CHAPTER 12 Differentiation

Section-A JEE Advanced/ IIT-JEE

A Fill in the Blanks

1. If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin x^2$, then $\frac{dy}{dx} =$ .... *(1982 - 2 Marks)*

2. If $f_r(x),\ g_r(x),\ h_r(x)$, $r = 1, 2, 3$ are polynomials in x such that $f_r(a) = g_r(a) = h_r(a)$, $r = 1, 2, 3$

and $F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$ then $F'(x)$ at $x = a$ is .... *(1985 - 2 Marks)*

3. If $f(x) = \log_x(\ln x)$, then $f'(x)$ at $x = e$ is .... *(1985 - 2 Marks)*

4. The derivative of $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ with respect to $\sqrt{1-x^2}$ at $x = \frac{1}{2}$ is .... *(1986 - 2 Marks)*

5. If $f(x) = |x-2|$ and $g(x) = f[f(x)]$, then $g'(x) =$ .... for $x > 20$ *(1990 - 2 Marks)*

6. If $xe^{xy} = y + \sin^2 x$, then at $x = 0$, $\frac{dy}{dx} =$ .... *(1996 - 1 Mark)*

B True/ False

1. The derivative of an even function is always an odd function. *(1983 - 1 Mark)*

C MCQs with One Correct Answer

1. If $y^2 = P(x)$, a polynomial of degree 3, then $2\frac{d}{dx}\left(y^3 \frac{d^2y}{dx^2}\right)$ equals *(1988 - 2 Marks)*
   (a) $P'''(x) + P'(x)$ (b) $P''(x)P'''(x)$
   (c) $P(x)P'''(x)$ (d) a constant

2. Let $f(x)$ be a quadratic expression which is positive for all the real values of x. If $g(x) = f(x) + f'(x) + f''(x)$, then for any real x, *(1990 - 2 Marks)*
   (a) $g(x) < 0$ (b) $g(x) > 0$
   (c) $g(x) = 0$ (d) $g(x) \geq 0$

3. If $y = (\sin x)^{\tan x}$, then $\frac{dy}{dx}$ is equal to *(1994)*
   (a) $(\sin x)^{\tan x}(1 + \sec^2 x \log \sin x)$
   (b) $\tan x (\sin x)^{\tan x - 1}.\cos x$
   (c) $(\sin x)^{\tan x} \sec^2 x \log \sin x$
   (d) $\tan x (\sin x)^{\tan x - 1}$

4. If $x^2 + y^2 = 1$ then *(2000)*
   (a) $yy'' - 2(y')^2 + 1 = 0$ (b) $yy'' + (y')^2 + 1 = 0$
   (c) $yy'' + (y')^2 - 1 = 0$ (d) $yy'' + 2(y')^2 + 1 = 0$

5. Let $f: (0, \infty) \to R$ and $F(x) = \int_0^x f(t)dt$. If $F(x^2) = x^2(1+x)$, then $f(4)$ equals *(2001S)*
   (a) 5/4 (b) 7 (c) 4 (d) 2

6. If y is a function of x and $\log(x+y) - 2xy = 0$, then the value of $y'(0)$ is equal to *(2004S)*
   (a) 1 (b) -1 (c) 2 (d) 0

7. If $f(x)$ is a twice differentiable function and given that $f(1) = 1; f(2) = 4, f(3) = 9$, then *(2005S)*
   (a) $f''(x) = 2$ for $\forall\ x \in (1, 3)$
   (b) $f''(x) = f'(x) = 5$ for some $x \in (2, 3)$
   (c) $f''(x) = 3$ for $\forall\ x \in (2, 3)$
   (d) $f''(x) = 2$ for some $x \in (1, 3)$

8. $\frac{d^2x}{dy^2}$ equals *(2007 -3 marks)*
   (a) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (b) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$
   (c) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (d) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

9. Let $g(x) = \log f(x)$ where $f(x)$ is twice differentible positive function on $(0, \infty)$ such that $f(x+1) = x f(x)$. Then, for $N = 1, 2, 3, ...$ *(2008)*

$$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$$

(a) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N-1)^2}\right\}$

(b) $4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N-1)^2}\right\}$

(c) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N+1)^2}\right\}$

(d) $4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N+1)^2}\right\}$

10. Let $f : [0, 2] \to R$ be a function which is continuous on $[0, 2]$ and is differentiable on $(0, 2)$ with $f(0) = 1$. Let

$F(x) = \int_0^{x^2} f\left(\sqrt{t}\right)dt$ for $x \in [0,2]$. If $F'(x) = f'(x)$ for all

$x \in (0,2)$, then $F(2)$ equals *(JEE Adv. 2014)*

(a) $e^2 - 1$ (b) $e^4 - 1$
(c) $e - 1$ (d) e^4

D MCQs with One or More than One Correct

1. Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in \mathbb{R}$. Then *(JEE Adv. 2016)*

(a) $g'(2) = \frac{1}{15}$ (b) $h'(1) = 666$

(c) $h(0) = 16$ (d) $h(g(3)) = 36$

E Subjective Problems

1. Find the derivative of $\sin(x^2 + 1)$ with respect to x from first principle. *(1978)*

2. Find the derivative of

$$f(x) = \begin{cases} \dfrac{x-1}{2x^2-7x+5} & \text{when } x \neq 1 \\ -\dfrac{1}{3} & \text{when } x = 1 \end{cases}$$

at $x = 1$ *(1979)*

3. Given $y = \frac{5x}{3\sqrt{(1-x)^2}} + \cos^2(2x+1)$; Find $\frac{dy}{dx}$. *(1980)*

4. Let $y = e^{x\sin x^3} + (\tan x)^x$. Find $\frac{dy}{dx}$ *(1981 - 2 Marks)*

5. Let f be a twice differentiable function such that

$f''(x) = -f(x)$, and $f'(x) = g(x)$, $h(x) = [f(x)]^2 + [g(x)]^2$

Find $h(10)$ if $h(5) = 11$ *(1982 - 3 Marks)*

6. If α be a repeated root of a quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$ and $C(x)$ be polynomials of degree 3, 4 and 5

respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is

divisible by $f(x)$, where prime denotes the derivatives. *(1984 - 4 Marks)*

7. If $x = \sec\theta - \cos\theta$ and $y = \sec^n\theta - \cos^n\theta$, then show

that $(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$ *(1989 - 2 Marks)*

8. Find $\frac{dy}{dx}$ at $x = -1$, when

$$(\sin y)^{\sin\left(\frac{\pi}{2}x\right)} + \frac{\sqrt{3}}{2}\sec^{-1}(2x) + 2^x \tan(\ln(x+2)) = 0$$

(1991 - 4 Marks)

9. If $y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$,

prove that $\frac{y'}{y} = \frac{1}{x}\left(\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right)$.

