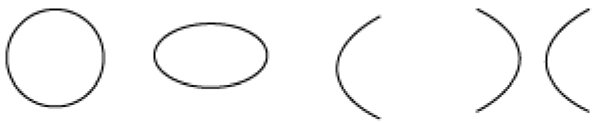
Conic Sections

Introduction to Conic Sections

- Conic sections or conics are the curves obtained by intersecting a double-napped right-circular cone with a plane.

- The concept of conic sections is widely used in astronomy, projectile motion of an object, etc.

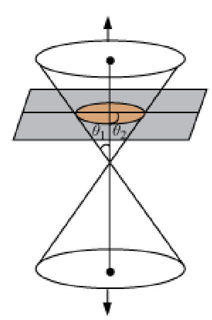
- The example of conic sections are circle (Figure **I**), ellipse (Figure **II**), parabola (Figure **III**) and hyperbola (Figure **IV**).

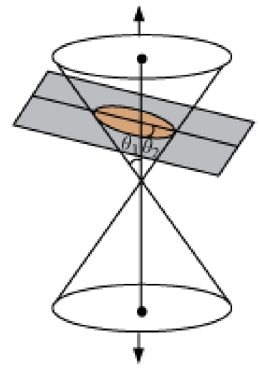
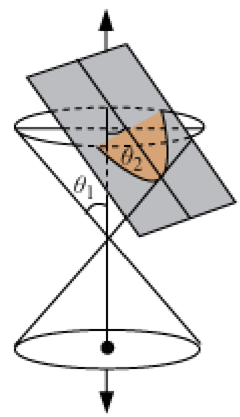
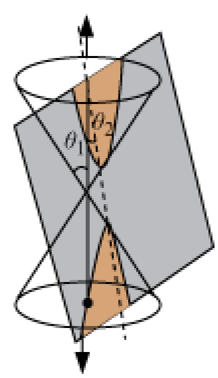


(I) (II) (III) (IV)

- Different types of conics can be formed by intersecting a plane with a double-napped cone (other than the vertex) by different ways.

- If θ_1 is the angle between the axis and the generator and θ_2 is the angle between the plane and the axis, then for different conditions of θ_1 and θ_2, we get different conics. These are described in the table shown below.

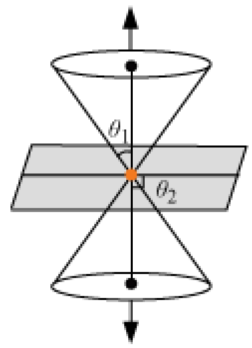
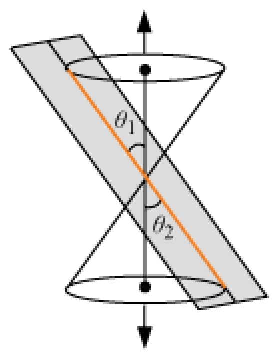
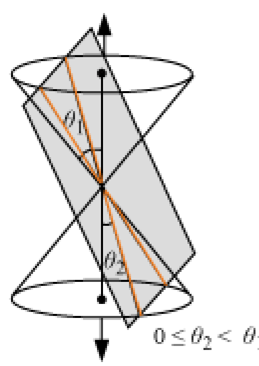
Condition	Conic Formed	Figure
$\theta_2 = 90°$ (The plane cuts only one nappe of the cone entirely)	Circle	

$\theta_1 < \theta_2 < 90°$ (The plane cuts only one nappe of the cone entirely)	Ellipse	
$\theta_1 = \theta_2$ (The plane cuts only one nappe of the cone entirely)	Parabola	
$0 \le \theta_2 < \theta_1$ (The plane cuts each nappe of the cone entirely)	Hyperbola	

The conic sections obtained by cutting a plane with a double-napped cone at its vertex are known as **degenerated conic sections**.

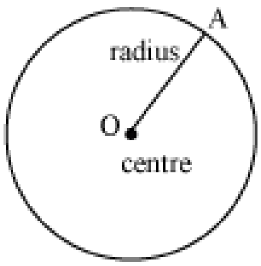
- If θ_1 is the angle between the axis and the generator and θ_2 is the angle between the plane and the axis, then for different conditions of θ_1 and θ_2, we get different conics.

These are described in the table shown below.

