 3q + 2$

Now $f(3^-) = f(3^+)$ (for continuity at $x = 3$)

$\Rightarrow 9p + 3q = 0$ (1)

$$f'(x) = \begin{cases} 2ax - a, & x < 1 \\ 1, & 1 < x < 3 \\ 2px + q, & x > 3 \end{cases}$$

Now given that $f'(1)$ does not exist

$\Rightarrow f'(1^+) \ne f'(1^-)$

$\Rightarrow 1 \ne 2a - a$

$\Rightarrow a \ne 1.$

Also given that $f'(3)$ exists.

$\Rightarrow f'(3^-) = f'(3^+)$

$\Rightarrow 1 = 6p + q$ (2)

Solving equations (1) and (2) for p and q, we get

$p = 1/3, q = -1.$

Example 3.52 Discuss the differentiability of

$$f(x) = \sin^{-1} \frac{2x}{1 + x^2}.$$

Sol. $f(x) = \sin^{-1}\left(\dfrac{2x}{1 + x^2}\right) = \begin{cases} 2\tan^{-1} x, & -1 \le x \le 1 \\ \pi - 2\tan^{-1} x, & \text{if } x > 1 \\ -\pi - 2\tan^{-1} x, & \text{if } x < -1 \end{cases}$

$$\Rightarrow f'(x) = \begin{cases} \dfrac{2}{1 + x^2}, & -1 < x < 1 \\ -\dfrac{2}{1 + x^2}, & \text{if } x > 1 \\ -\dfrac{2}{1 + x^2}, & \text{if } x < -1 \end{cases} \quad (1)$$

$\Rightarrow f'(-1^-) = -1, f'(-1^+) = 1$ and $f'(1^-) = 1$ and $f'(1^+) = -1.$

Hence, $f(x)$ is non-differentiable at $x = \pm 1.$

Graph of $f(x) = \sin^{-1}\left(\dfrac{2x}{1 + x^2}\right).$

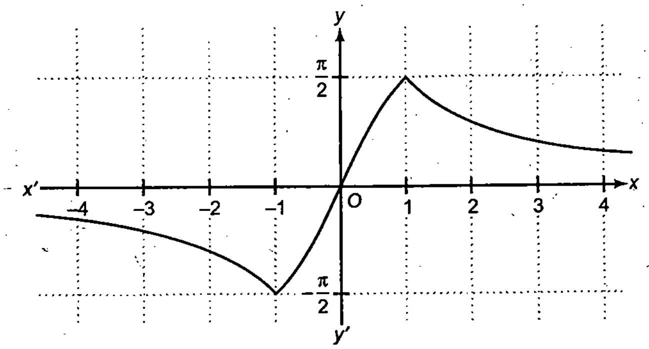


Fig. 3.23

Students find it difficult to remember all the cases of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ in equation (1).

Then, use the following short-cut method to check the differentiability.

Differentiate $f(x)$ w.r.t. x, we get

$$\frac{df(x)}{dx}=\frac{\frac{d\left(\frac{2x}{1+x^2}\right)}{dx}}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}}$$

$$=\frac{\frac{2(1+x^2)-2x(2x)}{(1+x^2)}}{\sqrt{(1+x^2)^2-4x^2}}$$

$$=\frac{2(1-x^2)}{(1+x^2)|1-x^2|}$$

Clearly, $\frac{df(x)}{dx}$ is discontinuous at $x^2=1$ or $x=\pm 1$.

Hence, $f(x)$ is non-differentiable at $x=\pm 1$.

Concept Application Exercise 3.4

1. Discuss the continuity and differentiability of $f(x)=|x+1|+|x|+|x-1|$, $\forall x \in R$; also draw the graph of $f(x)$.
2. Find x where $f(x)=$ maximum $\{\sqrt{x(2-x)}, 2-x\}$ is non-differentiable.
3. Discuss the differentiability of function $f(x)=x-|x-x^2|$.
4. Discuss the differentiability of $f(x)=|[x]x|$ in $-1<x\le 2$, where [.] represents the greatest integer function.
5. Discuss the differentiability of $f(x)=\cos^{-1}(\cos x)$.
6. Discuss the differentiability of $f(x)=\max\{\tan^{-1}x, \cot^{-1}x\}$.
7. Find the values of a and b if $f(x)=\begin{cases}ax^2+1, & x\le 1\\ x^2+ax+b, & x>1\end{cases}$ is differentiable at $x=1$.
8. Discuss the differentiability of $f(x)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.
9. Which of the following function is non-differentiable in its domain?
   a. $f(x)=\frac{x-2}{x^2+3}$ b. $f(x)=\log|x|$
   c. $f(x)=x^3\log x$ d. $f(x)=(x-3)^{3/5}$
10. Discuss the differentiability of $f(x)=||x^2-4|-12|$.
11. Which of the following function is not differentiable at $x=0$?
    (i) $f(x)=\min\{x, \sin x\}$ (ii) $f(x)=\begin{cases}0; & x\ge 0\\ x^2; & x<0\end{cases}$
    (iii) $f(x)=x^2\operatorname{sgn}(x)$

EXERCISES

Subjective Type

Solutions on page 3.33

1. A function $f(x)$ defined as
   $$f(x)=\begin{cases}x^2+ax+1, & x \text{ is rational}\\ ax^2+2x+b, & x \text{ is irrational}\end{cases}$$
   is continuous at $x=1$ and 2, then find the values of a and b.
2. Discuss the differentiability of $f(x)=[x]+\sqrt{\{x\}}$, where [.] and {.} denote the greatest integer function and the fractional part, respectively.
3. Consider $f(x)=\frac{x}{(1+x)}+\frac{x}{(1+x)(1+2x)}+\frac{x}{(1+2x)(1+3x)}+\cdots$ to infinity. Discuss the continuity at $x=0$.
4. If $f(x)$ be a continuous function for all real values of x and satisfies
   $$x^2+\{f(x)-2\}x+2\sqrt{3}-3-\sqrt{3}f(x)=0, \forall x \in R.$$
   Then find the value of $f(\sqrt{3})$.
5. If $g(x)=\begin{cases}[f(x)], & x\in(0,\pi/2)\cup(\pi/2,\pi)\\ 3, & x=\pi/2\end{cases}$
   and $f(x)=\frac{2(\sin x-\sin^n x)+|\sin x-\sin^n x|}{2(\sin x-\sin^n x)-|\sin x-\sin^n x|}$, $n\in N$
   where [.] denotes the greatest integer function. Prove that $g(x)$ is continuous at $x=\pi/2$ when $n>1$.
6. Let $y=f(x)$ be defined parametrically as $y=t^2+t|t|$, $x=2t-|t|$, $t\in R$.
   Then at $x=0$, find $f(x)$ and discuss the differentiability of $f(x)$.
7. If $f(x)$ be a continuous function in $[0, 2\pi]$ and $f(0)=f(2\pi)$, then prove that there exists point $c \in (0,\pi)$ such that $f(c)=f(c+\pi)$.
8. Test the continuity of $f(x)$ at $x=0$
   if $f(x)=\begin{cases}(x+1)^{2-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x\ne 0\\ 0, & x=0\end{cases}$

9. Discuss the differentiability of $\sin(\pi(x - [x]))$ in $(-\pi/2, \pi/2)$, where [.] denotes the greatest integral function less than or equal to x.

10. Let $f(x) = \begin{cases} \sqrt{x}(1 + x\sin(1/x)), & x > 0 \\ -\sqrt{(-x)}(1 + x\sin(1/x)), & x < 0. \\ 0, & x = 0 \end{cases}$

Show that $f'(x)$ exists everywhere and is finite except at $x = 0$.

11. Discuss the differentiability of $f(x) = \min\{|x|, |x-2|, 2-|x-1|\}$.

12. Let $f(x)$ be a function satisfying $f(x+y) = f(x) + f(y)$ and $f(x) = xg(x)$ for all $x, y \in R$, where $g(x)$ is continuous. Then prove that $f'(x) = g(0)$.

13. If $f(x) = \begin{cases} x-3, & x < 0 \\ x^2 - 3x + 2, & x \ge 0 \end{cases}$ and let $g(x) = f(|x|) + |f(x)|$. Discuss the differentiability of $g(x)$.

14. Discuss the continuity and differentiability in [0, 2] of $f(x)$

$= \begin{cases} |2x-3|[x], & x \ge 1 \\ \sin\left(\dfrac{\pi x}{2}\right), & x < 1 \end{cases}$

where [.] denotes the greatest integer function.

15. Let $f(x)$ is defined as follows:

$$f(x) = \begin{cases} (\cos x - \sin x)^{\operatorname{cosec} x}, & \dfrac{-\pi}{2} < x < 0 \\ a, & x = 0 \\ \dfrac{e^{1/x} + e^{2/x} + e^{3/x}}{ae^{2/x} + be^{3/x}}, & 0 < x < \pi/2 \end{cases}$$

If $f(x)$ is continuous at $x = 0$, find a and b.

16. Given a real-valued function $f(x)$ as follows:

$$f(x) = \begin{cases} \dfrac{x^2 + 2\cos x - 2}{x^4}, & \text{for } x < 0 \\ 1/12, & \text{for } x = 0 \\ \dfrac{\sin x - \log_e(e^x \cos x)}{6x^2}, & \text{for } x > 0 \end{cases}$$

Test the continuity and differentiability of $f(x)$ at $x = 0$.

17. Find the value of $f(0)$ so that the function

$$f(x) = \begin{cases} \left(\dfrac{e^{-x} + x^2 - a}{-x}\right)^{-1/x}, & -1 \le x < 0 \\ \dfrac{e^{1/x} + e^{2/x} + e^{3/|x|}}{ae^{2/x} + be^{3/|x|}}, & 0 < x < 1 \end{cases}$$ is continuous at $x = 0$

18. Find the value of a and b if

$$f(x) = \begin{cases} \dfrac{ae^{1/(|x+2|)} - 1}{2 - e^{1/(|x+2|)}}; & -3 < x < -2 \\ b; & x = -2 \\ \sin\left(\dfrac{x^4 - 16}{x^5 + 32}\right); & -2 < x < 0 \end{cases}$$ is continuous at $x = -2$.

19. Let $f: R \to R$ satisfying $|f(x)| \le x^2, \forall x \in R$, then show that $f(x)$ is differentiable at $x = 0$.

Objective Type

Solutions on page 3.36

Each question has four choices a, b, c, and d, out of which *only one* is correct.

1. Which of the following functions have finite number of points of discontinuity in R ([·] represents greatest integer function)?

   a. $\tan x$ b. $x[x]$ c. $\dfrac{|x|}{x}$ d. $\sin[\pi x]$

2. The function $f(x) = \dfrac{4-x^2}{4x-x^3}$ is

   a. Discontinuous at only one point
   b. Discontinuous exactly at two points
   c. Discontinuous exactly at three points
   d. None of these

3. If $f(x) = \dfrac{\tan\left(\dfrac{\pi}{4} - x\right)}{\cot 2x}$, $(x \ne \pi/4)$ is continuous at $x = \pi/4$, then the value of $f\left(\dfrac{\pi}{4}\right)$ is

   a. 1 b. 1/2 c. 1/3 d. −1

4. The function $f(x) = \dfrac{(3^x - 1)^2}{\sin x \cdot \ln(1+x)}$, $x \ne 0$, is continuous at $x = 0$. Then the value of $f(0)$ is

   a. $2\log_e 3$ b. $(\log_e 3)^2$
   c. $\log_e 6$ d. None of these

5. If $f(x) = \begin{cases} \dfrac{1-|x|}{1+x}; & x \ne -1 \\ 1; & x = -1 \end{cases}$, then $f([2x])$ where [·] represents the greatest integer function is

   a. discontinuous at $x = -1$ b. continuous at $x = 0$
   c. continuous at $x = 1/2$ d. continuous at $x = 1$

6. Let $f(x) = \begin{cases} \dfrac{x-4}{|x-4|} + a, & x < 4 \\ a + b, & x = 4 \\ \dfrac{x-4}{|x-4|} + b, & x > 4 \end{cases}$. Then $f(x)$ is continous at $x = 4$ when,

   a. $a = 0, b = 0$ b. $a = 1, b = 1$
   c. $a = -1, b = 1$ d. $a = 1, b = -1$

7. If $f(x) = \dfrac{x - e^x + \cos 2x}{x^2}$, $x \ne 0$, is continuous at $x = 0$, then

   a. $f(0) = 5/2$ b. $[f(0)] = -2$
   c. $\{f(0)\} = -0.5$ d. $[f(0)]\{f(0)\} = -1.5$

   where $[x]$ and $\{x\}$ denote the greatest integer and fractional part function, respectively.

8. Let $f(x)$ be defined in the interval [0, 4] such that

   $f(x) = \begin{cases} 1-x, & 0 \le x \le 1 \\ x+2, & 1 < x < 2 \\ 4-x, & 2 \le x \le 4 \end{cases}$

   then number of points where $f(f(x))$ is discontinuous is

   a. 1 b. 2
   c. 3 d. None of these

9. The value of $f(0)$, so that the function

$f(x) = \dfrac{2x - \sin^{-1} x}{2x + \tan^{-1} x}$ is continuous at each point in its domain, is equal to

a. 2 b. 1/3 c. 2/3 d. −1/3

10. Which of the following is true about

$$f(x)=\begin{cases}\dfrac{(x-2)}{|x-2|}\left(\dfrac{x^2-1}{x^2+1}\right); & x\neq 2\\ \dfrac{3}{5}; & x=2\end{cases}$$

a. $f(x)$ is continuous at $x = 2$.
b. $f(x)$ has removable discontinuity at $x = 2$.
c. $f(x)$ has non-removable discontinuity at $x = 2$.
d. Discontinuity at $x = 2$ can be removed by redefining function at $x = 2$.

11. $f(x) = \lim\limits_{n\to\infty} \dfrac{(x-1)^{2n}-1}{(x-1)^{2n}+1}$ is discontinuous at

a. $x = 0$ only b. $x = 2$ only
c. $x = 0$ and 2 d. None of these

12. If $f(x)=\begin{cases}\dfrac{8^x-4^x-2^x+1}{x^2}, & x>0\\ e^x\sin x+\pi x+\lambda\ln 4, & x\le 0\end{cases}$

is continuous at $x = 0$. Then the value of λ is

a. $4\log_e 2$ b. $2\log_e 2$
c. $\log_e 2$ d. None of these

13. If $f(x) = \dfrac{a\cos x - \cos bx}{x^2}$, $x \neq 0$ and $f(0) = 4$ is continuous at $x = 0$, then the ordered pair (a, b) is

a. $(\pm 1, 3)$ b. $(1, \pm 3)$ c. $(-1, -3)$ d. $(1, 3)$

14. If $f(x)=\begin{cases}x+2, & x<0\\ -x^2-2, & 0\le x<1\\ x, & x\ge 1\end{cases}$ then the number of points of discontinuity of $|f(x)|$ is

a. 1 b. 2
c. 3 d. None of these

15. Let $f: R \to R$ be given by $f(x) = 5x$, if $x \in Q$ and $f(x) = x^2 + 6$ if $x \in R \sim Q$, then

a. f is continuous at $x = 2$ and $x = 3$
b. f is not continuous at $x = 2$ and $x = 3$
c. f is continuous at $x = 2$ but not at $x = 3$
d. f is continuous at $x = 3$ but not at $x = 2$

16. The function $f(x) = |2 \operatorname{sgn} 2x| + 2$ has

a. Jump discontinuity b. Removal discontinuity
c. Infinite discontinuity d. No discontinuity at $x = 0$

17. Let $f(x) = \lim\limits_{n\to\infty} (\sin x)^{2n}$, then which of the following is not true?

a. discontinuous at infinite number of points
b. Discontinuous at $x = \dfrac{\pi}{2}$
c. Discontinuous at $x = -\dfrac{\pi}{2}$
d. None of these

18. Let f be a continuous function on R such that

$f(1/4n) = (\sin e^n)e^{-n^2} + \dfrac{n^2}{n^2+1}$. Then the value of $f(0)$ is

a. 1 b. 1/2 c. 0 d. None of these

19. If $f(x) = \dfrac{x^2 - bx + 25}{x^2 - 7x + 10}$ for $x \neq 5$ is continuous at $x = 5$, then the value of $f(5)$ is

a. 0 b. 5 c. 10 d. 25

20. Which of the following statements is always true? ([.] represents the greatest integer function)

a. If $f(x)$ is discontinuous, then $|f(x)|$ is discontinuous
b. If $f(x)$ is discontinuous, then $f(|x|)$ is discontinuous
c. $f(x) = [g(x)]$ is discontinuous when $g(x)$ is an integer
d. None of these

21. A function $f(x)$ is defined as

$f(x)=\begin{cases}\sin x, & x \text{ is rational}\\ \cos x, & x \text{ is irrational}\end{cases}$ is continuous at

a. $x = n\pi + \pi/4, n \in I$ b. $x = n\pi + \pi/8, n \in I$
c. $x = n\pi + \pi/6, n \in I$ d. $x = n\pi + \pi/3, n \in I$

22. The number of points $f(x)=\begin{cases}[\cos \pi x], & 0\le x\le 1\\ |2x-3|[x-2], & 1<x\le 2\end{cases}$

is discontinuous at ([.] denotes the greatest integral function)

a. two points b. three points
c. four points d. no points

23. A point where function $f(x)$ is not continuous where $f(x) = [\sin [x]]$ in $(0, 2\pi)$; [.] denotes the greatest integer $\le x$ is

a. (3, 0) b. (2, 0) c. (1, 0) d. None of these

24. The function $f(x) = \sin(\log_e |x|), x \neq 0$, and 1 if $x = 0$

a. is continuous at $x = 0$
b. has removable discontinuity at $x = 0$
c. has jump of discontinuity at $x = 0$
d. has oscillating discontinuity at $x = 0$

25. The function defined by $f(x) = (-1)^{[x^3]}$ ([.] denotes the greatest integer function) satisfies

a. discontinuous for $x = n^{1/3}$, where n is any integer
b. $f(3/2) = 1$
c. $f'(x) = 1$ for $-1 < x < 1$
d. None of these

26. The function $f(x) = \{x\}\sin(\pi[x])$, where [.] denotes the greatest integer function and {.} is the fractional part function, is discontinuous at

a. all x b. all integer points
c. no x d. x which is not an integer

27. The function $f(x)$ defined by

$$f(x)=\begin{cases}\log_{(4x-3)}(x^2-2x+5), & \frac{3}{4}<x<1 \text{ and } x>1\\ 4, & x=1\end{cases}$$

a. is continuous at $x = 1$.
b. is discontinuous at $x = 1$ since $f(1^+)$ does not exist though $f(1^-)$ exists.
c. is discontinuous at $x = 1$ since $f(1^-)$ does not exist though $f(1^+)$ exists.
d. is discontinuous at $x = 1$ since neither $f(1^+)$ nor $f(1^-)$ exists.

28. Let $f(x) = [x]$ and $g(x)=\begin{cases}0, & x\in Z\\ x^2, & x\in R-Z\end{cases}$. Then which of the following is not true ([.] represents greatest integer function)
a. $\lim_{x\to 1} g(x)$ exists but $g(x)$ is not continuous at $x = 1$.
b. $\lim_{x\to 1} f(x)$ does not exist and $f(x)$ is not continuous at $x = 1$.
c. gof is a discontinuous function.
d. fog is a discontinuous function.

29. $f(x)=\begin{cases}\dfrac{x}{2x^2+|x|}, & x\neq 0\\ 1, & x=0\end{cases}$ then $f(x)$ is
a. Continuous but non-differentiable at $x = 0$
b. Differentiable at $x = 0$
c. Discontinuous at $x = 0$
d. None of these

30. Let a function $f(x)$ be defined by $f(x) = \dfrac{x-|x-1|}{x}$, then which of the following is not true
a. Discontinuous at $x = 0$
b. Discontinuous at $x = 1$
c. Not differentiable at $x = 0$
d. Not differentiable at $x = 1$

31. If $f(x) = x^3 \operatorname{sgn} x$, then
a. f is derivable at $x = 0$
b. f is continuous but not derivable at $x = 0$
c. L.H.D. at $x = 0$ is 1
d. R.H.D. at $x = 0$ is 1

32. Let $f(x)=\begin{cases}\min(\{x, x^2\}) & x\geq 0\\ \max\{2x, x^2-1\} & x<0\end{cases}$. Then which of the following is not true.
a. $f(x)$ is continuous at $x = 0$
b. $f(x)$ is not differentiable at $x = 1$
c. $f(x)$ is not differnetiable at exactly three point
d. None of these

33. The function $f(x) = \sin^{-1}(\cos x)$ is
a. not differentiable at $x = \dfrac{\pi}{2}$
b. differentiable at $\dfrac{3\pi}{2}$
c. differentiable at $x = 0$
d. differentiable at $x = 2\pi$

34. Which of the following functions is non-differentiable?
a. $f(x) = (e^x - 1)|e^{2x} - 1|$ in R
b. $f(x) = \dfrac{x-1}{x^2+1}$ in R
c. $f(x)=\begin{cases}||x-3|-1|, & x<3\\ \dfrac{x}{3}[x]-2, & x\geq 3\end{cases}$ at $x = 3$

where [.] represents the greatest integer function
d. $f(x) = 3(x-2)^{1/3} + 3$ in R

35. The number of values of $x \in [0, 2]$ at which $f(x) = \left|x-\frac{1}{2}\right| + |x-1| + \tan x$ is not differentiable at
a. 0 b. 1
c. 3 d. None of these

36. The set of points where $x^2|x|$ is thrice differentiable is
a. R b. $R - \{0, \pm 1\}$
c. $R - \{0\}$ d. None of these

37. Which of the following function is not differentiable at $x = 1$?
a. $f(x) = (x^2 - 1)|(x-1)(x-2)|$
b. $f(x) = \sin(|x-1|) - |x-1|$
c. $f(x) = \tan(|x-1|) + |x-1|$
d. None of these

38. $f(x)=\begin{cases}xe^{-\left(\frac{1}{x}+\frac{1}{|x|}\right)}, & x\neq 0\\ a, & x=0\end{cases}$. The value of a, such that $f(x)$ is differentiable at $x = 0$, is equal to
a. 1 b. −1
c. 0 d. None of these

39. If $f(x)=\begin{cases}ax^2+1, & x\leq 1\\ x^2+ax+b, & x>1\end{cases}$ is differentiable at $x = 1$, then
a. $a = 1, b = 1$ b. $a = 1, b = 0$
c. $a = 2, b = 0$ d. $a = 2, b = 1$

40. If $f(x) = a|\sin x| + be^{|x|} + c|x|^3$ is differentiable at $x = 0$, then
a. $a = b = c = 0$ b. $a = 0, b = 0, c \in R$
c. $b = c = 0, a \in R$ d. $c = 0, a = 0, b \in R$

41. The number of points of non-differentiability for $f(x) = \max\{||x|-1|, 1/2\}$ is
a. 4 b. 3 c. 2 d. 5

42. Let $f(x)=\begin{cases}\sin 2x, & 0, x\leq \pi/6\\ ax+b, & \pi/6<x<1\end{cases}$. If $f(x)$ and $f'(x)$ are continuous, then
a. $a = 1, b = \dfrac{1}{\sqrt{2}} + \dfrac{\pi}{6}$ b. $a = \dfrac{1}{\sqrt{2}}, b = \dfrac{1}{\sqrt{2}}$
c. $a = 1, b = \dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}$ d. None of these

43. If $f(x)=\begin{cases} x^3, & x^2<1 \\ x, & x^2 \geq 1 \end{cases}$, then $f(x)$ is differentiable at

a. $(-\infty, \infty)-\{1\}$ b. $(-\infty, \infty) \sim \{1-1\}$
c. $(-\infty, \infty) \sim \{1-1, 0\}$ d. $(-\infty, \infty) \sim \{-1\}$

44. If $f(x)=(x^2-4)\,|x^3-6x^2+11x-6|+\dfrac{x}{1+|x|}$, then the set of points at which the function $f(x)$ is not differentiable is

a. $\{-2, 2, 1, 3\}$ b. $\{-2, 0, 3\}$
c. $\{-2, 2, 0\}$ d. $\{1, 3\}$

45. If $f(x) = \cos \pi\,(|x| + [x])$, (where [.] denotes the greatest integral function), then which is not true?

a. continuous at $x = 1/2$ b. continuous at $x = 0$
c. differentiable in $(-1, 0)$ d. differentiable in $(0, 1)$

46. If $f(x)=\begin{cases} e^{x^2+x}, & x>0 \\ ax+b, & x \leq 0 \end{cases}$ is differentiable at $x = 0$, then

a. $a=1, b=-1$ b. $a=-1, b=1$
c. $a=1, b=1$ d. $a=-1, b=-1$

47. If $f(x)=\begin{cases} e^{-1/x^2}, & x>0 \\ 0, & x \leq 0 \end{cases}$, then $f(x)$ is

a. Differentiable at $x=0$
b. Continuous but not differentiable at $x = 0$
c. Discontinuous at $x = 0$
d. None of these

48. If $f(x)=\begin{cases} x-1, & x<0 \\ x^2-2x, & x \geq 0 \end{cases}$, then

a. $f(|x|)$ is discontinuous at $x = 0$
b. $f(|x|)$ is differentiable at $x = 0$
c. $|f(x)|$ is non-differentiable at $x = 0, 2$
d. $|f(x)|$ is continuous at $x = 0$

49. If $f(x) = \begin{cases} |1-4x^2|, & 0 \leq x<1 \\ [x^2-2x], & 1 \leq x<2 \end{cases}$, where [.] denotes the greatest integer function, then $f(x)$ is

a. Differentiable for all x
b. Continuous at $x = 1$
c. $f(x)$ is non-differentiable at $x = 1$
d. None of these

Discuss the continuity and differentiability of $f(x)$ in $[0, 2)$

50. Let $f(x)=\lim\limits_{n \to \infty} \dfrac{\log(2+x)-x^{2n}\sin x}{1+x^{2n}}$. Then

a. f is continuous at $x = 1$ b. $\lim\limits_{x \to 1^+} f(x)=\log 3$
c. $\lim\limits_{x \to 1^+} f(x)=-\sin 1$ d. $\lim\limits_{x \to 1^-} f(x)$ does not exist

51. If $f(x) = \begin{cases} x^a \sin\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x=0 \end{cases}$ is continuous but non-differentiable at $x = 0$, then

a. $a \in (-1, 0)$ b. $a \in (0, 2]$ c. $a \in (0, 1]$ d. $a \in [1, 2)$

52. $f(x) = [\sin x] + [\cos x]$, $x \in [0, 2\pi]$, where [.] denotes the greatest integer function. The total number of points, where $f(x)$ is non-differentiable, is equal to

a. 2 b. 3 c. 5 d. 4

53. If $x + 4|y| = 6y$, then y as a function of x is

a. continuous at $x = 0$ b. derivable at $x = 0$
c. $\dfrac{dy}{dx}=\dfrac{1}{2}$ for all x d. None of these

54. Let $g(x)$ be a polynomial of degree one and $f(x)$ be defined by $f(x)=\begin{cases} g(x), & x \leq 0 \\ |x|^{\sin x}, & x>0 \end{cases}$. If $f(x)$ is continuous satisfying $f'(1)=f(-1)$, then $g(x)$ is

a. $(1+\sin 1)x+1$ b. $(1-\sin 1)x+1$
c. $(1-\sin 1)x-1$ d. $(1+\sin 1)x-1$

55. If $f(x) = |1 - x|$, then the points where $\sin^{-1}(f|x|)$ is non-differentiable are

a. $\{0, 1\}$ b. $\{0, -1\}$ c. $\{0, 1, -1\}$ d. None of these

56. Given that $f(x) = xg(x)/|x|$, $g(0) = g'(0) = 0$ and $f(x)$ is continuous at $x = 0$. Then the value of $f'(0)$

a. Does not exist b. is -1
c. is 1 d. is 0

57. The number of points, where the function $f(x)$ = max $(|\tan x|, \cos |x|)$ is non-differentiable in the interval $(-\pi, \pi)$, is

a. 4 b. 6 c. 3 d. 2

58. If $f(x) = \begin{cases} \sin x, & x<0 \\ \cos x-|x-1|, & x \geq 0 \end{cases}$ then $g(x) = f(|x|)$ is non-differentiable for

a. No value of x b. Exactly one value of x
c. Exactly three values of x d. None of these

59. If $f(x) = \begin{cases} 2x-[x]+x\sin(x-[x]); & x \neq 0 \\ 0; & x=0 \end{cases}$, where [.] denotes the greatest integer function, then n cannot be

a. 4 b. 2 c. 5 d. 6

60. $f(x)=\max\{x/n, |\sin \pi x|\}$, $n \in N$ has maximum points of non-differentiability for $x \in (0, 4)$, then n cannot be

a. 4
b. 2
c. 5
d. 6

61. $f(x) = [x^2] - \{x\}^2$, where [.] and {.} denote the greatest integer function and the fractional part, respectively, is

a. continuous at $x = 1, -1$
b. continuous at $x = -1$ but not at $x = 1$
c. continuous at $x = -1$ but not at $x = -1$
d. discontinuous at $x = 1$ and $x = -1$

62. If $f(x)=[\log_e x]+\sqrt{\{\log_e x\}}$, $x>1$, where [.] and {.} denote the greatest integer function and the fractional part, respectively, then

a. $f(x)$ is continuous but non-differentiable at $x = e$
b. $f(x)$ is differentiable at $x = e$
c. $f(x)$ is discontinuous at $x = e$
d. None of these

63. $f(x) = \lim_{n\to\infty} \sin^{2n}(\pi x) + \left[x + \frac{1}{2}\right]$, where [.] denotes the greatest integer function is
   a. continuous at $x = 1$ but discontinuous at $x = 3/2$
   b. continuous at $x = 1$ and $x = 3/2$
   c. discontinuous at $x = 1$ and $x = 3/2$
   d. discontinuous at $x = 1$ but continuous at $x = 3/2$

64. If $f(x) = \text{sgn}(\sin^2 x - \sin x - 1)$ has exactly four points of discontinuity for $x \in (0, n\pi)$, $n \in N$, then
   a. Minimum value of n is 5
   b. Maximum value of n is 6
   c. There are exactly two possible values of n
   d. None of these

65. If $f(x) = \begin{cases} x^2 - ax + 3, & x \text{ is rational} \\ 2 - x, & x \text{ is irrational} \end{cases}$ is continuous at exactly two points, then the possible values of a are
   a. $(2, \infty)$ b. $(-\infty, 3)$
   c. $(-\infty, -1) \cup (3, \infty)$ d. None of these

66. $f(x) = \{x\}^2 - \{x^2\}$ ({.} denotes the fractional part function)
   a. $f(x)$ is discontinuous at infinite number of integers but not all integers
   b. $f(x)$ is discontinuous at finite number of integers
   c. $f(x)$ is discontinuous at all integers
   d. $f(x)$ is continuous at all integers

67. Let $f(x) = \begin{cases} g(x)\cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, where $g(x)$ is an even function differentiable at $x = 0$, passing through the origin. The $f'(0)$
   a. is equal to 1 b. is equal to 0
   c. is equal to 2 d. does not exist

68. Let $f(x) = \begin{cases} 1 - \sqrt{1 - x^2}, & \text{if } -1 \le x \le 1 \\ 1 + \log\frac{1}{x}, & \text{if } x > 1 \end{cases}$ is
   a. Continuous and differentiable at $x = 1$
   b. Continuous but not differentiable at $x = 1$
   c. Neither continuous nor differentiable at $x = 1$
   d. None of these

69. If $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$, then $f(x)$ is
   a. Continuous on $[-1, 1]$ and differentiable on $(-1, 1)$
   b. Continuous $[-1, 1]$ and differentiable on $(-1, 0) \cup (0, 1)$
   c. Continuous and differentiable on $[-1, 1]$
   d. None of these

70. The set of all points, where $f(x) = \sqrt[3]{x^2 |x|} - |x| - 1$ is not differentiable, is
   a. $\{0\}$ b. $\{-1, 0, 1\}$
   c. $\{0, 1\}$ d. None of these

71. Let $f(x)$ be a function for all $x \in R$ and $f'(0) = 1$. Then $g(x) = f(|x|) - \sqrt{\frac{1 - \cos 2x}{2}}$, at $x = 0$,
   a. is differentiable at $x = 0$ and its value is 1
   b. is differentiable at $x = 0$ and its value is 0
   c. is non-differentiable at $x = 0$ as its graph has sharp turn at $x = 0$
   d. is non-differentiable at $x = 0$ as its graph has vertical tangent at $x = 0$

72. A function $f(x)$ is defined as

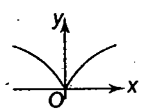
$$f(x) = \begin{cases} x^m \sin\frac{1}{x}, & x \neq 0,\ m \in N \\ 0, & \text{if } x = 0 \end{cases}$$

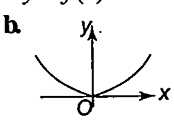
. The least value of m for which $f'(x)$ is continuous at $x = 0$ is
   a. 1 b. 2 c. 3 d. None

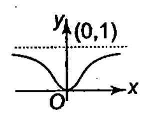
73. $f(x) = \begin{cases} x^2\left(\dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$. Then
   a. $f(x)$ is discontinuous at $x = 0$
   b. $f(x)$ is continuous but non-differentiable at $x = 0$
   c. $f(x)$ is differentiable at $x = 0$
   d. $f'(0) = 2$

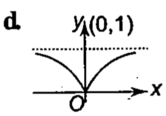
74. If $f(x) = \{x^2\} - (\{x\})^2$, where $\{x\}$ denotes the fractional part of x, then
   a. $f(x)$ is continuous at $x = -2$ but not at $x = 2$
   b. $f(x)$ is continuous at $x = 2$ but not at $x = -2$
   c. $f(x)$ is continuous at $x = 2$ and at $x = -2$
   d. $f(x)$ is discontinuous at $x = -2$ and at $x = 2$

75. Let $y = f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Then which of the following can best represent the graph of $y = f(x)$?

a. 

b. 

c. 

d. 

76. If $f(2 + x) = f(-x)$ for all $x \in R$, then differentiability at $x = 4$ implies differentiability at
   a. $x = 1$ b. $x = -1$
   c. $x = -2$ d. cannot say anything.