(1998 - 8 Marks)

H Assertion & Reason Type Questions

1. Let $f(x) = 2 + \cos x$ for all real x.

STATEMENT - 1 : For each real t, there exists a point c in $[t, t+\pi]$ such that $f'(c) = 0$ because

STATEMENT - 2 : $f(t) = f(t + 2\pi)$ for each real t.

(2007 -3 marks)

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True.

2. Let f and g be real valued functions defined on interval $(-1, 1)$ such that $g''(x)$ is continuous, $g(0) \neq 0$. $g'(0) = 0$, $g''(0) \neq 0$, and $f(x) = g(x)\sin x$

STATEMENT - 1 : $\lim_{x\to 0} [g(x)\cot x - g(0)\operatorname{cosec} x] = f''(0)$

and

STATEMENT - 2 : $f'(0) = g(0)$ ***(2008)***

(a) Statement - 1 is True, Statement - 2 is True; Statement - 2 is a correct explanation for Statement - 1
(b) Statement - 1 is True, Statement - 2 is True; Statement - 2 **is NOT** a correct explaination for Statement - 1
(c) Statement - 1 is True, Statement - 2 is False
(d) Statement - 1 is False, Statement - 2 is True

I Integer Value Correct Type

1. If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is ***(2009)***

2. Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.

Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is ***(2011)***

Section-B JEE Main / AIEEE

1. If $y = (x + \sqrt{1+x^2})^n$, then $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx}$ is **[2002]**

(a) n^2y (b) $-n^2y$ (c) $-y$ (d) $2x^2y$

2. If $f(y) = e^y$, $g(y) = y$; $y > 0$ and

$F(t) = \int_0^t f(t-y)g(y)\,dy$, then **[2003]**

(a) $F(t) = te^{-t}$ (b) $F(t) = 1 - te^{-t}(1+t)$

(c) $F(t) = e^t - (1+t)$ (d) $F(t) = te^t$.

3. If $f(x) = x^n$, then the value of **[2003[**

$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \dots\dots\dots \frac{(-1)^n f^n(1)}{n!}$ is

(a) 1 (b) 2^n (c) $2^n - 1$ (d) 0

4. Let $f(x)$ be a polynomial function of second degree. If $f(1) = f(-1)$ and a, b, c are in A. P, then $f'(a), f'(b), f'(c)$ are in **[2003]**

(a) Arithmetic -Geometric Progression
(b) A.P
(c) G.P
(d) H.P.

5. If $x = e^{y + e^{y + e^{y + \dots\infty}}}$, $x > 0$, then $\frac{dy}{dx}$ is **[2004]**

(a) $\frac{1+x}{x}$ (b) $\frac{1}{x}$ (c) $\frac{1-x}{x}$ (d) $\frac{x}{1+x}$

6. The value of a for which the sum of the squares of the roots of the equation $x^2 - (a-2)x - a - 1 = 0$ assume the least value is **[2005]**

(a) 1 (b) 0 (c) 3 (d) 2

7. If the roots of the equation $x^2 - bx + c = 0$ be two consecutive integers, then $b^2 - 4c$ equals **[2005]**

(a) -2 (b) 3 (c) 2 (d) 1

8. Let $f: R \to R$ be a differentiable function having $f(2) = 6$, $f'(2) = \left(\frac{1}{48}\right)$. Then $\lim_{x\to 2} \int_6^{f(x)} \frac{4t^3}{x-2}dt$ equals **[2005]**

(a) 24 (b) 36 (c) 12 (d) 18

9. The set of points where $f(x) = \frac{x}{1+|x|}$ is differentiable is

(a) $(-\infty, 0) \cup (0, \infty)$ (b) $(-\infty, -1) \cup (-1, \infty)$
(c) $(-\infty, \infty)$ (d) $(0, \infty)$ **[2006]**

10. If $x^m . y^n = (x+y)^{m+n}$, then $\frac{dy}{dx}$ is **[2006]**

(a) $\frac{y}{x}$ (b) $\frac{x+y}{xy}$ (c) xy (d) $\frac{x}{y}$

11. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then $y'(1)$ equals **[2009]**

(a) 1 (b) $\log 2$ (c) $-\log 2$ (d) -1

12. Let $f: (-1, 1) \to \mathbf{R}$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x) + 2)]^2$. Then $g'(0) =$ **[2010]**

(a) -4 (b) 0 (c) -2 (d) 4

13. $\dfrac{d^2x}{dy^2}$ equals : **[2011]**

(a) $-\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$ (b) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-2}$

(c) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$ (d) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$

14. If $y = \sec(\tan^{-1}x)$, then $\dfrac{dy}{dx}$ at $x = 1$ is equal to : **[JEE M 2013]**

(a) $\dfrac{1}{\sqrt{2}}$ (b) $\dfrac{1}{2}$ (c) 1 (d) $\sqrt{2}$

15. If g is the inverse of a function f and $f'(x) = \dfrac{1}{1+x^5}$, then $g'(x)$ is equal to: **[JEE M 2014]**

(a) $\dfrac{1}{1+\{g(x)\}^5}$ (b) $1+\{g(x)\}^5$

(c) $1+x^5$ (d) $5x^4$

16. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha\log|x| + \beta x^2 + x$ then **[JEE M 2014]**

(a) $\alpha = 2, \beta = -\dfrac{1}{2}$ (b) $\alpha = 2, \beta = \dfrac{1}{2}$

(c) $\alpha = -6, \beta = \dfrac{1}{2}$ (d) $\alpha = -6, \beta = -\dfrac{1}{2}$

12 Differentiation

Section-A : JEE Advanced/ IIT-JEE

A **1.** $\frac{2+2x-2x^2}{(x^2+1)^2}\sin\left(\frac{2x-1}{x^2+1}\right)^2$ **2.** zero **3.** $1/e$ **4.** 4 **5.** 1 **6.** 1

B **1.** T

C **1.** (c) **2.** (b) **3.** (a) **4.** (b) **5.** (c) **6.** (a)
7. (d) **8.** (d) **9.** (a) **10.** (b)

D **1.** (b, c)

E **1.** $2x\cos(x^2+1)$ **2.** $\frac{-2}{9}$

3. $\frac{5}{3}.\frac{1}{(1-x)^2}-2\sin(4x+2)$, if $x<1$; $-\frac{5}{3}.\frac{1}{(1-x)^2}-2\sin(4x+2)$, if $x>1$

4. $e^{x\sin x^3}\left[\sin x^3+3x^3\cos x^3\right]+(\tan x)^x\left[\frac{2x}{\sin 2x}+\log\tan x\right]$ **5.** 11 **8.** 0

H **1.** (b) **2.** (a)