Condition	Conic Formed	Figure
$\theta_1 < \theta_2 \leq 90°$	Point	
$\theta_1 = \theta_2$	Line	
$0 \leq \theta_2 < \theta_1$	Hyperbola	

Circle

- A **circle** is the set of all points in a plane that are equidistant from a fixed point in the plane.
- The fixed point is called the **centre** of the circle.
- The fixed distance is called the **radius** of the circle.



- To find the equation of a circle, let us watch the following video
- The equation of a circle with radius *r* units and centre at (*h*, *k*) is given by the equation $(x-h)^2+(y-k)^2=r^2$
- The equation of a circle with radius *r* units and centre at origin (0, 0) is given by the equation $x^2+y^2=r^2$

Solved Examples

Example 1:

Find the centre and the radius of the circle $x^2+y^2+2ax+2by+d=0$.

Solution:

The given equation of the circle is

$x^2+y^2+2ax+2by=-d$

This can be written as

$(x^2+2ax)+(y^2+2by)=-d$

Now, on completing the squares within the parentheses, we obtain

$$(x^2+2ax+a^2)+(y^2+2by+b^2)=a^2+b^2-d$$
$$\Rightarrow (x+a)^2+(y+b)^2=\left(\sqrt{a^2+b^2-d}\right)^2$$

Hence, the centre of the circle is at (−*a*, −*b*) and the radius of the circle is $\sqrt{a^2+b^2-d}$.

Example 2:

Find the equation of the circle that passes through the points (1, 11), (−11, −5) and (3, 9).

Also find the centre and the radius of this circle.

Solution:

Let the centre of the circle be at (*h*, *k*) and the radius of the circle be *r*.

We know that the distance between the centre of a circle and a point on it is equal to the radius of the circle.

Since the circle passes through the point (1, 11),

$$(h-1)^2+(k-11)^2=r^2$$
$$\Rightarrow h^2+k^2-2h-22k+122=r^2 \quad \ldots (1)$$

Since the circle passes through the point (−11, −5),

$$(h+11)^2+(k+5)^2=r^2$$
$$\Rightarrow h^2+k^2+22h+10k+146=r^2 \quad \ldots (2)$$

Since the circle passes through the point (3, 9),

$$(h-3)^2+(k-9)^2=r^2$$
$$\Rightarrow h^2+k^2-6h-18k+90=r^2 \quad \ldots (3)$$

Subtracting equation (1) from equation (2), we get

$24h + 32k = -24$

$\Rightarrow 3h + 4k = -3$ … (4)

Subtracting equation (3) from equation (2), we get

$28h + 28k = -56$

$\Rightarrow h + k = -2$ … (5)

Multiplying equation (5) by 3, and then subtracting this result from equation (4), we get

$k = 3$

Substituting the value of k in equation (5), we get

$h = -5$

So, the centre of the circle is at (−5, 3).

Substituting the values of h and k in equation (1), we get

$$r^2 = (-5-1)^2 + (3-11)^2 = 100$$
$$\Rightarrow r = 10$$

So, the radius of the given circle is 10 units.

Thus, the equation of the circle is

$$(x+5)^2 + (y-3)^2 = 10^2$$
$$\Rightarrow x^2 + y^2 + 10x - 6y - 66 = 0$$

Example 3:

Show that the circles $x^2 + y^2 + 22x + 4y + 25 = 0$ and $x^2 + y^2 - 18x - 26y + 25 = 0$ touch each other externally. Also find the point of contact.

Solution:

Equation of the first circle C_1 is

$$x^2 + y^2 + 22x + 4y + 25 = 0$$
$$\Rightarrow (x^2 + 22x) + (y^2 + 4y) = -25$$
$$\Rightarrow (x^2 + 22x + 121) + (y^2 + 4y + 4) = 121 + 4 - 25$$
$$\Rightarrow (x+11)^2 + (y+2)^2 = 10^2$$

Hence, radius r_1 of this circle = 10 units, and the centre O_1 is at (−11, −2)

Equation of the second circle C_2 is

$$x^2 + y^2 - 18x - 26y + 25 = 0$$
$$\Rightarrow (x^2 - 18x) + (y^2 - 26y) = -25$$
$$\Rightarrow (x^2 - 18x + 81) + (y^2 - 26y + 169) = 81 + 169 - 25$$
$$\Rightarrow (x-9)^2 + (y-13)^2 = 15^2$$

Hence, radius r_2 of this circle = 15 units, and the centre O_2 is at (9, 13).