77. $f(x) = \begin{cases} 3 - \left[\cot^{-1}\dfrac{2x^3 - 3}{x^2}\right] & \text{if } x > 0 \\ \{x^2\}\cos(e^{1/x}), & \text{if } x < 0 \end{cases}$ is continuous at $x = 0$, then the value of $f(0)$, (where [x] and {x} denotes the greatest integer and fractional part functions, respectively)
   a. 0 b. 1
   c. -1 d. none of these

78. If both $f(x)$ and $g(x)$ are differentiable functions at $x = x_0$, then the function defined as $h(x) = \text{maximum } \{f(x), g(x)\}$:
   a. is always differentiable at $x = x_0$
   b. is never differentiable at $x = x_0$
   c. is differentiable at $x = x_0$ provided $f(x_0) \neq g(x_0)$
   d. cannot be differentiable at $x = x_0$ if $f(x_0) = g(x_0)$.

79. Number of points where the function

$$f(x)=\begin{cases}1+\left[\cos\frac{\pi x}{2}\right], & 1<x\le 2\\ 1-\{x\}, & 0\le x<1\\ |\sin \pi x|, & -1\le x<0\end{cases}$$ and $f(1)=0$ is continuous

but non-differentiable is/are (where $[\cdot]$ and $\{\cdot\}$ represent greatest integer and fractional part function, respectively)

a. 0 b. 1
c. 2 d. none of these

80. Let $f(x)=\lim\limits_{n\to\infty}\frac{(x^2+2x+3+\sin \pi x)^n-1}{(x^2+2x+3+\sin \pi x)^n+1}$, then

a. $f(x)$ is continuous and differentiable for all $x\in R$.
b. $f(x)$ is continuous but not differentiable for all $x\in R$.
c. $f(x)$ is discontinuous at infinite number of points.
d. $f(x)$ is discontinuous at finite number of points.

81. Given that $\prod\limits_{n=1}^{n}\cos\frac{x}{2^n}=\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$ and

$$f(x)=\begin{cases}\lim\limits_{n\to\infty}\sum\limits_{n=1}^{n}\frac{1}{2^n}\tan\left(\frac{x}{2^n}\right), & x\in(0,\pi)-\left\{\frac{\pi}{2}\right\}\\ \frac{2}{\pi}, & x=\frac{\pi}{2}\end{cases}$$

Then which one of the following is true?

a. $f(x)$ has non-removable discontinuity of finite type at $x=\frac{\pi}{2}$.
b. $f(x)$ has removable discontinuity at $x=\frac{\pi}{2}$.
c. $f(x)$ is continuous at $x=\frac{\pi}{2}$.
d. $f(x)$ has non-removable discontinuity of infinite type at $x=\frac{\pi}{2}$.

Multiple Correct Answers Type

Solutions on page 3.46

Each question has four choices a, b, c, and d, out of which *one or more* answers are correct.

1. Which of the statement(s) is/are incorrect ?

a. If $f+g$ is continuous at $x=a$, then f and g are continuous at $x=a$.
b. If $\lim\limits_{x\to a}(fg)$ exists, then both $\lim\limits_{x\to a}f$ and $\lim\limits_{x\to a}g$ exist.
c. Discontinuity at $x=a \Rightarrow$ non-existence of limit.
d. All functions defined on a closed interval attain a maximum or a minimum value in that interval.

2. A function f is defined on an interval $[a, b]$. Which of the following statement(s) is/are incorrect ?

a. If $f(a)$ and $f(b)$ have opposite signs, then there must be a point $c\in(a, b)$ such that $f(c)=0$.
b. If f is continuous on $[a, b]$, $f(a)<0$ and $f(b)>0$, then there must be a point $c\in(a, b)$ such that $f(c)=0$.
c. If f is continuous on $[a, b]$, then there is a point c in (a, b) such that $f(c)=0$, then $f(a)$ and $f(b)$ have opposite signs.
d. If f has no zeros on $[a, b]$, then $f(a)$ and $f(b)$ have the same sign.

3. Which of the following functions f has/have a removable discontinuity at the indicated point ?

a. $f(x)=\frac{x^2-2x-8}{x+2}$ at $x=-2$
b. $f(x)=\frac{x-7}{|x-7|}$ at $x=7$
c. $f(x)=\frac{x^3+64}{x+4}$ at $x=-4$
d. $f(x)=\frac{3-\sqrt{x}}{9-x}$ at $x=9$

4. The function $f(x)=\begin{cases}5x-4 & \text{for } 0<x\le 1\\ 4x^2-3x & \text{for } 1<x<2\\ 3x+4 & \text{for } x\ge 2\end{cases}$ is

a. continuous at $x=1$ and $x=2$
b. continuous at $x=1$ but not derivable at $x=2$
c. continuous at $x=2$ but not derivable at $x=1$
d. continuous at $x=1$ and 2 but not derivable at $x=1$ and $x=2$

5. Which of the following is/are true for $f(x)=\text{sgn}(x)\times\sin x$

a. discontinuous no where
b. an even function
c. $f(x)$ is periodic
d. $f(x)$ is differentiable for all x

6. If $f(x)=\lim\limits_{t\to\infty}\frac{|a+\sin \pi x|^t-1}{|a+\sin \pi x|^t+1}$, $x\in(0, 6)$, then

a. if $a=1$, then $f(x)$ has 5 points of discontinuity.
b. if $a=3$, then $f(x)$ has no point of discontinuity.
c. if $a=0.5$, then $f(x)$ has 6 points of discontinuity.
d. if $a=0$, then $f(x)$ has 6 points of discontinuity.

7. If $f(x)=\text{sgn}(x^2-ax+1)$ has maximum number of points of discontinuity, then

a. $a\in(2,\infty)$ b. $a\in(-\infty,-2)$
c. $a\in(-2,2)$ d. None of these

8. If $f(x)=[|x|]$, where $[.]$ denotes the greatest integer function, then which of the following is not true?

a. $f(x)$ is continuous $\forall x\in R$.
b. $f(x)$ is continuous from right and discontinuous from left $\forall x\in N$.
c. $f(x)$ is continuous from left and discontinuous from right $\forall x\in I$.
d. $f(x)$ is continuous at $x=0$.

9. $f(x)$ is differentiable function and $(f(x)\,.\,g(x))$ is differentiable at $x=a$, then

a. $g(x)$ must be differentiable at $x = a$.
b. If $g(x)$ is discontinuous, then $f(a) = 0$.
c. $f(a) \neq 0$, then $g(x)$ must be differentiable.
d. None of these.

10. The function defined as

$$f(x) = \lim_{n\to\infty}\begin{cases}\cos^{2n}x & \text{if } x<0\\ \sqrt[n]{\sqrt{1+x^n}} & \text{if } 0\le x\le 1,\\ \dfrac{1}{1+x^n} & \text{if } x>1\end{cases}$$

which of the following does not hold good?
a. continuous at $x = 0$ but discontinuous at $x = 1$.
b. continuous at $x = 1$ but discontinuous at $x = 0$.
c. continuous both at $x = 1$ and $x = 0$.
d. discontinuous both at $x = 1$ and $x = 0$.

11. Which of the following function(s) has/have removable discontinuity at $x = 1$?

a. $f(x) = \dfrac{1}{\ln|x|}$ b. $f(x) = \dfrac{x^2-1}{x^3-1}$

c. $f(x) = 2^{-2^{\frac{1}{1-x}}}$ d. $f(x) = \dfrac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}$

12. $f(x) = \dfrac{[x]+1}{\{x\}+1}$ for $f: \left[0, \dfrac{5}{2}\right) \to \left(\dfrac{1}{2}, 3\right]$, where [.] represents the greatest integer function and {.} represents the fractional part of x, then which of the following is true.
a. $f(x)$ is injective discontinuous function.
b. $f(x)$ is surjective non-differentiable function.
c. $\min\left(\lim_{x\to 1^-} f(x), \lim_{x\to 1^+} f(x)\right) = f(1)$.
d. max (x values of point of discontinuity) $= f(1)$.

13. The function $f(x) = \begin{cases}1, & |x|\ge 1\\ \dfrac{1}{n^2}, & \dfrac{1}{n}<|x|<\dfrac{1}{n-1}, n = 2, 3, \ldots\\ 0, & x = 0\end{cases}$

a. is discontinuous at infinite points
b. is continuous everywhere
c. is discontinuous only at $x = \dfrac{1}{n}$, $n \in Z - \{0\}$
d. None of these

14. Let $f(x) = [x]$ and $g(x) = \begin{cases}0, & x\in Z\\ x^2, & x\in R-Z\end{cases}$ ([.] represents greatest integer function). Then
a. $\lim_{x\to 1} g(x)$ exists but $g(x)$ is not continuous at $x = 1$.
b. $f(x)$ is not continuous at $x = 1$.
c. gof is continuous for all x.
d. fog is continuous for all x.

15. If $f(x) = \begin{cases}\dfrac{x\log\cos x}{\log(1+x^2)}, & x\neq 0\\ 0, & x = 0\end{cases}$ then
a. $f(x)$ is not continuous at $x = 0$.
b. $f(x)$ is continuous at $x = 0$.
c. $f(x)$ is continuous at $x = 0$ but not differentiable at $x = 0$.
d. $f(x)$ is differentiable at $x = 0$.

16. If $f(x) = x + |x| + \cos([\pi^2]x)$ and $g(x) = \sin x$, where [.] denotes the greatest integer function, then
a. $f(x) + g(x)$ is continuous everywhere.
b. $f(x) + g(x)$ is differentiable everywhere.
c. $f(x) \times g(x)$ is differentiable everywhere.
d. $f(x) \times g(x)$ is continuous but not differentiable at $x = 0$.

17. If $f(x) = \begin{cases}(\sin^{-1}x)^2\cos(1/x), & x\neq 0\\ 0, & x = 0\end{cases}$ then
a. $f(x)$ is continuous everywhere in $x \in (-1, 1)$.
b. $f(x)$ is discontinuous in $x \in [-1, 1]$.
c. $f(x)$ is differentiable everywhere in $x \in (-1, 1)$.
d. $f(x)$ is non-differentiable nowhere in $x \in [-1, 1]$.

18. $f(x) = \begin{cases}x+a, & x\ge 0\\ 2-x, & x<0\end{cases}$ and $g(x) = \begin{cases}\{x\}, & x<0\\ \sin x + b, & x\ge 0\end{cases}$ and if $f(g(x))$ is continuous at $x = 0$ then which of the following is/are true (where $\{x\}$ represents the fractional part function)
a. if $b = 1$, then a can take any real value.
b. if $b < -1$, then $a + b = 1$.
c. no values of a and b are possible.
d. there exist finite ordered pairs (a, b).

19. If $f(x) = \begin{cases}|x|-3, & x<1\\ |x-2|+a, & x\ge 1\end{cases}$ and $g(x) = \begin{cases}2-|x|, & x<2\\ \text{sgn}(x)-b, & x\ge 2\end{cases}$ and $h(x) = f(x) + g(x)$ is discontinuous at exactly one point then which of the following values of a and b are possible
a. $a = -3, b = 0$ b. $a = 2, b = 1$
c. $a = 2, b = 0$ d. $a = -3, b = 1$

20. Let $f: R \to R$ be any function and $g(x) = \dfrac{1}{f(x)}$. Then which of following is/are not true
a. g is onto if f is onto.
b. g is one-one if f is one-to-one.
c. g is continuous if f is continuous.
d. g is differentiable if f is differentiable.

21. If $f(x) = \begin{cases}x^2(\text{sgn}[x]) + \{x\}, & 0\le x<2\\ \sin x + |x-3|, & 2\le x<4\end{cases}$, where [ ] and { } represent the greatest integer and the fractional part function, respectively.
a. $f(x)$ is differentiable at $x = 1$.
b. $f(x)$ is continuous but non-differentiable at $x = 1$.
c. $f(x)$ is non-differentiable at $x = 2$.
d. $f(x)$ is discontinuous at $x = 2$.

22. $f(x)=\begin{cases}\left(\frac{3}{2}\right)^{(\cot 3x)/(\cot 2x)}; & 0<x<\frac{\pi}{2}\\ b+3; & x=\frac{\pi}{2}\\ (1+|\cot x|)^{(a|\tan x|)/b}; & \frac{\pi}{2}<x<\pi\end{cases}$ is continuous at $x=\pi/2$, then

a. $a=0$ b. $a=2$ c. $b=-2$ d. $b=2$

23. Which of the following function is thrice differentiable at $x=0$?

a. $f(x)=|x^3|$ b. $f(x)=x^3|x|$
c. $f(x)=|x|\sin^3 x$ d. $f(x)=x|\tan^3 x|$

24. Let $f(x)=[\sin^4 x]$, then (where $[\cdot]$ represents the greatest integer function)

a. $f(x)$ is continuous at $x=0$
b. $f(x)$ is differentiable at $x=0$
c. $f(x)$ is non-differentiable at $x=0$
d. $f'(0)=1$

25. Let $f(x)=\text{sgn}(\cos 2x-2\sin x+3)$, where sgn $(\cdot)$ is the signum function, then $f(x)$

a. is continuous over its domain
b. has a missing point discontinuity
c. has isolated point discontinuity
d. irremovable discontinuity

26. A function $f(x)$ satisfies the relation $f(x+y)=f(x)+f(y)+xy(x+y)\ \forall\ x, y\in R$. If $f'(0)=-1$, then

a. $f(x)$ is a polynomial function
b. $f(x)$ is an exponential function
c. $f(x)$ is twice differentiable for all $x\in R$
d. $f'(3)=8$

27. Let $f(x)=\begin{cases}\dfrac{e^x-1+ax}{x^2}, & x>0\\ b, & x=0\\ \dfrac{\sin\frac{x}{2}}{x}, & x<0\end{cases}$, then

a. $f(x)$ is continuous at $x=0$ if $a=-1$, $b=\frac{1}{2}$
b. $f(x)$ is discontinuous at $x=0$ if $b\neq\frac{1}{2}$
c. $f(x)$ has irremovable discontinuity at $x=0$ if $a\neq-1$
d. $f(x)$ has removable discontinuity at $x=0$ if $a=-1$, $b\neq\frac{1}{2}$

Reasoning Type

Solutions on page 3.49

Each question has four choices a, b, c, and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. if both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.
b. if both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.
c. if STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.
d. if STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** $y=\sin x$ and $y=\sin^{-1}x$, both are differentiable functions.
Statement 2: Differentiability of $f(x)\Rightarrow$ differentiability of $y=f^{-1}(x)$.

2. **Statement 1:** $f(x)=(2x-5)^{3/5}$ is non-differentiable at $x=5/2$.
Statement 2: If the graph of $y=f(x)$ has sharp turn at $x=a$, then it is non-differentiable.

3. **Statement 1:** $f(x)=\text{sgn}(x^2-2x+3)$ is continuous for all x.
Statement 2: $ax^2+bx+c=0$ has no real roots if $b^2-4ac<0$.

4. **Statement 1:** $f(x)=\lim\limits_{x\to\infty}\dfrac{x^{2n}-1}{x^{2n}+1}$ is discontinuous at $x=1$.
Statement 2: If limit of function exists at $x=a$ but not equal to $f(a)$, then $f(x)$ is discontinuous at $x=a$.

5. **Statement 1:** $f(x)=[\sin x]-[\cos x]$ is discontinuous at $x=\pi/2$, where [.] represent the greatest integer function.
Statement 2: If $f(x)$ and $g(x)$ are discontinuous at $x=a$, then $f(x)+g(x)$ is discontinuous at $x=a$.

6. **Statement 1:** $f(x)=\text{sgn}\,x$ is discontinuous at $x=0\Rightarrow f(x)=|\text{sgn}x|$ is discontinuous at $x=0$.
Statement 2: Discontinuity of $f(x)\Rightarrow$ discontinuity of $|f(x)|$.

7. **Statement 1:** $f(x)=(\sin\pi x)(x-1)^{1/5}$ is differentiable at $x=1$.
Statement 2: Product of two differentiable function is always differentiable.

8. **Statements 1:** The function $f(x)=\begin{cases}\dfrac{e^{1/x}-1}{e^{1/x}+1}, & x\neq0\\ \cos x & x=0\end{cases}$ is discontinuous at $x=0$.
Statements 2: $f(0)=1$.

9. **Statement 1:** $f(x)=\sin x+[x]$ is discontinuous at $x=0$, where [.] denotes the greatest integer function.
Statement 2: If $g(x)$ is continuous and $h(x)$ is discontinuous at $x=a$, then $g(x)+h(x)$ will necessarily be discontinuous at $x=a$.

10. **Statement 1:** $f(x)=|x|\sin x$ is non-differentiable at $x=0$.
Statement 2: If $f(x)$ is not differentiable and $g(x)$ is differentiable at $x=a$, then $f(x)\,g(x)$ can still be differentiable at $x=a$.

11. **Statement 1:** If $f(x)$ is discontinuous at $x=e$ and $\lim\limits_{x\to a}g(x)=e$, then $\lim\limits_{x\to a}f(g(x))$ cannot be equal to $f\left(\lim\limits_{x\to a}g(x)\right)$.
Statement 2: If $f(x)$ is continuous at $x=e$ and $\lim\limits_{x\to a}g(x)=e$, then $\lim\limits_{x\to a}f(g(x))=f\left(\lim\limits_{x\to a}g(x)\right)$.

12. **Statement 1:** Both the functions $|\ln x|$ and $\ln x$ are both continuous for all x.
Statement 2: Continuity of $|f(x)|\Rightarrow$ continuity of $f(x)$.

13. **Statement 1:** $f(x)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ is non-differentiable at $x=\pm 1$.
Statement 2: Principal value of $\tan^{-1}x$ are $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

14. **Statement 1:** If $|f(x)|\le |x|$ for all $x\in R$, then $|f(x)|$ is continuous at 0.
Statement 2: If $f(x)$ is continuous, then $|f(x)|$ is also continuous.

15. **Statement 1:** $f(x)=||x|^2-3|x|+2|$ is not differentiable at 5 points.
Statement 2: If the graph of $f(x)$ crosses the x-axis at m distinct points, then $g(x)=|f(x)|$ is always non-differentiable at least at m distinct points.

16. **Statement 1:** The function $f(x)=a_1e^{|x|}+a_2|x|^5$, where a_1, a_2 are constants, is differentiable at $x=0$ if $a_1=0$.
Statement 2: $e^{|x|}$ is a many-one function.

17. Consider $[\cdot]$ and $\{\cdot\}$ denote the greatest integer function and the fractional part function, respectively.
Let $f(x)=\{x\}+\sqrt{\{x\}}$.
Statement 1: f is not differentiable at integral values of x.
Statement 2: f is not continuous at integral points.

18. **Statement 1:** Let $f(x)=\lim\limits_{m\to\infty}\left\{\lim\limits_{n\to\infty}\cos^{2m}(n!\pi x)\right\}$, and
$$g(x)=\begin{cases}0, \text{ if } x \text{ is rational}\\ 1, \text{ if } x \text{ is irrational}\end{cases}$$. Then $h(x)=f(x)+g(x)$ is continuous for all x.
Statement 2: $f(x)$ and $g(x)$ are discontinuous for all $x\in R$.

19. **Statement 1:** If $f'(x)$ exists then $f'(x)$ is continuous.
Statement 2: Every differentiable function is continuous.

20. Consider the functions $f(x)=x^2-2x$ and $g(x)=-|x|$.
Statement 1: The composite function $F(x)=f(g(x))$ is not derivable at $x=0$.
Statement 2: $F'(0^+)=2$ and $F'(0^-)=-2$.

21. **Statement 1:** If $f(x)$ and $g(x)$ are two differentiable functions $\forall\, x\in R$, then $y=\max\{f(x), g(x)\}$ is always continuous but not differentiable at the point of intersection of graphs of $f(x)$ and $g(x)$.
Statement 2: $y=\max\{f(x), g(x)\}$ is always differentiable in between the two consecutive roots of $f(x)-g(x)=0$ if both the functions $f(x)$ and $g(x)$ are differentiable $\forall\, x\in R$.

22. Consider the function
$$f(x)=\cot^{-1}\left(\operatorname{sgn}\left(\frac{[x]}{2x-[x]}\right)\right),$$ where $[\cdot]$ denotes the greatest integer function.
Statement 1: $f(x)$ is discontinuous at $x=1$.
Statement 2: $f(x)$ is non-differentiable at $x=1$.

23. Consider the function $f(x)=\operatorname{sgn}(x-1)$ and $g(x)=\cot^{-1}[x-1]$, where $[\cdot]$ denotes the greatest integer function.
Statement 1: The function $F(x)=f(x)\,.\,g(x)$ is discontinuous at $x=1$.
Statement 2: If $f(x)$ is discontinuous at $x=a$ and $g(x)$ is also discontinuous at $x=a$, then the product function $f(x)\,g(x)$ is discontinuous at $x=a$.

24. **Statement 1:** $f(x)=\min\{\sin x, \cos x\}$ is non-differentiable at $x=\pi/2$.
Statement 2: Non-differentiability of max $\{g(x), h(x)\}$ $\Rightarrow$ non-differentiability of min $\{g(x)\, h(x)\}$.

25. **Statement 1:** If $f(x)$ is a continuous function such that $f(0)=1$ and $f(x)\ne x, \forall\, x\in R$, then $f(f(x))>x$.
Statement 2: If $f: R\to R, f(x)$ is an onto function, then $f(x)=0$ has at least one solution.

26. **Statement 1:** The function $f(x)=\left[\sqrt{x}\right]$ is discontinuous for all integral values of x in its domain (where $[x]$ is the greatest integer $\le x$).
Statement 2: $[g(x)]$ will be discontinuous for all x given by $g(x)=k$, where k is any integer.

Linked Comprehension Type

Solutions on page 3.51

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c, and d, out of which *only one* is correct.

For Problems 1 – 3

$$\text{Let } f(x)=\begin{cases}\dfrac{a(1-x\sin x)+b\cos x+5}{x^2}, & x<0\\ 3, & x=0\\ \left\{1+\left(\dfrac{P(x)}{x^2}\right)\right\}^{1/x}, & x>0\end{cases}, \text{ where } P(x)$$
is a cubic function and f is continuous at $x=0$.

1. The range of function $g(x)=3a\sin x-b\cos x$ is
   a. $[-10, 10]$ b. $[-5, 5]$
   c. $[-12, 12]$ d. None of these
2. The value of $P''(0)$ is
   a. $\log_e 9$ b. $\log_e 2$
   c. 2 d. 1
3. If the leading co-efficient of $P(x)$ is positive, then the equation $P(x)=b$ has
   a. Only one real, positive root
   b. Only one real negative root
   c. Three real roots
   d. None of these

For Problems 4–6

$$\text{Let } f(x)=\begin{cases}x+2, & 0\le x<2\\ 6-x, & x\ge 2\end{cases},$$

$$g(x)=\begin{cases}1+\tan x, & 0\le x<\dfrac{\pi}{4}\\ 3-\cot x, & \dfrac{\pi}{4}\le x<\pi\end{cases}$$

4. $f(g(x))$ is
   a. discontinuous at $x = \pi/4$.
   b. differentiable at $x = \pi/4$.
   c. continuous but non-differentiable at $x = \pi/4$.
   d. differentiable at $x = \pi/4$, but derivative is not continuous.
5. The number of points of non-differentiability of $h(x) = |f(g(x))|$ is
   a. 1 b. 2 c. 3 d. 4
6. The range of $h(x) = f(g(x))$ is
   a. $(-\infty, \infty)$ b. $(4, \infty)$
   c. $(-\infty, 4]$ d. None of these

For Problems 7–9

Consider $f(x) = x^2 + ax + 3$ and $g(x) = x + b$ and $F(x) = \lim_{n\to\infty} \dfrac{f(x) + x^{2n} g(x)}{1 + x^{2n}}$.

7. If $F(x)$ is continuous at $x = 1$, then
   a. $b = a + 3$ b. $b = a - 1$
   c. $a = b - 2$ d. None of these
8. If $F(x)$ is continuous at $x = -1$, then
   a. $a + b = -2$ b. $a - b = 3$
   c. $a + b = 5$ d. None of these
9. If $F(x)$ is continuous at $x = \pm 1$, then $f(x) = g(x)$ has
   a. imaginary roots b. both the roots positive
   c. both the roots negative d. roots of opposite signs

For Problems 10 – 12

Let $f(x) = \begin{cases} [x], & -2 \le x \le -\dfrac{1}{2} \\ 2x^2 - 1, & -\dfrac{1}{2} < x \le 2 \end{cases}$ and $g(x) = f(|x|) + |f(x)|$,

where $[\cdot]$ represents greatest integer function.

10. The number of points where $|f(x)|$ is non-differentiable is
    a. 3 b. 4 c. 2 d. 5
11. The number of points where $g(x)$ is non-differentiable is
    a. 4 b. 5 c. 2 d. 3
12. The number of points where $g(x)$ is discontinuous is
    a. 1 b. 2
    c. 3 d. None of these

For problems 13 – 15

Given the continuous function

$$y = f(x) = \begin{cases} x^2 + 10x + 8, & x \le -2 \\ ax^2 + bx + c, & -2 < x < 0,\ a \ne 0 \\ x^2 + 2x, & x \ge 0 \end{cases}$$

If a line L touches the graph of $y = f(x)$ at three points, then

13. The slope of the line 'L' is equal to
    a. 1 b. 2 c. 4 d. 6
14. The value of $(a + b + c)$ is equal to
    a. $5\sqrt{2}$ b. 5 c. 6 d. 7
15. If $y = f(x)$ is differentiable at $x = 0$, then the value of b
    a. is -1 b. is 2
    c. is 4 d. Cannot be determined

Matrix-Match Type

Solutions on page 3.53

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct match is a-p, a-s, b-q, b-r, c-p, c-q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows

	p	q	r	s
a	(p)	(q)	(r)	(s)
b	(p)	(q)	(r)	(s)
c	(p)	(q)	(r)	(s)
d	(p)	(q)	(r)	(s)

1.

Column 1	Column 2
a. $f(x) = \lvert x^3 \rvert$ is	p. continuous in $(-1, 1)$
b. $f(x) = \sqrt{\lvert x \rvert}$ is	q. differentiable in $(-1, 1)$
c. $f(x) = \lvert \sin^{-1} x \rvert$ is	r. differentiable in $(0, 1)$
d. $f(x) = \cos^{-1} \lvert x \rvert$ is	s. not differentiable at least at one point in $(-1, 1)$

2.

Column 1	Column 2
a. $f(x) = \begin{cases} \dfrac{1}{\lvert x \rvert} & \text{for } \lvert x \rvert \ge 1 \\ ax^2 + b & \text{for } \lvert x \rvert < 1 \end{cases}$ is differentiable everywhere and $\lvert k \rvert = a + b$, then the value of k is	p. 2
b. If $f(x) = \operatorname{sgn}(x^2 - ax + 1)$ has exactly one point of discontinuity, then the value of a can be	q. -2
c. $f(x) = [2 + 3\lvert n \rvert \sin x]$, $n \in N$, $x \in (0, \pi)$ has exactly 11 points of discontinuity, then the value of n is	r. 1
d. $f(x) = \lvert\lvert\lvert x \rvert - 2\rvert + a\rvert$ has exactly three points of non-differentiability, then the value of a is	s. -1

3.
Consider the function $f(x) = x^2 + bx + c$, where $D = b^2 - 4c > 0$

Column 1 Condition on b and c	Column 2 Number of points of non-differentiability of $g(x) = \lvert f(\lvert x \rvert) \rvert$
a. $b < 0, c > 0$	p. 1
b. $c = 0, b < 0$	q. 2
c. $c = 0, b > 0$	r. 3
d. $b = 0, c < 0$	s. 5

4. Let $f(x)=\begin{cases}\dfrac{5e^{1/x}+2}{3-e^{1/x}}, & x\neq 0\\ 0, & x=0\end{cases}$

Column 1	Column 2
a. $y=f(x)$ is	p. continuous at $x=0$
b. $y=xf(x)$ is	q. discontinuous at $x=0$
c. $y=x^2f(x)$ is	r. differentiable at $x=0$
d. $y=x^{-1}f(x)$ is	s. non-differentiable at $x=0$

5.

Column 1	Column 2
a. $f(x)=\lim\limits_{n\to\infty}\cos^{2n}(2\pi x)+\left\{x+\dfrac{1}{2}\right\}$, where {.} denotes the fractional part function at $x=\dfrac{1}{2}$	p. continuous
b. $f(x)=(\log_e x)(x-1)^{1/5}$ at $x=1$	q. discontinuous
c. $f(x)=[\cos 2\pi x]+\sqrt{\left\{\sin\pi\dfrac{x}{2}\right\}}$, where [.] and {.} denote the greatest integer and the fractional part function, respectively at $x=1$	r. differentiable
d. $f(x)=\begin{cases}\cos 2x, & x\notin Q\\ \sin x, & x\notin Q\end{cases}$ at $x=\dfrac{\pi}{6}$	s. non-differentiable

Integer Type

Solutions on page 3.56

1. Number of points of discontinuity for $f(x)=\text{sgn}(\sin x)$, $x\in[0,4\pi]$ is

2. If $f(x)$ is a continuous function $\forall\ x\in R$ and the $f(x)\in(1,\sqrt{30})$, and $g(x)=\left[\dfrac{f(x)}{a}\right]$, where $[\cdot]$ denotes the greatest integer function, is continuous $\forall\ x\in R$, then the least positive integral value of a is.

3. Number of points where $f(x)=\text{sgn}(x^2-3x+2)+[x-3]$, $x\in[0,4]$ is discontinuous is (where $[\cdot]$ denotes the greatest integer function)

4. Let $g(x)=\begin{bmatrix}a\sqrt{x+1} & \text{if } 0<x<3\\ bx+2 & \text{if } 3\le x<5\end{bmatrix}$, if $g(x)$ is differentiable on (0, 5) then $(a+b)$ equals

5. Let $f(x)=\lim\limits_{n\to\infty}\dfrac{x^{2n-1}+ax^2+bx}{x^{2n}+1}$. If $f(x)$ is continuous for all $x\in R$, then the value of $a+8b$ is

6. Let $f(x)=\begin{cases}\dfrac{x}{2}-1, & 0\le x<1\\ \dfrac{1}{2}, & 1\le x\le 2\end{cases}$ and $g(x)=(2x+1)(x-k)+3$, $0\le x\,\infty$. Then $g(f(x))$ is continuous at $x=1$ if $12k$ is equal to

7. A differentiable function f satisfying a relation $f(x+y)=f(x)+f(y)+2xy(x+y)-\dfrac{1}{3}\ \forall\ x,y\in R$ and $\lim\limits_{h\to 0}\dfrac{3f(h)-1}{6h}=\dfrac{2}{3}$. Then the value of $[f(2)]$ is (where $[x]$ represents greatest integer function)

8. The least integral value of p for which $f''(x)$ is everywhere continuous where $f(x)=\begin{cases}x^p\sin\left(\dfrac{1}{x}\right)+x|x|, & x\neq 0\\ 0, & x=0\end{cases}$

9. Number of points where $f(x)=[x]+[x+1/3]+[x+2/3]$, then ($[\cdot]$ denotes the greatest integer function) is discontinuous for $x\in(0,3)$.

10. Let $f(x)$ and $g(x)$ be two continuous functions and $h(x)=\lim\limits_{n\to\infty}\dfrac{x^{2n}\cdot f(x)+x^{2m}\cdot g(x)}{(x^{2n}+1)}$. If limit of $h(x)$ exists at $x=1$, then one root of $f(x)-g(x)=0$ is

11. Given $\dfrac{\int_{f(y)}^{f(x)}e^t\,dt}{\int_y^x(1/t)\,dt}=1,\ \forall\ x,y\in\left(\dfrac{1}{e^2},\infty\right)$ where $f(x)$ is continuous and differentiable function and $f\left(\dfrac{1}{e}\right)=0$. If $g(x)=\begin{cases}e^x, & x\ge k\\ e^{x^2}, & 0<x<k\end{cases}$; then the value of '$k$' for which $f(g(x))$ is continuous $\forall\ x\in R^+$ is

12. $f(x)=\dfrac{x}{1+(\ln x)(\ln x)\cdots\infty}\ \forall\ x\in[1,3]$ is non-differentiable at $x=k$. Then the value of $[k^2]$ is (where $[\cdot]$ represents greatest integer function)

13. If the function $f(x)=\dfrac{\tan(\tan x)-\sin(\sin x)}{\tan x-\sin x}$ $(x\neq 0)$ is continuous at $x=0$, then the value of $f(0)$ is

14. Number of points of non-differentiability of function $f(x)=\max\{\sin^{-1}|\sin x|,\cos^{-1}|\sin x|\}$, $0<x<2\pi$ is

Archives

Solutions on page 3.58

Subjective

1. Determine the values of a, b, c for which the function $f(x)$ is continuous at $x=0$, where

$$f(x)=\begin{cases}\dfrac{\sin[(a+1)x]+\sin x}{x}; & x<0\\ c; & x=0.\\ \dfrac{(x+bx^2)^{1/2}-x^{1/2}}{bx^{3/2}}; & x>0\end{cases}$$ **(IIT-JEE, 1982)**

2. Let $f(x)=\begin{cases}1+x, & 0\le x\le 2\\ 3-x, & 2<x\le 3\end{cases}$. Determine the function $g(x)=f(f(x))$, and hence find the points of discontinuity of g, if any.

3. Let $f(x)=\begin{cases}\dfrac{x^2}{2} & ,0\le x<1\\ 2x^2-3x+\dfrac{3}{2} & ,1\le x\le 2\end{cases}$ **(IIT-JEE, 1983)**
discuss the continuity of f, f' and f'' on [0, 2].

4. Let $f(x)=x^3-x^2+x+1$ and
$g(x)=\begin{cases}\max.f(t); & 0\le t\le x \text{ for } 0\le x\le 1\\ 3-x; & 1<x\le 2\end{cases}$
Discuss the continuity and differentiability of $g(x)$ in (0, 2).