I **1.** 2 **2.** 1

Section-B : JEE Main/ AIEEE

1. (a) **2.** (c) **3.** (d) **4.** (b) **5.** (c) **6.** (a) **7.** (d)
8. (d) **9.** (c) **10.** (a) **11.** (d) **12.** (a) **13.** (c) **14.** (a)
15. (b) **16.** (a)

Section-A JEE Advanced/ IIT-JEE

A. Fill in the Blanks

1. $\frac{2+2x-2x^2}{(x^2+1)^2}\sin\left(\frac{2x-1}{x^2+1}\right)^2$

Given: $y=f\left(\frac{2x-1}{x^2+1}\right); f'(x)=\sin x^2$

$$\therefore \quad \frac{dy}{dx}=f'\left(\frac{2x-1}{x^2+1}\right).\frac{d}{dx}\left(\frac{2x-1}{x^2+1}\right)$$

$$=\sin\left(\frac{2x-1}{x^2+1}\right)^2.\frac{2(x^2+1)-2x(2x-1)}{(x^2+1)^2}$$

$$=\frac{2+2x-2x^2}{(x^2+1)^2}\sin\left(\frac{2x-1}{x^2+1}\right)^2$$

2. Given that $F(x)=\begin{vmatrix} f_1(x) & f_2(x) & f_3(x)\\ g_1(x) & g_2(x) & g_3(x)\\ h_1(x) & h_2(x) & h_3(x)\end{vmatrix}$ ...(1)

Where $f_r(x), g_r(x), h_r(x), r=1,2,3$, are polynominals in x and hence differentiable and

$f_r(a)=g_r(a)=h_r(a), r=1,2,3$ ...(2)

Differentiating eq. (1) with respect to x, we get

$$F'(x)=\begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x)\\ g_1(x) & g_2(x) & g_3(x)\\ h_1(x) & h_2(x) & h_3(x)\end{vmatrix}$$

$$+\begin{vmatrix} f_1(x) & f_2(x) & f_3(x)\\ g_1'(x) & g_2'(x) & g_3'(x)\\ h_1(x) & h_2(x) & h_3(x)\end{vmatrix}+\begin{vmatrix} f_1(x) & f_2(x) & f_3(x)\\ g_1(x) & g_2(x) & g_3(x)\\ h_1'(x) & h_2'(x) & h_3'(x)\end{vmatrix}$$

$$\therefore \quad F'(a)=\begin{vmatrix} f_1'(a) & f_2'(a) & f_3'(a)\\ g_1(a) & g_2(a) & g_3(a)\\ h_1(a) & h_2(a) & h_3(a)\end{vmatrix}$$

$$+\begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1'(a) & g_2'(a) & g_3'(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h_1'(a) & h_2'(a) & h_3'(a) \end{vmatrix}$$

$F'(a) = D_1 + D_2 + D_3$

Using eq. (2) we get $D_1 = D_2 = D_3 = 0$ [By the property of determinants that $D = 0$ if two rows in D are identical]

$\therefore \quad F'(a) = 0.$

3. Given that

$$f(x) = \log_x(ln\,x) = \frac{\log_e(\log_e x)}{(\log_e x)}$$

$$f'(x) = \frac{\frac{1}{\log_e x} \times \frac{1}{x} \times \log_e x - \frac{1}{x}\log_e(\log_e x)}{(\log_e x)^2}$$

$$= \frac{\frac{1}{x}[1 - \log_e(\log_e x)]}{(\log_x x)^2}$$

$$f'(e) = \frac{\frac{1}{e}[1 - \log_e(\log_e e)]}{(\log_e e)^2} = \frac{\frac{1}{e}[1 - \log_e 1]}{(1)^2} = \frac{1}{e}(1-0) = \frac{1}{e}.$$

4. Let $u = \sec^{-1}\left(\frac{1}{2x^2 - 1}\right); v = \sqrt{1 - x^2}$

Then to find $\left.\frac{du}{dv}\right|_{x=1/2}$, we have

$u = \cos^{-1}(2x^2 - 1) = 2\cos^{-1} x$

$\therefore \quad \frac{du}{dx} = \frac{-2}{\sqrt{1-x^2}}$ and $\quad v = \sqrt{1-x^2}$

$\therefore \quad \frac{dv}{dx} = \frac{-x}{\sqrt{1-x^2}} \quad \therefore \quad \frac{du}{dv} = \frac{\frac{-2}{\sqrt{1-x^2}}}{\frac{-x}{\sqrt{1-x^2}}} = \frac{2}{x}$

$\therefore \quad \left.\frac{du}{dv}\right|_{x=\frac{1}{2}} = 4$

5. $f(x) = |x - 2|$

$\Rightarrow \quad g(x) = f(f(x)) = |f(x) - 2|$ as $x > 20$

$= ||x-2| - 2| = |x - 2 - 2|$ as $x > 20$

$= |x - 4| = x - 4$ as $x > 20 \qquad \therefore \quad g'(x) = 1$

6. Given : $xe^{xy} = y + \sin^2 x$

Differentiating both sides w. r.to x, we get

$$e^{xy}.1 + xe^{xy}\left(y + x\frac{dy}{dx}\right) = \frac{dy}{dx} + 2\sin x \cos x$$

Put $x = 0 \Rightarrow \; 1 + 0 = \frac{dy}{dx} + 0 \;\Rightarrow\; \frac{dy}{dx} = 1$

B. True/ False

1. Consider $\phi(x) = \frac{f(x) + f(-x)}{2}$, which is an even function

Now, $\psi(x) = \phi'(x) = \frac{f'(x) - f'(-x)}{2}$

$\psi(-x) = \frac{f'(-x) - f'(x)}{2} = -\psi(x) \;\therefore\; \psi$ is odd.