Two circles touch each other externally if the sum of their radii is equal to the distance between their centres.

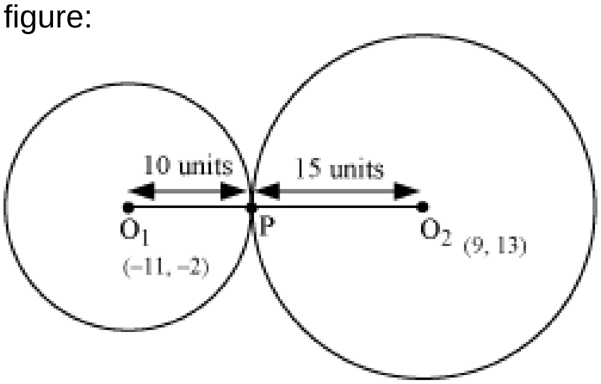
We can prove that circles C_1 and C_2 touch other externally if we can prove $O_1O_2 = r_1 + r_2$.

Now,

$$O_1O_2 = \sqrt{(9+11)^2 + (13+2)^2} = 25$$

$$r_1 + r_2 = 10 + 15 = 25$$
$$\therefore O_1O_2 = r_1 + r_2$$

Therefore, the given circles touch each other externally, which is shown in the following figure:



Let the given circles touch each other externally at point P.

Clearly, P will be on the line segment O_1O_2, and $\frac{O_1P}{O_2P} = \frac{10}{15} = \frac{2}{3}$

So, the coordinates of P are $\left(\frac{2\times 9+3(-11)}{2+3}, \frac{2\times 13+3(-2)}{2+3}\right)=(-3,4)$.

Example 4:

Find the value of λ for which the line $3x-4y=\lambda-3$ is a tangent to the circle $x^2+y^2-10x-4y+13=0$, where λ is a constant.

Solution:

A line will be tangent to a circle if it touches the circle at exactly one point.

For this, the perpendicular distance between the centre and the line should be equal to the radius of the circle.

Equation of the given circle is

$$x^2+y^2-10x-4y+13=0$$
$$\Rightarrow (x^2-10x)+(y^2-4y)=-13$$
$$\Rightarrow (x^2-10x+25)+(y^2-4y+4)=25+4-13$$
$$\Rightarrow (x-5)^2+(y-2)^2=4^2$$

Hence, the centre of the given circle is at (5, 2) and the radius of the circle is 4 units.

Since the line $3x-4y=\lambda-3$ is a tangent to the circle $x^2+y^2-10x-4y+13=0$, the perpendicular distance between (5, 2) and the line should be equal to 4 units.

$$\left|\frac{3\times 5-4\times 2+3-\lambda}{\sqrt{3^2+4^2}}\right|=4$$
$$\therefore \left|\frac{10-\lambda}{5}\right|=4$$
$$\Rightarrow |10-\lambda|=20$$
$$\Rightarrow 10-\lambda=\pm 20$$
$$\Rightarrow 10-\lambda=-20 \text{ or } 10-\lambda=20$$
$$\Rightarrow \lambda=30 \text{ or } \lambda=-10$$

So, for λ = 30 or −10, the line $3x-4y=\lambda-3$ is a tangent to the circle $x^2+y^2-10x-4y+13=0$.

Example 5:

If the extreme points of the diameter of a circle are (2, −11) and (12, 13), then find the equation of the circle passing through these points. Check whether (19, −4) lies on this circle or not.

Solution:

Let A (2, −11) and B (12, 13) be the extreme points of the diameter of the circle. Clearly, its centre O is the mid-point of AB, and the radius is half of AB.

So, the centre O of the circle (*h*, *k*) is at $\left(\frac{2+12}{2}, \frac{-11+13}{2}\right) = (7,1)$

Radius of the circle is $r = \frac{1}{2}\text{AB} = \frac{1}{2}\sqrt{(12-2)^2 + (13+11)^2} = \frac{1}{2} \times 26 = 13$

Hence, the equation of the circle with centre at $(7,1)$ and radius 13 units is given as

$$(x-7)^2 + (y-1)^2 = 13^2 \qquad \text{... (1)}$$
$$\Rightarrow x^2 + y^2 - 14x - 2y - 119 = 0$$

This is the equation of the required circle.