5. Let $f(x)$ be defined in the interval [–2, 2] such that $f(x)=\begin{cases}-1, & -2\le x\le 0\\ x-1, & 0<x\le 2\end{cases}$ and $g(x)=f(|x|)+|f(x)|$.
Test the differentiability of $g(x)$ in (–2, 2). **(IIT-JEE, 1986)**

6. Let $f(x)$ be a continuous and $g(x)$ is a discontinuous function, then prove that $f(x)+g(x)$ is discontinuous at $x=a$. **(IIT-JEE, 1987)**

7. Let $f(x)$ be a function satisfying the condition $f(-x)=f(x)$ for all real x. If $f'(0)$ exists, find its value. **(IIT-JEE, 1987)**

8. Let $g(x)$ be a polynomial of degree one and $f(x)$ be defined by $f(x)=\begin{cases}g(x), & x\le 0\\ \left(\dfrac{1+x}{2+x}\right)^{1/x}, & x>0\end{cases}$, find the continuous function satisfying $f'(1)=f(-1)$. **(IIT-JEE, 1987)**

9. Find the values of a and b so that the function
$f(x)=\begin{cases}x+a\sqrt{2}\sin x, & 0\le x<\pi/4\\ 2x\cot x+b, & \pi/4\le x\le \pi/2\\ a\cos 2x-b\sin x, & \pi/2<x\le\pi\end{cases}$ is continuous for $0\le x\le\pi$. **(IIT-JEE, 1989)**

10. Draw a graph of the function $y=[x]+|1-x|, -1\le x\le 3$. Determine the points, if any, where this function is not differentiable. **(IIT-JEE, 1989)**

11. Let $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2}, & x<0\\ a, & x=0\\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & x>0\end{cases}$ **(IIT-JEE, 1990)**

Determine the value of a, if possible, so that the function is continuous at $x=0$.

12. Let $f(x)=\begin{cases}\{1+|\sin x|\}^{a/|\sin x|}; & -\dfrac{\pi}{6}<x<0\\ b; & x=0\\ e^{\tan 2x/\tan 3x}; & 0<x<\dfrac{\pi}{6}\end{cases}$
Determine a and b such that $f(x)$ is continuous at $x=0$. **(IIT-JEE, 1994)**

13. Let $f(x)=\begin{cases}xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x\ne 0\\ 0, & x=0\end{cases}$ **(IIT-JEE, 1997)**
Test whether
a. $f(x)$ is continuous at $x=0$
b. $f(x)$ is differentiable at $x=0$

14. Determine the values of x for which the following function fails to be continuous or differentiable:
$f(x)=\begin{cases}1-x & x<1\\ (1-x)(2-x), & 1\le x\le 2\\ 3-x & x>2\end{cases}$
Justify your answer. **(IIT-JEE, 1997)**

15. Let $\alpha\in R$. Prove that a function $f: R\to R$ is differentiable at $x=\alpha$ if and only if there is a function $g: R\to R$ which is continuous at a and satisfies $f(x)-f(\alpha)=g(x)(x-\alpha)$ for all $\alpha\in R$. **(IIT-JEE, 2001)**

16. Let $f(x)=\begin{cases}x+a & \text{if } x<0\\ |x-1| & \text{if } x\ge 0\end{cases}$ and
$g(x)=\begin{cases}x+1 & \text{if } x<0\\ (x-1)^2+b & \text{if } x\ge 0,\end{cases}$ where a and b are non-negative real numbers. Determine the composite function gof. If $(gof)(x)$ is continuous for all real x, determine the values of a and b. Further, for these values of a and b, is gof differentiable at $x=0$? Justify your answer. **(IIT-JEE, 2002)**

17. If a function $f: [-2a, 2a]\to R$ is an odd function such that $f(x)=f(2a-x)$ for $x\in[a, 2a]$ and the left-hand derivative at $x=a$ is 0, then find the left-hand derivative at $x=-a$. **(IIT-JEE, 2003)**

18. $f'(0)=\lim\limits_{n\to\infty} nf\left(\dfrac{1}{n}\right)$ and $f(0)=0$. Using this, find $\lim\limits_{n\to\infty}\left((n+1)\dfrac{2}{\pi}\cos^{-1}\left(\dfrac{1}{n}\right)-n\right), \left|\cos^{-1}\dfrac{1}{n}\right|<\dfrac{\pi}{2}$. **(IIT-JEE, 2004)**

19. If $|c|\le\dfrac{1}{2}$ and $f(x)$ is a differentiable function at $x=0$ given by $f(x)=\begin{cases}b\sin^{-1}\left(\dfrac{c+x}{2}\right), & -\dfrac{1}{2}<x<0\\ \dfrac{1}{2}, & x=0\\ \dfrac{e^{ax/2}-1}{x}, & 0<x<\dfrac{1}{2}\end{cases}$

Find the value of a and prove that $64b^2 = 4 - c^2$.
(IIT-JEE, 2004)

20. If $f(x-y) = f(x)g(y) - f(y)g(x)$ and $g(x-y) = g(x)g(y) - f(x)f(y)$ for all $x, y \in R$.
If right-hand derivative at $x = 0$ exists for $f(x)$. Find the derivative of $g(x)$ at $x = 0$. **(IIT-JEE, 2005)**

Objective

Fill in the blanks

1. Let $f(x) = \begin{cases} (x-1)^2 \sin \dfrac{1}{(x-1)} - |x|, & \text{if } x \neq 1 \\ -1, & \text{if } x = 1 \end{cases}$ be a real-valued function, then the set of points where $f(x)$ is not differentiable is ________. **(IIT-JEE, 1981)**

2. Let $f(x) = \begin{cases} \dfrac{(x^3 + x^2 - 16x + 20)}{(x-2)^2}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$ if $f(x)$ is continuous for all x, then $k =$ ________. **(IIT-JEE, 1981)**

3. A discontinuous function $y = f(x)$ satisfying $x^2 + y^2 = 4$ is given by $f(x) =$ ________. **(IIT-JEE, 1982)**

4. Let $f(x) = x|x|$. The set of points, where $f(x)$ is twice differentiable, is ________. **(IIT-JEE, 1992)**

5. Let $f(x) = [x]\sin\left(\dfrac{\pi}{[x+1]}\right)$, where $[\cdot]$ denotes the greatest integer function. The domain of f is ________ and the points of discontinuity of f in the domain are ________. **(IIT-JEE, 1996)**

6. Let $f(x)$ be a continuous function defined for $1 \le x \le 3$. If $f(x)$ takes rational values for all x and $f(2) = 10$, then $f(1.5) =$ ________. **(IIT-JEE, 1997)**

Multiple choice questions with one correct answer

1. For a real number y, let $[y]$ denotes the greatest integer less than or equal to y. Then the function

$$f(x) = \frac{\tan(\pi[x-\pi])}{1+[x]^2} \text{ is}$$ **(IIT-JEE, 1981)**

a. discontinuous at some x
b. continuous at all x, but the derivative $f'(x)$ does not exist for some x
c. $f'(x)$ exists for all x, but the derivative $f'(x_0)$ does not exist second for some x
d. $f'(x)$ exists for all x

2. Let [.] denote the greatest integer function and $f(x) = [\tan^2 x]$, then **(IIT-JEE, 1993)**

a. $\lim\limits_{x\to 0} f(x)$ does not exist
b. $f(x)$ is continuous at $x = 0$
c. $f(x)$ is not differentiable at $x = 0$
d. $f'(0) = 1$

3. The function $f(x) = [x]\cos\left(\dfrac{2x-1}{2}\right)\pi$, where [.] denotes the greatest integer function, is discontinuous at **(IIT-JEE, 1995)**

a. all x
b. all integer points
c. no x
d. x which is not an integer

4. The function $f(x) = [x]^2 - [x^2]$ (where $[y]$ is the greatest integer less than or equal to y), is discontinuous at **(IIT-JEE, 1999)**

a. all integers
b. all integers except 0 and 1
c. all integers except 0
d. all integers except 1

5. The function $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos(|x|)$ is NOT differentiable at

a. -1 b. 0 c. 1 d. 2

6. The left-hand derivatives of $f(x) = [x]\sin(\pi x)$ at $x = k$, k an integer, is **(IIT-JEE, 2001)**

a. $(-1)^k(k-1)\pi$ b. $(-1)^{k-1}(k-1)\pi$
c. $(-1)^k k\pi$ d. $(-1)^{k-1}k\pi$

7. Let $f: R \to R$ be a function defined by $f(x) = \max\{x, x^3\}$. The set of all point where $f(x)$ is NOT differentiable is **(IIT-JEE, 2001)**

a. $\{-1, 1\}$ b. $\{-1, 0\}$
c. $\{0, 1\}$ d. $\{-1, 0, 1\}$

8. Which of the following functions is differentiable at $x = 0$? **(IIT-JEE, 2001)**

a. $\cos(|x|) + |x|$ b. $\cos(|x|) - |x|$
c. $\sin(|x|) + |x|$ d. $\sin(|x|) - |x|$

9. The domain of the derivative of the function

$$f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \le 1 \\ \dfrac{1}{2}(|x|-1) & \text{if } |x| > 1 \end{cases} \text{ is}$$ **(IIT-JEE, 2002)**

a. $R - \{0\}$ b. $R - \{1\}$
c. $R - \{-1\}$ d. $R - \{-1, 1\}$

10. The function given by $y = ||x| - 1|$ is differentiable for all real numbers except the points **(IIT-JEE, 2005)**

a. $\{0, 1, -1\}$ b. ± 1
c. 1 d. -1

11. If $f(x)$ is a continuous and differentiable function and $f(1/n) = 0\ \forall\ n \ge 1$ and $n \in I$, then **(IIT-JEE, 2005)**

a. $f(x) = 0, x \in (0, 1]$
b. $f(0) = 0, f'(0) = 0$
c. $f(0) = 0 = f'(0), x \in (0, 1]$
d. $f(0) = 0$ and $f'(0)$ need not to be zero

Multiple choice question with one or more than one correct answer

1. If $x + |y| = 2y$, then y as a function of x is **(IIT-JEE, 1984)**

a. defined for all real x
b. continuous at $x=0$
c. differentiable for all x
d. such that $\frac{dy}{dx}=\frac{1}{3}$ for $x<0$

2. The function $f(x)=1+|\sin x|$ is (IIT-JEE, 1986)
a. continuous nowhere
b. continuous everywhere
c. differentiable nowhere
d. not differentiable at $x=0$
e. not differentiable at infinite number of points

3. Let $[x]$ denotes the greatest integer less than or equal to x. If $f(x)=[x\sin\pi x]$, then $f(x)$ is (IIT-JEE, 1986)
a. Continuous at $x=0$ b. Continuous in $(-1,0)$
c. Differentiable at $x=1$ d. Differentiable in $(-1,1)$
e. None of these

4. The set of all points, where the function $f(x)=\frac{x}{1+|x|}$ is differentiable is (IIT-JEE, 1987)
a. $(-\infty,\infty)$ b. $[0,\infty)$
c. $(-\infty,0)\cup(0,\infty)$ d. $(0,\infty)$

5. The function $f(x)=\begin{cases}|x-3|, & x\geq 1\\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, & x<1\end{cases}$ is (IIT-JEE, 1988)
a. continuous at $x=1$ b. differentiable at $x=1$
c. continuous at $x=3$ d. differentiable at $x=3$

6. If $f(x)=\frac{x-2}{2}$, then in $[0,\pi]$ (IIT-JEE, 1989)
a. both $\tan(f(x))$ and $\frac{1}{f(x)}$ are continuous
b. $\tan(f(x))$ is continuous but $f^{-1}(x)$ is not continuous
c. $\tan(f^{-1}(x))$ and $f^{-1}(x)$ are discontinuous
d. None of these

7. The following functions are continuous on $(0,\pi)$ (IIT-JEE, 1991)
a. $\tan x$
b. $\int_0^x t\sin\frac{1}{t}\,dt$
c. $\begin{cases}1, & 0<x\leq\frac{3\pi}{4}\\ 2\sin\frac{2}{9}x, & \frac{3\pi}{4}<x<\pi\end{cases}$
d. $\begin{cases}x\sin x, & 0<x\leq\frac{\pi}{2}\\ \frac{\pi}{2}\sin(\pi+x), & \frac{\pi}{2}<x<\pi\end{cases}$

8. Let $h(x)=\min\{x,x^2\}$, for every real number of x, then (IIT-JEE, 1998)
a. h is continuous for all x
b. h is differentiable for all x
c. $h'(x)=1$, for all $x>1$
d. h is not differentiable at two values of x

9. Let $f(x)=\begin{cases}0, & x<0\\ x^2, & x\geq 0\end{cases}$ then for all x (IIT-JEE, 1994)
a. f' is differentiable b. f is differentiable
c. f' is continuous d. f is continuous

10. Let $g(x)=xf(x)$, where $f(x)=\begin{cases}x\sin\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$. At $x=0$ (IIT-JEE, 1994)
a. g is differentiable but g' is not continuous
b. g is differentiable while f is not
c. both f and g are differentiable
d. g is differentiable and g' is continuous

11. The function $f(x)=\max\{(1-x),(1+x),2\}, x\in(-\infty,\infty)$ is
a. continuous at all points
b. differentiable at all points
c. differentiable at all points except at $x=1$ and $x=-1$
d. continuous at all points except at $x=1$ and $x=-1$, where it is discontinuous

12. If $f(x)=\min\{1,x^2,x^3\}$, then (IIT-JEE, 2006)
a. $f(x)$ is continuous $\forall x\in R$
b. $f'(x)>0, \forall x>1$
c. $f(x)$ is continuous but not differentiable $\forall x\in R$
d. $f(x)$ is not differentiable at two points

13. If $f(x)=\begin{cases}-x-\frac{\pi}{2}, & x\leq-\frac{\pi}{2}\\ -\cos x, & -\frac{\pi}{2}<x\leq 0\\ x-1, & 0<x\leq 1\\ \ln x, & x>1\end{cases}$, then
a. $f(x)$ is continuous at $x=-\pi/2$
b. $f(x)$ is not differentiable at $x=0$
c. $f(x)$ is differentiable at $x=1$
d. $f(x)$ is differentiable at $x=-3/2$ (IITJEE 2011)

14. Let $f: R\rightarrow R$ be a function such that $f(x+y)=f(x)+f(y)$, $\forall x,y\in R$. If $f(x)$ is differentiable at $x=0$, then
a. $f(x)$ is differentiable only in a finite interval containing zero
b. $f(x)$ is continuous $\forall x\in R$
c. $f(x)$ is constant $\forall x\in R$
d. $f(x)$ is differentiable except at finitely many points (IITJEE 2011)

Match the following type

Match the functions in Column I with the properties in Column II

1. In the following, $[x]$ denotes the greatest integer less than or equal to x. (IIT-JEE, 2007)

Column I	Column II
a. $x\lvert x\rvert$	p. continuous in $(-1,1)$
b. $\sqrt{\lvert x\rvert}$	q. differentiable in $(-1,1)$
c. $x+[x]$	r. strictly increasing in $(-1,1)$
d. $\lvert x-1\rvert$	s. not differentiable at least at one point in $(-1,1)$

ANSWERS AND SOLUTIONS

Subjective Type

1. $f(x)=\begin{cases} x^2+ax+1, & x \text{ is rational} \\ ax^2+2x+b, & x \text{ is irrational} \end{cases}$

is continuous at $x = 1$ and 2
$\Rightarrow$ $x = 1$ and 2 are the roots of the equation $x^2 + ax + 1 = ax^2 + 2x + b$
or $(a-1)x^2 + (2-a)x + b - 1 = 0$

$$\Rightarrow \frac{a-2}{a-1}=3 \text{ and } \frac{b-1}{a-1}=2$$

$\Rightarrow$ $a = 1/2$ and $b = 0$

2. Let k be an integer
$f(k)=k, f(k-0)=k-1+1=k, f(k+0)=k+0=k$

$$\Rightarrow f'(k-0)=\lim_{h\to 0}\frac{f(k-h)-f(k)}{-h}$$

$$=\lim_{h\to 0}\frac{(k-1)+\sqrt{1-h}-k}{-h}$$

$$=\lim_{h\to 0}\frac{1-h-1}{-h(1+\sqrt{1-h})}=\frac{1}{2}$$

$$f'(k+0)=\lim_{h\to 0}\frac{f(k+h)-f(k)}{h}=\lim_{h\to 0}\frac{k+\sqrt{h}-k}{h}=+\infty$$

Thus, $f(x)$ is continuous for all x but non-differentiable at all integral values of x.

3. For $x \neq 0$

$$f(x)=\left(1-\frac{1}{1+x}\right)+\left(\frac{1}{1+x}-\frac{1}{1+2x}\right)+\left(\frac{1}{1+2x}-\frac{1}{1+3x}\right)$$

$$+\cdots+\left(\frac{1}{1+(n-1)x}-\frac{1}{1+nx}\right)=1-\frac{1}{1+nx}$$

$$\Rightarrow f(x)=\lim_{n\to\infty}\left(1-\frac{1}{1+nx}\right)=1-0=1 \text{ and for } x=0, f(0)=0$$

$$\Rightarrow f(x)=\begin{cases}1, & x\neq 0\\ 0, & x=0\end{cases}$$

Clearly, $f(x)$ is discontinuous at $x = 0$.

4. As $f(x)$ is continuous for all $x \in R$.

Thus, $\lim_{x\to\sqrt{3}} f(x)=f(\sqrt{3})$

where $f(x)=\dfrac{x^2-2x+2\sqrt{3}-3}{\sqrt{3}-x}, x\neq\sqrt{3}$

Now, $\lim_{x\to\sqrt{3}} f(x)=\lim_{x\to\sqrt{3}}\dfrac{x^2-2x+2\sqrt{3}-3}{\sqrt{3}-x}$

$$=\lim_{x\to\sqrt{3}}\frac{(2-\sqrt{3}-x)(\sqrt{3}-x)}{(\sqrt{3}-x)}$$

$$=2(1-\sqrt{3})$$

$$\Rightarrow f(\sqrt{3})=2(1-\sqrt{3})$$

5. When x is in a neighbourhood of $\pi/2$, $\sin x$ is very close to 1 but less than 1, then

$$\lim_{x\to\frac{\pi}{2}^+}\frac{2(\sin x-\sin^n x)+|\sin x-\sin^n x|}{2(\sin x-\sin^n x)-|\sin x-\sin^n x|}$$

$$=\lim_{x\to\frac{\pi}{2}^+}\frac{2(\sin x-\sin^n x)+(\sin x-\sin^n x)}{2(\sin x-\sin^n x)-(\sin x-\sin^n x)}=3 \text{ (exactly 3)}$$

Also, $\lim_{x\to\frac{\pi}{2}^-}\dfrac{2(\sin x-\sin^n x)+|\sin x-\sin^n x|}{2(\sin x-\sin^n x)-|\sin x-\sin^n x|}$

$$=\lim_{x\to\frac{\pi}{2}^+}\frac{2(\sin x-\sin^n x)+(\sin x-\sin^n x)}{2(\sin x-\sin^n x)-(\sin x-\sin^n x)}=3 \text{ (exactly 3)}$$

Then, $g(x)$ is continuous at $x = \pi/2$.

6. As $y = t^2 + t|t|$ and $x = 2t - |t|$.
Thus when $t \geq 0$
$\Rightarrow x = 2t - t = t, y = t^2 + t^2 = 2t^2$
$\therefore$ $x = t$ and $y = 2t^2$
$\Rightarrow y = 2x^2, \forall x \geq 0$
when $t < 0$
$\Rightarrow x = 2t + t = 3t$ and $y = t^2 - t^2 = 0$
$\Rightarrow y = 0$ for all $x < 0$

Hence, $f(x)=\begin{cases}2x^2, & x\geq 0\\ 0, & x<0\end{cases}$

which is clearly continuous for all x as shown graphically in Fig. 3.23.

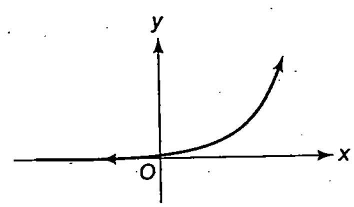


Fig. 3.24

Also $f'(x)=\begin{cases}4x, & x>0\\ 0, & x<0\end{cases}$

$\Rightarrow f'(0^+) = 0$ and $f'(0^-) = 0$
$\Rightarrow$ $f(x)$ is differentiable at $x = 0$.

7. Let $g(x) = f(x) - f(x + \pi)$ (1)
at $x = \pi$, $g(\pi) = f(\pi) - f(2\pi)$ (2)
at $x = 0$, $g(0) = f(0) - f(\pi)$ (3)
Adding equations (2) and (3), we get
$g(0) + g(\pi) = f(0) - f(2\pi)$
$\Rightarrow$ $g(0) + g(\pi) = 0$ [Given $f(0) = f(2\pi)$]
$\Rightarrow$ $g(0) = -g(\pi)$
$\Rightarrow$ $g(0)$ and $g(\pi)$ are opposite in sign.
$\Rightarrow$ There exists a point c between 0 and π such that $g(c) = 0$ as shown in the graph

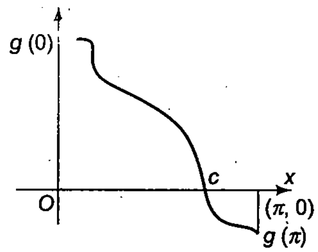


Fig. 3.25

From equation (1) putting $x = c$

$g(c) = f(c) - f(c + \pi) = 0$

Hence, $f(c) = f(c + \pi)$.

8. L.H.L. $= \lim_{x \to 0-} f(x)$

$= \lim_{h \to 0} f(0 - h)$

$= \lim_{h \to 0} (0 - h + 1)^{2 - \left(\frac{1}{|0-h|} + \frac{1}{(0-h)}\right)}$

$= \lim_{h \to 0} (1 - h)^{2 - \left(\frac{1}{h} - \frac{1}{h}\right)}$

$= \lim_{h \to 0} (1 - h)^2 = (1 - 0)^2 = 1$

R.H.L. $= \lim_{x \to 0+} f(x)$

$= \lim_{h \to 0} f(0 + h)$

$= \lim_{h \to 0} (h + 1)^{2 - \left(\frac{1}{|h|} + \frac{1}{h}\right)}$

$= \lim_{h \to 0} (h + 1)^{2 - \frac{2}{h}}$

$= \dfrac{\lim_{h \to 0} (h + 1)^2}{\lim_{h \to 0} (1 + h)^{2/h}} = \dfrac{1}{e^2} = e^{-2}$

Also $f(0) = 0$

$\Rightarrow$ L.H.L. = R.H.L. $\neq f(0)$

Hence, $f(x)$ is discontinuous at $x = 0$.

9.

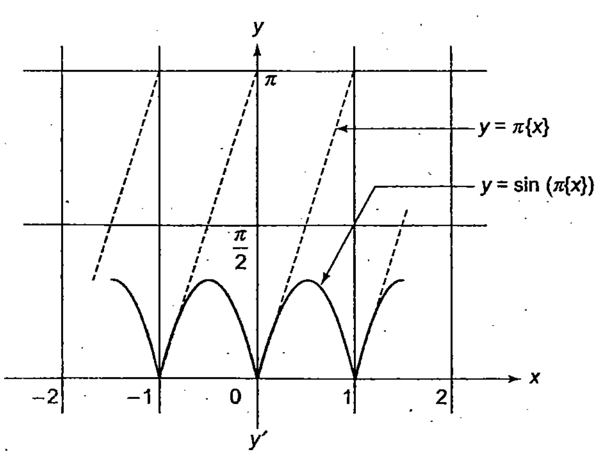


Fig. 3.26

From the graph, $f(x)$ is non-differentiable at $x = 0, \pm 1$.

10. $f(0^+) = \lim_{h \to 0} \sqrt{h}\left(1 + h \sin \frac{1}{h}\right)$

$= 0 \times [1 + 0 \times (\text{any value between } -1 \text{ and } 1)] = 0$

$f(0^-) = \lim_{h \to 0} \left[-\sqrt{-(-h)}\left(1 - h \sin\left(-\frac{1}{h}\right)\right)\right]$

$= \lim_{h \to 0} \left[-\sqrt{h}\left(1 + h \sin \frac{1}{h}\right)\right]$

$= -0 \times [1 + 0 \times (\text{any value between } -1 \text{ and } 1)] = 0$

$f'(0^+) = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h}$

$= \lim_{h \to 0} \dfrac{\sqrt{h}\left[1 + h \sin \frac{1}{h}\right] - 0}{h}$

$= \lim_{h \to 0} \dfrac{1 + h \sin \frac{1}{h}}{\sqrt{h}}$

$= \lim_{h \to 0} \left[\dfrac{1}{\sqrt{h}} + \sqrt{h} \sin \dfrac{1}{h}\right] = \infty + 0 = \infty$

Hence, $f(x)$ is non-differentiable at $x = 0$.

11. $\because f(x) = \min\{|x|, |x - 2|, 2 - |x - 1|\}$

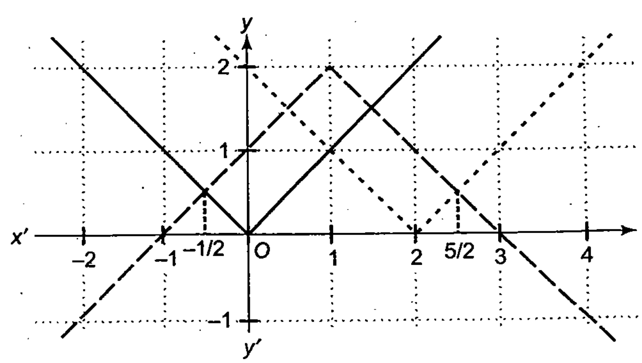
Draw the graphs of

$y = |x|$, $y = |x - 2|$ and $y = 2 - |x - 1|$

——— $y = |x|$

.... $y = |x - 2|$

– – – – $y = 2 - |x - 1|$



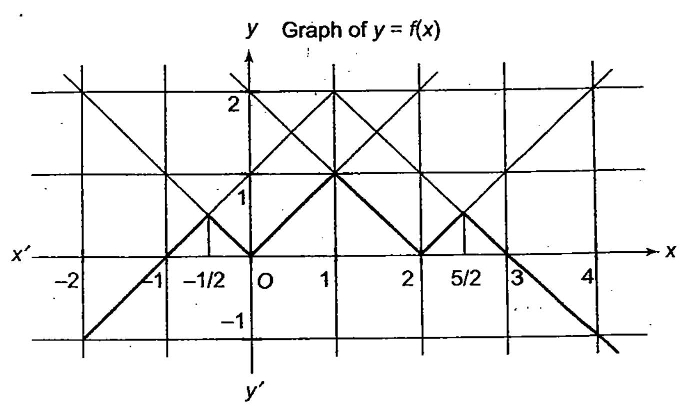


Fig. 3.27

It is clear from the graph,

$f(x) = \min\{|x|, |x-2|, 2-|x-1|\}$ is continuous $\forall x \in R$ and non-differentiable at $x = -\frac{1}{2}, 0, 1, 2, \frac{5}{2}$.

12. $f(x+y) = f(x) + f(y)$ and $f(x) = xg(x)$ for all $x, y \in R$, where $g(x)$ is continuous.

We have $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x)+f(h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)}{h}$$

$$= \lim_{h\to 0} \frac{hg(h)}{h} = \lim_{h\to 0} g(h) = g(0).$$

[$\because$ g is continuous at $x = 0$]

13. $f(|x|) = \begin{cases} |x|-3, & |x|<0 \\ |x|^2-3|x|+2, & |x|\ge 0 \end{cases}$

where $|x| < 0$ is not possible, thus neglecting, we get

$\Rightarrow f(|x|) = |x|^2 - 3|x| + 2, |x| \ge 0$

$$\Rightarrow f(|x|) = \begin{cases} x^2+3x+2, & x<0 \\ x^2-3x+2, & x\ge 0 \end{cases} \quad (1)$$

Also, $|f(x)| = \begin{cases} |x-3|, & x<0 \\ |x^2-3x+2|, & x\ge 0 \end{cases}$

$$= \begin{cases} (3-x), & x<0 \\ (x^2-3x+2), & 0\le x<1 \\ -(x^2-3x+2), & 1\le x<2 \\ (x^2-3x+2), & 2\le x \end{cases} \quad (2)$$

Now from equations (1) and (2), we get

$g(x) = f(|x|) + |f(x)|$

$$g(x) = \begin{cases} x^2+2x+5, & x<0 \\ 2x^2-6x+4, & 0\le x<1 \\ 0, & 1\le x<2 \\ 2x^2-6x+4, & x\ge 2 \end{cases}$$

$$\Rightarrow g'(x) = \begin{cases} 2x+2, & x<0 \\ 4x-6, & 0<x<1 \\ 0, & 1<x<2 \\ 4x-6, & x>2 \end{cases}$$

$\Rightarrow$ $g(x)$ is continuous in $R - \{0\}$ and $g(x)$ is differentiable in $R - \{0, 1, 2\}$.

14. $f(x) = \begin{cases} \sin\left(\frac{\pi x}{2}\right), & 0\le x<1 \\ |2x-3|[x], & 1\le x\le 2 \end{cases}$

$$= \begin{cases} \sin\left(\frac{\pi x}{2}\right), & 0\le x<1 \\ (3-2x)[x], & 1\le x<3/2 \\ (2x-3)[x], & \frac{3}{2}\le x\le 2 \end{cases}$$

$$= \begin{cases} \sin\left(\frac{\pi x}{2}\right), & 0\le x<1 \\ 3-2x, & 1\le x<3/2 \\ (2x-3), & \frac{3}{2}\le x<2 \\ 2, & x=2 \end{cases}$$

Graph of $y = f(x)$

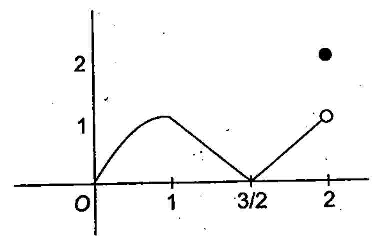


Fig. 3.28

From the graph it is clear that $f(x)$ is discontinuous at $x = 2$. Also, $f(x)$ is non-differentiable at $x = 1, 3/2, 2$.

15. Here, $f(x)$ is continuous at $x = 0$.

$\Rightarrow$ R.H.L. (at $x = 0$) = L.H.L. (at $x = 0$) = $f(0)$

R.H.L. (at $x=0$) = $\lim_{h\to 0} \frac{e^{1/h}+e^{2/h}+e^{3/h}}{ae^{2/h}+be^{3/h}}$ $\left\{\frac{\infty}{\infty} \text{ form}\right\}$

$$= \lim_{h\to 0} \frac{e^{3/h}\left\{\frac{1}{e^{2/h}}+\frac{1}{e^{1/h}}+1\right\}}{e^{3/h}\left\{\frac{a}{e^{1/h}}+b\right\}}$$

$$= \frac{1}{b} \quad (1)$$

again, L.H.L. (at $x = 0$)

$$= \lim_{h\to 0} (\cos h + \sin h)^{-\operatorname{cosec} h}$$

$$= \lim_{h\to 0} \{1 + (\cos h + \sin h - 1)\}^{\frac{-1}{\sin h}} \quad \{(1)^\infty \text{ form}\}$$

$$= e^{\lim_{h\to 0}\{\cos h+\sin h-1\}\left(-\frac{1}{\sin h}\right)}$$

$$= e^{\lim_{h\to 0}\{-2\sin^2 h/2+2\sin h/2\cos h/2\}\left\{-\frac{1}{2\sin h/2\cos h/2}\right\}}$$

$$= e^{\lim_{h\to 0}\frac{\sin h/2-\cos h/2}{\cos h/2}} = e^{-1}, \quad (2)$$

and $f(0) = a$

$\Rightarrow a = e^{-1} = \frac{1}{b}$ or $a = e^{-1}$ and $b = e$

16. $f(0^+) = \lim_{h\to 0} \frac{\sin h - \log(e^h \cos h)}{6h^2}$

$$= \lim_{h\to 0} \frac{\cos h - \frac{e^h(\cos h - \sin h)}{e^h \cos h}}{12h} \quad \text{(Using L'Hopital's rule)}$$

$$= \lim_{h\to 0} \frac{\cos h - (1 - \tan h)}{12h}$$

$= \lim_{h\to 0}\dfrac{-\sin h+\sec^2 h}{12} = \dfrac{1}{12}$ (Using L'Hopital's rule)

$f(0^-) = \lim_{h\to 0}\dfrac{h^2+2\cos h-2}{h^4}$ (Using expansion formula of cos h)

$= \lim_{h\to 0}\dfrac{h^2+2\left[1-\dfrac{h^2}{2!}+\dfrac{h^4}{4!}\right]-2}{h^4} = \dfrac{1}{12}$

$\Rightarrow f(x)$ is continuous at $x = 0$

$f'(0^+) = \lim_{h\to 0}\dfrac{f(h)-f(0)}{h}$

$= \lim_{h\to 0}\dfrac{\dfrac{\sin h-\log(e^h\cos h)}{6h^2}-\dfrac{1}{12}}{h}$

$= \lim_{h\to 0}\dfrac{2\sin h-2\log(e^h\cos h)-h^2}{12h^3}$

$\lim_{h\to 0}\dfrac{2\cos h-2(1-\tan h)-2h}{36h^2}$ (Using L' Hopital's rule)

$= \lim_{h\to 0}\dfrac{\cos h-(1-\tan h)-h}{18h^2}$

$= \lim_{h\to 0}\dfrac{-\sin h+\sec^2 h-1}{36h}$ (Using L'Hopital's rule)

$= \lim_{h\to 0}\dfrac{-\cos h+2\sec^2 h\tan h}{36} = -\dfrac{1}{36}$ (Using L'Hopital's rule)

$f'(0^-) = \lim_{h\to 0}\dfrac{f(-h)-f(0)}{-h} = \lim_{h\to 0}\dfrac{\dfrac{h^2+2\cos h-2}{h^4}-\dfrac{1}{12}}{-h}$

$= \lim_{h\to 0}\dfrac{12h^2+24\cos h-24-h^4}{-12h^5}$

$= \lim_{h\to 0}\dfrac{12h^2+24\left[1-\dfrac{h^2}{2!}+\dfrac{h^4}{4!}\right]-24-h^4}{-12h^5} = 0$

Hence, $f(x)$ is continuous but non-differentiable at $x = 0$.

17. At $x = 0$, R.H.L.

$= \lim_{h\to 0}\dfrac{e^{1/h}+e^{2/h}+e^{3/h}}{ae^{2/h}+be^{3/h}} = \lim_{h\to 0}\dfrac{e^{-2/h}+e^{-1/h}+1}{ae^{-1/h}+b} = \dfrac{1}{b}$

and L.H.L.

$= \lim_{h\to 0}\left(\dfrac{e^h+h^2-a}{h}\right)^{1/h}$

$= \lim_{h\to 0}\left(h+\dfrac{e^h-a}{h}\right)^{1/h} \Rightarrow a = 1$ (for 1^∞ for m)

$= e^{\lim_{h\to 0}\frac{1}{h}\left(h+\frac{e^h-1}{h}-1\right)} = e^{3/2}$ (using expansion of e^x)

$\Rightarrow f(0) = e^{3/2} = \dfrac{1}{b}$.