C. MCQs with ONE Correct Answer

1. **(c)** We have $y^2 = P(x)$, ...(1)

where $P(x)$ is a polynomial of degree 3 and hence thrice differentiable. Differentiating (1) w.r. to x, we get

$$2y\frac{dy}{dx} = P'(x) \qquad ...(2)$$

Again differentiating with respect to x, we get

$$2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = P''(x)$$

$$\Rightarrow \frac{[P'(x)]^2}{2y^2} + 2y\frac{d^2y}{dx^2} = P''(x) \qquad \text{[Using (2)]}$$

$$\Rightarrow 4y^3\frac{d^2y}{dx^2} = 2y^2P''(x) - [P'(x)]^2$$

$$\Rightarrow 4y^3\frac{d^2y}{dx^2} = 2P(x)P''(x) - [P'(x)]^2 \quad \text{[Using (1)]}$$

$$\Rightarrow 2y^3\frac{d^2y}{dx^2} = P(x)P''(x) - \frac{1}{2}[P'(x)]^2$$

Again differentiating w.r. to x, we get $2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$

$= P'''(x)P(x) + P''(x)P'(x) - P'(x)P''(x) = P'''(x)P(x)$

2. **(b)** Let $f(x) = ax^2 + bx + c$

As given that $f(x) > 0, \; \forall \; x \in R$

$\therefore \quad a > 0$ and $D < 0$

$\Rightarrow \quad a > 0$ and $b^2 - 4ac < 0$ .....(1)

Now, $g(x) = f(x) + f'(x) + f''(x)$

$= ax^2 + bx + c + 2ax + b + 2a$

$= ax^2 + (2a + b)x + (2a + b + c)$

Here, $D = (2a + b)^2 - 4a(2a + b + c)$

$= 4a^2 + b^2 + 4ab - 8a^2 - 4ab - 4ac$

$= b^2 - 4a^2 - 4ac = -4a^2 + b^2 - 4ac$

$= (-ve) + (-ve) = -ve$ [Using eq. (1)]

Also $a > 0$ from (1),

$\therefore \quad g(x) > 0, \; \forall \; x \in R$

3. **(a)** $y = (\sin x)^{\tan x} \Rightarrow \log y = \tan x.\log\sin x$

Differentiating w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \log\sin x + \tan x.\frac{1}{\sin x}.\cos x$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\tan x}[1 + \sec^2 x \log\sin x]$$

4. **(b)** $x^2 + y^2 = 1 \Rightarrow 2x + 2yy' = 0 \Rightarrow x + yy' = 0$

$\Rightarrow \quad 1 + yy'' + (y')^2 = 0 \Rightarrow yy'' + (y')^2 + 1 = 0$

5. **(c)** $F(x) = \int_0^x f(t)dt$ and $F(x^2) = x^2(1+x)$

$F'(x) = f(x)$ ...... (1)

But $F'(x^2).2x = 2x + 3x^2$

$\Rightarrow F'(x^2) = \left(\frac{2+3x}{2}\right) \Rightarrow f(x^2) = \frac{2+3x}{2}$

$\Rightarrow f(4) = \frac{2+3\times 2}{2} = \frac{8}{2} = 4$

6. **(a)** $\log(x+y) = 2xy$ when $x = 0$ then $y = 1$

Differentiating w.r.t. x

$$\frac{1}{x+y}\left[1+\frac{dy}{dx}\right] = 2y + \frac{2xdy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{1}{x+y} - 2y}{2x - \frac{1}{x+y}} \Rightarrow y'(0) = \frac{1-2}{0-1} = 1$$

7. **(d)** Let us consider the function $g(x) = f(x) - x^2$ so that

$g(1) = f(1) - 1^2 = 1 - 1 = 0$

$g(2) = f(2) - 2^2 = 4 - 4 = 0$

$g(3) = f(3) - 3^2 = 9 - 9 = 0$

Since $f(x)$ is twice differentiable we can say $g(x)$ is continuous and differentiable everywhere and

$g(1) = g(2) = g(3) = 0$

$\therefore$ By Rolle's theorem, $g'(c) = 0$ for some $c \in (1,2)$

and $g'(d) = 0$ for some $d \in (2,3)$

Again by Rolle's theorem,

$g''(e) = 0$ for some $e \in (c,d) \Rightarrow e \in (1,3)$

$\Rightarrow f''(e) - 2 = 0$ or $f''(e) = 2$ for some $x \in (1,3)$

$f''(x) = 2$ for some $x \in (1,3)$

8. **(d)** $\frac{d^2x}{d^2y} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dx}\left(\frac{dx}{dy}\right) \times \frac{dx}{dy}$

$$= \left\{\frac{d}{dx}\left[\frac{1}{\left(\frac{dy}{dx}\right)}\right]\right\} \times \frac{1}{\frac{dy}{dx}} = -\frac{1}{\left(\frac{dy}{dx}\right)^2} \times \frac{d^2y}{dx^2} \times \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= -\left(\frac{dy}{dx}\right)^{-3} \frac{d^2y}{dx^2}$$

9. **(a)** Given that $g(x) = \log f(x) \Rightarrow g(x+1) = \log f(x+1)$

$\Rightarrow g(x+1) = \log x f(x)$ $[\because f(x+1) = xf(x)]$

$\Rightarrow g(x+1) = \log x + \log f(x) \Rightarrow g(x+1) - g(x) = \log(x)$

$\Rightarrow g'(x+1) - g'(x) = \frac{1}{x}$

$\Rightarrow g''(x+1) - g''(x) = -\frac{1}{x^2}$

Putting, $x = x - \frac{1}{2}$, we get

$$\Rightarrow g''\left(x+\frac{1}{2}\right) - g''\left(x-\frac{1}{2}\right) = -\frac{1}{\left(x-\frac{1}{2}\right)^2} = \frac{-2^2}{(2x-1)^2}$$

Putting $x = 1, 2, 3$ ...., N we get

$g''\left(\frac{3}{2}\right) - g''\left(\frac{1}{2}\right) = -\frac{2^2}{1^2}$ ...(1)

$g''\left(\frac{5}{2}\right) - g''\left(\frac{3}{2}\right) = \frac{-2^2}{3^2}$ ...(2)

$g''\left(\frac{7}{2}\right) - g''\left(\frac{5}{2}\right) = \frac{-2^2}{5^2}$ ...(3)

$g''\left(N+\frac{1}{2}\right) - g''\left(N-\frac{1}{2}\right) = -\frac{2^2}{(2N-1)^2}$ ...(N)

Adding all the above equations, we get

$$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) = -4\left[1+\frac{1}{3^2}+\frac{1}{5^2}+.....+\frac{1}{(2N-1)^2}\right]$$

10. **(b)** $F(x) = \int_0^{x^2} f(\sqrt{t})dt$ for $x \in [0,2]$

$\Rightarrow F'(x) = f(x).2x$

Now $F'(x) = f'(x) \forall x \in (0,2)$

$\Rightarrow f(x).2x = f'(x) \Rightarrow \frac{f'(x)}{f(x)} = 2x$

$\Rightarrow \ln f(x) = x^2 + c \Rightarrow f(x) = e^{x^2+c} = e^{x^2}.e^c$

As $f(0) = 1 \Rightarrow 1 = e^c$

$\therefore f(x) = e^{x^2}$

So $F(x) = \int_0^{x^2} e^x dx = e^{x^2} - 1$ $\quad\therefore F(2) = e^4 - 1$

D. MCQs with ONE or MORE THAN one Correct

1. **(b, c)** $f(x) = x^3 + 3x + 2 \Rightarrow f'(x) = 3x^2 + 3$

Also $f(0) = 2, f(1) = 6, f(2) = 16, f(3) = 38, f(6) = 236$

And $g(f(x)) = x \Rightarrow g(2) = 0, g(6) = 1, g(16) = 2, g(38) = 3, g(236) = 6$