The point (19, −4) lies on the circle if it point satisfies equation (1).

Now, for the point (19, −4), the LHS of equation (1) changes to

$$(19-7)^2 + (-4-1)^2 = 12^2 + 5^2 = 169 = 13^2$$

= R.H.S.

This shows that the point (19, −4) lies on the circle.

Parabola

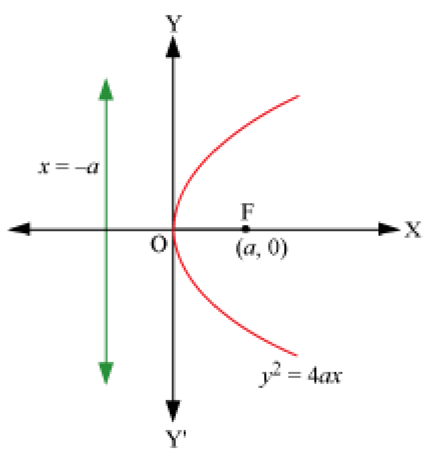
Key Concepts

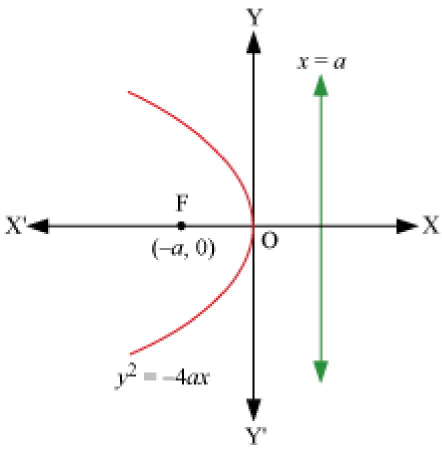
- A **parabola** is defined as the set of all points in a plane which are equidistant from a fixed line and a fixed point (not on the line) in the plane.

- If the fixed point lies on the fixed line, then the resulting set of points is a straight line. This straight line passes through the fixed point and is perpendicular to the fixed line. This is the degenerated case of a parabola.

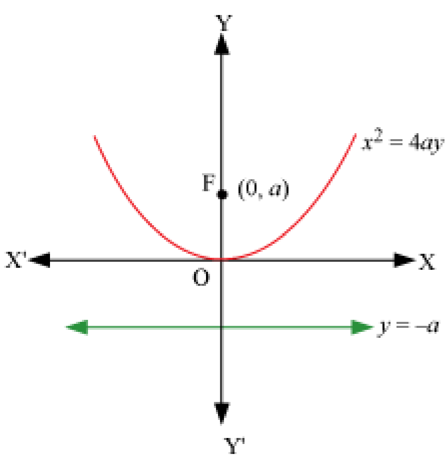
Standard Equations

- There are four possible orientations of a parabola.

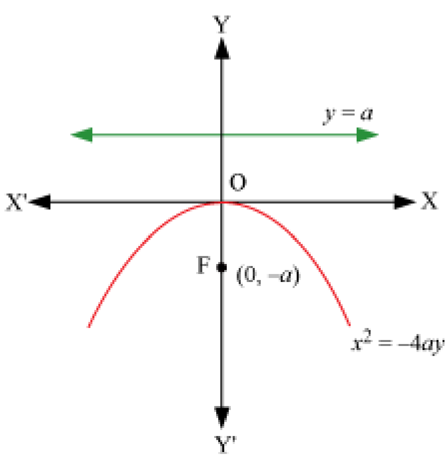
	**Open towards**	Right
	Standard Equation	$y^2 = 4ax$, $a > 0$
	Coordinates of Focus	$(a, 0)$
	Coordinates of Vertex	$(0, 0)$
	Equation of Directrix	$x = -a$
	Length of Latus Rectum	$4a$
	Axis of parabola	Positive x-axis
	Open towards	Left
	Standard Equation	$y^2 = -4ax$, $a > 0$



Coordinates of Focus	$(-a, 0)$
Coordinates of Vertex	(0, 0)
Equation of Directrix	$x = a$
Length of Latus Rectum	$4a$
Axis of parabola	Negative x-axis



Open towards	Upward
Standard Equation	$x^2 = 4ay$, $a > 0$
Coordinates of Focus	$(0, a)$
Coordinates of Vertex	(0, 0)
Equation of Directrix	$y = -a$