18. At $x = -2$,

$f(-2) = b$ (1)

R.H.L. $= \lim_{x\to -2+} f(x) = \lim_{h\to 0} f(-2+h)$

$= \lim_{h\to 0}\sin\left(\dfrac{(-2+h)^4-16}{(-2+h)^5+32}\right)$

$= \sin\left\{\lim_{h\to 0}\dfrac{(h-2)^4-2^4}{2^5+(-2+h)^5}\right\}$

$= \sin\left\{\lim_{h\to 0}\dfrac{(h-2)^4-(-2)^4}{(h-2)^5-(-2)^5}\right\}$

$= \sin\left\{\lim_{h\to 0}\dfrac{\dfrac{(h-2)^4-(-2)^4}{(h-2)-(-2)}}{\dfrac{(h-2)^5-(-2)^5}{(h-2)-(-2)}}\right\}$

$= \sin\left\{\dfrac{4(-2)^{4-1}}{5(-2)^{5-1}}\right\}$

$= \sin\left\{\dfrac{4(-8)}{5(16)}\right\} = \sin\left(-\dfrac{2}{5}\right)$ (2)

L.H.L. $= \lim_{x\to 2-} f(x) = \lim_{h\to 0} f(-2-h)$

$= \lim_{h\to 0}\dfrac{ae^{1/(|-2-h+2|)}-1}{2-e^{1/(|-2-h+2|)}}$

$= \lim_{h\to 0}\dfrac{ae^{1/h}-1}{2-e^{1/h}}$

$= \lim_{h\to 0}\dfrac{a-e^{-1/h}}{2e^{-1/h}-1} = \dfrac{a-0}{0-1} = -a$ (3)

From equations (1), (2), and (3), we get

$a = \sin\left(\dfrac{2}{5}\right)$ and $b = -\sin\left(\dfrac{2}{5}\right)$

19. Since $|f(x)| \le x^2, \forall x \in R$

$\therefore$ at $x = 0$, $|f(0)| \le 0$

$\Rightarrow f(0) = 0$ (1)

$\therefore f'(0) = \lim_{h\to 0}\dfrac{f(h)-f(0)}{h} = \lim_{h\to 0}\dfrac{f(h)}{h}$ (2)

Now, $\left|\dfrac{f(h)}{h}\right| \le |h|$ $(\because\ |f(x)| \le x^2)$

$\Rightarrow -|h| \le \dfrac{f(h)}{h} \le |h|$

$\Rightarrow \lim_{h\to 0}\dfrac{f(h)}{h} \to 0$ (3)

(Using Sandwich theorem)

$\therefore$ From equations (2) and (3), we get $f'(0) = 0$, i.e., $f(x)$ is differentiable at $x = 0$.

Objective Type

1.c. $f(x) = \tan x$ is discontinuous when $x = (2n+1)\pi/2, n \in Z$

$f(x) = x[x]$ is discontinuous when $x = k, k \in Z$

$f(x) = \sin[n\pi x]$ is discontinuous when $n\pi x = k, k \in Z$

Thus, all the above functions have infinite number of points of discontinuity.

But $f(x) = \dfrac{|x|}{x}$ is discontinuous when $x = 0$ only.

2.c. We have $f(x)=\dfrac{4-x^2}{x(4-x^2)}$

Clearly, there are three points of discontinuity, viz., 0, 2, –2.

3.b. $f(x)=\dfrac{\tan\left(\frac{\pi}{4}-x\right)}{\cot 2x}$, $(x\neq \pi/4)$ is continuous at $x=\pi/4$

$$\Rightarrow\ f\left(\frac{\pi}{4}\right)=\lim_{x\to\frac{\pi}{4}} f(x)$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\tan\left(\frac{\pi}{4}-x\right)}{\cot 2x}$$

Now by applying L'Hopital's rule,

$$=\lim_{x\to\frac{\pi}{4}}\frac{-\sec^2\left(\frac{\pi}{4}-x\right)}{-2\operatorname{cosec}^2(2x)}=\frac{1}{2}$$

4.b. Given $f(x)$ is continuous at $x=0$

$$\Rightarrow\ \lim_{x\to 0} f(x)=f(0)$$

$$\Rightarrow\ \lim_{x\to 0}\frac{(3^x-1)^2}{\sin x\ln(1+x)}=f(0)$$

$$\Rightarrow f(0)=\lim_{x\to 0}\frac{\left(\frac{3^x-1}{x}\right)^2}{\left(\frac{\sin x}{x}\right)\left(\frac{\ln(1+x)}{x}\right)}=(\ln 3)^2$$

5.b. We have $f(x)=\begin{cases}\dfrac{1-|x|}{1+x}, & x\neq -1\\ 1, & x=-1\end{cases}$

$$=\begin{cases}1, & x<0, \quad (\because\ f(-1)=1 \text{ is given})\\ \dfrac{1-x}{1+x}, & x\geq 0\end{cases}$$

$$\Rightarrow f([2x])=\begin{cases}1, & [2x]<0\\ \dfrac{1-[2x]}{1+[2x]}, & [2x]\geq 0\end{cases}$$

$$=\begin{cases}1, & x<0\\ 1, & 0\leq x<1/2\\ 0, & 1/2\leq x<1\\ -1/3, & 1\leq x<\frac{3}{2}\end{cases}$$

Clearly, $f(x)$ is continuous for all $x<\frac{1}{2}$ and discontinuous at $x=\frac{1}{2}, 1$.

6.d. We have,

$$\text{L.H.L.}=\lim_{x\to 4^-} f(x)$$

$$=\lim_{h\to 0} f(4-h)$$

$$=\lim_{h\to 0}\frac{4-h-4}{|4-h-4|}+a$$

$$=\lim_{h\to 0}\left(-\frac{h}{h}+a\right)=a-1$$

$$\text{R.H.L.}=\lim_{x\to 4^+} f(x)$$

$$=\lim_{h\to 0} f(4+h)$$

$$=\lim_{h\to 0}\frac{4+h-4}{|4+h-4|}+b=b+1$$

$\Rightarrow f(4)=a+b$

Since $f(x)$ is continuous at $x=4$, therefore

$$\lim_{x\to 4^-} f(x)=f(4)=\lim_{x\to 4^+} f(x)$$

$\Rightarrow a-1=a+b=b+1\Rightarrow b=-1$ and $a=1$.

7.d. $$\lim_{x\to 0}\frac{x-e^x+1-(1-\cos 2x)}{x^2}$$

$$=\lim_{x\to 0}\left[\frac{x-e^x+1}{x^2}-\frac{(1-\cos 2x)}{x^2}\right]$$

$$=\lim_{x\to 0}\left[\frac{x+1-\left(1+x+\frac{x^2}{2}\right)}{x^2}-\frac{2\sin^2 x}{x^2}\right]$$

(Using expansion of e^x)

$$=-\frac{1}{2}-2$$

$=-\frac{5}{2}$; hence for continuity $f(0)=-\frac{5}{2}$

Now $[f(0)]=-3$; $\{f(0)\}=\left\{-\frac{5}{2}\right\}=\frac{1}{2}$.

Hence, $[f(0)]\,\{f(0)\}=-\frac{3}{2}=-1.5$.

8.b. $f(x)$ is discontinuous at $x=1$ and $x=2$

$\Rightarrow f(f(x))$ may be discontinuous when $f(x)=1$ or 2

Now $1-x=1 \Rightarrow x=0$, where $f(x)$ is continuous

$x+2=1 \Rightarrow x=-1\notin(1,2)$

$4-x=1 \Rightarrow x=3\in[2,4]$

now $1-x=2 \Rightarrow x=-1\notin[0,1]$

$x+2=2 \Rightarrow x=0\notin(1,2]$

$4-x=2 \Rightarrow x=2\in[2,4]$

Hence $f(f(x))$ is discontinuous at $x=2, 3$

Note that it can be discontinuous at $x=1$ also.

9.b. The function f is clearly continuous at each point in its domain except possibly at $x=0$. Given that $f(x)$ is continuous at $x=0$.

Therefore, $f(0)=\lim_{x\to 0} f(x)$

$$=\lim_{x\to 0}\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x}$$

$$=\lim_{x\to 0}\frac{2-(\sin^{-1}x)/x}{2+(\tan^{-1}x)/x}=\frac{1}{3}$$

10.c. $$\lim_{x\to 2^+}\frac{(x-2)}{|x-2|}\left(\frac{x^2-1}{x^2+1}\right)=\lim_{x\to 2^+}\frac{(x-2)}{(x-2)}\left(\frac{x^2-1}{x^2+1}\right)$$

$$=\lim_{x\to 2^+}\left(\frac{x^2-1}{x^2+1}\right)=\frac{3}{5}$$

$$= \lim_{x\to 2^-} \frac{(x-2)}{|x-2|}\left(\frac{x^2-1}{x^2+1}\right)$$

$$= \lim_{x\to 2^-} \frac{(x-2)}{(2-x)}\left(\frac{x^2-1}{x^2+1}\right) = -\frac{3}{5}$$

Thus, L.H.L. $\neq$ R.H.L.

Hence, the function has non-removable discontinuity at $x = 2$.

11.c. $f(x) = \lim_{n\to\infty} \frac{[(x-1)^2]^n - 1}{[(x-1)^2]^n + 1}$

$$= \lim_{n\to\infty} \frac{1 - \frac{1}{[(x-1)^2]^n}}{1 + \frac{1}{[(x-1)^2]^n}}$$

$$= \begin{cases} -1, & 0 \le (x-1)^2 < 1 \\ 0, & (x-1)^2 = 1 \\ 1, & (x-1)^2 > 1 \end{cases}$$

$$= \begin{cases} 1, & x < 0 \\ 0, & x = 0 \\ -1, & 0 < x < 2 \\ 0, & x = 2 \\ 1, & x > 2 \end{cases}$$

Thus, $f(x)$ is discontinuous at $x = 0, 2$.

12.c. $f(0) = 0 + 0 + \lambda \ln 4 = \lambda \ln 4$ (1)

R.H.L. $= \lim_{x\to 0+} f(x) = \lim_{h\to 0} f(0+h)$

$$= \lim_{h\to 0} \frac{8^h - 4^h - 2^h + 1^h}{h^2}$$

$$= \lim_{h\to 0} \frac{(4^h-1)(2^h-1)}{h.h}$$

$$= \lim_{h\to 0}\left(\frac{4^h-1}{h}\right)\lim_{h\to 0}\left(\frac{2^h-1}{h}\right)$$

$= \ln 4 \ln 2$ (2)

$\therefore f(0) =$ R.H.L.

$\Rightarrow \lambda = \ln 2$

13.b. We must have $\lim_{x\to 0} \frac{a\cos x - \cos bx}{x^2} = 4$

$$\Rightarrow \lim_{x\to 0} \frac{a\left(1-\frac{x^2}{2!}\right) - \left(1-\frac{b^2x^2}{2!}\right)}{x^2} = 4$$

$$\Rightarrow \lim_{x\to 0}\left[\frac{(a-1)}{x^2} - \left(\frac{a}{2} - \frac{b^2}{2}\right)\right] = 4$$

$\Rightarrow a = 1$ and $\frac{a}{2} - \frac{b^2}{2} = -4$

$\Rightarrow a = 1$ and $b^2 = 9$

$\Rightarrow a = 1$ and $b = \pm 3$

14.a. $f(x) = \begin{cases} x+2, & x < 0 \\ -x^2-2, & 0 \le x < 1 \\ x, & x \ge 1 \end{cases}$

$$\therefore |f(x)| = \begin{cases} -x-2, & x < -2 \\ x+2, & -2 \le x < 0 \\ x^2+2, & 0 \le x < 1 \\ x, & x \ge 1 \end{cases}$$

discontinuous at $x = 1$ $\therefore$ number of points of discont. 1

15.a. $f(x)$ is continuous when $5x = x^2 + 6 \Rightarrow x = 2, 3$.

16.a. $f(x) = 2|\text{sgn}(2x)| + 2 = \begin{cases} 4, & x > 0 \\ 2, & x = 0 \\ 0, & x < 0 \end{cases}$

Thus, $f(x)$ has non-removable discontinuity at $x = 0$

17.d. Since $\lim_{n\to\infty} x^{2n} = \begin{cases} 0, & \text{if } |x| < 1 \\ 1, & \text{if } |x| = 1 \end{cases}$

$$\therefore f(x) = \lim_{x\to\infty} (\sin x)^{2n} = \begin{cases} 0, & \text{if } |\sin x| < 1 \\ 1 & \text{if } |\sin x| = 1 \end{cases}$$

Thus, $f(x)$ is continuous at all x, except for those values of x for which $|\sin x| = 1$, i.e., $x = (2k+1)\frac{\pi}{2}, k \in Z$

18.a. As f is continuous so $f(0) = \lim_{x\to 0} f(x)$

$\Rightarrow f(0) = \lim_{n\to\infty} f(1/4n)$

$$= \lim_{n\to\infty}\left((\sin e^n)\, e^{-n^2} + \frac{1}{1+1/n^2}\right) = 0 + 1 = 1.$$

19.a. $f(x) = \frac{x^2 - bx + 25}{x^2 - 7x + 10}, x \neq 5$

$f(x)$ is continuous at $x = 5$, only if $\lim_{x\to 5} \frac{x^2 - bx + 25}{x^2 - 7x + 10}$ is finite.

Now $x^2 - 7x + 10 \to 0$ when $x \to 5$.

Then we must have $x^2 - bx + 25 \to 0$ for which $b = 10$

Hence, $\lim_{x\to 5} \frac{x^2 - 10x + 25}{x^2 - 7x + 10} = \lim_{x\to 5} \frac{x-5}{x-2} = 0.$

20.d. Refer theory.

21.a. $f(x)$ is continuous at some x where $\sin x = \cos x$ or $\tan x = 1$ or $x = n\pi + \pi/4, n \in I$.

22.b. Consider $x \in [0, 1]$.

From the graph given in Fig. 3.28, it is clear that $[\cos \pi x]$ is discontinuous at

$x = 0, 1/2$ (1)

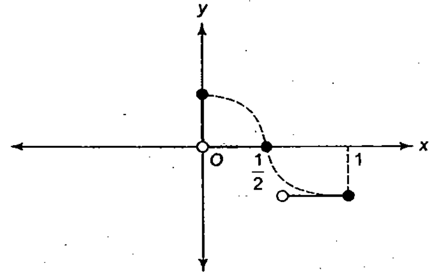


Fig. 3.29

Now consider $x \in (1, 2]$

$f(x) = [x-2]\,|2x-3|$

For $x \in (1, 2)$; $[x-2] = -1$ and for $x = 2$; $[x-2] = 0$

Also $|2x-3| = 0 \quad \Rightarrow x = 3/2$

$\Rightarrow$ $x = 3/2$ and 2 may be the points at which $f(x)$ is discontinuous (2)

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & 0 < x \le \dfrac{1}{2} \\ -1, & \dfrac{1}{2} < x \le 1 \\ -(3-2x), & 1 < x \le 3/2 \\ -(2x-3), & 3/2 < x \le 2 \\ 0, & x = 2 \end{cases}$$

Thus, $f(x)$ is continuous when $x \in [0, 2] - \{0, 1/2, 2\}$.

23.d. For $0 \le x < 1, f(x) = [\sin 0] = 0$, $\quad 1 \le x < 2, f(x) = [\sin 1] = 0$

$2 \le x < 3, f(x) = [\sin 2] = 0$, $\quad 3 \le x < 4, f(x) = [\sin 3] = 0$

$4 \le x < 5, f(x) = [\sin 4] = -1$

Hence, there is discontinuity at point $(4, -1)$

24.d. We have $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{h \to 0} \sin(\log_e |-h|) = \lim\limits_{h \to 0} \sin(\log_e h)$

which does not exist and oscillates between -1 and 1.

Similarly, $\lim\limits_{x \to 0^+} f(x)$ lies between -1 and 1.

25.a. $f(x) = (-1)^{[x^3]}$ is discontinuous

when $x^3 = n, n \in Z \Rightarrow x = n^{1/3}$

$f\left(\dfrac{3}{2}\right) = (-1)^3 = -1$

For $x \in (-1, 0)$, $f(x) = (-1)^{-1} = -1$

$\Rightarrow \ f'(x) = 0$

For $x \in [0, 1)$, $f(x) = (-1)^0 = 1$

$\Rightarrow \ f'(x) = 0$

26.c. $f(x) = \{x\}\sin(\pi[x])$

$= \{x\}\sin(\text{integral multiple of } \pi)$

$= 0$

Hence, $f(x)$ is continuous for all x.

27.d. We have $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{h \to 0} f(1-h)$

$$= \lim_{h \to 0} \frac{\log(4+h^2)}{\log(1-4h)} = -\infty$$

and, $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{h \to 0} f(1+h) = \lim\limits_{h \to 0} \dfrac{\log(4+h^2)}{\log(1+4h)} = \infty$

So, $f(1^-)$ and $f(1^+)$ do not exist.

28.c. Since, $\lim\limits_{x \to 1^-} g(x) = \lim\limits_{x \to 1^+} g(x) = 1$ and $g(1) = 0$.

So, $g(x)$ is not continuous at $x = 1$ but $\lim\limits_{x \to 1} g(x)$ exists.

We have $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{h \to 0} f(1-h) = \lim\limits_{h \to 0}[1-h] = 0$

and, $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{h \to 0} f(1+h) = \lim\limits_{h \to 0}[1+h] = 1$

So, $\lim\limits_{x \to 1} f(x)$ does not exist and so $f(x)$ is not continuous at $x = 1$.

We have $gof(x) = g(f(x)) = g([x]) = 0, \forall x \in R$

So, gof is continuous for all x.

We have $fog(x) = f(g(x))$

$$= \begin{cases} f(0), & x \in Z \\ f(x^2), & x \in R - Z \end{cases} = \begin{cases} 0, & x \in z \\ [x^2], & x \in R - Z \end{cases}$$

which is clearly not continuous.

29.c. $f(0+0) = \lim\limits_{h \to 0} f(h)$

$$= \lim_{h \to 0} \frac{h}{2h^2 + h} = \lim_{h \to 0} \frac{1}{2h+1} = 1$$

and $f(0-0) = \lim\limits_{h \to 0} f(-h) = \lim\limits_{h \to 0} \dfrac{-h}{2h^2 + |-h|}$

$$= \lim_{h \to 0} \frac{-h}{2h^2 + h} = \lim_{h \to 0} \frac{-1}{2h+1} = -1$$

30.b. We have

$$f(x) = \frac{x - |x-1|}{x} = \begin{cases} \dfrac{x + x - 1}{x}, & x < 1, x \ne 0 \\ \dfrac{x-(x-1)}{x}, & x \ge 1 \end{cases}$$

$$= \begin{cases} \dfrac{2x-1}{x}, & x < 1, x \ne 0 \\ \dfrac{1}{x}, & x \ge 1 \end{cases}$$

Clearly, $f(x)$ is discontinuous at $x = 0$ as it is not defined at $x = 0$. Since $f(x)$ is not defined at $x = 0$, therefore $f(x)$ cannot be differentiable at $x = 0$. Clearly $f(x)$ is continuous at $x = 1$, but it is not differentiable at $x = 1$, because $Lf'(1) = 1$ and $Rf'(1) = -1$.

31.a. We have $f(x) = \begin{cases} x^3, & x > 0 \\ 0, & x = 0 \\ -x^3, & x < 0 \end{cases}$

Clearly, $f(x)$ is continuous at $x = 0$

(L.H.D. at $x = 0$) $= \left[\dfrac{d}{dx}(-x^3)\right]_{x=0} = [-3x^2]_{x=0} = 0$

Similarly (R.H.D. at $x = 0$) $= 0$

So, $f(x)$ is differentiable at $x = 0$.

32.d.

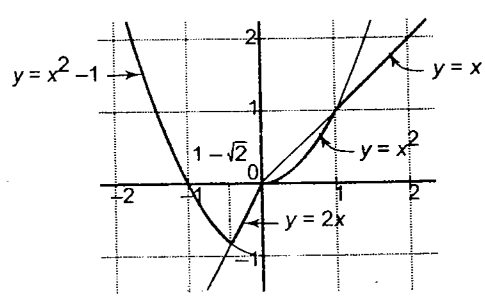


Fig. 3.30

From the graph it is clear that $f(x)$ is everywhere continuous but not differentiable at $x = 1 - \sqrt{2}$, 0, 1.

33.b. Since both $\cos x$ and $\sin^{-1}x$ are continuous function, $f(x) = \sin^{-1}(\cos x)$ is also a continuous function.
Now,

$$f'(x) = \frac{-\sin x}{\sqrt{1-\cos^2 x}} = \frac{-\sin x}{|\sin x|}$$

Hence, $f(x)$ is non-differentiable at $x = n\pi, n \in Z$.

34.d. $f(x) = (e^x - 1)|e^{2x} - 1|$
$= (e^x - 1)|e^x - 1||e^x + 1|$
$= (e^x + 1)(e^x - 1)|e^x - 1|$
Now, both $e^x + 1$ and $(e^x - 1)|e^x - 1|$ are differentiable [as $g(x)|g(x)|$ is differentiable when $g(x) = 0$]
Hence, $f(x)$ is differentiable.

$f(x) = \dfrac{x-1}{x^2+1}$ is rational function in which denominator never becomes zero.
Hence, $f(x)$ is differentiable.

$$f(x) = \begin{cases} ||x-3|-1|, & x<3 \\ \frac{x}{3}[x]-2, & x \geq 3 \end{cases}$$

$$= \begin{cases} |3-x-1|, & x<3 \\ \frac{x}{3}3-2, & 3 \leq x < 4 \end{cases}$$

$$= \begin{cases} |x-2|, & x<3 \\ x-2, & 3 \leq x < 4 \end{cases}$$

$= x-2, x \in [2, 4)$
Hence, $f(x)$ is differentiable at $x = 3$.

$$f(x) = 3(x-2)^{3/4} + 3 \Rightarrow f'(x) = \frac{9}{4}(x-2)^{-1/4}$$

which is non-differentiable at $x = 2$.
Here $f(x)$ is continuous and the graph has vertical tangent at $x = 2$; however, graph is smooth in neighbourhood of $x = 2$.

35.c. $\left|x - \frac{1}{2}\right|$ is continuous everywhere but not differentiable at $x = \frac{1}{2}$, $|x - 1|$ is continuous everywhere but not differentiable at $x = 1$, and $\tan x$ is continuous in $[0, 2]$ except at $x = \frac{\pi}{2}$.

Hence $f(x)$ is not differentiable at $x = \frac{1}{2}, 1, \frac{\pi}{2}$.

36.c. Let $f(x) = x^2|x|$ which could be expressed as

$$f(x) = \begin{cases} -x^3, & x<0 \\ 0, & x=0 \\ x^3, & x>0 \end{cases} \Rightarrow f'(x) = \begin{cases} -3x^2, & x<0 \\ 0, & x=0 \\ 3x^2, & x>0 \end{cases}$$

So, $f'(x)$ exists for all real x.

$$f''(x) = \begin{cases} -6x, & x<0 \\ 0, & x=0 \\ 6x, & x>0 \end{cases}$$

So, $f''(x)$ exists for all real x.

$$f'''(x) = \begin{cases} -6, & x<0 \\ 0, & x=0 \\ 6, & x>0 \end{cases}$$

However, $f'''(0)$ does not exist since $f'''(0^-) = -6$ and $f'''(0^+) = 6$ which are not equal. Thus, the set of points where $f(x)$ is thrice differentiable is $R - \{0\}$.

37.c $f(x) = (x^2 - 1)|(x-1)(x-2)|$
$f(x) = (x^2 - 1)|(x-1)(x-2)|$
$= (x+1)[(x-1|x-1|]|x-2|$
which is differentiable at $x = 1$
For $f(x) = \sin(|x-1|) - |x-1|$

$$f'(1^+) = \lim_{h\to 0} \frac{\sin h - h - 0}{h} = 0$$

$$f'(1^-) = \lim_{h\to 0} \frac{\sin|-h| - |-h|}{-h} = \lim_{h\to 0} \frac{\sin h - h}{-h} = 0$$

Hence, $f(x)$ is differentiable at $x = 1$.
For $f(x) = \tan(|x-1|) + |x-1|$

$$f'(1^+) = \lim_{h\to 0} \frac{\tan h + h - 0}{h} = 2$$

$$f'(1^-) = \lim_{h\to 0} \frac{\tan|-h| + |-h|}{-h} = \lim_{h\to 0} \frac{\tan h + h}{-h} = -2.$$

Hence, $f(x)$ is non-differentiable at $x = 1$.

38.d. Clearly $f(x)$ is continuous at $x = 0$ if $a = 0$

$$\text{Now } f'(0+0) = \lim_{h\to 0} \frac{he^{-\left(\frac{1}{h}+\frac{1}{h}\right)} - 0}{h}$$

$$= \lim_{h\to 0} \frac{he^{-2/h} - 0}{h} = 0$$

$$f'(0-0) = \lim_{h\to 0} \frac{-he^{-\left(-\frac{1}{h}+\frac{1}{h}\right)} - 0}{-h} = 1$$

Thus, no values of a exists.

39.c. $f(x) = \begin{cases} ax^2+1, & x \leq 1 \\ x^2+ax+b, & x>1 \end{cases}$ is differentiable at $x = 1$

Then $f(x)$ is continuous at $x = 1$
$\Rightarrow f(1^-) = f(1^+) \Rightarrow a + 1 = 1 + a + b \Rightarrow b = 0.$

$$\text{Also } f'(x) = \begin{cases} 2ax, & x<1 \\ 2x+a, & x>1 \end{cases}$$

We must have $f'(1^-) = f'(1^+) \Rightarrow 2a = 2 + a \Rightarrow a = 2.$

40.b. $|\sin x|$ and $e^{|x|}$ are not differentiable at $x = 0$ and $|x|^3$ is differentiable at $x = 0$.
Therefore, for $f(x)$ to be differentiable at $x = 0$, we must have $a = 0$, $b = 0$ and c can be any real number.

41.d.

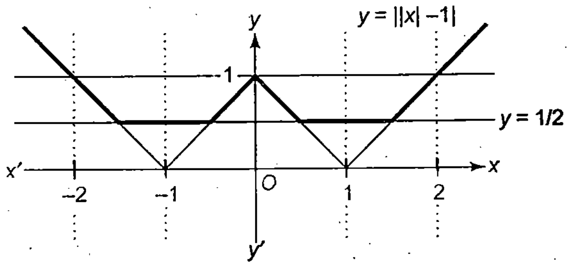


Fig. 3.31

Clearly from the graph, $f(x)$ is non-differentiable at five points, $x = -2, -1, 0, 1, 2$.

42.c. Clearly, $f(x)$ is continuous for all x except possibly at $x = \pi/6$.

For $f(x)$ to be continuous at $x = \pi/6$, we must have

$$\lim_{x\to\pi/6^-} f(x) = \lim_{x\to\pi/6^+} f(x)$$

$$\Rightarrow \lim_{x\to\pi/6} \sin 2x = \lim_{x\to\pi/6} ax + b$$

$$\Rightarrow \sin(\pi/3) = (\pi/6)a + b$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{\pi}{6}a + b \qquad (1)$$

For $f(x)$ to be differentiable at $x = \pi/6$, we must have L.H.D. at $x = \pi/6$ = R.H.D. at $x = \pi/6$

$$\Rightarrow \lim_{x\to\pi/6} 2\cos 2x = \lim_{x\to\pi/6} a$$

$$\Rightarrow 2\cos\pi/3 = a \Rightarrow a = 1$$

Putting $a = 1$ in equation (1), we get $b = (\sqrt{3}/2) - \pi/6$.

43.b. $f(x)$ is clearly continuous for $x \in R$.

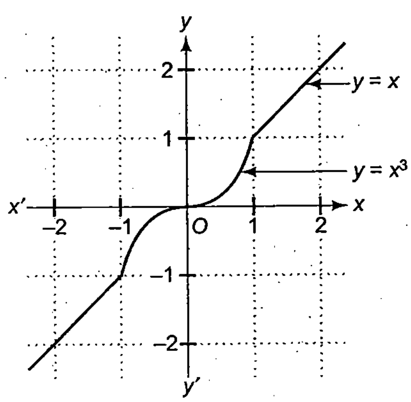


Fig. 3.32

$$f'(x) = \begin{cases} 3x^2, & x^2 < 1 \\ 1, & x^2 > 1 \end{cases}$$

thus $f(x)$ is non-differentiable at $x = 1, -1$.

44.d. $\dfrac{x}{1+|x|}$ is always differentiable (also at $x = 0$)

Also $(x-2)(x+2)\,|(x-1)(x-2)(x-3)|$

is not differentiable at $x = 1, 3$.

So, $f(x)$ is not differentiable at $x = 1, 3$.

45.b. $f(x) = \cos\pi(|x| + [x])$

$$= \begin{cases} \cos\pi(-x+(-1)), & -1 \le x < 0 \\ \cos\pi(x+0), & 0 \le x < 1 \end{cases}$$

$$= \begin{cases} -\cos\pi x, & -1 \le x < 0 \\ \cos\pi x, & 0 \le x < 1 \end{cases}$$

Obviously, $f(x)$ is discontinuous at $x = 0$, otherwise $f(x)$ is continuous and differentiable in $(-1, 0)$ and $(0, 1)$.

46.c. For $f(x)$ to be continuous at $x = 0$, we have

$$f(0^-) = f(0^+) \Rightarrow a(0) + b = 1 \quad \Rightarrow \quad b = 1$$

$$f'(0^+) = \lim_{h\to 0}\frac{f(h) - f(0)}{h} = \lim_{h\to 0}\frac{e^{h^2+h} - b}{h}$$

$$= \lim_{h\to 0}\frac{e^{h^2+h} - 1}{h} = \lim_{h\to 0}\frac{e^{h^2+h} - 1}{h(h+1)}(h+1) = 1$$

$\therefore\ f'(0^-) = a$

Hence, $a = 1$

47.a. Clearly $f(x)$ is continuous at $x = 0$

$$\text{Now } f'(0^+) = \lim_{h\to 0}\frac{e^{-1/h^2} - 0}{h} = \lim_{h\to 0}\frac{1/h}{e^{1/h^2}}$$

$$= \lim_{h\to 0}\frac{-1/h^2}{-2/h^3\, e^{1/h^2}} \quad \text{(applying L'Hopital's rule)}$$

$$= \frac{1}{2}\lim_{h\to 0}\frac{h}{e^{1/h^2}} = 0$$

Also $f(0^-) = 0$

Thus, $f(x)$ is differentiable at $x = 0$.

48.c. $f(x) = \begin{cases} |x| - 1, & |x| < 0 \\ |x|^2 - 2|x|, & |x| \ge 0 \end{cases}$

where $|x| < 0$ is not possible thus, neglecting we get,

$f(|x|) = |x|^2 - 2|x|, |x| \ge 0$

$$f(|x|) = \begin{cases} x^2 + 2x, & x < 0 \\ x^2 - 2x, & x \ge 0 \end{cases} \qquad (1)$$

$$\Rightarrow \quad f'(|x|) = \begin{cases} 2x + 2, & x < 0 \\ 2x - 2, & x > 0 \end{cases}$$

Clearly $f(|x|)$ is continuous at $x = 0$, but non-differentiable at $x = 0$.

$$f(|x|) = \begin{cases} |x| - 1, & |x| < 0 \\ |x|^2 - 2|x|, & |x| \ge 0 \end{cases}$$

$$g(x) = |f(x)| = \begin{cases} 1 - x, & x < 0 \\ -x^2 + 2x, & 0 \le x < 2 \\ x^2 - 2x, & x \ge 2 \end{cases} \qquad (2)$$

Clearly $|f(x)|$ is discontinuous at $x = 0$, but continuous at $x = 2$

Also, $g'(x)=\begin{cases}-1, & x<0\\ -2x+2, & 0<x<2\\ 2x-2, & x>2\end{cases}$

$|f(x)|$ is non-differentiable at $x=0$ and $x=2$.

49.c. Since $1\le x<2 \Rightarrow 0\le x-1<1$

$\Rightarrow\ [x^2-2x]=[(x-1)^2-1]=[(x-1)^2]-1=0-1=-1$

$\therefore\ f(x)=\begin{cases}1-4x^2, & 0\le x<\frac{1}{2}\\ 4x^2-1, & \frac{1}{2}\le x<1\\ -1 & 1\le x<2\end{cases}$

$\therefore$ graph of $f(x)$:

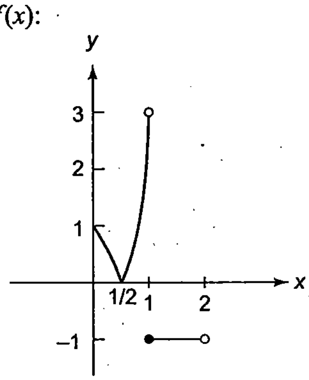


Fig. 3.33

It is clear from graph that $f(x)$ is discontinuous at $x=1$ and not differentiable at $x=\frac{1}{2}$ and $x=1$.

50.c. For $|x|<1, x^{2n}\to 0$ as $n\to\infty$ and for $|x|>1$, $1/x^{2n}\to 0$ as $n\to\infty$. So

$$f(x)=\begin{cases}\log(2+x), & |x<1|\\ \lim\limits_{n\to\infty}\dfrac{x^{-2n}\log(2+x)-\sin x}{x^{-2n}+1}=-\sin x, & \text{if } |x|>1\\ \dfrac{1}{2}[\log(2+x)-\sin x], & |x|=1\end{cases}$$

Thus, $\lim\limits_{x\to 1+} f(x)=\lim\limits_{x\to 1}(-\sin x)=-\sin 1$

and $\lim\limits_{x\to 1-} f(x)=\lim\limits_{x\to 1}\log(2+x)=\log 3.$

51.c. $f'(0)=\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h}$

$$=\lim_{h\to 0}\frac{h^a\sin\left(\frac{1}{h}\right)}{h}=\lim_{h\to 0}h^{a-1}\sin\left(\frac{1}{h}\right)$$

This limit will not exist if $a-1\le 0\Rightarrow a\le 1$.

Now $\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}x^a\sin\left(\frac{1}{x}\right)=0$ if $a>0$.

Thus, $a\in(0,1]$.

52.c. $[\sin x]$ is non-differentiable at $x=\frac{\pi}{2}, \pi, 2\pi$

and $[\cos x]$ is non-differentiable at $x=0, \frac{\pi}{2}; \frac{3\pi}{2}, 2\pi$.