(a) $g(f(x)) = x \Rightarrow g'(f(x)). f'(x) = 1$

For $g'(2)$, $f(x) = 2 \Rightarrow x = 0$

$\therefore$ Putting $x = 0$, we get $g'(f(0))\ f'(0) = 1$

$\Rightarrow g'(2) = \frac{1}{3}$

(b) $h(g(g(x))) = x \Rightarrow h'(g(g(x))). g'(g(x)). g'(x) = 1$

For $h'(1)$, we need $g(g(x)) = 1$

$\Rightarrow g(x) = 6 \Rightarrow x = 236$

$\therefore$ Putting $x = 236$, we get

$$h'[g(g(236))] = \frac{1}{g'(g(236)).g'(236)}$$

$$\Rightarrow h'(g(6)) = \frac{1}{g'(6)\, g'(236)}$$

$$\Rightarrow h'(1) = \frac{1}{g'(f(1))\, g'(f(6))} = f'(1)\, f'(6)$$

$= 6 \times 111 = 666$

(c) $h[g(g(x))] = x$
For h (0), $g(g(x)) = 0 \Rightarrow g(x) = 2 \Rightarrow x = 16$
$\therefore$ Putting x = 16, we get
$h(g(g(16))) = 16$
$\Rightarrow h(0) = 16$

(d) $h[g(g(x))] = x$
For h(g(3)), we need $g(x) = 3 \Rightarrow x = 38$
$\therefore$ Putting x = 38, we get
$h[g(g(38))] = 38 \Rightarrow h(g(3)) = 38$

E. Subjective Problems

1. Let $f(x) = \sin(x^2 + 1)$ then

$$f'(x) = \lim_{\delta x \to 0} \frac{\sin[(x+\delta x)^2 + 1] - \sin[x^2+1]}{\delta x}$$

$$\Rightarrow f'(x) = \lim_{\delta x \to 0} 2\cos\left(\frac{(x^2 + (\delta x)^2 + 2x\delta x + 1 + x^2 + 1)}{2}\right) . \frac{\sin\left(\frac{x^2 + (\delta x)^2 + 2x\delta x + 1 - x^2 - 1}{2}\right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cos\left[x^2 + 1 + x\delta x + \frac{(\delta x)^2}{2}\right] \sin\left[\frac{(\delta x)^2 + 2x\delta x}{2}\right]}{\delta x\left[\frac{\delta x + 2x}{2}\right]} \times \left(\frac{\delta x + 2x}{2}\right)$$

$$= 2\cos(x^2+1) \lim_{\delta x \to 0} \frac{\sin\left[\frac{(\delta x)^2 + 2x\delta x}{2}\right]}{\left[\frac{(\delta x)^2 + 2x\delta x}{2}\right]} \times \left(\frac{\delta x + 2x}{2}\right)$$

$$= 2\cos(x^2 + 1) \times 1 \times \frac{2x}{2} = 2x\cos(x^2+1)$$

2. $f(x)\begin{cases} \dfrac{x-1}{2x^2 - 7x + 5}, & x \ne 1 \\ -\dfrac{1}{3}, & x = 1 \end{cases}$

$$\therefore \quad f'(x)|_{x=1} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \left[\frac{\frac{1+h-1}{2(1+h)^2 - 7(1+h) + 5} + \frac{1}{3}}{h}\right] = \lim_{h \to 0} \frac{\frac{h}{2h^2 - 3h} + \frac{1}{3}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2h-3} + \frac{1}{3}}{h} = \lim_{h \to 0} \frac{2h}{3h(2h-3)} = \lim_{h \to 0} \frac{2}{3(2h-3)}$$

$= -2/9$

3. We have, $y = \dfrac{5x}{3|1-x|} + \cos^2(2x+1)$

(Clearly y is not defined at $x = 1$)

$$\Rightarrow \quad y = \begin{cases} \dfrac{5x}{3(1-x)} + \cos^2(2x+1), & x < 1 \\ \dfrac{5x}{3(x-1)} + \cos^2(2x+1), & x > 1 \end{cases}$$

$$\Rightarrow \quad \frac{dy}{dx} = \begin{cases} \dfrac{5}{3}\left(\dfrac{(1-x) - x(-1)}{(1-x)^2}\right) - 2\sin(4x+2), & x < 1 \\ \dfrac{5}{3}\left(\dfrac{(x-1) - x}{(x-1)^2}\right) - 2\sin(4x+2), & x > 1 \end{cases}$$

$$\text{or} \quad \frac{dy}{dx} = \begin{cases} \dfrac{5}{3}\dfrac{1}{(1-x)^2} - 2\sin(4x+2), & x < 1 \\ -\dfrac{5}{3}\dfrac{1}{(x-1)^2} - 2\sin(4x+2), & x > 1 \end{cases}$$

4. We are given $y = e^{x\sin x^3} + (\tan x)^x$
Here y is the sum of two functions and in the second function base as well as power are functions of x. Therefore we will use logarithmic differentiation here.
Let $y = u + v$

where $u = e^{x\sin x^3}$ ...(1)
and $v = (\tan x)^x$ ...(2)

$\therefore \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ ...(3)

Differentiating (1) with respect to x, we get

$$\frac{du}{dx} = e^{x\sin x^3} . \frac{d}{dx}(x\sin x^3)$$

$$= e^{x\sin x^3} . [3x^3 . \cos x^3 + \sin x^3]$$

Taking log on both sides on equ^n (2), we get
$\log v = x \log\tan x$
Differentiating the above with respect to x, we get

$$\frac{1}{v}\frac{dv}{dx} = x . \frac{1}{\tan x} . \sec^2 x + 1 . \log\tan x$$

$$\therefore \quad \frac{dv}{dx} = (\tan x)^x \left[\frac{2x}{\sin 2x} + \log\tan x\right]$$

Substituting the value of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in eqⁿ (3), we get

$$\frac{dy}{dx} = e^{x\sin x^3}[\sin x^3 + 3x^3\cos x^3]$$

$$+(\tan x)^x\left[\frac{2x}{\sin 2x} + \log\tan x\right]$$

5. Given that f is twice differentiable such that

$f''(x) = -f(x)$ and $f'(x) = g(x)$

$h(x) = [f(x)]^2 + [g(x)]^2$

To find $h(10)$ when $h(5) = 11$.