	Length of Latus Rectum	$4a$
	Axis of parabola	Positive y-axis
	**Open towards**	Downward
	Standard Equation	$x^2 = -4ay,\ a > 0$
	Coordinates of Focus	$(0, -a)$
	Coordinates of Vertex	$(0, 0)$
	Equation of Directrix	$y = a$
	Length of Latus Rectum	$4a$
	Axis of parabola	Negative y-axis

Solved Examples

Example 1:

Find the equation of a parabola whose vertex is at (0, 0), passing through the point (−3, 10) and symmetrical along the

1. x-axis
2. y-axis

Solution:

1. Since the parabola is symmetric about the x-axis, and has its vertex at the origin, the equation is of the form $y^2 = 4ax$ or $y^2 = -4ax$, where the sign depends on whether the parabola open towards right or left.

   Since the parabola passes through (−3, 10), which lies in the second quadrant, it must open towards left. Thus, the equation of the parabola is of the form $y^2 = -4ax$. As the parabola passes through the point (−3, 10), we have

   $(10)^2 = -4a\,(-3) \Rightarrow a = \frac{100}{12} = \frac{25}{3}$

   Hence, the equation of the required parabola is

   $$y^2 = -4 \times \frac{25}{3} x \Rightarrow 3y^2 = -100x$$

2. Since the parabola is symmetric about the y-axis, and has its vertex at the origin, the equation is of the form $x^2 = 4ay$ or $x^2 = -4ay$, where the sign depends on whether the parabola opens upward or downward.

   Since the parabola passes through (−3, 10), which lies in the second quadrant, it must open upward. Thus, the equation of the parabola is of the form $x^2 = 4ay$. As the parabola passes through the point (−3, 10), we have

   $(-3)^2 = 4a\,(10) \Rightarrow a = \frac{9}{40}$

   Hence, the equation of the required parabola is

   $$x^2 = 4 \times \frac{9}{40} y \Rightarrow 10x^2 = 9y$$

Example 2:

In a parabolic mirror, an isosceles right-angled triangle is inscribed. The vertex related to right angle is the vertex of the parabola, and the other vertices are the extremities of one of its diameter. If the area of the isosceles right triangle is 484 cm^2, then find:

1. Radius of the parabolic mirror

2. Length of the semi-latus rectum

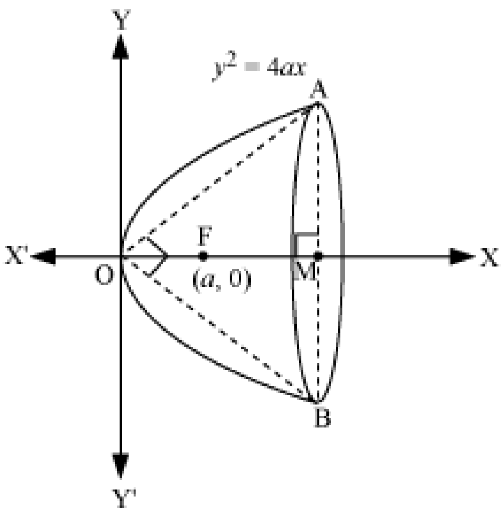
3. Its depth

Solution:

Let the parabolic mirror have its vertex at the origin and be symmetrical to the *x*-axis. So, its equation is given by

$$y^2 = 4ax \qquad \text{... (1)}$$

The given information can be represented with the help of a figure as shown below.



We have the equation of the parabola as

$$y^2 = 4ax$$
$$\Rightarrow y = \pm 2\sqrt{ax}$$

So, the coordinates of any point on the parabola is of the form $\left(x, 2\sqrt{ax}\right)$ or $\left(x, -2\sqrt{ax}\right)$

Since ΔOAB is isosceles and is right-angled at O, OM ⊥AB.

Let the coordinates of points A and B on the parabolic mirror given by the equation $y^2 = 4ax$ be $\left(x, 2\sqrt{ax}\right)$ and $\left(x, -2\sqrt{ax}\right)$ respectively.