Thus, $f(x)$ is definitely non-differentiable at $x=\pi, \frac{3\pi}{2}, 0$

Also, $f\left(\frac{\pi}{2}\right)=1,\ f\left(\frac{\pi}{2}-0\right)=0$

$f(2\pi)=1, f(2\pi-0)=-1$

Thus, $f(x)$ is also non-differentiable at $x=\frac{\pi}{2}$ and 2π.

53.a We have $x+4|y|=6y$

$\Rightarrow\begin{cases}x-4y=6y, & \text{if } y<0\\ x+4y=6y, & \text{if } y\ge 0\end{cases}$

$\Rightarrow y=\begin{cases}\frac{1}{2}x, & \text{if } x\ge 0\\ \frac{1}{10}x, & \text{if } x<0\end{cases}\Rightarrow f'(x)=\begin{cases}\frac{1}{2}, & x>0\\ \frac{1}{10}, & x<0\end{cases}$

Clearly, $f(x)$ is continuous at $x=0$ but non-differentiable at $x=0$.

54.b. $f(0^+)=\lim\limits_{x\to 0+}|x|^{\sin x}=e^{\lim\limits_{x\to 0}\sin x\log|x|}$

$=e^{\lim\limits_{x\to 0}\frac{\log x}{\operatorname{cosec} x}}=e^0=1$ (Using L' Hopital's rule)

$f(0^-)=g(0)=1$

Let $g(x)=ax+b$

$\Rightarrow b=1\Rightarrow g(x)=ax+1$

For $x>0, f'(x)=e^{\sin x\ln(|x|)}\left[\cos x\ln(|x|)+\dfrac{\sin x}{x}\right]$

$f'(1)=1[0+\sin 1]=\sin 1$

$f(-1)=-a+1\Rightarrow a=1-\sin 1$

$\Rightarrow\ g(x)=(1-\sin 1)x+1$

55.c. Given that $f(x)=|1-x|$

$\Rightarrow f(|x|)=\begin{cases}x-1, & x>1\\ 1-x, & 0<x\le 1\\ 1+x, & -1\le x\le 0\\ -x-1, & x<-1.\end{cases}$

Clearly, the domain of $\sin^{-1}(f|x|)$ is $[-2,2]$.

$\Rightarrow$ It is non-differentiable at the points $\{-1, 0, 1\}$.

56.d $f(x)$ is continuous at $x=0\Rightarrow\lim\limits_{x\to 0}f(x)=f(0)$

$$\Rightarrow f(0)=\lim_{x\to 0}f(0+h)=\lim_{h\to 0}\frac{hg(h)}{|h|}=\lim_{h\to 0}g(h)=g(0)=0$$

Now $f'(0^+)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{\frac{hg(h)}{|h|}}{h}$

$$=\lim_{h\to 0}\frac{g(h)}{h}=\lim_{h\to 0}\frac{g(h)-g(0)}{h}$$

$=g'(0)$ (as $g(0)=0$) $=0$

$f'(0^-)=\lim\limits_{h\to 0}\dfrac{f(0-h)-f(0)}{-h}$

$$=\lim_{h\to 0}\frac{\frac{-hg(-h)}{|-h|}}{-h}=\lim_{h\to 0}\frac{g(-h)}{h}$$

$$= -\lim_{h\to 0}\frac{g(-h)-g(0)}{-h} = -g'(0) = 0$$

Hence, $f'(0)$ exists and $f'(0) = 0$.

57.a.

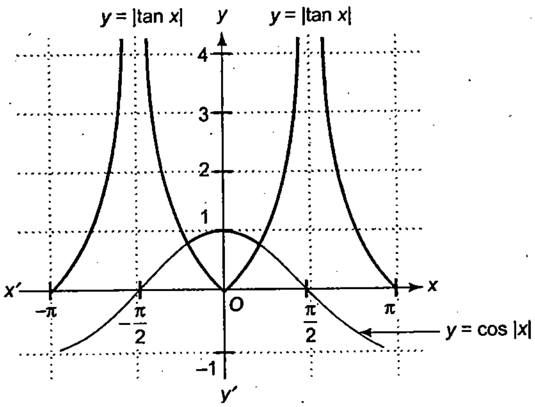


Fig. 3.34

The functions is not differentiable and continuous at two points between $x = -\pi/2$ and $x = \pi/2$. Also the function is not continuous at $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$. Hence, at four points, the function is not differentiable.

58.c. $f(|x|) = \begin{cases} \sin|x|, & |x| < 0 \\ \cos(x) - ||x|-1|, & |x| \geq 0 \end{cases}$

$\Rightarrow f(|x|) = \cos(x) - ||x|-1|, x \in R$

[as $|x| < 0$ is not possible and $|x| \geq 0$ is true $\forall x \in R$]

which is non-differentiable at $x = 0$ and when $|x| - 1 = 0$ or $x = \pm 1$.

Hence, $f(|x|)$ has exactly three points of non-differentiability.

59.d. $f(2^+) = 2 + 2\sin(0) = 2$

$f(2^-) = 3 + 2\sin(1)$

Hence, $f(x)$ is discontinuous at $x = 2$.

Also $f(0^+) = 2(0) - 0 - 0\sin(0-0) = 0$

and $f(0^-) = 2(0) - (-1) - 0\sin(0-(-1)) = 1$

Hence, $f(x)$ is discontinuous at $x = 0$.

60.b. $f(x) = \max\left\{\frac{x}{n}, |\sin \pi x|\right\}$

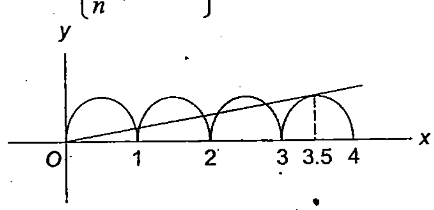


Fig. 3.35

Thus, for the maximum points of non-differentiability, graphs of $y = \frac{x}{n}$ and $y = |\sin \pi x|$ must intersect at maximum number of points which occurs when $n > 3.5$.

Hence, the least value of n is 4.

61.d. $f(x) = [x^2] - \{x\}^2$

$f(-1) = 1, f(-1^-) = 1 - 1 = 0$

$f(1) = 1, f(1^+) = 1 - 0 = 1$

$f(1^-) = 0 - 1 = -1$

Thus, $f(x)$ is discontinuous at $x = 1, -1$.

62.a. $f(e) = [\log_e e] + \sqrt{\{\log_e e\}} = [1] + \sqrt{\{1\}} = 1 + 0 = 1$

$f(e^+) = [\log_e e^+] + \sqrt{\{\log_e e^+\}}$

$= \lim_{h\to 0}[1+h] + \sqrt{\{1+h\}} = 1 + 0 = 1$

$f(e^-) = [\log_e e^-] + \sqrt{\{\log_e e^-\}}$

$= \lim_{h\to 0}[1-h] + \sqrt{\{1-h\}} = 0 + 1 = 1$

Hence, $f(x)$ is continuous at $x = e$.

Now $f'(e^+) = \lim_{h\to 0}\frac{f(e+h) - f(e)}{h}$

$= \lim_{h\to 0}\frac{[1+h] + \sqrt{\{1+h\}} - 1}{h}$

$= \lim_{h\to 0}\frac{1 + \sqrt{h} - 1}{h} = \lim_{h\to 0}\frac{1}{\sqrt{h}} \to \infty.$

Hence, $f(x)$ is non-differentiable at $x = 0$.

63.a. $f(x) = \lim_{n\to\infty}\left(\sin^2(\pi x)\right)^n + \left[x + \frac{1}{2}\right]$

Now $g(x) = \lim_{n\to\infty}\left(\sin^2(\pi x)\right)^n$ is discontinuous when

$\sin^2(\pi x) = 1$ or $\pi x = (2n+1)\frac{\pi}{2}$ or $x = \frac{(2n+1)}{2}, n \in z$

Thus, $g(x)$ is discontinuous at $x = 3/2$.

Also $h(x) = \left[x + \frac{1}{2}\right]$ is discontinuous at $x = 3/2$.

But $f(3/2) = \lim_{n\to\infty}\left(\sin^2(3\pi/2)\right)^n + \left[\frac{3}{2} + \frac{1}{2}\right] = 1 + 2 = 3.$

$f(3/2^+) = \lim_{n\to\infty}\left(\sin^2((3\pi/2)^+)\right)^n + \left[\left(\frac{3}{2}\right)^+ + \frac{1}{2}\right] = 0 + 2 = 2.$

Hence, $f(x)$ is discontinuous at $x = 3/2$.

Both $g(x)$ and $h(x)$ are continuous at $x = 1$, hence $f(x)$ is continuous at $x = 1$.

64.c. $f(x) = \text{sgn}(\sin^2 x - \sin x - 1)$ is discontinuous when $\sin^2 x - \sin x - 1 = 0$

or $\sin x = \frac{1 \pm \sqrt{5}}{2}$ or $\sin x = \frac{1 - \sqrt{5}}{2}$.

For exactly four point of discontinuity, n can take value 4 or 5 as shown in the diagram

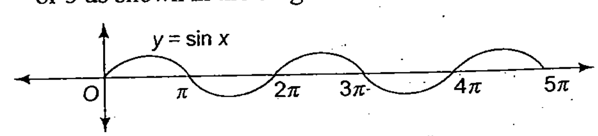


Fig. 3.36

65.c. $f(x)=\begin{cases} x^2-ax+3, & x \text{ is rational} \\ 2-x, & x \text{ is irrational} \end{cases}$

is continuous when $x^2-ax+3=2-x$ or

$x^2-(a-1)x+1=0$

which must have two distinct roots for $(a-1)^2-4>0$

$\Rightarrow (a-1-2)(a-1+2)>0$

$\Rightarrow a \in (-\infty,-1)\cup(3,\infty)$

66.a. Hence check continuity at $x=k, k \in Z$

For positive integers.

$f(k)=\{k\}^2-\{k^2\}=0$

$f(k^+)=\{k^+\}^2-\{(k^+)^2\}=0-0$

$f(k^-)=\{k^-\}^2-\{(k^-)^2\}=1-1=0$

For negative integers,

$f(k)=\{k\}^2-\{k^2\}=0$

$f(k^+)=\{k^+\}^2-\{(k^+)^2\}=0-1=-1$

$f(k^-)=\{k^-\}^2-\{(k^-)^2\}=1-0=1$

Hence, $f(x)$ is continuous at positive integers and discontinuous at negative integers.

67.b. $g(x)$ is an even function, then $g(x)=g(-x)$

$\Rightarrow g'(x)=-g'(-x)\Rightarrow g'(0)=-g'(0)\Rightarrow g'(0)=0$

Now $f'(0)=\lim\limits_{h\to 0}\dfrac{g(0+h)\cos(1/h)-0}{h}$

$=\lim\limits_{h\to 0}\dfrac{g(h)\cos(1/h)}{h}=\lim\limits_{h\to 0}g'(0)\cos(1/h)=0$

68.b. $f(1)=1-\sqrt{1-1^2}=1$

$f(1^-)=\lim\limits_{x\to 1^+}\left(1-\sqrt{1-x^2}\right)=1$

$f(1^+)=\lim\limits_{x\to 1^-}\left(1+\log\dfrac{1}{x}\right)=1+\log\dfrac{1}{1}=1$

Hence, $f(x)$ is continuous at $x=1$

$f'(1^+)=\lim\limits_{h\to 0}\dfrac{f(1+h)-f(1)}{h}$

$=\lim\limits_{h\to 0}\dfrac{1+\log\dfrac{1}{1+h}-1}{h}$

$=-\lim\limits_{h\to 0}\dfrac{\log(1+h)}{h}=-1$

$f'(1^-)=\lim\limits_{h\to 0}\dfrac{f(1-h)-f(1)}{-h}$

$=\lim\limits_{h\to 0}\dfrac{1-\sqrt{1-(1-h)^2}-1}{-h}=\lim\limits_{h\to 0}\dfrac{\sqrt{2-h}}{\sqrt{h}}=\infty$

Hence, $f(x)$ is non-differentiable at $x=1$.

69.b. We have $f(x)=\sqrt{1-\sqrt{1-x^2}}$.

The domain of definition of $f(x)$ is $[-1, 1]$.

For $x\neq 0, x\neq\pm 1$, we have

$f'(x)=\dfrac{1}{\sqrt{1-\sqrt{1-x^2}}}\times\dfrac{x}{\sqrt{1-x^2}}$

Since $f(x)$ is not defined on the right side of $x=1$ and on the left side of $x=-1$.

Also, $f'(x)\to\infty$ when $x\to -1^+$ or $x\to 1^-$.

So, we check the differentiability at $x=0$.

Now, L.H.D. at $x=0$

$=\lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x-0}$

$=\lim\limits_{h\to 0}\dfrac{f(0-h)-f(0)}{-h}$

$=\lim\limits_{h\to 0}\dfrac{\sqrt{1-\sqrt{1-h^2}}-0}{-h}$

$=-\lim\limits_{h\to 0}\dfrac{\sqrt{1-(1-(1/2)h^2+(3/8)h^4+\cdots)}}{h}$

$=-\lim\limits_{h\to 0}\sqrt{\dfrac{1}{2}-\dfrac{3}{8}h^2+\cdots}=-\dfrac{1}{\sqrt{2}}$

Similarly, R.H.D. at $x=0$ is $\dfrac{1}{\sqrt{2}}$.

Hence, $f(x)$ is not differentiable at $x=0$.

70.d. $f(x)=\sqrt[3]{|x|^3}-|x|-1$

$\Rightarrow |x|-|x|-1=-1$

Hence, differentiable for all x.

71.b. $g'(0^+)=\lim\limits_{h\to 0}\dfrac{f(|h|)-|\sin h|-f(0)}{h}$

$=\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h}-\lim\limits_{h\to 0}\dfrac{\sin h}{h}$

$=1-1=0$

$g'(0^-)=\lim\limits_{h\to 0}\dfrac{f(|-h|)-|\sin(-h)|-f(0)}{-h}$

$=\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{-h}+\lim\limits_{h\to 0}\dfrac{\sin h}{h}$

$=-1+1=0$

Thus $g(x)$ is differentiable and $g'(0)=0$.

72.c. $f'(0^+)=\lim\limits_{h\to 0}\dfrac{h^m\sin\dfrac{1}{h}}{h}$ must exist $\Rightarrow m>1$

for $m>1$, $h'(x)=\begin{cases} m\,x^{m-1}\sin\dfrac{1}{x}-x^{m-2}\cos\dfrac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$

Now $\lim\limits_{h\to 0}h(x)=\lim\limits_{h\to 0}\left(m\,h^{m-1}\sin\dfrac{1}{h}-h^{m-2}\cos\dfrac{1}{h}\right)$

limit exists if $m>2$

$\therefore\ m\in N\Rightarrow m=3$

73.c. At $x=0$,

L.H.L. $=\lim\limits_{x\to 0-}f(x)=\lim\limits_{h\to 0}f(0-h)$

$$= \lim_{h\to 0} h^2\left(\frac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}}\right)$$

$$= \lim_{h\to 0} h^2\left(\frac{e^{-2/h}-1}{e^{-2/h}+1}\right)$$

$$=0\left(\frac{0-1}{0+1}\right)=0$$

$$\text{R.H.L.} = \lim_{x\to 0+} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} h^2\left(\frac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}\right)$$

$$= \lim_{h\to 0} h^2\left(\frac{1-e^{-2/h}}{1+e^{-2/h}}\right)$$

$$= 0\left(\frac{1-0}{1+0}\right)=0$$

and $f(0)=0$

$\Rightarrow$ L.H.L. = R.H.L. = $f(0)$

Hence, $f(x)$ is continuous at $x=0$.

$$\text{Also L.H.D.} = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{h^2\dfrac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}}-0}{-h}$$

$$= -\lim_{h\to 0} h\frac{e^{-2/h}-1}{e^{-2/h}+1}=0.$$

$$\text{And R.H.D.} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h^2\dfrac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}-0}{-h}$$

$$= -\lim_{h\to 0} h\frac{1-e^{-2/h}}{1+e^{-2/h}}=0.$$

Hence, $f(x)$ is differentiable at $x=0$ and $f'(0)=0$,

74. b $f(2)=0$,

$f(2^+)=\{4^+\}-\{2^+\}^2=0-0=0$

$f(2^-)=\{4^-\}-\{2^-\}^2=1-1=0$

Hence $f(x)$ is continuous at $x=2$

$f(-2)=0$,

$f(-2^+)=\{4^-\}-\{-2^+\}^2=1-0=1$

Hence $f(x)$ is discontinuous at $x=-2$

75. c Obviously $\lim_{x\to 0+} e^{-1/x^2} = \lim_{x\to 0-} e^{-1/x^2} = 0$,

hence $f(x)$ is continuous at $x=0$

$$f'(0) = \lim_{h\to 0}\frac{e^{-1/h^2}}{h} = \lim_{h\to 0}\frac{1/h}{e^{1/h^2}}$$

$$= \lim_{h\to 0}\frac{-1/h^2}{e^{1/h^2}\dfrac{2}{h^3}} = \lim_{h\to 0}\frac{2h^3}{h^2 e^{1/h^2}}=0$$

Hence f is differentiable at $x=0$. Also $\lim_{x\to\pm\infty} e^{-\frac{1}{x^2}} \to 1$

76. c $f(2+x)=f(-x)$ (1)

Replace x by $x-1$, we have $f(2+x-1)=f(-x+1)$ or $f(1+x)=f(1-x)$

Hence $f(x)$ is symmetrical about line $x=1$

Now put $x=2$ in (1), we get $f(4)=f(-2)$, hence differentiability at $x=4$ implies differentiability at $x\to 2$

77. a $\lim_{x\to 0+}\left(3-\left[\cot^{-1}\frac{2x^3-3}{x^2}\right]\right)=(3-[\cot^{-1}(-\infty)])=(3-[\pi])$

$$\lim_{x\to 0-}\{x^2\}\cos(e^{1/x}) = \left(\lim_{x\to 0-}\{x^2\}\right)\left(\lim_{x\to 0-}\cos(e^{1/x})\right)$$

$$= (0)(\cos(e^{-\infty})) = 0$$

Thus $f(x)$ has irremovable discontinuity at $x=0$, hence $f(0)$ does not exist.

78. c

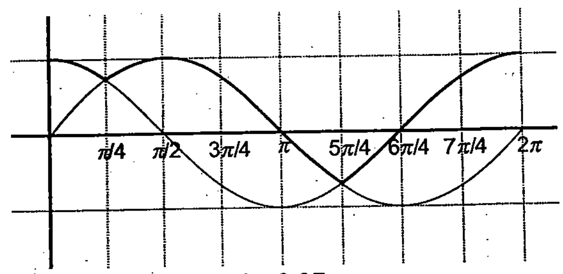


Fig. 3.37

Consider the graph of $f(x)=\max(\sin x, \cos x)$, which is non-differentiable at $x=\pi/4$, hence statement (a) is false

From the graph $y=f(x)$ is differentiable at $x=\pi/2$, hence statement (b) is false.

Statement (c) is always true.

Statement (d) is false as consider $g(x)=\max(x, x^2)$ at $x=0$, for which $x=x^2$ at $x=0$, but $f(x)$ is differentiable at $x=0$

79. b
$$f(x)=\begin{cases}1+\left[\cos\dfrac{\pi x}{2}\right], & 1<x\le 2\\ 1-\{x\}, & 0\le x<1\\ |\sin \pi x|, & -1\le x<0\end{cases} = \begin{cases}1-1, & 1<x\le 2\\ 1-x, & 0\le x<1\\ -\sin \pi x, & -1\le x<0\end{cases}$$

$f(x)$ is continuous at $x=1$ but not differentiable

80. a $x^2+2x+3+\sin \pi x=(x+1)^2+2+\sin \pi x>1$

$\therefore f(x)=1 \quad \forall\ x\in R$

81. c Given that $\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cdots\cos\frac{x}{2^n}=\dfrac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$ (1)

Taking logarithm to the base 'e' on both sides of equation (1) and then differentiating w.r.t. x, we get

$$\sum_{n=1}^{n}\frac{1}{2^n}\tan\frac{x}{2^n}=\left(\frac{1}{2^n}\cot\frac{x}{2^n}-\cot x\right)$$

$$\therefore \lim_{n\to\infty}\sum_{n=1}^{n}\frac{1}{2^n}\tan\frac{x}{2^n}=\lim_{n\to\infty}\left(\frac{1}{x}\times\frac{\frac{x}{2^n}}{\tan\frac{x}{2^n}}-\cot x\right)=\left(\frac{1}{x}-\cot x\right)$$

$$\therefore \quad \text{We have } f(x)=\begin{cases}\frac{1}{x}-\cot x, & x\in(0,\pi)-\left\{\frac{\pi}{2}\right\}\\ \frac{2}{\pi}, & x=\frac{\pi}{2}\end{cases}$$

Clearly $\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\left(\frac{1}{x}-\cot x\right)=\frac{2}{\pi}=f\left(\frac{\pi}{2}\right)$

Hence $f(x)$ is continuous at $x=\frac{\pi}{2}$.

Multiple Correct Answers Type

1. **a, b, c, d.**
   a, b, and **c** are false. Refer to definitions.
   for **d,** f must be continuous $\Rightarrow$ False
2. **a, c, d.**
   a is wrong as continuity is a must for $f(x)$.
   b is the correct form of intermediate value theorem.

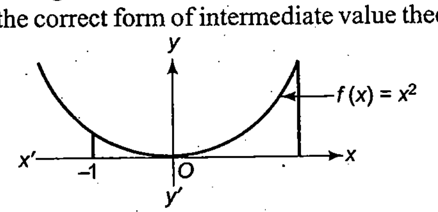


Fig. 3.38

c, as per the graph (in Fig. 3.34), is incorrect.

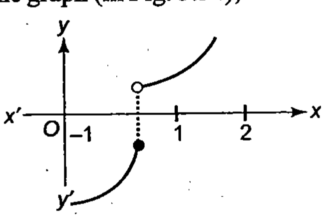


Fig. 3.39

d is wrong if f is discontinuous.

3. **a, c, d.**

$$f(x)=\frac{x^2-2x-8}{x+2}=\frac{(x+2)(x-4)}{x+2}=x-4,\ x\neq-2$$

Hence $f(x)$ has removable discontinuity at $x=-2$.
Similarly $f(x)$ in options (c) and (d) has also removable discontinuity.

$$f(x)=\frac{x-7}{|x-7|}=\begin{cases}-1, & x<7\\ 1, & x>7\end{cases}$$

Hence $f(x)$ has non-removable discontinuity at $x=7$.

4. **a, b.**
   $f(1^-)=1;\ f(1^+)=1;\ f(1)=1$
   $f'(1^-)=5;\ f'(1^+)=5$
   $f(2^+)=10,\ f(2^-)=10$
   $f'(2^+)=3;\ f'(2^-)=13.$
5. **a, b.**
   $f(x)=\text{sgn}(x)\sin x$
   $f(0^+)=\text{sgn}(0^+)\sin(0^+)=1\times(0)=0$
   $f(0^-)=\text{sgn}(0^-)\sin(0^-)=(-1)\times(0)=0$
   Also $f(0)=0$
   Hence, $f(x)$ is continuous everywhere.
   Both sgn(x) and $\sin(x)$ are odd functions.
   Hence, $f(x)$ is an even function.
   Obviously, $f(x)$ is non-periodic.

$$\text{Now } f'(0^+)=\lim_{h\to0}\frac{f(h)-f(0)}{h}$$
$$=\lim_{h\to0}\frac{\text{sgn}(h)\sin h-0}{h}=\lim_{h\to0}\frac{\sin h}{h}=1$$
$$\text{and } f'(0^+)=\lim_{h\to0}\frac{\text{sgn}(-h)\sin(-h)-0}{-h}$$
$$=\lim_{h\to0}\frac{-1\times(-\sin h)}{-h}=-1$$

   Hence, $f(x)$ is non-differentiable at $x=0$.
6. **a, b, c, d.**
   Given function is discontinuous when $a+\sin\pi x=1$
   Now if $a=1\Rightarrow\sin\pi x=0\Rightarrow x=1,2,3,4,5$
   If $a=3\Rightarrow\sin\pi x=-2$ not possible.
   If $a=0.5\Rightarrow\sin\pi x=0.5$
   $\Rightarrow x$ has 6 values, 2 each for one cycle of period 2.
   If $a=0\Rightarrow\sin\pi x=+1\Rightarrow x=\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2},\frac{9}{2},\frac{11}{2}$
   Hence, all the options are correct.
7. **a, b.**
   For maximum points of discontinuity of
   $f(x)=\text{sgn}(x^2-ax+1)$,
   $x^2-ax+1=0$ must have two distinct roots,
   for which $D=a^2-4>0$
   $\Rightarrow a\in(-\infty,-2)\cup(2,\infty)$.
8. **b, d.**

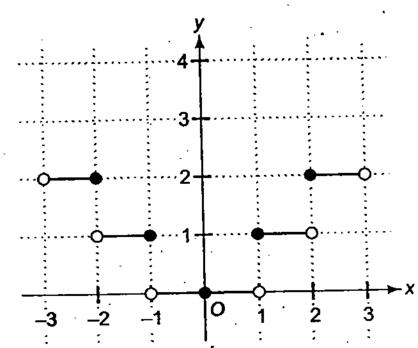


Fig. 3.40

9. **b, c.**
   Option (a) is wrong as $f(x)=\sin x$ and $g(x)=|x|$,
   $g(x)$ is non-differentiable at $x=0$, but $f(x)\ g(x)$ is differentiable at $x=0$.
10. **b, c.**

$$f(0^-)=\lim_{n\to\infty}\left[\lim_{x\to0^-}\left(\cos^2 x\right)^n\right]$$
$$=(\text{a value lesser than }1)^\infty=0$$

$f(0^+)= \lim_{n\to\infty}\left[\lim_{x\to 0^+}\left(1+x^n\right)^{1/n}\right]=1$

Also $f(0) = 1 \Rightarrow$ discontinuous at $x = 0$

Further, $f(1^-)=1; f(1^+)=0; f(1)=1$

$\Rightarrow$ discontinuous at $x = 1$.

11. b,d.

a. $\lim_{x\to 1^+}\frac{1}{\ln|x|}=\infty$ and $\lim_{x\to 1^-}\frac{1}{\ln|x|}=-\infty$,

hence $f(x)$ has non-removable discontinuity.

b. $\lim_{x\to 1}\frac{x^2-1}{x^3-1}=\frac{2}{3}$

$\therefore$ $f(x)$ has removable discontinuity at $x = 1$

c. $\lim_{x\to 1^+}\left(2^{-2^{\frac{1}{1-x}}}\right)=1$ and $\lim_{x\to 1^-}\left(2^{-2^{\frac{1}{1-x}}}\right)=0$.

Hence, the limit does not exist.

d. $\lim_{x\to 1}\frac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}=\frac{-1}{2\sqrt{2}}$ (Rationalizing)

$\therefore$ $f(x)$ has removable discontinuity at $x = 1$.

12. a,b,d.

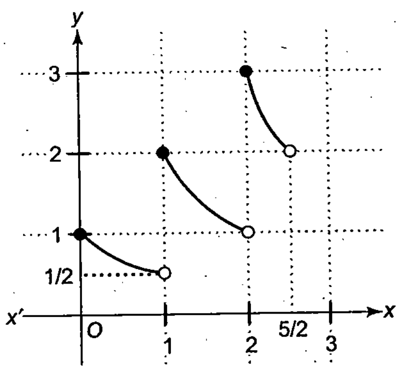


Fig. 3.41

$$f(x)\begin{cases}\frac{1}{x+1}, & 0\le x<1\\ \frac{2}{x}, & 1\le x<2\\ \frac{3}{x-1}, & 2\le x<\frac{5}{2}\end{cases}$$

Clearly, $f(x)$ is discontinuous and bijective function

$\lim_{x\to 1^-} f(x)=\frac{1}{2},\ \lim_{x\to 1^+} f(x)=2$

$\min\left(\lim_{x\to 1^-} f(x), \lim_{x\to 1^+} f(x)\right)=\frac{1}{2}\ne f(1)$

$\max(1,2)=2=f(1)$.

13. a, c.

$$f(x)=\begin{cases}1, & |x|\ge 1\\ \frac{1}{n^2}, & \frac{1}{n}<|x|<\frac{1}{n-1},\ n=2,3,\ldots\\ 0, & x=0\end{cases}$$

$$=\begin{cases}1, & x\le \text{ or } x\ge 1\\ \frac{1}{4}, & x\in\left(-1,-\frac{1}{2}\right)\cup\left(\frac{1}{2},1\right)\\ \frac{1}{9}, & x\in\left(\frac{-1}{2},\frac{-1}{3}\right)\cup\left(\frac{1}{3},\frac{1}{2}\right)\\ \vdots\end{cases}$$

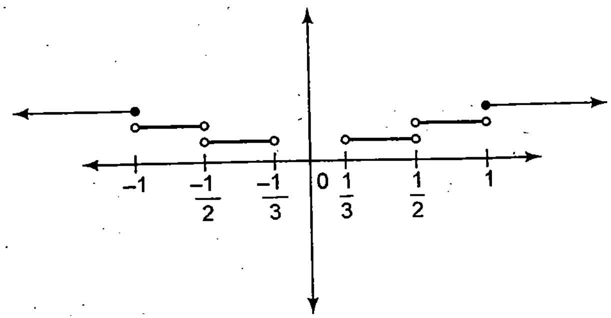


Fig. 3.42

The function f is clearly continuous for $|x| > 1$.

We observe that

$\lim_{x\to -1+} f(x)=1,\ \lim_{x\to -1^-} f(x)=\frac{1}{4}$

Also, $\lim_{x\to \frac{1+}{n}} f(x)=\frac{1}{n^2}$ and $\lim_{x\to \frac{1-}{n}} f(x)=\frac{1}{(n+1)^2}$

Thus f is discontinuous for $x=\pm\frac{1}{n},\ n=1, 2, 3, \ldots$

Hence **a** and **c** are the correct answers.

14. a, b, c.

Since, $\lim_{x\to 1^-} g(x)=\lim_{x\to 1^+} g(x)=1$ and $g(1)=0$.

So, $g(x)$ is not continuous at $x = 1$ but $\lim_{x\to 1} g(x)$ exists.

We have $\lim_{x\to 1^-} f(x)=\lim_{h\to 0} f(1-h)=\lim_{h\to 0}[1-h]=0$

and $\lim_{x\to 1^+} f(x)=\lim_{h\to 0} f(1+h)=\lim_{h\to 0}[1+h]=1$

So, $\lim_{x\to 1} f(x)$ does not exist and hence $f(x)$ is not continuous at $x = 1$

We have $gof(x)=g(f(x))=g([x])=0, \forall x, \in R$

So, gof is continuous for all x.

We have $fog(x)=f(g(x))=\begin{cases}f(0), & x\in Z\\ f(x^2), & x\in R-Z\end{cases}$

$=\begin{cases}0, & x\in Z\\ [x^2], & x\in R-Z\end{cases}$

which is clearly not continuous.

15. b,d.

We have $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{\log\cos x}{\log(1+x^2)}$

$=\lim_{x\to 0}\frac{\log(1-1+\cos x)}{\log(1+x^2)}\frac{1-\cos x}{1-\cos x}$

$$= \lim_{x\to 0} \frac{\log\{1-(1-\cos x)\}}{1-\cos x} \cdot \frac{1-\cos x}{\log(1+x^2)}$$

$$= -\lim_{x\to 0} \frac{\log[1-(1-\cos x)]}{-(1-\cos x)} \cdot \frac{2\sin^2 \frac{x}{2}}{4\left(\frac{x}{2}\right)^2} \cdot \frac{x^2}{\log(1+x^2)} = -\frac{1}{2}$$

Hence, $f(x)$ is differentiable at $x = 0$.

Hence, **b** and **d** are the correct answers.

16. a, c.

$f(x) = x + |x| + \cos 9x$, $g(x) = \sin x$

Since both $f(x)$ and $g(x)$ are continuous everywhere,

$f(x) + g(x)$ is also continuous everywhere

$f(x)$ is non-differentiable and $x = 0$.

Hence $f(x) + g(x)$ is non-differentiable at $x = 0$

Now $h(x) = f(x) \times g(x)$

$$= \begin{cases} (\cos 9x)(\sin x), & x < 0 \\ (2x + \cos 9x)(\sin x), & x \ge 0 \end{cases}$$

Clearly, $h(x)$ is continuous at $x = 0$

Also

$$h'(x) = \begin{cases} \cos x \cos 9x - 9\sin x \sin 9x, & x < 0 \\ (2 - 9\sin 9x)\sin x + \cos x(2x + \cos 9x), & x > 0 \end{cases}$$

$h'(0^-) = 1, h'(0^+) = 1$

$\Rightarrow f(x) \times g(x)$ is differentiable everywhere.

17. a, c.

$$f(x) = \begin{cases} (\sin^{-1} x)^2 \cos\left(\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (\sin^{-1} x)^2 \cos\left(\frac{1}{x}\right)$$

$= 0 \times$ (any value between -1 to 1) $= 0$

Hence $f(x)$ is continuous at $x = 0$

$$f'(0^+) = \lim_{h\to 0} \frac{(\sin^{-1} h)^2 \cos\left(\frac{1}{h}\right) - 0}{h}$$

$$= \left(\lim_{h\to 0} \frac{\sin^{-1} h}{h}\right)\left(\lim_{h\to 0} \sin^{-1} h\right)\left(\lim_{h\to 0} \cos\left(\frac{1}{h}\right)\right)$$

$= 1 \times (0) \times$ (any value between -1 to 1) $= 0$

Similarly, $f'(0^-) = 0$.

Hence, $f(x)$ is continuous and differentiable in $[-1, 1]$ and $(-1, 1)$, respectively.

18. a, b.

For $b = 1$, we have $f(g(0)) = f(\sin(0) + 1) = f(1) = 1 + a$

Also $f(g(0^+)) = \lim_{x\to 0^+} f(\sin x + 1) = f(1) = 1 + a$

and $f(g(0^-)) = \lim_{x\to 0^-} f(\{x\}) = f(1^-) = 1 + a$

Hence, $f(g(x))$ is continuous for $b = 1$

For $b < 0$,

$f(g(0)) = f(\sin(0) + b) = f(b) = 2 - b$

$f(g(0^+)) = \lim_{x\to 0^+} f(\sin x + b) = f(b) = 2 - b$

and $f(g(0^-)) = \lim_{x\to 0^-} f(\{x\}) = f(1) = 1 + a$

For continuity at $x = 0$, we must have $2 - b = 1 + a$ or $a + b = 1$.