Consider $h'(x) = 2ff' + 2gg' = 2f(x)\,g(x) + 2g(x)f''(x)$

$[\because\ g(x) = f'(x) \Rightarrow g'(x) = f''(x)]$

$= 2f(x)\,g(x) + 2g(x)(-f(x))$

$= 2f(x)\,g(x) - 2f(x)\,g(x) = 0$

$\therefore\quad h'(x) = 0, \forall x$

$\Rightarrow$ h is a constant function

$\therefore\quad h(5) = 11 \Rightarrow h(10) = 11.$

6. Let $F(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

Given that α is a repeated root of quadratic equation $f(x) = 0$

$\therefore$ We must have $f(x) = k(x-\alpha)^2$; where k is a non- zero real no.

If we put $x = \alpha$ on both sides of eq. (1); we get

$$F(\alpha) = \begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0$$

$[\because\ R_1$ and R_2 are identical$]$

$\therefore\quad F(\alpha) = 0$

Hence $(x-\alpha)$ is a factor of $F(x)$

Differentiating eq. (1) w.r. to x, we get

$$F'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

Putting $x = \alpha$ on both sides, we get

$$F'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0$$

[as R_1 and R_3 are identical]

$\Rightarrow$ $(x-\alpha)$ is a factor of $F'(x)$ also. Or we can say $(x-\alpha)^2$ is a factor of $F(x)$.

$\Rightarrow$ $F(x)$ is divisible by $f(x)$.

7. We have, $x = \sec\theta - \cos\theta$, $y = \sec^n\theta - \cos^n\theta$

$$\Rightarrow \frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta$$

$= \sec\theta\tan\theta + \tan\theta\cos\theta = \tan\theta(\sec\theta + \cos\theta)$

and $\frac{dy}{d\theta} = n\sec^{n-1}\theta\sec\theta\tan\theta - n\cos^{n-1}\theta(-\sin\theta)$

$= n\sec^n\theta\tan\theta + n\tan\theta\cos^n\theta = n\tan\theta(\sec^n\theta + \cos^n\theta)$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{n\tan\theta(\sec^n\theta + \cos^n\theta)}{\tan\theta(\sec\theta + \cos\theta)}$$

or $\frac{dy}{dx} = \frac{n(\sec^n\theta + \cos^n\theta)}{(\sec\theta + \cos\theta)}$ ...(1)

Also $x^2 + 4 = (\sec\theta - \cos\theta)^2 + 4$

$= \sec^2\theta + \cos^2\theta - 2\sec\theta\cos\theta + 4$

$= \sec^2\theta + \cos^2\theta + 2$

$= (\sec\theta + \cos\theta)^2$ ...(2)

and $y^2 + 4 = \sec^n\theta - \cos^n\theta)^2 + 4$

$= \sec^{2n}\theta + \cos^{2n}\theta - 2\sec^n\theta\cos^n\theta + 4$

$= \sec^{2n}\theta + \cos^{2n}\theta + 2$

$= (\sec^n\theta + \cos^n\theta)^2$ ...(3)

Now we have to prove

$$(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$$

$$\text{LHS} = (\sec\theta + \cos\theta)^2.\frac{n^2(\sec^n\theta + \cos^n\theta)^2}{(\sec\theta + \cos\theta)^2}$$

[Using (1) and (2)]

$= n^2(\sec^n\theta + \cos^n\theta)^2$

$= n^2(y^2+4)$ [From eq. (3)]

$=$ RHS

8. We have given the function

$$(\sin y)^{\sin\left(\frac{\pi x}{2}\right)} + \frac{\sqrt{3}}{2}\sec^{-1}(2x) + 2^x\tan[\ln(x+2)] = 0$$

...(1)

For $x = -1$, we have

$$(\sin y)^{\sin\left(-\frac{\pi}{2}\right)} + \frac{\sqrt{3}}{2}\sec^{-1}(-2) + 2^{-1}\tan[\ln(-1+2)] = 0$$

$$\Rightarrow (\sin y)^{-1} + \frac{\sqrt{3}}{2}\left(\frac{2\pi}{3}\right) + \frac{1}{2}\tan 0 = 0 \Rightarrow \frac{1}{\sin y} = -\frac{\pi}{\sqrt{3}}$$

$\Rightarrow \sin y = -\frac{\sqrt{3}}{\pi}$, when $x = -1$ ....(2)

Now Let $u = (\sin y)^{\sin\left(\frac{\pi x}{2}\right)}$

Taking ln on both sides; we get

$$\ln u = \sin\left(\frac{\pi x}{2}\right)\ln\sin y$$

Differentiating both sides with respect to x, we get

$$\frac{1}{u}\frac{du}{dx} = \frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\ln\sin y + \cot y\frac{dy}{dx}\sin\left(\frac{\pi x}{2}\right)$$

$$\Rightarrow \frac{du}{dx} = (\sin y)^{\sin\left(\frac{\pi x}{2}\right)}$$

$$\times\left[\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\ln\sin y + \sin\left(\frac{\pi x}{2}\right)\cot y\frac{dy}{dx}\right] \quad ...(3)$$

Now differentiating eq. (1), we get

$$\frac{d}{dx}\left[(\sin y)^{\sin\left(\frac{\pi x}{2}\right)}\right] + \frac{\sqrt{3}}{2}\frac{1}{2x\sqrt{4x^2-1}}.2$$

$$+ 2^x(\ln 2)\tan[(\ln(x+2)]$$

$$+2^x\sec^2[\ln(x+2)]\frac{1}{x+2} = 0$$

$$\Rightarrow (\sin y)^{\sin\left(\frac{\pi x}{2}\right)}\left[\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\ln\sin y\right.$$

$$\left.+\sin\left(\frac{\pi x}{2}\right)\cot y\frac{dy}{dx}\right]$$

$$+\frac{\sqrt{3}}{2x\sqrt{4x^2-1}} + 2^x\ln 2\tan(\ln(x+2))$$

$$+\frac{2^x\sec^2[\ln(x+2)]}{x+2} = 0$$

At $x = -1$ and $\sin y = -\frac{\sqrt{3}}{\pi}$, we get

$$\Rightarrow \left(-\frac{\sqrt{3}}{\pi}\right)^{-1}\left[0-(-1)\sqrt{\frac{\pi^2}{3}-1}\left(\frac{dy}{dx}\right)_{x=-1}\right]$$

$$+\frac{\sqrt{3}}{-2\sqrt{3}} + 0 + 2^{-1} = 0$$

$$\Rightarrow -\frac{\pi}{\sqrt{3}\sqrt{3}}\sqrt{\pi^2-3}\left(\frac{dy}{dx}\right)_{x=-1} - \frac{1}{2}+\frac{1}{2} = 0 \Rightarrow \left.\frac{dy}{dx}\right]_{x=-1} = 0$$

9. $$y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$$

$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{x}{x-c}$$

$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \left(\frac{b}{x-b}+1\right)\frac{x}{x-c}$$