$\therefore \text{AB} = 4\sqrt{ax}$

Since ΔOAB is right-angled at O,

$$\begin{aligned}
&\text{OA}^2 + \text{OB}^2 = \text{AB}^2 \\
&\Rightarrow \{x^2 + (2\sqrt{ax})^2\} + \{x^2 + (-2\sqrt{ax})^2\} = (4\sqrt{ax})^2 \\
&\Rightarrow 2(x^2 + 4ax) = 16ax \\
&\Rightarrow x^2 = 4ax \\
&\Rightarrow x = 4a
\end{aligned}$$

Hence, the coordinates of points A and B are $(4a, 4a)$ and $(4a, -4a)$.

$$\begin{aligned}
&\therefore\ \text{OA} = \text{OB} = \sqrt{(4a)^2 + (4a)^2} = 4\sqrt{2}a \\
&\text{AB} = 4a \times 2a = 8a
\end{aligned}$$

It is given that the area of triangle ABC is 484 cm^2.

$$\begin{aligned}
&\therefore \frac{1}{2}\text{OA} \times \text{OB} = 484\ \text{cm}^2 \\
&\Rightarrow \frac{1}{2} 4\sqrt{2}a \times 4\sqrt{2}a = 16a^2 = 484\ \text{cm}^2 \\
&\Rightarrow a^2 = \frac{484\ \text{cm}^2}{16} = \frac{121\ \text{cm}^2}{4} \\
&\Rightarrow a = \frac{11}{2}\ \text{cm} \\
&\therefore \text{OA} = 4\sqrt{2} \times \frac{11}{2}\ \text{cm} = 22\sqrt{2}\ \text{cm}
\end{aligned}$$

(a) So, the radius (AM) of the parabolic mirror is given by

$$\text{AM} = \frac{1}{2}\text{AB} = \frac{1}{2} \times 8 \times \frac{11}{2}\ \text{cm} = 22\ \text{cm}$$

(b) Semi-latus rectum = 2*a* = 2 $\times \frac{11}{2}$ = 11 cm

(c) Depth (OM) of the mirror is given by

$$OM = \sqrt{OA^2 - AM^2} = \sqrt{(22\sqrt{2}\ \text{cm})^2 - (22\ \text{cm})^2} = 22\ \text{cm}$$

Ellipse

Key Concepts

- An **ellipse** is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant.
- The two fixed points are called the **foci** of the ellipse.

Standard Equations

- There are two possible orientations of an ellipse

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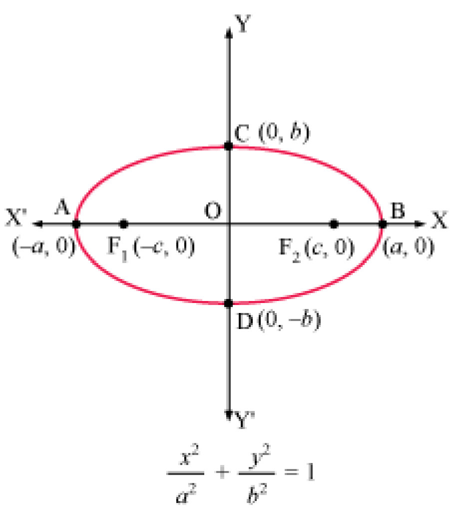
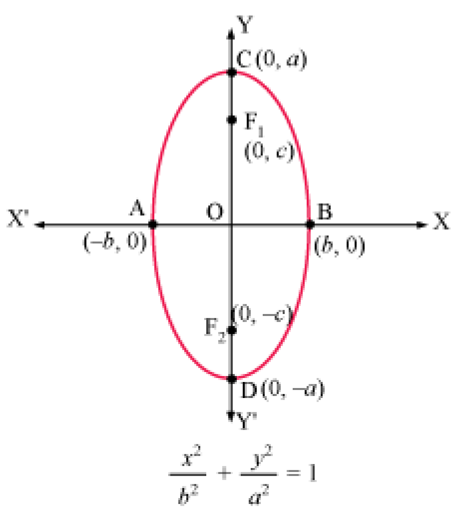
<tr><td rowspan="6">
</td><td>Standard Equation</td><td>$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b$</td></tr>
<tr><td>Centre</td><td>(0, 0)</td></tr>
<tr><td>Vertex</td><td>(±a, 0)</td></tr>
<tr><td>End points of minor axis</td><td>(0, ±b)</td></tr>
<tr><td>Foci</td><td>(±c, 0)</td></tr>
<tr><td>Length of major axis</td><td>2a along x-axis</td></tr>
</table>