19. a, b

$f(x)$ is continuous for all x if it is continuous at $x = 1$ for which $|1| - 3 = |1 - 2| + a$ or $a = -3$

$g(x)$ is continuous for all x if it is continuous at $x = 2$ for which $2 - |2| = \text{sgn}(2) - b = 1 - b$ or $b = 1$

Thus, $f(x) + g(x)$ is continuous for all x if $a = -3$, $b = 1$.

Hence, $f(x)$ is discontinuous at exactly one point for options **a** and **b**.

20. a, c, d.

a is not correct as $f(x) = x$ from R to R is onto but its reciprocal function $g(x) = \frac{1}{x}$ is not onto on R.

b is obviously true.

Also $g(x)$ is not continuous, hence not differentiable though $f(x)$ is continuous and differentiable in the above case.

21. a, c, d.

For continuity at $x = 1$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2 \text{sgn}[x] + \{x\}) = 1 + 0 = 1$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2 \text{sgn}[x] + \{x\}) = 1 \text{ sgn}(0) + 1 = 1$$

Also, $f(1) = 1$

$\therefore$ L.H.L. = R.H.L $= f(1)$. Hence, $f(x)$ is continuous at $x = 1$.

Now for differentiability,

$$f'(1^+) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)^2 \text{ sgn}[1+h] + \{1+h\} - 1}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)^2 + h - 1}{h} = \lim_{h\to 0} \frac{h^2 + 3h}{h} = 3$$

$$\text{and } f'(1^-) = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{(1-h)^2 \text{ sgn}[1-h] + \{1-h\} - 1}{-h}$$

$$= \lim_{h\to 0} \frac{(1-h)^2 + 1 - h - 1}{-h}$$

$$= \lim_{h\to 0} \frac{h^2 - 3h}{-h} = 3$$

$f'(1^+) = f'(1^-)$

Hence, $f(x)$ is differentiable at $x = 1$.

Now at $x = 2$,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 \text{sgn}[x] + \{x\}) = 4 \times 0 + 1$$

$$\lim_{h\to 2^+} f(x) = \lim_{h\to 2^+} (\sin x + |x - 3|) = 1 + \sin 2$$

Hence, L.H.L $\ne$ R.H.L.

Hence, $f(x)$ is discontinuous at $x = 2$ and then $f(x)$ is also non-differentiable at $x = 2$.

22. a, c.

$$f\left(\frac{\pi^-}{2}\right)=\lim_{h\to0}\left(\frac{3}{2}\right)^{\cot\left(3\left(\frac{\pi}{2}-h\right)\right)/\cot\left(2\left(\frac{\pi}{2}-h\right)\right)}$$

$$=\lim_{h\to0}\left(\frac{3}{2}\right)^{\frac{\tan 3h}{-\cot 2h}}$$

$$=\lim_{h\to0}\left(\frac{3}{2}\right)^{-(\tan 3h)(\tan 2h)}=1$$

$$f\left(\frac{\pi^+}{2}\right)=\lim_{h\to0}\left[1+\left|\cot\left(\frac{\pi}{2}+h\right)\right|\right]^{\left[a\left|\tan\left(\frac{\pi}{2}+h\right)\right|\right]/b}$$

$$=\lim_{h\to0}(1+\tan h)^{\frac{a\cot h}{b}}$$

$$=e^{\lim_{h\to0}(1+\tan h-1)\frac{a\cot h}{b}}=e^{a/b}$$

Also $f\left(\frac{\pi}{2}\right)=b+3$

$f(x)$ is continuous at $x=\pi/2$

$\Rightarrow 1=b+3=e^{a/b}\Rightarrow b=-2$ and $a=0$.

23. b, c, d

$$f(x)=|x^3|=\begin{cases}-x^3, & x<0\\ x^3, & x\ge0\end{cases}\Rightarrow f'''(x)\begin{cases}-6, & x<0\\ 6, & x>0\end{cases}.$$

Hence $f'''(0)$ does not exist

$$f(x)=x^3|x|=\begin{cases}-x^4, & x<0\\ x^4, & x\ge0\end{cases}\Rightarrow f'''(x)=\begin{cases}-24x, & x<0\\ 24x, & x>0\end{cases}.$$

Hence $f'''(0)=0$ and exists.

Similarly for $f(x)=|x|\sin^3x$ and $f(x)=x|\tan^3x|$, also $f'''(0)=0$ and exists.

24. a, b

$\sin^4x\in(0,1)$ for $x\in(-\pi/2,\pi/2)$,

$\Rightarrow f(x)=0$ for $x\in(-\pi/2,\pi/2)$

Hence $f(x)$ is continuous and differentiable at $x=0$

25. b, d

$f(x)=\text{sgn}(\cos 2x-2\sin x+3)$

$=\text{sgn}(1-2\sin^2x-2\sin x+3)$

$=\text{sgn}(-2\sin^2x-2\sin x+4)$

$f(x)$ is discontinuous when $-2\sin^2x-2\sin x+4=0$ or $\sin^2x+\sin x-2=0$

or $(\sin x-1)(\sin x+2)=0$ or $\sin x=1$

Hence $f(x)$ is discontinuous.

26. a, c, d

Differentiating w.r.t. x, keeping y as constant, we get

$f'(x+y)=f'(x)+2xy+y^2$

Now put $x=0$

$f'(y)=f'(0)+y^2=y^2-1$

$\therefore f'(x)=x^2-1$

$\therefore f(x)=\frac{x^3}{3}-x+c$

Also $f(0+0)=f(0)+f(0)+0 \therefore f(0)=0$

$\therefore f(x)=\frac{x^3}{3}-x$, $f(x)$ is twice differentiable for all $x\in R$ and

$f'(3)=3^2-1=8$

27. a, b, c, d

a. $\lim_{x\to0^+}f(x)=\lim_{x\to0^+}\frac{e^x+a}{2x}=\frac{1}{2}\Rightarrow a=-1$

If $a=-1$, then $\lim_{x\to0^+}f(x)=\frac{1}{2}$, $\lim_{x\to0^-}f(x)=\frac{1}{2}$

$\therefore f(x)$ is continuous at $x=0$ if $b=\frac{1}{2}$

c. If $a\ne-1$, then $\lim_{x\to0}\frac{e^x+a}{2x}$ does not exist

$\therefore x=0$ is a point of irremovable type of discontinuity

d. if $a=-1$, then $\lim_{x\to0}f(x)=\frac{1}{2}$

$\therefore b\ne\frac{1}{2}\Rightarrow$ removable type of discontinuity at $x=0$

Reasoning Type

1.c. Statement 1 is obviously true.
But statement 2 is false as $f(x)=x^3$ is differentiable, but $f^{-1}(x)=x^{1/3}$ is non-differentiable at $x=0$.
$f^{-1}(x)=x^{1/3}$ has vertical tangent at $x=0$.

2.b. $f(x)=(2x-5)^{3/5}\Rightarrow f'(x)=\frac{3}{5(2x-5)^{2/5}}$

Statement 2 as it is fundamental concept for non-differentiability.
But given function is non-differentiable at $x=5/2$, as it has vertical tangent at $x=5/2$, but not due to sharp turn.
The graph of the function is smooth in the neighbourhood of $x=5/2$.

3.a. Statement 2 is true as it is a fundamental concept.
Also $f(x)=\text{sgn}(g(x))$ is discontinuous when $g(x)=0$.
Now the given function $f(x)=\text{sgn}(x^2-2x+3)$ may be discontinuous when $x^2-2x+3=0$, which is not possible: it has imaginary roots as its discriminant is <0.

4.b. $f(x)=\lim_{n\to\infty}\frac{x^{2n}-1}{x^{2n}+1}$ is discontinuous at $x=1$

$$=\begin{cases}-1, & x^2<1\\ 1, & x^2>1\\ 0, & x^2=1\end{cases}$$

$\Rightarrow f(1^+)=1$ and $f(1^{-1})=-1$

Hence, $f(x)$ is discontinuous at $x=1$ as the limit of the function does not exist.

5.c. We know that both $[\sin x]$ and $[\cos x]$ are discontinuous at $x=\pi/2$.
Also $f(x)=[\sin x]-[\cos x]$ is discontinuous at $x=\pi/2$.
As $f(\pi/2)=1-0=1$ and $f(\pi/2^+)=0-(-1)=1$
$f(\pi/2^-)=0-0=0$.
But the difference of two discontinuous function is not necessarily discontinuous.

6.c. We know that sgn(x) is discontinuous at $x = 0$.

Also $f(x) = |\text{sgn}x| = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$ which is discontinuous at $x = 0$.

Consider $g(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$. Here $g(x)$ is discontinuous at $x = 0$ but $|g(x)| = 1$ for all x is continuous at $x = 0$.

Hence, answer is **c**.

7.b. $f(x) = (\sin\pi x)(x-1)^{1/5}$ is continuous function as both $(\sin\pi x)$ and $(x-1)^{1/5}$ are continuous.

But $(x-1)^{1/5}$ is not differentiable at $x = 1$.

$$\text{However, } f'(1^-) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$
$$= \lim_{h\to 0}\frac{\sin[\pi(1-h)](1-h-1)^{1/5}-0}{-h}$$
$$= \lim_{h\to 0}\frac{\sin(\pi h)(-h)^{1/5}}{h} = 0$$
$$\text{And } f'(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0}\frac{\sin[\pi(1+h)](1+h-1)^{1/5}-0}{h}$$
$$= \lim_{h\to 0}\frac{-\sin(\pi h)(h)^{1/5}}{h} = 0$$

Hence, $f(x)$ is differentiable at $x = 1$, though $(x-1)^{1/5}$ is not differentiable at $x = 1$.

However, statement 2 is correct but it is not a correct explanation of statement 1.

8.b. Statement 2 is true as $\cos 0 = 1$

$$\text{Now } \lim_{x\to 0+}\frac{e^{1/x}-1}{e^{1/x}+1} = \lim_{h\to 0}\frac{e^{1/h}-1}{e^{1/h}+1} = \lim_{h\to 0}\frac{1-e^{-1/h}}{1+e^{-1/h}} = 1$$
$$\text{and } \lim_{x\to 0-}\frac{e^{1/x}-1}{e^{1/x}+1} = \lim_{h\to 0}\frac{e^{-1/h}-1}{e^{-1/h}+1} = -1$$

Thus L.H.L. $\neq$ R.H.L.

Hence, the function has non-removable discontinuity at $x = 0$.

Hence, statement 2 is not a correct explanation of statement 1.

9.a. $\lim_{x\to 0^+}(\sin x + [x]) = 0$, $\lim_{x\to 0^-}(\sin x + [x]) = -1$

Thus, limit does not exist, hence $f(x)$ is discontinuous at $x = 0$.

Statement 2 is a fundamental property and is a correct explanation of statement 1.

10.d. $f(x) = |x|\sin x$

$$\text{L.H.D.} = \lim_{h\to 0}\frac{|0-h|\sin(0-h)-0}{-h} = \lim_{h\to 0}\frac{-h\sin h}{-h} = 0$$
$$\text{R.H.D} = \lim_{h\to 0}\frac{|0+h|\sin(0+h)-0}{h} = \lim_{h\to 0}\frac{h\sin h}{h} = 0$$

$\Rightarrow f(x)$ is differentiable at $x = 0$.

11.d. Statement 1 is incorrect because if $\lim_{x\to a} g(x)$ and $\lim_{x\to a} f(g(x))$ approach e from the same side of e (say right side), and $\lim_{x\to e} f(x) = f(e) \neq \lim_{x\to e} f(x)$, then $\lim_{x\to a} f(g(x)) = f(e^+) = f(e)$.

Statement 2 is correct.

12.c. Consider $f(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0 \end{cases}$

Hence $|f(x)| = 1$ for all x is continuous at $x = 0$ but $f(x)$ is discontinuous at $x = 0$.

13.b. Statement 2 is obviously true.

But $f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ is non-differentiable at $x = \pm 1$ as $\frac{2x}{1-x^2}$ is not defined at $x = \pm 1$. Hence statement 1 is true but statement 2 is not the correct explanation of statement 1.

14.b. $|f(x)| \leq |x|$

$\Rightarrow 0 \leq |f(x)| \leq |x|$

$\Rightarrow$ Graph of $y = |f(x)|$ lies between the graph of $y = 0$ and $y = |x|$

Also $|f(0)| \leq 0 \Rightarrow f(0) = 0$

Also from Sandwich theorem, $\lim_{x\to 0} 0 \leq \lim_{x\to 0}|f(x)| \leq \lim_{x\to 0}|x|$

$\Rightarrow \lim_{x\to 0}|f(x)| = 0$

$\Rightarrow y = f(x)$ is continuous at $x = 0$.

Also statement 2 is correct but it has no link with statement 1.

15.c. See the graph of $f(x) = ||x|^2 - 3|x| + 2|$,

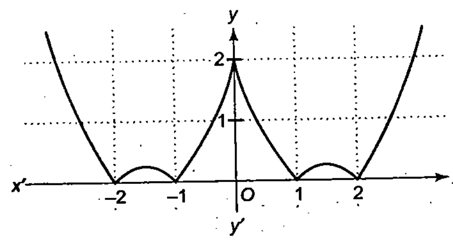


Fig. 3.43

which is non-differentiable at 5 points, $x = 0, \pm 1, \pm 2$.

However, statement 2 is false,

as $f(x) = x^3$ crosses x-axis at $x = 0$,

but $|f(x)| = |x^3|$ is differentiable at $x = 0$.

16.b. Statement 1 is correct as $e^{|x|}$ is non-differentiable at $x = 0$.

17.a. Let $x = k$, $k \in Z \Rightarrow f(k) = \{k\} + \sqrt{\{k\}} = 0$

$f(k^+) = 0 + 0 = 0$, $f(k^-) = 1 + 1 = 2$.

Hence, $f(x)$ is not continuous at integral points.

Hence, correct answer is **a**.

18.b. We know that $0 \leq \cos^2(n!\,\pi x) \leq 1$

Hence, $\lim_{m\to\infty}\cos^{2m}(n!\pi x) = 0$ or 1, as

$0 \leq \cos^2(n!\,\pi x) < 1$ or $\cos^2(n!\,\pi x) = 1$

Also, since $n \to \infty$, then $n!\,x$ = integer if $x \in Q$ and $n!\,x \neq$ integer, if $x \in$ irrational.

Hence, $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

$\Rightarrow h(x) = 1$ when $\forall x \in R$ which is continuous for all x; however, statement 2 does not correctly explain statement 1 as the addition of discontinuous functions may be continuous.

19.d. Consider $f(x) = \begin{cases} x^2\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ which is differentiable at $x = 0$, but derivative is not continuous at $x = 0$.

However, statement 2 is correct.

20.a. $F(x) = f(g(x))$,

$\Rightarrow F(x) = x^2 + 2|x|$

$$\Rightarrow F'(x) = \begin{cases} 2x - 2, & x < 0 \\ 2x + 2, & x > 0 \end{cases}$$

Hence, $F'(0^+) = 2$ and $F'(0^-) = -2$.

Hence, both statements are correct and statement 2 is a correct explanation of statement 1.

21.d. Statement 1 is false, as consider the function $f(x) = \max\{0, x^3\}$ which is equivalent to

$$f(x) = \begin{cases} 0, & x < 0 \\ x^3, & x \ge 0 \end{cases}.$$

Here $f(x)$ is continuous and differentiable at $x = 0$.

However, statement 2 is obviously true.

22.b. $$f(x) = \begin{bmatrix} \pi/4, & x > 1 \\ \pi/4, & x = 1 \text{ [in the interval } (1 - 8, 1 + 8)] \\ \pi/2, & x < 1 \end{bmatrix}$$

Hence, f is discontinuous and non-derivable, but non-derivability does not imply discontinuity.

23.c. $F(1) = 0, F(1^+) = \dfrac{\pi}{2}$ and $F(1^-) = -\dfrac{3\pi}{4}$

$\Rightarrow F$ is discontinuous

But for $f(x) = \begin{bmatrix} 1, & \text{if } x \ge 0 \\ -1, & \text{if } x < 0 \end{bmatrix}$ and $g(x) = \begin{bmatrix} -1, & \text{if } x \ge 0 \\ 1, & \text{if } x < 0 \end{bmatrix}$

then $f(x)\,g(x)$ is continuous at $x = 0$.

24.c.

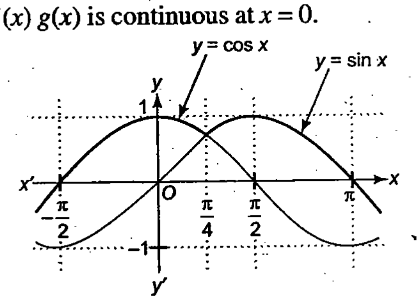
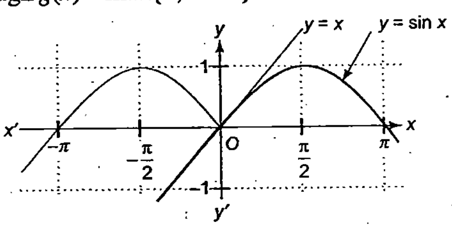


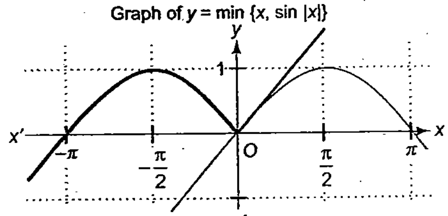
Fig. 3.44

From the graph, statement 1 is true.

Consider $f(x) = \min\{x, \sin|x|\}$ is differentiable at $x = 0$, though $g(x) = \max\{x, \sin|x|\}$ is non-differentiable at $x = 0$



Graph of $y = \min\{x, \sin|x|\}$



Graph of $y = \max\{x, \sin|x|\}$

Fig. 3.45

25.b.

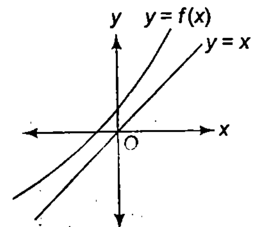


Fig. 3.46

Since $f(x)$ is a continuous function such that $f(0) = 1$ and $f(x) \ne x, \forall x \in R$

The graph of $y = f(x)$ always lies above the graph of $y = x$.

Hence $f(x) > x$.

Hence, $f(f(x)) > x$ (as $f(x)$ is onto function, $f(x)$ takes all real values which acts as x).

Statement 2 is a fundamental property of continuous function, but does not explain statement 1.

26.c. Statement 1 is true as $\sqrt{x}$ is monotonic function. But statement 2 is false as $f(x) = [\sin x]$ is continuous at $x = 3\pi/2$, though $\sin(3\pi/2) = -1$ (integer).

Linked Comprehension Type

For Problems 1 – 3

1. b, 2. a, 3. b.

Sol. $$f(x) = \begin{cases} \dfrac{a(1 - x\sin x) + b\cos x + 5}{x^2}, & x < 0 \\ 3, & x = 0 \\ \left\{1 + \left(\dfrac{P(x)}{x}\right)\right\}^{1/x}, & x > 0 \end{cases}$$

where $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

$f(0) = 3$

R.H.L. $= \lim\limits_{x \to 0^+} f(x)$

$$= \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \left\{1 + \left(\frac{P(h)}{h}\right)\right\}^{1/h}$$

$\because f$ is continuous at $x = 0$

$\therefore$ R.H.L. exists.

For the existence of R.H.L., $a_0, a_1 = 0$

$\Rightarrow$ R.H.L. $= \lim\limits_{h \to 0}(1 + a_2h + a_3h^2)^{1/h}$ (1^∞ form)

$$= e^{\lim\limits_{h \to 0}(1 + a_2h + a_3h^2 - 1)(1/h)} = e^{a_2}$$

L.H.L. $= \lim\limits_{x \to 0^-} f(x) = \lim\limits_{h \to 0} f(0 - h)$

$$= \lim_{h \to 0} \frac{a(1 - (-h)\sin(-h)) + b\cos(-h) + 5}{(-h)^2}$$

$$= \lim_{h \to 0} \frac{a(1 - h(h)) + b\left(1 - \dfrac{h^2}{2!}\right) + 5}{h^2}$$

For finite value of L.H.L., $a + b + 5 = 0$ and $-a - \dfrac{b}{2} = 3$

Solving, we get $a = -1, b = -4$.

Now $g(x) = 3a\sin x - b\cos x = -3\sin x + 4\cos x$ which has the range $[-5, 5]$.

Also $P(x) = a_3x^3 + (\log_e 3)x^2$

$P''(x) = 6a_3x + 2\log_e 3$

$\Rightarrow P''(0) = 2\log_e 3$

Further, $P(x) = b \Rightarrow a_3x^3 + (\log_e 3)x^2 = -4$ has only one real root, as the graph of $P(x) = a_3x^3 + (\log_e 3)x^2$ meets $y = -4$ only once for negative value of x.

For Problems 4 – 6

4. c, 5. b, 6. c.

Sol. For $0 \le x < \frac{\pi}{4}$, $g(x) = 1 + \tan x$

$x \in \left[0, \frac{\pi}{4}\right) \Rightarrow 1 + \tan x \in [1, 2)$

so $f(g(x)) = f(1 + \tan x) = 1 + \tan x + 2$

and for $x \in \left[\frac{\pi}{4}, \pi\right)$, $g(x) = 3 - \cot x$

$x \in \left[\frac{\pi}{4}, \pi\right) \Rightarrow 3 - \cot x \in [2, \infty)$

so $f(g(x)) = f(3 - \cot x) = 6 - (3 - \cot x)$

$$\text{Let } h(x) = f(g(x)) = \begin{cases} 3 + \tan x, & 0 \le x < \frac{\pi}{4} \\ 3 + \cot x, & \frac{\pi}{4} \le x < \pi \end{cases}$$

Clearly, $f(g(x))$ is continuous in $[0, \pi)$

Now $h'\left(\frac{\pi^+}{4}\right) = \lim\limits_{x \to \frac{\pi^+}{4}} (-\operatorname{cosec}^2 x) = -2$

$h'\left(\frac{\pi^-}{4}\right) = \lim\limits_{x \to \frac{\pi^-}{4}} (\sec^2 x) = 2$

So $f(g(x))$ is differentiable everywhere in $[0, \pi)$ other than at $x = \frac{\pi}{4}$

$$|f(g(x))| = \begin{cases} |3 + \tan x|, & 0 \le x < \frac{\pi}{4} \\ |3 + \cot x|, & \frac{\pi}{4} \le x < \pi \end{cases}$$

which is non-differentiable at $x = \pi/4$ and where $3 + \cot x = 0$ or $x = \cot^{-1}(-3)$

For $x \in \left[0, \frac{\pi}{4}\right)$, $3 + \tan x \in [3, 4)$

For $x \in \left[\frac{\pi}{4}, \pi\right)$, $3 + \cot x \in (-\infty, 4]$

Hence, the range is $(-\infty, 4]$.

For Problems 7 – 9

7.a, 8.c, 9.d.

Sol. $F(x) = \lim\limits_{n \to \infty} \dfrac{f(x) + x^{2n} g(x)}{1 + x^{2n}}$

$$= \begin{cases} f(x), & 0 \le x^2 < 1 \\ \dfrac{f(x) + g(x)}{2}, & x^2 = 1 \\ g(x), & x^2 > 1 \end{cases}$$

$$= \begin{cases} g(x), & x < -1 \\ \dfrac{f(-1) + g(-1)}{2}, & x = -1 \\ f(x), & -1 < x < 1 \\ \dfrac{f(1) + g(1)}{2}, & x = 1 \\ g(x) & x > 1 \end{cases}$$

If $F(x)$ is continuous $\forall x \in R$, $F(x)$ must be made continuous at $x = \pm 1$.

For continuity at $x = -1$, $f(-1) = g(-1) \Rightarrow 1 - a + 3 = b - 1 \Rightarrow a + b = 5$ (1)

For continuity at $x = 1$, $f(1) = g(1) \Rightarrow 1 + a + 3 = 1 + b \Rightarrow a - b = -3$ (2)

Solving equations (1) and (2), we get $a = 1$ and $b = 4$

$f(x) = g(x) \Rightarrow x^2 + x + 3 = x + 4 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$.

For Problems 10 – 12

10.a, 11.d, 12.b.

Sol.

$$f(x) = \begin{cases} [x], & -2 \le x \le -\frac{1}{2} \\ 2x^2 - 1, & -\frac{1}{2} < x \le 2 \end{cases} = \begin{cases} -2, & -2 \le x < -1 \\ -1, & -1 \le x \le -\frac{1}{2} \\ 2x^2 - 1, & -\frac{1}{2} < x \le 2 \end{cases}$$

$$|f(x)| = \begin{cases} 2, & -2 \le x < -1 \\ 1, & -1 \le x \le -\frac{1}{2} \\ |2x^2 - 1|, & -\frac{1}{2} < x \le 2 \end{cases}$$

$$= \begin{cases} 2, & -2 \le x < -1 \\ 1, & -1 \le x \le -\frac{1}{2} \\ 1 - 2x^2, & -\frac{1}{2} < x \le \frac{1}{\sqrt{2}} \\ 2x^2 - 1, & \frac{1}{\sqrt{2}} < x \le 2 \end{cases}$$

$$f(|x|) = \begin{cases} -2, & -2 \le |x| < -1 \\ -1, & -1 \le |x| \le -\frac{1}{2} \\ 2|x|^2 - 1, & -\frac{1}{2} < |x| \le 2 \end{cases} = 2x^2 - 1,\ -2 \le x \le 2$$

$$\Rightarrow g(x) = f(|x|) + |f(x)| = \begin{cases} 2x^2 + 1, & -2 \le x < -1 \\ 2x^2, & -1 \le x \le -\frac{1}{2} \\ 0, & -\frac{1}{2} < x < \frac{1}{\sqrt{2}} \\ 4x^2 - 2, & \frac{1}{\sqrt{2}} \le x \le 2 \end{cases}$$

$g(-1^-) = \lim\limits_{x \to -1} (2x^2 + 1) = 3$, $g(-1^+) = \lim\limits_{x \to -1} 2x^2 = 2$

$$g\left(-\frac{1}{2}^-\right)=\lim_{x\to-\frac{1}{2}} 2x^2=\frac{1}{2},\ g\left(-\frac{1}{2}^+\right)=\lim_{x\to-\frac{1}{2}} 0=0$$

$$g\left(\frac{1}{\sqrt{2}}^-\right)=\lim_{x\to\frac{1}{\sqrt{2}}} 0=0,\ g\left(\frac{1}{\sqrt{2}}^+\right)=\lim_{x\to\frac{1}{\sqrt{2}}}(4x^2-2)=0.$$

Hence, $g(x)$ is discontinuous at $x=-1,-\frac{1}{2}$.

$g(x)$ is continuous at $x=\frac{1}{\sqrt{2}}$

Now, $g'\left(\frac{1}{\sqrt{2}}^-\right)=0,\ g'\left(\frac{1}{\sqrt{2}}^+\right)=8\left(\frac{1}{\sqrt{2}}\right)=\frac{8}{\sqrt{2}}$

Hence, $g(x)$ is non-differentiable at $x=\frac{1}{\sqrt{2}}$.

For problems 13 – 15

13. c, 14.d, 15.b

Sol.

$$f(x)=\begin{cases}x^2+10x+8, & x\le -2\\ ax^2+bx+c, & -2<x<0, a\ne 0\\ x^2+2x, & x\le 0\end{cases}$$

For continuous at $x=0 \Rightarrow c=0$

Continuous at $x=-2 \Rightarrow 4-20+8=4a-2b$

$\Rightarrow 2a-b=-4$ (1)

Now let the line $y=mx+p$ is tangent to all the three curves

Solving $y=mx+p$ and $y=x^2+2x$

$x^2+2x=mx+p$

$x^2+(2-m)x-p=0$

$D=0$

$(2-m)^2+4p=0$ (2)

Again solving $y=mx+p$ and $y=x^2+10x+8$

$x^2+10x+8=mx+p$

$\Rightarrow x^2+(10-m)x+8-p=0$

$D=0 \Rightarrow (10-m)^2-4(8-p)=0$

$\Rightarrow (10-m)^2-(2-m)^2=42$

$\Rightarrow (100-20m)-(4-4m)=32$

$\Rightarrow m=4$ and $p=-1$

Hence equation of the tangent to first and last curves is

$y=4x-1$ (3)

Now solving this with $y=ax^2+bx$ (as $c=0$)

$ax^2+bx=4x-1 \Rightarrow ax^2+(b-4)x+1=0$

$D=0$

$\Rightarrow (b-4)^2=4a$

Also $b=2a+4$ (from (1))

$\therefore 4a^2=4a \Rightarrow a=1$ and $b=6$ (as $a\ne 0$)

$f'(0^-)=\lim_{x\to 0}(2ax+b)=b$

$f'(0^+)=\lim_{x\to 0}(2ax+2)=2 \Rightarrow b=2$

Matrix-Match Type

1. a → p, q, r; b → p, r, s; c → p, r, s; d → p, r, s.

a. $f(x)=|x^3|=x(x|x|)$ is continuous and differentiable.

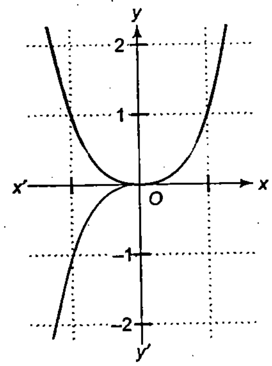


Fig. 3.47

b. $f(x)=\sqrt{|x|}$ is continuous

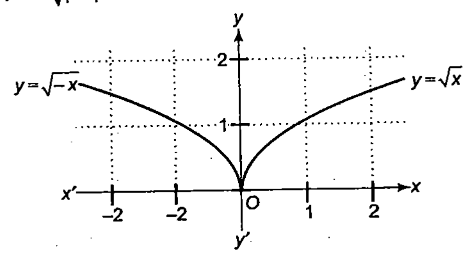


Fig. 3.48

Clearly from the graph, $f(x)$ is non-differentiable at $x=0$.

c. $f(x)=|\sin^{-1}x|$ is continuous

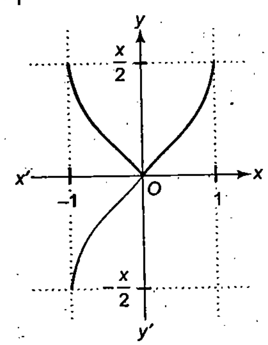


Fig. 3.49

Clearly from the graph, $f(x)$ is non-differentiable at $x=0$.

d. $f(x)=\cos^{-1}|x|$ is continuous

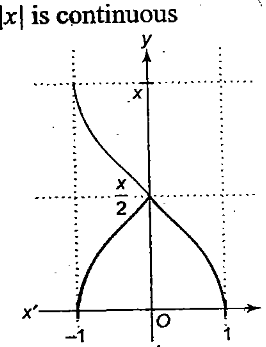


Fig. 3.50

Clearly from the graph, $f(x)$ is non-differentiable at $x=0$.

2. a → r, s; b → p, q; c → p, q; d → p, r.

a. The given function is clearly continuous at all points except possibly at $x=\pm 1$.

As $f(x)$ is an even function, so we need to check its continuity only at $x=1$.

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = f(1)$

$\Rightarrow \lim\limits_{x\to 1^-} (ax^2 + b) = \lim\limits_{x\to 1^+} \dfrac{1}{|x|} \Rightarrow a + b = 1$ (1)

Clearly, $f(x)$ is differentiable for all x, except possibly at $x = \pm 1$. As $f(x)$ is an even function, so we need to check its differentiability at $x = 1$ only.

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$$

$$\Rightarrow \lim_{x\to 1} \frac{ax^2+b-1}{x-1} = \lim_{x\to 1^+} \frac{\frac{1}{|x|}-1}{x-1}$$

$$\Rightarrow \lim_{x\to 1} \frac{ax^2-a}{x-1} = \lim_{x\to 1} \frac{-1}{x} \Rightarrow 2a = -1 \Rightarrow a = -\frac{1}{2}$$

Putting $a = -1/2$ in (1) we get $b = 3/2 \Rightarrow |k| = 1 \Rightarrow k = \pm 1$

b. If $f(x) = \text{sgn}(x^2 - ax + 1)$ is discontinuous then $x^2 - ax + 1 = 0$ must have only one real root. Hence $a = \pm 2$.

c. $f(x) = [2 + 3|n| \sin x]$, $n \in N$ has exactly 11 points of discontinuity in $x \in (0, \pi)$.

The required number of points are $1 + 2\,(3|n| - 1) = 6|n| - 1 = 11 \Rightarrow n = \pm 2$.

d. $f(x) = ||x| - 2| + a|$ has exactly three points of non-differentiability.

$f(x)$ is non-differentiable at $x = 0$, $|x| - 2 = 0$ or $x = 0, \pm 2$.

Hence, the value of a must be positive, as negative value of a allows $||x| - 2| + a = 0$ to have real roots, which gives more points of non-differentiability.

3. **a** $\to$ **s**, **b** $\to$ **r**, **c** $\to$ **p**, **d** $\to$ **q**.