$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{x^2}{(x-b)(x-c)}$$

$$= \left(\frac{a}{x-a}+1\right)\frac{x^2}{(x-b)(x-c)} = \frac{x^3}{(x-a)(x-b)(x-c)}$$

$$\Rightarrow \log y = 3\log x - \log(x-a) - \log(x-b) - \log(x-c)$$

$$\Rightarrow \frac{y'}{y} = \frac{3}{x} - \frac{1}{x-a} - \frac{1}{x-b} - \frac{1}{x-c}$$

$$= \left(\frac{1}{x}-\frac{1}{x-a}\right) + \left(\frac{1}{x}-\frac{1}{x-b}\right) + \left(\frac{1}{x}-\frac{1}{x-c}\right)$$

$$= \frac{a}{x(a-x)} + \frac{b}{x(b-x)} + \frac{c}{x(c-x)}$$

$$= \frac{1}{x}\left[\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right]$$

H. Assertion & Reason Type Questions

1. **(b)** Given that $f(x) = 2 + \cos x$ which is continuous and differentiable every where.

Also $f'(x) = -\sin x \Rightarrow f'(x) = 0 \Rightarrow x = n\pi$

$\Rightarrow$ There exists $c \in [t, t+\pi]$ for $t \in R$

Such that $f'(c) = 0$

$\therefore$ Statement-1 is true.

Also $f(x)$ being periodic of period 2π, statement-2 is true, but statement-2 is not a correct explanation of statement-1.

2. **(a)** We have $f(x) = g(x)\sin x$

$\Rightarrow f'(x) = g'(x)\sin x + g(x)\cos x$

$\Rightarrow f'(0) = g'(0)\times 0 + g(0) = g(0) \quad [\because g'(0) = 0]$

$\therefore$ Statement 2 is correct.

Also $f''(0) = \lim\limits_{x\to 0}\frac{f'(x)-f'(0)}{x}$

$$= \lim_{x\to 0}\frac{g(x)\cos x + g'(x)\sin x - g(0)}{x}$$

$$= \lim_{x\to 0}\frac{g(x)\cos x - g(0)}{x} + \lim_{x\to 0}\frac{g'(x)\sin x}{x}$$

$$= \lim_{x\to 0}\frac{g(x)\cos x - g(0)}{x\times\frac{\sin x}{x}} + \lim_{x\to 0}g'(x)$$

$$= \lim_{x\to 0}\frac{g(x)\cos x - g(0)}{\sin x} + g'(0)$$

$$= \lim_{x\to 0}[g(x)\cot(x) - g(0)\operatorname{cosec} x] + 0$$

$$= \lim_{x\to 0}[g(x)\cot x - g(0)\operatorname{cosec} x]$$

$\therefore$ Statement 1 is also true and is a correct explanation for statement 2.

I. Integer Value Correct Type

1. **(2)** Given that $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$

then we should have $gof(x) = x$

$\Rightarrow g(f(x)) = x \Rightarrow g(x^3 + e^{x/2}) = x$

Differentiating both sides with respect to x, we get

$$g'(x^3 + e^{x/2}).\left(3x^2 + e^{x/2}.\frac{1}{2}\right) = 1$$

$$\Rightarrow g'(x^3 + e^{x/2}) = \frac{1}{3x^2 + e^{x/2}.\frac{1}{2}}$$

For $x = 0$, we get $g'(1) = \frac{1}{1/2} = 2$

GP_3480

2. **(1)** $f(\theta)=\sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$

$= \sin\left[\sin^{-1}\left(\frac{\sin\theta}{\sqrt{\sin^2\theta+\cos 2\theta}}\right)\right]$ $\left[\because \tan^{-1}\frac{x}{y}=\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}\right]$

$= \sin\left[\sin^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)\right] = \tan\theta$

$\therefore \frac{df(\theta)}{d\tan\theta}=1.$

Section-B JEE Main/ AIEEE

1. **(a)** $y=(x+\sqrt{1+x^2})^n$

$\frac{dy}{dx}=n(x+\sqrt{1+x^2})^{n-1}\left(1+\frac{1}{2}(1+x^2)^{-1/2}.\,2x\right);$

$\frac{dy}{dx}=n(x+\sqrt{1+x^2})^{n-1}\frac{(\sqrt{1+x^2}+x)}{\sqrt{1+x^2}}$

$=\frac{n(\sqrt{1+x^2}+x)^n}{\sqrt{1+x^2}}$

or $\sqrt{1+x^2}\frac{dy}{dx}=ny$ or $\sqrt{1+x^2}\,y_1=ny$

$(y_1=\frac{dy}{dx})$ Squaring, $(1+x^2)y_1^2=n^2y^2$

Differentiating, $(1+x^2)2y_1y_2+y_1^2.2x=n^2.2yy_1$

or $(1+x^2)y_2+xy_1=n^2y$

2. **(c)** $F(t)=\int_0^t f(t-y)g(y)dy$

$=\int_0^t e^{t-y}y\,dy=e^t\int_0^t e^{-y}y\,dy$

$=e^t\left[-ye^{-y}-e^{-y}\right]_0^t=-e^t\left[ye^{-y}+e^{-y}\right]_0^t$

$=-e^t\left[te^{-t}+e^{-t}-0-1\right]=-e^t\left[\frac{t+1-e^t}{e^t}\right]$

$=e^t-(1+t)$

3. **(d)** $f(x)=x^n\Rightarrow f(1)=1$

$f'(x)=nx^{n-1}\Rightarrow f'(1)=n$

$f''(x)=n(n-1)x^{n-2}\Rightarrow f''(1)=n(n-1)$

....

.... $f^n(x)=n!\Rightarrow f^n(1)=n!$

$=1-\frac{n}{1!}+\frac{n(n-1)}{2!}-\frac{n(n-1)(n-2)}{3!}+....+(-1)^n\frac{n!}{n!}$

$={}^nC_0-{}^nC_1+{}^nC_2-{}^nC_3+......+(-1)^n\,{}^nC_n=0$

4. **(b)** $f(x)=ax^2+bx+c$

$f(1)=f(-1)\Rightarrow a+b+c=a-b+c$ or $b=0$

$\therefore f(x)=ax^2+c$ or $f'(x)=2ax$

Now $f'(a); f'(b)$; and $f'(c)$ are $2a(a); 2a(b); 2a(c)$ i.e. $2a^2, 2ab, 2ac.$

$\Rightarrow$ If a,b,c are in A.P. then $f'(a); f'(b)$ and $f'(c)$ are also in A.P.

5. **(c)** $x=e^{y+e^{y+.....\infty}}\Rightarrow x=e^{y+x}.$

Taking log.