	Length of minor axis	2*b* along *y*-axis
	Length between foci	2*c* along *x*-axis
	Relation between *a*, *b* and *c*	$a^2 = b^2 + c^2$
	Length of latus rectum	$\frac{2b^2}{a}$
	Eccentricity (*e*< 1)	$\frac{c}{a}$
	**Standard Equation**	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b$
	Centre	(0, 0)
	Vertex	(0, ±*a*)
	End points of minor axis	(±*b*, 0)

	Foci	(0, ±*c*)
	Length of major axis	2*a* along *y*-axis
	Length of minor axis	2*b* along *x*-axis
	Length between foci	2*c* along *y*-axis
	Relation between *a*, *b* and *c*	$a^2 = b^2 + c^2$
	Length of latus rectum	$\frac{2b^2}{a}$
	Eccentricity (*e*< 1)	$\frac{c}{a}$

Solved Examples

Example 1:

The centre of an ellipse is at the origin. If one of its vertices and one of the end points of the minor axis lie on the line $5x - 4y = 20$, then find:

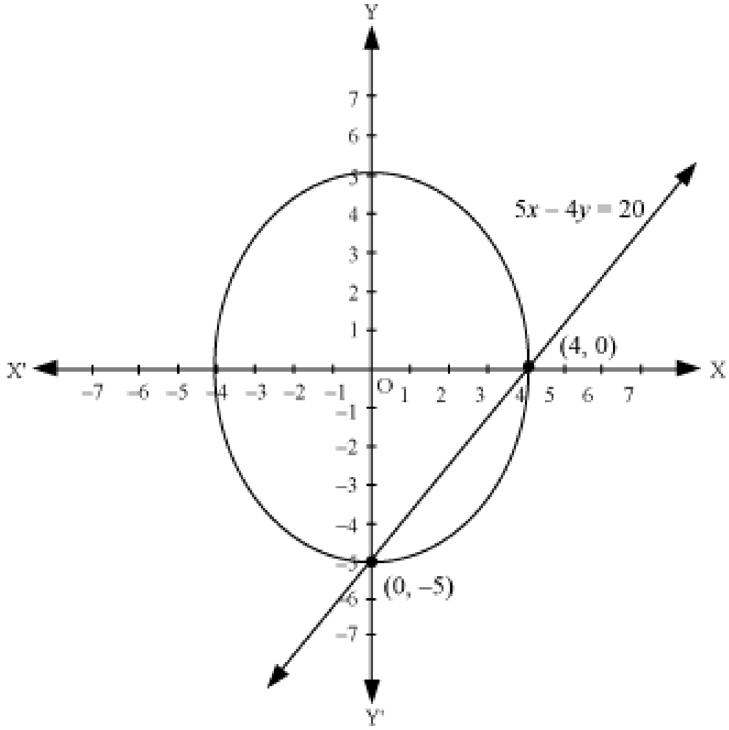
1. Coordinates of the end points of the major axis
2. Coordinates of the end points of the minor axis
3. Length of the major axis
4. Length of the minor axis
5. Major axis
6. Minor axis
7. Equation
8. Coordinates of the foci
9. Distance between the foci
10. Eccentricity
11. Length of the latus rectum

Solution:

Since the centre of the ellipse is at the origin and one of its vertices and one of the end points of the minor axis lie on the line, $5x - 4y = 20$, they lie on the coordinate axis.

The given equation of the line is $5x - 4y = 20$

This line intersects the x-axis and y-axis at (4, 0) and (0, −5) respectively.



Since 5 > 4, the said vertex is (0, −5) and the end point of the minor axis is (4, 0).

1. Since one of the vertices is (0, −5), the coordinates of the vertices are (0, ±5)

2. Since one of the end points of the minor axis is (4, 0), the coordinates of the end points of the minor axis are (±4, 0).

3. Length of the major axis, $2a = 2 \times 5 = 10$ units

4. Length of the minor axis, $2b = 2 \times 4 = 8$ units

5. Since the coordinates of the vertices are (0, ±5), the major axis of the ellipse is along the y-axis.

6. Since the coordinates of the end points of minor axis are (±4, 0), the minor axis of the ellipse is along the x-axis.

7. We have $a = 5$ and $b = 4$. Also, the major and the minor axes of the ellipse are along the y-axis and the x-axis respectively. So, the equation of the ellipse is given by

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

$$\Rightarrow \frac{x^2}{4^2}+\frac{y^2}{5^2}=1$$

$$\Rightarrow \frac{x^2}{16}+\frac{y^2}{25}=1$$

8. We have $a = 5$ and $b = 4$

   $$c=\sqrt{a^2-b^2}=\sqrt{5^2-4^2}=3$$

   Hence, the coordinates of the foci are $(0, \pm c) = (0, \pm 3)$.