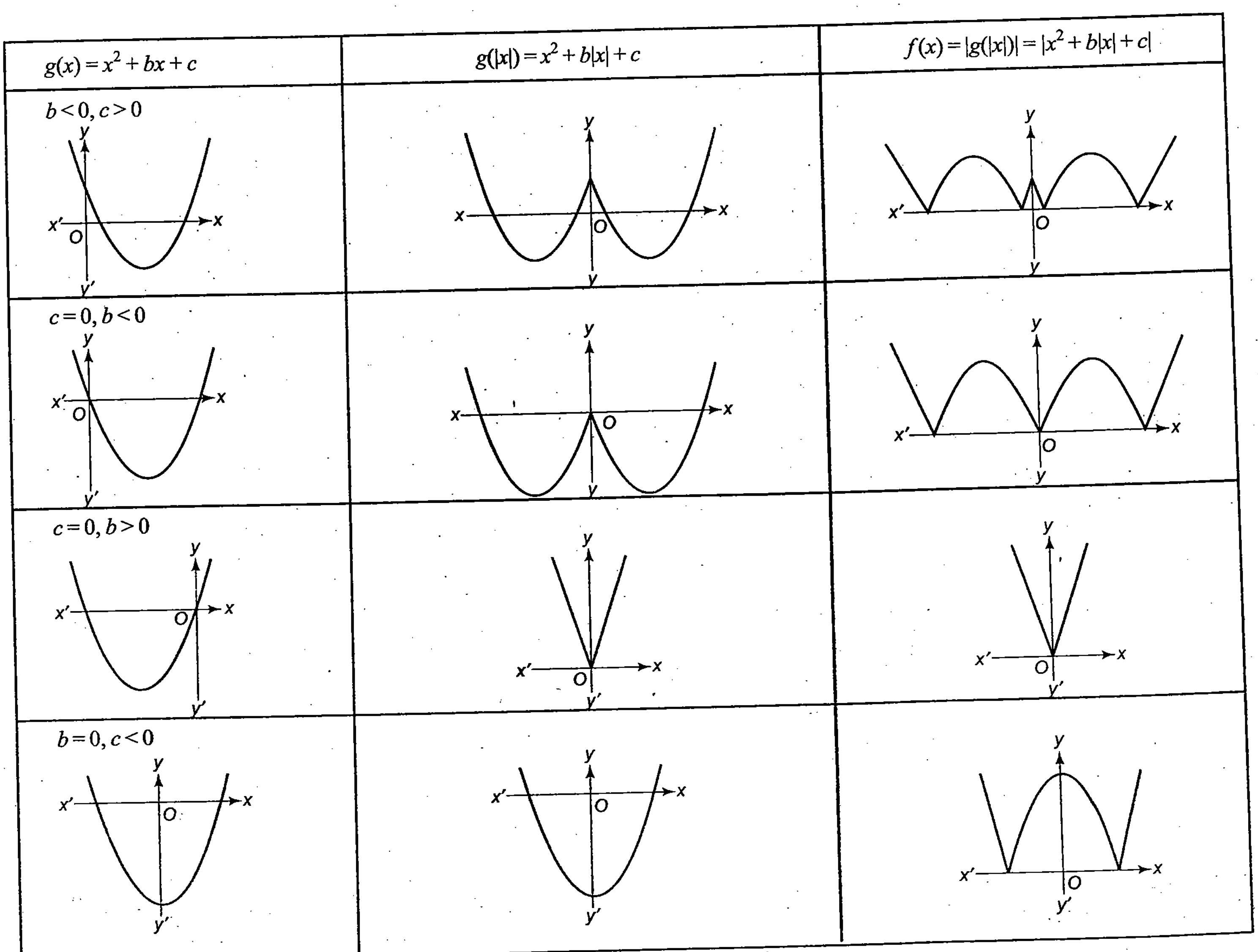


Fig. 3.51

4. a → q, s; b → p, s; c → p, r; d → q, s.

a. $f(x)=\begin{cases}\dfrac{5e^{1/x}+2}{3-e^{1/x}}, & x\neq 0\\ 0, & x=0\end{cases}$

$$f(0^+)=\lim_{h\to 0}\frac{5e^{1/h}+2}{3-e^{1/h}}=\lim_{h\to 0}\frac{5+2e^{-1/h}}{3e^{-1/h}-1}=-5.$$

Hence, $f(x)$ is discontinuous and non-differentiable at $x=0$.

b. $g(x)=xf(x)=\begin{cases}x\dfrac{5e^{1/x}+2}{3-e^{1/x}}, & x\neq 0\\ 0, & x=0\end{cases}$

$$f(0^+)=\lim_{h\to 0}h\frac{5e^{1/h}+2}{3-e^{1/h}}=\lim_{h\to 0}h\frac{5+2e^{-1/h}}{3e^{-1/h}-1}=0\times(-5)=0.$$

$$f(0^-)=\lim_{h\to 0}h\frac{5e^{-1/h}+2}{3-e^{-1/h}}=0\times(2/3)=0$$

Hence, $f(x)$ is continuous at $x=0$.

$$Lg'(0)=\lim_{h\to 0}\frac{g(0-h)-g(0)}{-h}$$

$$=\lim_{h\to 0}\frac{-hf(-h)-0}{-h}$$

$$=\lim_{h\to 0}f(-h)$$

$$=\lim_{h\to 0}\frac{5e^{-1/h}+2}{3-e^{-1/h}}=\frac{0+2}{3-0}=\frac{2}{3}$$

$$Rg'(0)=\lim_{h\to 0}\frac{g(0+h)-g(0)}{h}$$

$$=\lim_{h\to 0}\frac{g(h)-0}{h}$$

$$=\lim_{h\to 0}f(h)=\lim_{h\to 0}\frac{5e^{1/h}+2}{3-e^{1/h}}$$

$$=\lim_{h\to 0}\frac{5+2e^{-1/h}}{3e^{-1/h}-1}$$

$$=\frac{5+0}{0-1}=-5$$

$\because LF'(0)\neq RF'(0)$

Hence, $F(x)$ is not differentiable, but continuous at $x=0$.

c. For $x^2f(x)$,

Let $F(x)=x^2f(x)$

$$\therefore LF'(0)=\lim_{h\to 0}\frac{F(0-h)-F(0)}{-h}$$

$$=\lim_{h\to 0}\frac{h^2f(-h)-0}{-h}=0$$

$$RF'(0)=\lim_{h\to 0}\frac{F(0+h)-F(0)}{h}$$

$$=\lim_{h\to 0}\frac{h^2f(h)-0}{h}=0$$

$\therefore LF'(0)=RF'(0)$

Hence, $F(x)$ is differentiable at $x=0$, then it is always continuous at $x=0$.

d. Clearly from the above discussion $y=x^{-1}f(x)$ is discontinuous and hence non-differentiable at $x=0$.

5. a → q, s; b → p, r; c → p, r; d → p, s.

a. $f(x)=\lim_{n\to\infty}\left[\cos^2(2\pi x)\right]^n+\left\{x+\frac{1}{2}\right\}$

Obviously, $\lim_{x\to\frac{1}{2}^+}f(x)=0+0=0$

And $\lim_{x\to\frac{1}{2}^-}f(x)=0+1$

$\therefore f(x)$ is discontinuous at $x=\frac{1}{2}$.

b. $f(x)=(\log x)(x-1)^{1/5}$

Obviously, $f(x)$ is continuous at $x=1$

$$f'(1^+)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{\log(1+h)h^{1/5}}{h}=0$$

$$f'(1^-)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{h}=\lim_{h\to 0}\frac{\log(1-h)(-h)^{1/5}}{-h}=0$$

Hence, $f(x)$ is differentiable at $x=1$.

c. $f(x)=[\cos 2\pi x]+\sqrt{\left\{\sin\left(\frac{\pi x}{2}\right)\right\}}$

$$\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}[\cos 2\pi x]+\lim_{x\to 1^-}\sqrt{\left\{\sin\left(\frac{\pi x}{2}\right)\right\}}$$

$$=0+1=1$$

$$\lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}[\cos(2\pi x)]+\lim_{x\to 1^+}\sqrt{\left\{\sin\left(\frac{\pi x}{2}\right)\right\}}$$

$$=0+1=1$$

Also $f(1)=1+0=1$.

$f(x)$ is continuous at $x=1$

$$f'(1^+)=\lim_{h\to 0}\frac{\left[\cos 2\pi(1+h)\right]+\sqrt{\left\{\sin\left(\frac{\pi(1+h)}{2}\right)\right\}}-1}{h}$$

$$=\lim_{h\to 0}\frac{\left[\cos 2\pi h\right]+\sqrt{\left\{\cos\left(\frac{\pi h}{2}\right)\right\}}-1}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{\cos\left(\frac{\pi h}{2}\right)-1}}{h} = \lim_{h\to 0}\frac{-\frac{\pi}{2}\sin\left(\frac{\pi h}{2}\right)}{2\sqrt{\cos\left(\frac{\pi h}{2}\right)}} = 0$$

Similarly, $f'(1^-)=0$.

d. $f(x)=\begin{cases}\cos 2x, & x\in Q\\ \sin x, & x\notin Q\end{cases}$ at $\frac{\pi}{6}$

$f(x)$ is continuous when $\cos 2x = \sin x$ which has $x=\frac{\pi}{6}$ as one of the solutions. Hence, it is continuous.

Also in the neighbourhood of $x=\frac{\pi}{6}$,

$$f'(x)=\begin{cases}-2\sin 2x, & \frac{\pi}{6}-\delta<x<\frac{\pi}{6}\\ \cos x, & \frac{\pi}{6}<x<\frac{\pi}{6}+\delta\end{cases}$$

Here, $f'\left(\frac{\pi^-}{6}\right)\neq f'\left(\frac{\pi^+}{6}\right)$.

$\Rightarrow f(x)$ is not differentiable at $x=\frac{\pi}{6}$

Integer Type

1. (5) $f(x)=\text{sgn}(\sin x)$ is discontinuous when $\sin x = 0$
$\Rightarrow x=0,\pi,2\pi,3\pi,4\pi$

2. (6) $g(x)=\left[\frac{f(x)}{a}\right]$ is continuous if $\left[\frac{f(x)}{a}\right]=0$ for $\forall f(x)\in$ $(1,\sqrt{30})$, for which we must have $a>\sqrt{30}$
Hence the least value of a is 6.

3. (4) $\text{sgn}(x^2-3x+2)$ is discontinuous when $x^2-3x+2=0$ or $x=1,2$
$[x-3]=[x]-3$ is discontinuous at $x=1,2,3,4$
Thus $f(x)$ is discontinuous at $x=3,4$
Now both $\text{sgn}(x^2-3x+2)$ and $[x-3]$ are discontinuous at $x=1$ and 2.
Then $f(x)$ may be continuous at $x=1$ and 2.
But $f(1)=-2$ and $f(1^+)=-1+0-3=-4$
Thus $f(x)$ is discontinuous at $x=1$
Also $f(2)=-1$ and $f(2^+)=1-1=0$
Hence $f(x)$ is discontinuous at $x=2$ also.

4. (2) $g'(3^-)=\lim_{h\to 0}\frac{g(3-h)-g(3)}{-h}=\lim_{h\to 0}\frac{a\sqrt{4-h}-(3b+2)}{-h}$ (1)

for existence of limit $\lim_{h\to 0}N^r=0$

$\therefore\ 2a-3b=2$ (2)

Now $g'(3^+)=\lim_{h\to 0}\frac{b(3+h)+2-(3b+2)}{h}=b$ (3)

Substituting $3b+2=2a$ in equation (1)

$$g'(3^-)=\lim_{h\to 0}\frac{a\sqrt{4-h}-2a}{-h}=\lim_{h\to 0}\left(\frac{(4-h)-4}{(-h)(\sqrt{4-h}+2)}\right)=\frac{a}{4}$$

Hence $g'(3^-)=g'(3^+)$

$\frac{a}{4}=b\Rightarrow a=4b$ (4)

From equations (2) and (4)
$8b-3b=2$
$\Rightarrow b=\frac{2}{5}$ and $a=\frac{8}{5}$
$\Rightarrow a+b=2$

5. (8) $f(x)\begin{cases}ax^2+bx & \text{for } -1<x<1\\ \frac{a-b-1}{2} & x=-1\\ \frac{a+b+1}{2} & x=1\\ \frac{1}{x} & \text{for } x>1 \text{ or } x<-1\end{cases}$

for continuity at $x=1$ we have $a+b=\frac{a+b+1}{2}$
hence, $a+b=1$ (1)
for continuity at $x=-1$
$a-b=-1$ $a-b=-1$ (2)
hence $a=0$ and $b=1$

6. (6)

$$g(f(x))=\begin{cases}g\left(\frac{x}{2}-1\right), & 0\le x<1\\ g\left(\frac{1}{2}\right), & 1\le x\le 2\end{cases}=\begin{cases}\frac{(x-1)(x-2-2k)}{2}+3, & 0\le x<1\\ 4-2k, & 1\le x<2\end{cases}$$

$\lim_{x\to 1^-}g(f(x))=3, g(f(1))=4-2k$ and $\lim_{x\to 1^+}g(f(x))=4-2k$ for $g(f(x))$ to be continuous at $x=1$, $4-2k=3\Rightarrow k=\frac{1}{2}$

7. (8) $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{f(x)+f(h)+2xh(x+h)-\frac{1}{3}-\left(f(x)+f(0)-\frac{1}{3}\right)}{h}$$

$$=\lim_{h\to 0}\frac{f(h)-f(0)}{h}+2x^2=f'(0)+2x^2$$

$$\lim_{h\to 0}\frac{3f(h)-1}{6h}=\lim_{h\to 0}\frac{f(h)-\frac{1}{3}}{2h}=\lim_{h\to 0}\frac{f(h)-f(0)}{2h}$$

$$=\frac{f'(0)}{2}=\frac{2}{3}\ \Rightarrow\ f'(0)=\frac{4}{3}$$

$\therefore\ f'(x)=\frac{4}{3}+2x^2$

$f(x)=\lambda+\frac{4}{3}x+\frac{2x^3}{3}\ \Rightarrow\ f(0)=\lambda=\frac{1}{3}$

$\therefore\ f(x)=\frac{2x^3}{3}+\frac{4}{3}x+\frac{1}{3} \Rightarrow f(2)=\frac{25}{3}$

8. (5) $\because f(x)=\begin{cases} x^p \sin\left(\frac{1}{x}\right)+x^2, & x>0 \\ x^p \sin\left(\frac{1}{x}\right)-x^2, & x<0 \\ 0, & x=0 \end{cases}$

$$f''(x)=\begin{cases} -x^{p-4}\sin\left(\frac{1}{x}\right)-(p-2)x^{p-3}\cos\left(\frac{1}{x}\right) \\ -px^{p-3}\cos\left(\frac{1}{x}\right) \\ +p(p-1)x^{p-2}\sin\left(\frac{1}{x}\right)+2, & x>0 \\ -x^{p-4}\sin\left(\frac{1}{x}\right)-(p-2)x^{p-3}\cos\left(\frac{1}{x}\right) \\ px^{p-3}\cos\left(\frac{1}{x}\right) \\ +p(p-1))x^{p-2}\sin\left(\frac{1}{x}\right)-2, & x<0 \\ 0, & x=0 \end{cases}$$

RHL = LHL = $f(0)=0$

$\because$ $\sin\infty$ and $\cos\infty$ lie between –1 to 1. For $p\ge 5$, RHL = 2

LHL = –2

$f(0)=0$

For $p\in[5,\infty)$, $f''(x)$ is not continuous.

9. (8) We have $f(x)=[x]+[x+1/3]+[x+2/3]=[3x]$

Which is discontinuous when $3x=k$ or $x=k/3$, $k\in I$

Hence points of discontinuity are 1/3, 2/3, 3/3, 4/3, 5/3, 6/3, 7/3, 8/3.

10. (1) $\lim\limits_{x\to1^-} h(x)=\lim\limits_{x\to1^-}\lim\limits_{n\to\infty}\frac{x^{2n}\cdot f(x)+x^{2m}\cdot g(x)}{(1+x^{2n})}=g(1)$

$\lim\limits_{x\to1^-} h(x)=\lim\limits_{x\to1^+}\lim\limits_{n\to\infty}\frac{x^{2n}\cdot f(x)+x^{2m}\cdot g(x)}{(1+x^{2n})}=f(1)$

$\because$ $\lim\limits_{x\to1}h(x)$ exists $\Rightarrow f(1)=g(1)$

$\Rightarrow f(x)-g(x)=0$ has a root at $x=1$

11. (1) Given $\dfrac{\int_{f(y)}^{f(x)} e^t dt}{\int_y^x (1/t)dt}=1$

$\Rightarrow e^{f(x)}-e^{f(y)}=\ln x-\ln y$

$\Rightarrow e^{f(x)}-\ln x=c \Rightarrow f(x)=\ln(\ln x+c)$

Since $f\left(\frac{1}{e}\right)=0 \Rightarrow c=2$

Now $f(g(x))=\begin{cases}\ln(x+2); & x\ge k \\ \ln(2+x^2); & 0<x<k\end{cases}$

For continuity at $x=k$,

$\ln(k+c)=\ln(k^2+c) \Rightarrow$ either $k=0$ or $k=1$.

$\because k>0 \Rightarrow k=1$

12. (7) Let $g(x)=(\ln x)(\ln x)\cdots\infty$.

$g(x)=\begin{cases}0, & 1<x<e \\ 1, & x=e \\ \infty, & x>e\end{cases}$

Therefore $f(x)=\begin{cases}x, & 1<x<e \\ x/2, & x=e \\ 0, & e<x<3\end{cases}$

Hence $f(x)$ is non-differentiable at $x=e$.

13. (2) $f(0)=\lim\limits_{x\to0}\frac{\tan(\tan x)-\sin(\sin x)}{\tan x-\sin x}$

$=\lim\limits_{x\to0}\frac{\tan(\tan x)-\sin(\sin x)}{\frac{\tan x}{x}\left(\frac{1-\cos x}{x^2}\right)x^3}$

$=2\lim\limits_{x\to0}\frac{\tan(\tan x)-\sin(\sin x)}{x^3}$

$=2\lim\limits_{x\to0}\frac{\left(\tan x+\frac{\tan^3 x}{3}+\frac{2}{15}\tan^5 x+\cdots\right)-\left(\sin x-\frac{\sin^3 x}{3!}+\frac{\sin^5 x}{5!}\cdots\right)}{x^3}$

$=2\lim\limits_{x\to0}\left(\left(\frac{\tan x-\sin x}{x^3}\right)+\frac{\left(\frac{\tan^3 x}{3}+\frac{\sin^3 x}{3!}\right)}{x^3}+\cdots\right)$

$=2\lim\limits_{x\to0}\left(\left(\frac{\tan x}{x}\right)\left(\frac{1-\cos x}{x^2}\right)+\frac{1}{3}+\frac{1}{6}\right)=2\left[\frac{1}{2}+\frac{1}{2}\right]=2$

14. (7) $\sin^{-1}|\sin x|$ is periodic with period π

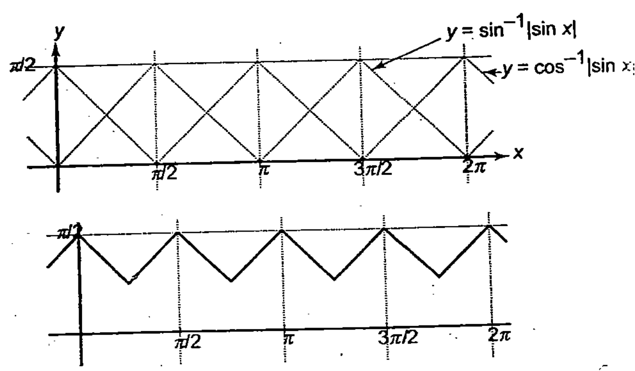


Fig. 3.52

Archives

Subjective

1. At $x=0, f(0)=c$ (1)

$f(0^-)=\text{L.H.L.}=\lim_{x\to 0-} f(x)=\lim_{h\to 0} f(0-h)$

$= \lim_{h\to 0} \frac{\sin[(a+1)(0-h)]+\sin(0-h)}{(0-h)}$

$= \lim_{h\to 0} \frac{-\sin[(a+1)h]-\sin h}{-h}$

$= \lim_{h\to 0} \left\{(a+1)\frac{\sin[(a+1)h]}{(a+1)h}+\frac{\sin h}{h}\right\}$

$= (a+1)\lim_{h\to 0} \frac{\sin[(a+1)h]}{(a+1)h}+\lim_{h\to 0}\frac{\sin h}{h}$

$=(a+1)\times 1+1=a+2$ (2)

$f(0^+)=\text{R.H.L.}=\lim_{x\to 0+} f(x)=\lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0} \frac{(h+bh^2)^{1/2}-h^{1/2}}{bh^{3/2}}$

$= \lim_{h\to 0} \frac{(1+bh)^{1/2}-1}{bh}$

$= \lim_{h\to 0} \frac{(1+bh)^{1/2}-(1)^{1/2}}{(1+bh)-1}=\frac{1}{2}(1)^{1/2-1}=\frac{1}{2}$ (3)

$\because f(x)$ is continuous at $x=0$.

$\Rightarrow$ L.H.L.=R.H.L.=$f(0)$ [from equations (1), (2) and (3)]

$\Rightarrow a+2=\frac{1}{2}=c \Rightarrow a=-\frac{3}{2}, b\in R, c=\frac{1}{2}$

2. $f(x)=\begin{cases}1+x, & 0\le x\le 2\\ 3-x, & 2<x\le 3\end{cases}$

$f(f(x))=\begin{cases}1+f(x), & 0\le f(x)\le 2\\ 3-f(x), & 2<f(x)\le 3\end{cases}$

$=\begin{cases}1+(1+x), & 0\le 1+x\le 2, & 0\le x\le 2\\ 1+(3-x), & 0\le 3-x\le 2, & 2<x\le 3\\ 3-(1+x), & 2<1+x\le 3, & 0\le x\le 2\\ 3-(3-x) & 2<3-x\le 2, & 2<x\le 3\end{cases}$

$=\begin{cases}2+x, & 0\le x\le 1\\ 2-x, & 1<x\le 2\\ 4-x, & 2<x\le 3\end{cases}$

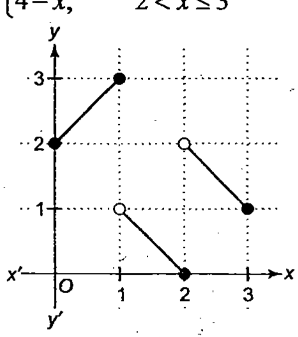


Fig. 3.53

At $x=1, x=2$, $f(f(x))$ is discontinuous.

3.

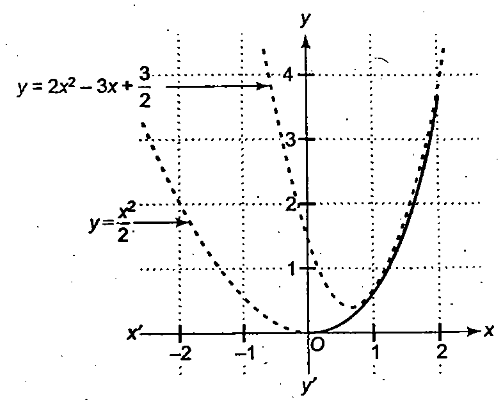


Fig. 3.54

We have $f(x)=\begin{cases}\frac{x^2}{2}, & 0\le x<1\\ 2x^2-3x+\frac{3}{2}, & 1\le x\le 2\end{cases}$

$\Rightarrow f'(x)=\begin{cases}x, & 0\le x<1\\ 4x-3, & 1<x\le 2\end{cases}$

Here $f(x)$ is continuous everywhere,

as $f(1^+)=\lim_{x\to 1}\left(2x^2-3x+\frac{3}{2}\right)$

$=2(1)-3(1)+\frac{3}{2}=\frac{1}{2}$

and $f(1^+)=\lim_{x\to 1}\left(\frac{x^2}{2}\right)=\frac{1}{2}$.

At $x=1, Lf'=1; Rf'=4(1)-3=1$

$\Rightarrow f$ is differentiable and hence f' is continuous at $x=1$.

Also $f''(x)=\begin{cases}1, & 0\le x<1\\ 4, & 1\le x\le 2\end{cases}$,

which is discontinuous at $x=1$.

4. Here $f(x)=x^3-x^2+x+1$

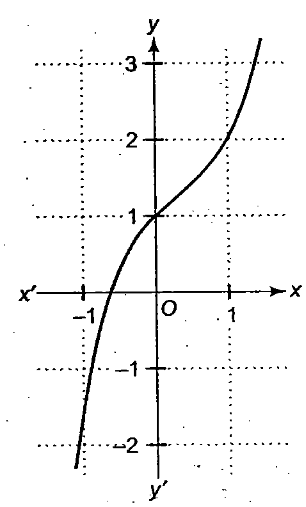


Fig. 3.55

$\Rightarrow f'(x) = 3x^2 - 2x + 1$ which is strictly increasing in $(0, 2)$

$$\therefore\ g(x) = \begin{cases} f(x); & 0 \le x \le 1 \\ 3 - x; & 1 < x \le 2 \end{cases}$$

[as $f(x)$ is increasing, $f(x)$ is maximum when $0 \le t \le x$]

$$\text{So, } g(x) = \begin{cases} x^3 - x^2 + x + 1, & 0 \le x \le 1 \\ 3 - x, & 1 < x \le 2 \end{cases}$$

$$\text{also, } g'(x) = \begin{cases} 3x^2 - 2x + 1; & 0 \le x \le 1 \\ -1; & 1 < x \le 2 \end{cases},$$

which clearly shows $g(x)$ is continuous for all $x \in [0, 2]$, but $g(x)$ is not differentiable at $x = 1$.

5. $f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x - 1, & 0 < x \le 2 \end{cases}$

$$\text{Now } f(|x|) = \begin{cases} -1, & -2 \le |x| \le 0 \\ |x| - 1, & 0 < |x| \le 2 \end{cases}$$

$\Rightarrow f(|x|) = |x| - 1,\ 0 \le |x| \le 2$

$$\Rightarrow f(|x|) = \begin{cases} -x - 1, & -2 \le x \le 0 \\ x - 1, & 0 < x \le 2 \end{cases} \qquad (1)$$

$$\text{Also } |f(x)| = \begin{cases} 1, & -2 \le x \le 0 \\ |x - 1|, & 0 < x \le 2 \end{cases}$$

$$\Rightarrow |f(x)| = \begin{cases} 1, & -2 \le x \le 0 \\ 1 - x, & 0 < x \le 1 \\ x - 1, & 1 < x \le 2 \end{cases} \qquad (2)$$

Hence, $g(x) = f(|x|) + |f(x)|$

$$\Rightarrow g(x) = \begin{cases} -x, & -2 \le x \le 0 \\ 0, & 0 < x \le 1 \\ 2x - 2, & 1 < x \le 2 \end{cases} \quad \text{[from equations (1) and (2)]}$$

$$\Rightarrow g'(x) = \begin{cases} -1, & -2 < x < 0 \\ 0, & 0 < x < 1 \\ 2, & 1 < x < 2 \end{cases}$$

Clearly, $g(x)$ is continuous but non-differentiable at $x = 0$ and 1.

6. Given that $f(x)$ is a continuous function, and $g(x)$ is a discontinuous function, then for some arbitrary real number a, we must have

$$\lim_{x \to a} f(x) = f(a) \qquad (1)$$

$$\text{and } \lim_{x \to a} g(x) \ne g(a) \qquad (2)$$

Now, $\lim_{x \to a} [f(x) + g(x)]$

$= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \ne f(a) + g(a)$

[Using equations (1) and (2)]

$\Rightarrow f(x) + g(x)$ is discontinuous.

7. Given that $f(x)$ is a function satisfying

$f(-x) = f(x), \forall x \in R$ (1)

Also $f'(0)$ exists

$\Rightarrow f'(0) = Rf'(0) = Lf'(0)$

Now, $Rf'(0) = f'(0)$

$$\Rightarrow \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = f'(0)$$

$$\Rightarrow \lim_{h \to 0} \frac{f(h) - f(0)}{h} = f'(0) \qquad (2)$$

again $Lf'(0) = f'(0)$

$$\Rightarrow \lim_{h \to 0} \frac{f(0 - h) - f(0)}{-h} = f'(0)$$

$$\Rightarrow \lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = f'(0)$$

$$\Rightarrow \lim_{h \to 0} \frac{f(h) - f(0)}{h} = -f'(0) \quad (3) \text{ [Using equation (1)]}$$

$\Rightarrow f'(0) = 0$

8. Let $g(x) = ax + b$ $\quad f(x) = \begin{cases} ax + b, & x \le 0 \\ \left(\dfrac{1 + x}{2 + x}\right)^{1/x}, & x > 0 \end{cases}$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) \Rightarrow \left(\frac{1}{2}\right)^{\infty} = b \Rightarrow b = 0$$

$$\Rightarrow f(x) = \left(\frac{1 + x}{2 + x}\right)^{1/x}, \quad f(1) = \frac{2}{3}$$

$$\Rightarrow \ln f(x) = \frac{1}{x}[\ln(1 + x) - \ln(2 + x)]$$

$$\Rightarrow \frac{f'(x)}{f(x)} = -\frac{1}{x^2}\ln\left(\frac{1 + x}{2 + x}\right) + \frac{1}{x(x + 1)(x + 2)}$$

$$\Rightarrow \frac{f'(1)}{f(1)} = \ln\frac{3}{2} + \frac{1}{6}$$

$$\Rightarrow f'(1) = \frac{2}{3}\ln\frac{3}{2} + \frac{1}{9}$$

$f(-1) = b - a$

$$\therefore\ b - a = \frac{2}{3}\ln\frac{3}{2} + \frac{1}{9}$$

$$\Rightarrow b = 0, a = -\frac{2}{3}\ln\frac{3}{2} - \frac{1}{9}$$

Hence, function $f(x) = -\left(\dfrac{2}{3}\ln\dfrac{3}{2} + \dfrac{1}{9}\right)x.$

9. Given that, $f(x) = \begin{cases} x + a\sqrt{2}\sin x, & 0 \le x < \pi/4 \\ 2x\cot x + b, & \pi/4 \le x \le \pi/2 \\ a\cos 2x - b\sin x, & \pi/2 < x \le \pi \end{cases}$ is

continuous for $0 \le x \le \pi$

$$\therefore\ \lim_{x \to \pi/4^-} f(x) = \lim_{x \to \pi/4^+} f(x)$$

$$\Rightarrow \lim_{x \to \pi/4^-} \left(x + a\sqrt{2}\sin x\right) = \lim_{x \to \pi/4^+} (2x\cot x + b)$$

$\Rightarrow \frac{\pi}{4}+a=\frac{\pi}{2}+b$ (1)

Also, $\lim_{x\to\pi/2^-} f(x)=\lim_{x\to\pi/2^+} f(x)$

$\Rightarrow \lim_{x\to\pi/2^-}(2x\cot x+b)=\lim_{x\to\pi/2^+}(a\cos 2x-b\sin x)$

$\Rightarrow 0+b=-a-b$ or $a+2b=0$ (2)

Solving (1) and (2) we have $a=\pi/6$ and $b=-\pi/12$

10. See Solution to Example 3.41.

11. Given that

$$f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2}, & x<0\\ a, & x=0\\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & x>0\end{cases}$$

Here, L.H.L. at $x=0$

$=\lim_{h\to0}\frac{1-\cos 4(0-h)}{(0-h)^2}=\lim_{h\to0}\frac{1-\cos 4h}{h^2}$

$=\lim_{h\to0}\frac{2\sin^2 2h}{4h^2}\times4=8$ R.H.L at $x=0$

$=\lim_{h\to0}\frac{\sqrt{0+h}}{\sqrt{16+\sqrt{0+h}}-4}$

$=\lim_{h\to0}\frac{\sqrt{h}\left(\sqrt{16+\sqrt{h}}+4\right)}{16+\sqrt{h}-16}$

$=\lim_{h\to0}\left(\sqrt{16+\sqrt{h}}+4\right)$

$=\sqrt{16}+4=8$

For continuity of function $f(x)$, we must have L.H.L. = R.H.L. = $f(0)$.

$\Rightarrow f(0)=8\Rightarrow a=8$

12. Given that,

$$f(x)=\begin{cases}(1+|\sin x|)^{a/|\sin x|}, & -\pi/6<x<0\\ b, & x=0\\ e^{\tan 2x/\tan 3x}, & 0<x<\pi/6\end{cases}$$

is continuous at $x=0$

$\therefore \lim_{h\to0} f(0-h)=f(0)=\lim_{h\to0} f(0+h)$

We have

$\lim_{h\to0} f(0-h)=\lim_{h\to0}\left[1+|\sin(-h)|\right]^{\frac{1}{|\sin(-h)|}}$

$=\lim_{h\to0}\left[1+\sin h\right]^{\frac{a}{\sin h}}=e^a$

and $f(0)=b$

$\therefore e^a=b$ (1)

Also $\lim_{h\to0} f(0+h)=\lim_{h\to0} e^{\tan 2h/\tan 3h}$

$=e^{\lim_{h\to0}\frac{\tan 2h}{2h}\times\frac{3h}{\tan 3h}\times\frac{2}{3}}=e^{2/3}$

$\therefore e^{2/3}=b$ (2)

From equations (1) and (2), $e^a=b=e^{2/3}$

$\Rightarrow a=2/3$ and $b=e^{2/3}$

13. See Solution to Question 38 in Objective Type Problems.

14.

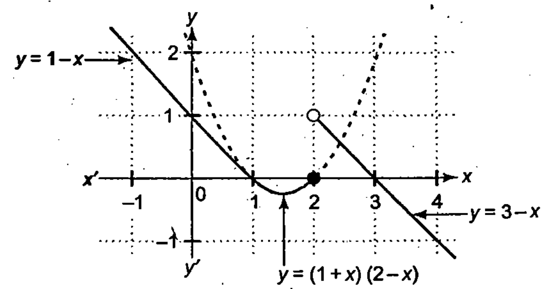


Fig. 3.56

$$f(x)=\begin{cases}1-x, & x<1\\ (1-x)(2-x), & 1\le x\le 2\\ 3-x, & x>2\end{cases} \quad (1)$$

$$\Rightarrow f'(x)\begin{cases}-1, & x<1\\ 2x-3, & 1<x<2\\ -x, & x>2\end{cases} \quad (2)$$

$f(1^-)=0, f(1^+)=0$ (from equation (1))

$f(2^-)=0, f(2^+)=1$ (from equation (2))

Hence, $f(x)$ is continuous at $x=1$, but discontinuous at $x=2$

Also $f'(1^-)=-1$ and $f'(1^+)=-1$ (from equation (2))

Hence, $f(x)$ is differentiable at $x=1$.

Hence, f is continuous and differentiable at all points except at $x=2$.

15. Let $f: R\to R$ be differentiable at $x=\alpha\in R$, then

$\lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{(x-\alpha)}=f'(\alpha)$ exists and is finite.

i.e., $Lf'(\alpha)=Rf'(\alpha)=f'(\alpha)$

$\Rightarrow \lim_{x\to\alpha-}\frac{f(x)-f(\alpha)}{(x-\alpha)}=\lim_{x\to\alpha+}\frac{f(x)-f(\alpha)}{(x-\alpha)}=f'(\alpha)$

$\lim_{x\to\alpha-} g(x)=\lim_{x\to\alpha+} g(x)=f'(\alpha)$ (1)

$[\because f(x)-f(\alpha)=g(x)(x-\alpha)]$

Again $f'(\alpha)=\lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{(x-\alpha)}$

$=\lim_{x\to\alpha} g(x)=g(\alpha)$ (2)

From equations (1) and (2), we get

$\lim_{x\to\alpha-} g(x)=\lim_{x\to\alpha+} g(x)=g(\alpha)$

L.H.L. = R.H.L. = $g(\alpha)$

$\Rightarrow g(x)$ is continuous function at $x=\alpha\in R$.

Conversely,

Assume $g(x)$ is continuous at $x=\alpha$ on R.