$\log x=y+x\Rightarrow\frac{1}{x}=\frac{dy}{dx}+1\Rightarrow\frac{dy}{dx}=\frac{1}{x}-1=\frac{1-x}{x}$

6. **(a)** $x^2-(a-2)x-a-1=0$

$\Rightarrow \alpha+\beta=a-2;\ \alpha\beta=-(a+1)$

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=a^2-2a+6=(a-1)^2+5$

For min. value of $\alpha^2+\beta^2$ where α is an integer

$\Rightarrow a=1.$

7. **(d)** Let α, $\alpha+1$ be roots

Then $\alpha+\alpha+1=b=$ sum of roots $\alpha(\alpha+1)=c$ = product of roots

$\therefore\ b^2-4c=(2\alpha+1)^2-4\alpha(\alpha+1)=1.$

8. (d) $\lim_{x\to 2}\int_0^{f(x)}\frac{4t^3}{x-2}dt = \lim_{x\to 0}\frac{\int_0^{f(x)}4t^3\,dt}{x-2}$

Applying L Hospital rule

$\lim_{x\to 2}\frac{[4f(x)^3 f'(x)]}{1} = 4(f(2))^3 f'(2) = 4\times 6^3\times\frac{1}{48} = 18$

9. (c) $f(x)=\begin{cases}\frac{x}{1-x}, & x<0\\ \frac{x}{1+x}, & x\geq 0\end{cases}$

$\Rightarrow\ f'(x)=\begin{cases}\frac{x}{(1-x)^2}, & x<0\\ \frac{x}{(1+x)^2}, & x\geq 0\end{cases}$

$\therefore$ $f'(x)$ exist at everywhere.

10. (a) $x^m.y^n=(x+y)^{m+n}$

$\Rightarrow\ m\ln x+n\ln y=(m+n)\ln(x+y)$

Differentiating both sides.

$\therefore\ \frac{m}{x}+\frac{n}{y}\frac{dy}{dx}=\frac{m+n}{x+y}\left(1+\frac{dy}{dx}\right)$

$\Rightarrow\ \left(\frac{m}{x}-\frac{m+n}{x+y}\right)=\left(\frac{m+n}{x+y}-\frac{n}{y}\right)\frac{dy}{dx}$

$\Rightarrow\ \frac{my-nx}{x(x+y)}=\left(\frac{my-nx}{y(x+y)}\right)\frac{dy}{dx}\ \Rightarrow\ \frac{dy}{dx}=\frac{y}{x}$

11. (d) $x^{2x}-2x^x\cot y-1=0$

$\Rightarrow 2\cot y=x^x-x^{-x}\Rightarrow 2\cot y=u-\frac{1}{u}$ where $u=x^x$

Differentiating both sides with respect to x, we get

$\Rightarrow\ -2\text{cosec}^2 y\frac{dy}{dx}=\left(1+\frac{1}{u^2}\right)\frac{du}{dx}$

where $u=x^x\Rightarrow\log u=x\log x$

$\Rightarrow\ \frac{1}{u}\frac{du}{dx}=1+\log x\ \Rightarrow\ \frac{du}{dx}=x^x(1+\log x)$

$\therefore$ We get $-2\,\text{cosec}^2 y\,\frac{dy}{dx}=(1+x^{-2x}).x^x(1+\log x)$

$\Rightarrow\ \frac{dy}{dx}=\frac{\left(x^x+x^{-x}\right)(1+\log x)}{-2(1+\cot^2 y)}$ ....(i)

Now when $x=1$, $x^{2x}-2x^x\cot y-1=0$, gives

$1-2\cot y-1=0\Rightarrow\ \cot y=0$

$\therefore$ From equation (i), at $x=1$ and $\cot y=0$, we get

$y'(1)=\frac{(1+1)(1+0)}{-2(1+0)}=-1$

12. (a) $g'(x)=2\big(f(2f(x)+2)\big)\left(\frac{d}{dx}\big(f(2f(x)+2)\big)\right)$

$=2f(2f(x)+2)f'(2f(x))+2).(2f'(x))$

$\Rightarrow\ g'(0)=2f(2f(0)+2).f'(2f(0)+2).2f'(0)=4f(0)(f'(0))^2$

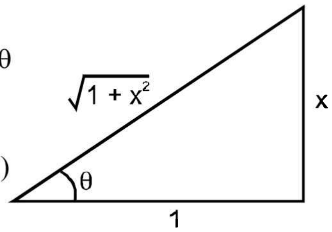
$=4(-1)(1)^2=-4$

13. (c) $\frac{d^2x}{dy^2}=\frac{d}{dy}\left(\frac{dx}{dy}\right)=\frac{d}{dx}\left(\frac{dx}{dy}\right)\frac{dx}{dy}$

$=\frac{d}{dx}\left(\frac{1}{dy/dx}\right)\frac{dx}{dy}=-\frac{1}{\left(\frac{dy}{dx}\right)^2}.\frac{d^2y}{dx^2}.\frac{1}{\frac{dy}{dx}}=-\frac{1}{\left(\frac{dy}{dx}\right)^3}\frac{d^2y}{dx^2}$

14. (a) Let $y=\sec(\tan^{-1}x)$ and $\tan^{-1}x=\theta$.

$\Rightarrow\ x=\tan\theta$



Thus, we have $y=\sec\theta$

$\Rightarrow\ y=\sqrt{1+x^2}$

$(\because\ \sec^2\theta=1+\tan^2\theta)$

$\Rightarrow\ \frac{dy}{dx}=\frac{1}{2\sqrt{1+x^2}}\cdot 2x$

At $x=1$, $\frac{dy}{dx}=\frac{1}{\sqrt{2}}$.

15. (b) Since $f(x)$ and $g(x)$ are inverse of each other

$\therefore\ g'(f(x))=\frac{1}{f'(x)}$

$\Rightarrow\ g'(f(x))=1+x^5\qquad\left(\because\ f'(x)=\frac{1}{1+x^5}\right)$

Here $x=g(y)$

$\therefore\ g'(y)=1+\{g(y)\}^5$

$\Rightarrow\ g'(x)=1+\{g(x)\}^5$

16. (a) Let $f(x)=\alpha\log|x|+\beta x^2+x$

Differentiating both sides,

$f'(x)=\frac{\alpha}{x}+2\beta x+1$

Since $x=-1$ and $x=2$ are extreme points therefore $f'(x)=0$ at these points.

Put $x=-1$ and $x=2$ in $f'(x)$, we get

$-\alpha-2\beta+1=0\Rightarrow\alpha+2\beta=1$ ...(i)

$\frac{\alpha}{2}+4\beta+1=0\Rightarrow\alpha+8\beta=-2$ ...(ii)

On solving (i) and (ii), we get

$6\beta=-3\Rightarrow\beta=-\frac{1}{2}\quad\therefore\ \alpha=2$