$\therefore \lim_{x\to\alpha} g(x)=g(\alpha)$ = a finite quantity (3)

and given $f(x)-f(\alpha)=g(x)(x-\alpha)$

for $x \neq \alpha$, $g(x) = \dfrac{f(x)-f(\alpha)}{(x-\alpha)}$ (4)

From equations (3) and (4), we get

$$\lim_{x\to\alpha} \frac{f(x)-f(\alpha)}{(x-\alpha)} = g(\alpha)$$

$\Rightarrow f'(\alpha) = g(\alpha) =$ a finite quantity

$\therefore f(x)$ is differentiable at $x = \alpha \in R$.

16. $(gof)(x) = g(f(x))$

$$= \begin{cases} f(x)+1, & \text{if } f(x)<0 \\ (f(x)-1)^2+b, & \text{if } f(x)\geq 0 \end{cases}$$

$$= \begin{cases} x+a+1, & \text{if } x+a<0 \text{ and } x<0 \\ (x+a-1)^2+b, & \text{if } x+a\geq 0 \text{ and } x<0 \\ |x-1|+1, & \text{if } |x-1|<0 \text{ and } x\geq 0 \\ (|x-1|-1)^2+b, & \text{if } |x-1|\geq 0 \text{ and } x\geq 0 \end{cases}$$

$$= \begin{cases} x+a+1, & \text{if } x<-a \\ (x+a-1)^2+b, & \text{if } -a\leq x<0 \\ |x-1|+1, & \text{if } x\in\phi \\ (|x-1|-1)^2+b, & \text{if } x\geq 0 \end{cases}$$

$$= \begin{cases} x+a+1, & \text{if } x<-a \\ (x+a-1)^2+b, & \text{if } -a\leq x<0 \\ (|x-1|-1)^2+b, & \text{if } x\geq 0 \end{cases}$$

$\Rightarrow$ Since $(gof)(x)$ is continuous for all real x, as $(gof)(x)$ is continuous at $x = -a$.

$\Rightarrow -a+a+1 = (-a+a-1)^2+b$

$\Rightarrow b = 0$

Also $(gof)(x)$ is continuous at $x = 0$.

$\Rightarrow (0+a-1)^2+b = 0+b$

$\Rightarrow a = 1$

Hence, $a = 1$ and $b = 0$

$$\text{Now, } (gof)(x) = \begin{cases} x+2, & \text{if } x<-1 \\ x^2, & \text{if } -1\leq x<0 \\ (|x-1|-1)^2, & \text{if } x\geq 0 \end{cases}$$

$$= \begin{cases} x+2, & \text{if } x<-1 \\ x^2, & \text{if } -1\leq x<0 \\ x^2, & \text{if } 0\leq x<1 \\ (x-2)^2, & \text{if } x\geq 1 \end{cases}$$

$$= \begin{cases} x+2, & \text{if } x<-1 \\ x^2, & \text{if } -1\leq x<1 \\ (x-2)^2, & \text{if } x\geq 1 \end{cases}$$

In the interval $(-1, 1)$, $(gof) = x^2$, which is differentiable at $x = 0$.

17. Given $f(2a-x) = f(x)$, $\forall x \in (a, 2a)$ (1)

and $f'(a^-) = 0 \Rightarrow \displaystyle\lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = 0$

Now $f'(-a^-) = \displaystyle\lim_{h\to 0} \frac{f(-a-h)-f(-a)}{-h}$

$$= \lim_{h\to 0} \frac{-f(a+h)+f(a)}{-h}$$

[$\because f(x)$ is an odd function]

$$= -\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Now in equation (1) replacing x by $a-h$, we get

$\Rightarrow f(a-h) = f(2a-(a-h)) = f(a+h)$

$$\Rightarrow f'(-a^-) = -\lim_{h\to 0} \frac{f(a-h)-f(a)}{h} = -f'(a^-) = 0$$

18. To find,

$$\lim_{n\to\infty} \left[(n+1)\frac{2}{\pi}\cos^{-1}\left(\frac{1}{n}\right) - n\right]$$

$$= \lim_{n\to\infty} n\left[\left(1+\frac{1}{n}\right)\frac{2}{\pi}\cos^{-1}\left(\frac{1}{n}\right) - 1\right]$$

$$= \lim_{n\to\infty} nf\left(\frac{1}{n}\right),$$

where $f(x) = \left[(1+x)\dfrac{2}{\pi}\cos^{-1} x - 1\right]$

such that $f(0) = \left[(1+0)\dfrac{2}{\pi}\cos^{-1} 0 - 1\right] = \dfrac{2}{\pi}\cdot\dfrac{\pi}{2} - 1 = 0.$

$\therefore$ Using the given relation as $\displaystyle\lim_{n\to\infty} nf\left(\frac{1}{n}\right) = f'(0)$,

the given limit becomes

$$= f'(0) = \frac{d}{dx}\left[(1+x)\frac{2}{\pi}\cos^{-1} x - 1\right]\Bigg|_{x=0}$$

$$= \frac{2}{\pi}\left[\cos^{-1} x - \frac{1+x}{\sqrt{1-x^2}}\right]\Bigg|_{x=0}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2} - 1\right] = 1 - \frac{2}{\pi} = \frac{\pi-2}{\pi}$$

19. Given that,

$$f(x) = \begin{cases} b\sin^{-1}\left(\dfrac{c+x}{2}\right), & -\dfrac{1}{2}<x<0 \\ \dfrac{1}{2} & x=0 \\ \dfrac{e^{ax/2}-1}{x}, & 0<x<1/2 \end{cases}$$

where $|c| \leq 1/2$

$f(x)$ is differentiable at $x = 0$, then $f(x)$ will also be continuous at $x = 0$.

$\Rightarrow \displaystyle\lim_{h\to 0} f(0+h) = f(0)$

$$\Rightarrow \lim_{h\to 0} \frac{e^{ah/2}-1}{h} = \frac{1}{2}$$

$$\Rightarrow \lim_{h\to 0} \frac{e^{ah/2}-1}{\frac{ah}{2}} \times \frac{a}{2} = \frac{1}{2} \Rightarrow a = 1$$

Also $Lf'(0) = Rf'(0)$

$$\Rightarrow \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{b\sin^{-1}\left(\frac{c-h}{2}\right)-\frac{1}{2}}{-h} = \lim_{h\to 0} \frac{e^{\frac{ah}{2}}-1-\frac{1}{2}}{h}$$

$$= \lim_{h\to 0} \frac{2e^{ah/2}-2-h}{2h^2}$$

For these limits to exist, we must have the 0/0 form and hence using L' Hopital's rule, we get

$$\lim_{h\to 0} \frac{\left(-\frac{1}{2}\right)\frac{b}{\sqrt{1-\left(\frac{c-h}{2}\right)^2}}}{-1} = \lim_{h\to 0} \frac{2e^{ah/2}(a/2)-1}{4h}$$

$$= \lim_{h\to 0} \frac{e^h/2-1}{8(h/2)}$$

[Putting $a = 1$]

$$\Rightarrow \frac{b}{2\sqrt{1-\frac{c^2}{4}}} = \frac{1}{8}$$

$$\Rightarrow 4b = \sqrt{1-\frac{c^2}{4}} \Rightarrow 16b^2 = \frac{4-c^2}{4} \Rightarrow 64b^2 = 4-c^2$$

20. Given that

$f(x-y) = f(x)g(y) - f(y)g(x)$ (1)

$g(x-y) = g(x)g(y) + f(x)f(y)$ (2)

In equation (1) putting $x = y$, we get

$f(0) = f(x)g(x) - f(x)g(x) \Rightarrow f(0) = 0$

Putting $y = 0$ in equation (1), we get

$f(x) = f(x)g(0) - f(0)g(x)$

$\Rightarrow f(x) = f(x)g(0)$ [using $f(0) = 0$]

$\Rightarrow g(0) = 1$

Putting $x = y$ in equation (2), we get

$g(0) = g(x)g(x) + f(x)f(x)$

$\Rightarrow 1 = [g(x)]^2 + [f(x)]^2$ [using $f(0) = 0$]

$\Rightarrow [g(x)]^2 = 1 - [f(x)]^2$ (3)

Clearly, $g(x)$ will be differentiable only if $f(x)$ is differentiable.

$\therefore$ First, let us check the differentiability of $f(x)$.

Given that $Rf'(0)$ exists,

$$\Rightarrow \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \text{ exists}$$

$$\Rightarrow \lim_{h\to 0} \frac{f(0)g(-h)-f(-h)\cdot g(0)}{h} \text{ exists}$$

$$\Rightarrow \lim_{h\to 0} \frac{-f(-h)}{h} \text{ exists} \quad (\text{using } f(0) = 0 \text{ and } g(0) = 1),$$

which can be written as $\lim_{h\to 0} \frac{f(-h)-f(0)}{-h} = Lf'(0)$

$\Rightarrow Lf'(0) = Rf'(0)$

$\therefore$ f is differentiable at $x = 0$.

By differentiating equation (3), we get

$2g(x)g'(x) = -2f(x)f'(x)$

For $x = 0$,

$\Rightarrow g(0)g'(0) = -f(0)f'(0) \quad \Rightarrow g'(0) = 0$

Objective

Fill in the blanks

1. Given $f(x) = \begin{cases} (x-1)^2 \sin\frac{1}{x-1} - |x|, & x \neq 1 \\ -1 & x = 1 \end{cases}$

The doubtful points where $f(x)$ is non-differentiable are $x = 0$ and $x = 1$

At $x = 0$, $(x-1)^2 \sin\frac{1}{x-1}$ is differentiable, but $|x|$ is not,

Hence $f(x)$ is non-differentiable at $x = 0$.

At $x = 1$, $\lim_{x\to 1^+}\left[(x-1)^2 \sin\frac{1}{x-1} - |x|\right]$

$$= \lim_{h\to 0}\left[h^2 \sin\frac{1}{h} - |1+h|\right] = -1$$

And $\lim_{x\to 1^-}\left[(x-1)^2 \sin\frac{1}{x-1} - |x|\right]$

$$\lim_{h\to 0}\left[(-h)^2 \sin\frac{1}{-h} - |1-h|\right] = -1$$

Also $f'(1^+) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$

$$= \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h} - |1+h| - (-1)}{h}$$

$$= \lim_{h\to 0} h\sin\frac{1}{h} = -1$$

Similarly $f'(1^-) = -1$.

Hence, $f(x)$ is non-differentiable at $x = 0$ only.

2. We have $f(x) = \begin{cases} \frac{x^3+x^2-16x+20}{(x-2)^2}, & x \neq 2 \\ k, & x = 2 \end{cases}$

Clearly, $f(x)$ is continuous for all values of x except possibly at $x = 2$.

It will be continuous at $x = 2$ if $\lim_{x\to 2} f(x) = f(2)$

$$\Rightarrow \lim_{x\to 2} \frac{x^3+x^2-16x+20}{(x-2)^2} = k$$

$$\Rightarrow k = \lim_{x\to 2} \frac{(x-2)^2(x+5)}{(x-2)^2} = \lim_{x\to 2}(x+5) = 7$$

$\Rightarrow k = 7$

3. By choosing any arc of the circle $x^2+y^2=4$, we can define a discontinuous function, one of which is

$f(x)=\sqrt{4-x^2}\ -2\le x\le 0.$

Hence, $f(x)=\begin{cases}\sqrt{4-x^2}, & -2\le x\le 0\\ -\sqrt{4-x^2}, & 0\le x\le 2\end{cases}$

4. We have $f(x)=x|x|=\begin{cases}-x^2, & x<0\\ x^2, & x\ge 0\end{cases}$

$\Rightarrow f'(x)=\begin{cases}-2x, & x<0\\ 2x, & x\ge 0\end{cases} \Rightarrow f''(x)=\begin{cases}-2, & x<0\\ 2, & x\ge 0\end{cases}$

Clearly, $f''(x)$ exists at every point except at $x=0$.
Thus, $f(x)$ is twice differentiable on $R-\{0\}$.

5. The domain of the given function is $x\in R\sim[-1,0)$.
Possible points of discontinuity of the function are $x=\text{integer}\sim\{-1\}$.
$f(0)=0, f(0+0)=0$. That means $f(x)$ is continuous at $x=0$.
Let $x=I_0$, where $I_0\ne -1, 0$,

then $f(I_0)=I_0\sin\dfrac{\pi}{(I_0+1)}$,

$f(I_0-0)=(I_0-1)\sin\dfrac{\pi}{I_0}$,

$f(I_0+0)=I_0\sin\dfrac{\pi}{I_0+1}$

Thus, $f(x)$ is discontinuous at $x=I_0$.

6. As $f(x)$ is continuous in [1, 3], $f(x)$ will attain all values between $f(1)$ and $f(3)$. As $f(x)$ takes rational values for all x and there are innumerable irrational values between $f(1)$ and $f(3)$ which implies that $f(x)$ can take rational values for all x if $f(x)$ has a constant value at all points between $x=1$ and $x=3$. Given that $f(2)=10$, then $f(1.5)=10$.

Multiple choice questions with one correct answer

1.d. $f(x)=\dfrac{\tan(\pi[x-\pi])}{1+[x]^2}$

By definition, $[x-\pi]$ is an integer whatever be the value of x and so $\pi[x-\pi]$ is an integral multiple of π.
Consequently, $\tan(\pi[x-\pi])=0, \forall x$.
And since $1+[x]^2\ne 0$ for any x, we conclude that $f(x)=0$.
Thus $f(x)$ is constant function and so it is continuous and differentiable.

2.b. $0\le\tan^2 x<1$ when $-\dfrac{\pi}{4}<x<\dfrac{\pi}{4}$

$\Rightarrow f(x)=0\ -\dfrac{\pi}{4}<x<\dfrac{\pi}{4}$

Hence, $f(x)$ is continuous and differentiable at $x=0$, also $f'(0)=0$.

3.c. When x is not an integer, both the functions $[x]$ and $\cos\left(\dfrac{2x-1}{2}\right)\pi$ are continuous.

$\therefore f(x)$ is continuous on all non-integral points.
For $x=n\in I$,

$$\lim_{x\to n-} f(x)=\lim_{x\to n-}[x]\cos\left(\frac{2x-1}{2}\right)\pi$$
$$=(n-1)\cos\left(\frac{2n-1}{2}\right)\pi=0.$$
$$\lim_{x\to n+} f(x)=\lim_{x\to n+}[x]\cos\left(\frac{2x-1}{2}\right)\pi$$
$$=n\cos\left(\frac{2n-1}{2}\right)\pi=0$$

Also $f(n)=n\cos\dfrac{(2n-1)\pi}{2}=0.$

$\therefore f$ is continuous at all integral points as well. Thus, f is continuous everywhere.

4.d. Let k is integer
$f(k)=0, f(k-0)=(k-1)^2-(k^2-1)=2-2k$
$f(k+0)=k^2-(k^2)=0$
If $f(x)$ is continuous at $x=k$, then $2-2k=0$
$\Rightarrow k=1$

5.d. $f(x)=(x^2-1)|x^2-3x+2|+\cos(|x|)$
$=[(x-1)|x-1|]|x-2|+\cos x$
$(x-1)|x-1|$ and $\cos x$ are differentiable for all x.
But $|x-2|$ is non-differentiable at $x=2$
Hence, $f(x)$ is non-differentiable at $x=2$.

6.a. L.H.D. at $x=k$

$$=\lim_{h\to 0}\frac{f(k)-f(k-h)}{h} \qquad (k=\text{integer})$$
$$=\lim_{h\to 0}\frac{[k]\sin k\pi-[k-h]\sin(k-h)\pi}{h}$$
$$=\lim_{h\to 0}\frac{-(k-1)\sin(k\pi-h\pi)}{h} \qquad [\because \sin k\pi=0]$$
$$=\lim_{h\to 0}\frac{-(k-1)(-1)^{k-1}\sin h\pi}{h\pi}\times\pi=\pi(k-1)(-1)^k.$$

7.d.

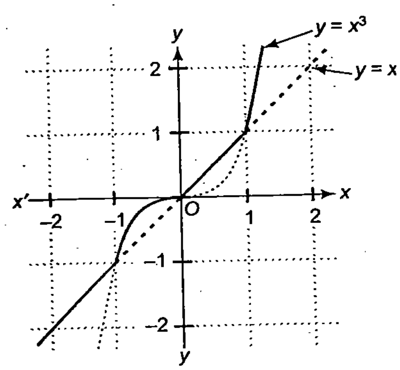


Fig. 3.57

From the graph $f(x)=\max\{x,x^3\}=\begin{cases}x, & x<-1\\ x^3, & -1\le x\le 0\\ x, & 0<x<1\\ x^3, & x\ge 1\end{cases}$

Clearly, f is not differentiable at -1, 0 and 1.

8.d. $f(x) = \cos(|x|) + |x| = \cos x + |x|$ is non-differentiable at $x = 0$ as $|x|$ is non-differentiable at $x = 0$. Similarly $f(x) = \cos(|x|) - |x| = \cos x - |x|$ is non-differentiable at $x = 0$.

$$f(x) = \sin|x| + |x| = \begin{cases} -\sin x - x, & x < 0 \\ +\sin x + x, & x \ge 0 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -\cos x - 1, & x < 0 \\ +\cos x + 1, & x \ge 0 \end{cases}$$

which is not differentiable at $x = 0$.

$$f(x) = \sin|x| - |x| = \begin{cases} -\sin x + x, & x < 0 \\ \sin x - x, & x \ge 0 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -\cos x + 1, & x < 0 \\ +\cos x - 1, & x \ge 0 \end{cases}$$

$\therefore$ f is differentiable at $x = 0$.

9.d. The given function is $f(x) = \begin{cases} \tan^{-1} x, & \text{if } |x| \le 1 \\ \frac{1}{2}(|x| - 1), & \text{if } |x| > 1 \end{cases}$

$$\Rightarrow f(x) = \begin{cases} \frac{1}{2}(-x-1), & \text{if } x < -1 \\ \tan^{-1} x, & \text{if } -1 \le x \le 1 \\ \frac{1}{2}(x-1), & \text{if } x > 1 \end{cases}$$

Clearly, $f(x)$ is discontinuous at $x = 1$ and -1 and hence non-differentiable at $x = 1$ and -1. Hence, $f(x)$ is differentiable for $R - \{-1, 1\}$.

10.a. $f(x) = ||x| - 1|$ is non-differentiable when $|x| = 0$ and when $|x| - 1 = 0$ or $x = 0$ and $x = \pm 1$.

Alternative method

The graph of $y = ||x| - 1|$ is as follows

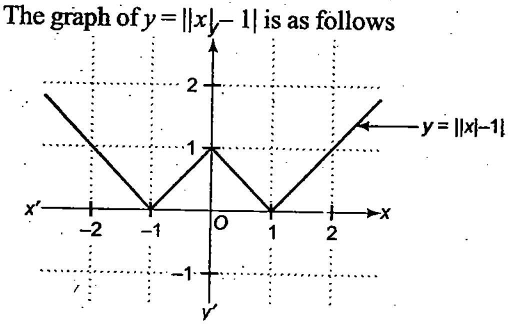


Fig. 3.58

Which has sharp turn at $x = -1$, 0, and 1 and hence not differentiable at $x = -1, 0, 1$.

11.b. Given that $f(x)$ is a continuous and differentiable function and $f\left(\frac{1}{x}\right) = 0$, $x = n$, $n \in I$

$\therefore$ $f(0^+) = f\left(\frac{1}{\infty}\right) = 0$

Since R.H.L. = 0, $\therefore$ $f(0) = 0$ for $f(x)$ to be continuous

Also $f'(0) = \lim\limits_{h \to 0} \dfrac{f(h) - f(0)}{h - 0} = \lim\limits_{h \to 0} \dfrac{f(h)}{h} = 0$

[Using $f(0) = 0$]
[$\because f(0^+) = 0$]

Hence, $f(0) = 0$, $f'(0) = 0$

Multiple choice question with one or more than one correct answer

1. a, b, d.

Given that $x + |y| = 2y$

If $y < 0$, then $x - y = 2y \Rightarrow y = x/3 \Rightarrow x < 0$

If $y = 0$, then $x = 0$.

If $y > 0$, then $x + y = 2y \Rightarrow y = x \Rightarrow x > 0$

Thus, we can define $f(x) = y = \begin{cases} x/3, & x < 0 \\ x, & x \ge 0 \end{cases}$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} 1/3, & x < 0 \\ 1, & x > 0 \end{cases}$$

Clearly, y is continuous but non-differentiable at $x = 0$.

2. b, d, e.

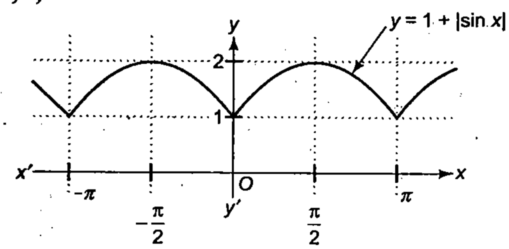


Fig. 3.59

$|\sin x|$ is continuous for all but not differentiable when $\sin x = 0$ (where $\sin x$ crosses x-axis) or $x = n\pi$, $n \in Z$.

3. a, b, d.

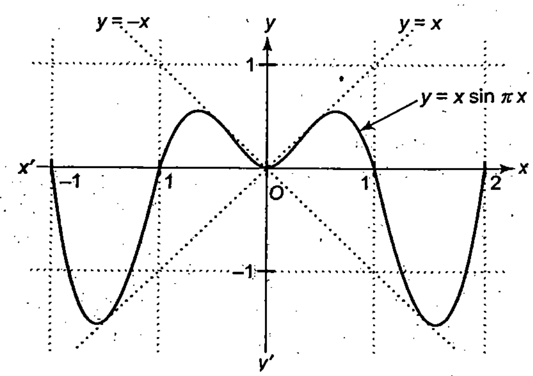


Fig. 3.60

From the graph, $0 \le x \sin \pi x < 1$, for $x \in [-1, 1]$.

Hence, $f(x) = 0$, $x \in [-1, 1]$.

4. a.

$f(x) = \dfrac{x}{1 + |x|}$ is differentiable everywhere except probably at $x = 0$.

For $x = 0$

$$Lf'(0) = \lim_{h \to 0} \frac{f(0 - h) - f(0)}{-h} = \lim_{h \to 0} \frac{\frac{-h}{1+h} - 0}{-h} = 1$$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{h}{1+h} - 0}{h} = 1$$

$Lf'(0) = Rf'(0)$

$\Rightarrow$ f is differentiable at $x = 0$.

Hence, f is differentiable in $(-\infty, \infty)$.

5. a, b, c.

$$f(x)=\begin{cases}|x-3|, & x\ge 1\\ \dfrac{x^2}{4}-\dfrac{3x}{2}+\dfrac{13}{4}, & x<1\end{cases}=\begin{cases}\dfrac{x^2}{4}-\dfrac{3x}{2}+\dfrac{13}{4}, & x<1\\ 3-x, & 1\le x<3\\ x-3, & x\ge 3\end{cases}$$

$$\Rightarrow f'(x)=\begin{cases}\dfrac{x}{2}-\dfrac{3}{2}, & x<1\\ -1, & 1<x<3\\ 1, & x>3\end{cases}$$

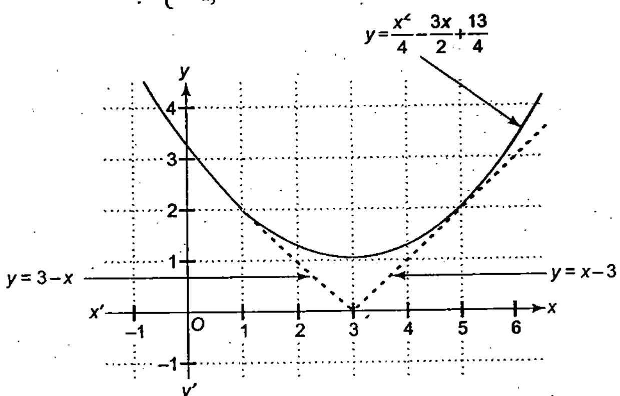


Fig. 3.61

Clearly, $f(x)$ is non-differentiable at $x=3$.
For $x=1$, where function changes its definition

$$f(1^-)=\lim_{x\to 1}\left[\frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}\right]=\frac{1}{4}-\frac{3}{2}+\frac{13}{4}=2$$

$$f(1^+)=\lim_{x\to 1}|x-3|=2$$

$Lf'(1^-)=-1, Rf'(1^+)=-1$

Hence, $f(x)$ is differentiable at $x=1$.
Hence, $f(x)$ is continuous for all x but non-differentiable at $x=3$.

6. d

$$x\in[0,\pi]\Rightarrow\frac{x-2}{2}\in\left[-1,\frac{\pi}{2}-1\right]$$

$\dfrac{1}{f(x)}=\dfrac{2}{x-2}$, which is continuous in $(-\infty,\infty)\sim\{2\}$.

$\tan(f(x))$ is continuous in $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$.

$f^{-1}(x)=2(x+1)$ which is clearly continuous but $\tan(f^{-1}(x))$ is not continuous.

7. b, c.

On $(0,\pi)$,

a. $\tan x=f(x)$
we know $\tan x$ is discontinuous at $x=\pi/2$.

b. $f(x)=\int_0^x t\sin\left(\dfrac{1}{t}\right)dt$

$\Rightarrow f'(x)=x\sin\left(\dfrac{1}{x}\right)$ which is well-defined on $(0,\pi)$

$\therefore f(x)$ being differentiable is continuous on $(0,\pi)$

c. $f(x)=\begin{cases}1, & 0<x\le 3\pi/4\\ 2\sin\dfrac{2x}{9}, & 3\pi/4<x<\pi.\end{cases}$

Clearly, $f(x)$ is continuous on $(0,\pi)$ except possibly at $x=3\pi/4$, where

$$\text{L.H.L.}=\lim_{h\to 0}f\left(\frac{3\pi}{4}-h\right)=\lim_{x\to 0}1=1$$

$$\text{R.H.L.}=\lim_{h\to 0}f\left(\frac{3\pi}{4}+h\right)=\lim_{x\to 0}2\sin\frac{2}{9}\left(\frac{3\pi}{4}+h\right)$$

$$=\lim_{h\to 0}2\sin\left(\frac{\pi}{6}+\frac{2h}{9}\right)=2\sin\frac{\pi}{6}=2\times\frac{1}{2}=1$$

Also $f\left(\dfrac{3\pi}{4}\right)=1.$

As L.H.L. = R.H.L. $=f\left(\dfrac{3\pi}{4}\right)$ $\therefore f(x)$ is continuous on $(0,\pi)$.

d. $f(x)=\begin{cases}x\sin x, & 0<x\le\pi/2\\ \dfrac{\pi}{2}\sin(\pi+x), & \dfrac{\pi}{2}<x<\pi\end{cases}$

Here $f(x)$ will be continuous on $(0,\pi)$ if it is continuous at $x=\pi/2$. At $x=\pi/2$,

$$\text{L.H.L.}=\lim_{h\to 0}f\left(\frac{\pi}{2}-h\right)$$

$$=\lim_{h\to 0}\left(\frac{\pi}{2}-h\right)\sin\left(\frac{\pi}{2}-h\right)=\frac{\pi}{2}\sin\frac{\pi}{2}=\frac{\pi}{2}$$

$$\text{R.H.L.}=\lim_{h\to 0}f\left(\frac{\pi}{2}+h\right)=\lim_{h\to 0}\frac{\pi}{2}\sin\left(\pi+\frac{\pi}{2}+h\right)$$

$$=\frac{\pi}{2}\sin\left(\pi+\frac{\pi}{2}\right)=\frac{-\pi}{2}\sin\frac{\pi}{2}=-\frac{\pi}{2}$$

As L.H.L. $\ne$ R.H.L. $\therefore f(x)$ is not continuous.

8. a, c, d.

From the figure, it is clear that $h(x)=\begin{cases}x, & \text{if } x\le 0\\ x^2, & \text{if } 0<x<1\\ x, & \text{if } x\ge 1\end{cases}$

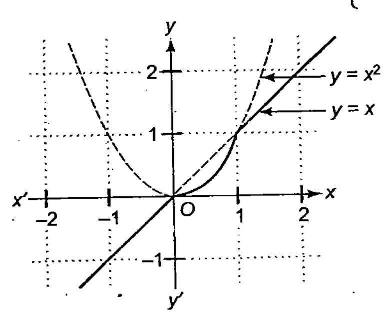


Fig. 3.62

From the graph, it is clear that $h(x)$ is continuous for all $x\in R$, $h'(x)=1$ for all $x>1$, and h is not differentiable at $x=0$ and 1.

9. b, c, d.

$$f(x)=\begin{cases}0, & x<0\\ x^2, & x\ge 0\end{cases}\Rightarrow f'(x)=\begin{cases}0, & x<0\\ 2x, & x>0\end{cases}$$

which exists $\forall\, x$ except possibly at $x=0$.

At $x=0, Lf'=0=Rf'$

$\Rightarrow f$ is differentiable.

Clearly, f' is non-differentiable.

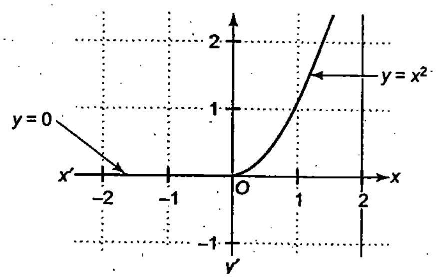


Fig. 3.63

10. a, b.

We have $g(x)=\begin{cases} x^2\sin\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0 \end{cases}$

If $x\neq 0$, $g'(x)=x^2\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)+2x\sin\left(\frac{1}{x}\right)$

$$=-\cos\left(\frac{1}{x}\right)+2x\sin\left(\frac{1}{x}\right)$$

which exists for $\forall\, x\neq 0$

If $x=0$,

then $g'(0)=\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}=\lim_{x\to 0}\frac{x^2\sin(1/x)-0}{x-0}$

$$=\lim_{x\to 0}x\sin\left(\frac{1}{x}\right)=0$$

$$\Rightarrow g'(x)=\begin{cases} -\cos\left(\frac{1}{x}\right)+2x\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

At $x=0$, $\cos\left(\frac{1}{x}\right)$ is not continuous, therefore $g'(x)$ is not continuous at $x=0$. At $x=0$,

$$Lf'=\lim_{x\to 0}\frac{0-(-x)\sin\sin\left(-\frac{1}{x}\right)}{x}=\sin\left(\frac{1}{x}\right)$$

which does not exist.

11. a, c.

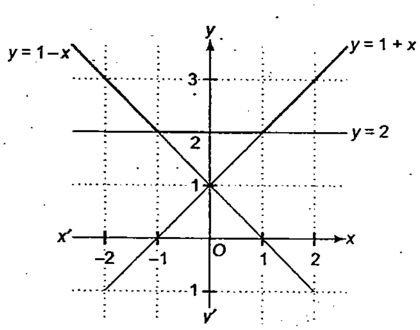


Fig. 3.64

From the graph, it is clear that $f(x)$ is continuous everywhere and also differentiable everywhere except at $x=1$ and -1.

12. a, c.

From the graph, $f(x)$ is continuous everywhere, but not differentiable at $x=1$.

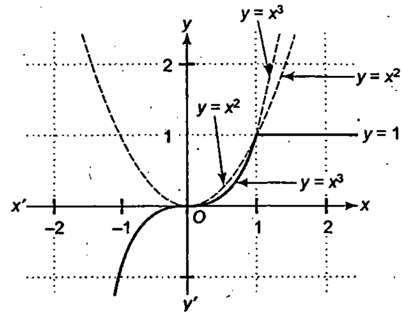


Fig. 3.65

[Using $f(0)=0$ and $g(0)=1$]

13. a, b, c, d

$$\lim_{x\to -\frac{\pi}{2}^-} f(x)=0$$

$$\lim_{x\to -\frac{\pi}{2}^+} f(x)=\cos\left(-\frac{\pi}{2}\right)=0$$

$$f'(x)=\begin{cases} -1, & x<-\pi/2 \\ \sin x, & -\pi/2<x<0 \\ 1, & 0<x<1 \\ 1/x, & x>1 \end{cases}$$

Clearly, $f(x)$ is not differentiable at $x=0$ as $f'(0^-)=0$ and $f'(0^+)=1$.

$f(x)$ is differentiable at $x=1$ as $f'(1^-)=f'(1^+)=1$.

14. b, c.

$\because f(0)=0$

and $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{f(h)}{h}=f'(0)=k\text{(say)}$$

$\Rightarrow f(x)=kx+c\Rightarrow f(x)=kx \qquad (\because f(0)=0)$

Match the following type questions

1. a. p, q, r. $y=x|x|$

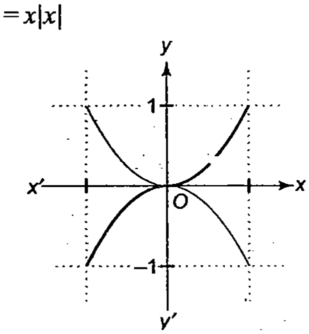


Fig. 3.66

From the graph, $f(x)$ is continuous and differentiable in $(-1, 1)$. Also $f(x)$ is strictly increasing.

b. p, s. $y=\sqrt{|x|}$

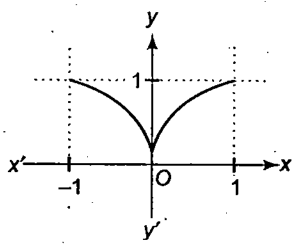


Fig. 3.67

From the graph, $f(x)$ is continuous in $(-1, 1)$, but non-differentiable at $x=0$.

c. r, s. $y=x+[x]$

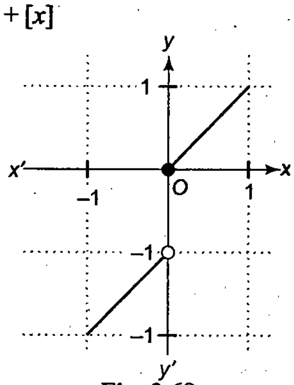


Fig. 3.68

From the graph, $f(x)$ is discontinuous at $x=0$. Also $f(x)$ is increasing.

d. p, q. $y=|x-1|$

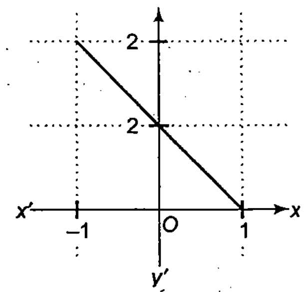


Fig. 3.69

From the graph, $f(x)$ is continuous and differentiable in $(-1, 1)$.