9. Distance between the two foci is given by
   $2c = 2 \times 3 = 6$ units

10. Eccentricity (e) of the ellipse is given by

    $$e=\frac{c}{a}=\frac{3}{5}=0.6$$

11. Length of the latus rectum of the ellipse is given by

    $\frac{2b^2}{a}=\frac{2\times 4^2}{5}=6.4$ units

Example 2:
The centre of an ellipse is at the origin and the major axis is on the *x*-axis. If the distance between the foci and the length of the minor axis are 10 units each, then find the equation, eccentricity and the length of the semi-latus rectum of the ellipse. Also, show that the point $(6, -\sqrt{7})$ lies on this ellipse.

Solution:

Since the centre of the ellipse is at the origin and the major axis is on the *x*-axis, the equation of the ellipse is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots (1)$$

Where, $a > b$

It is also given that the distance between the foci and the length of the minor axis are 10 units each. So, $2b = 10$ and $2c = 10$.

⇒ $b = 5$, $c = 5$

We know that

$$c^2 = a^2 - b^2$$
$$\Rightarrow a = \sqrt{b^2 + c^2} = \sqrt{5^2 + 5^2} = 5\sqrt{2}$$

On substituting the values of *a* and *b* in equation (1), we obtain the equation of the ellipse as

$$\frac{x^2}{(5\sqrt{2})^2} + \frac{y^2}{5^2} = 1$$
$$\Rightarrow \frac{x^2}{50} + \frac{y^2}{25} = 1$$

Eccentricity (e) of the ellipse is given by

$$e = \frac{c}{a} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Length of the semi-latus rectum (l) of the ellipse is given by

$l = \frac{b^2}{a} = \frac{(5)^2}{5\sqrt{2}} = \frac{5}{\sqrt{2}}$ units.

If the point (6, $-\sqrt{7}$) lies on the ellipse $\frac{x^2}{50} + \frac{y^2}{25} = 1$, then this point should satisfy this equation. In order to check this, let us put $x = 6$ and $y = -\sqrt{7}$, in the L.H.S of this equation.

Now, $\frac{6^2}{50} + \frac{(\sqrt{7})^2}{25} = \frac{18}{25} + \frac{7}{25} = 1$ = R.H.S

This shows that the point 6, $-\sqrt{7}$) lies on the ellipse $\frac{x^2}{50} + \frac{y^2}{25} = 1$.

Example 3:

The centre of an ellipse is at the origin. If the sum of the lengths of the latus recta and the coordinates of the foci are 36 units and (0, ±20), then find the equation of the ellipse and its eccentricity.

Solution:

Since the coordinates of the foci(0, ±20) of the ellipse lie on the *y*-axis, the equation of the ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad \text{... (1)}$$

Where, $a > b$

And $c = 20$

It is also given that the sum of the lengths of the latus recta is 36 units.

$$\therefore \frac{4b^2}{a} = 36$$
$$\Rightarrow b^2 = 9a$$

We also know that

$$c^2 = a^2 - b^2$$
$$\Rightarrow (20)^2 = a^2 - 9a$$
$$\Rightarrow a^2 - 9a - 400 = 0$$
$$\Rightarrow (a-25)(a+16) = 0$$
$$\Rightarrow a = 25, -16$$

If $a = -16$, then $b^2 = 16(-9) = -144$, which is not possible.

If $a = 25$, then

$$b^2 = 9 \times 25 = 225$$

On substituting the values of *a* and *b* in equation (1), we obtain the equation of the ellipse as

$$\frac{x^2}{225}+\frac{y^2}{25^2}=1$$
$$\Rightarrow \frac{x^2}{225}+\frac{y^2}{625}=1$$

Eccentricity (*e*) of the ellipse is given by

$$e=\frac{c}{a}=\frac{20}{25}=\frac{4}{5}$$

Hyperbola

Key Concepts

- **Hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.**
- The two fixed points are called the **foci** of the hyperbola.

Standard Equations

- There are two possible orientations of a hyperbola.

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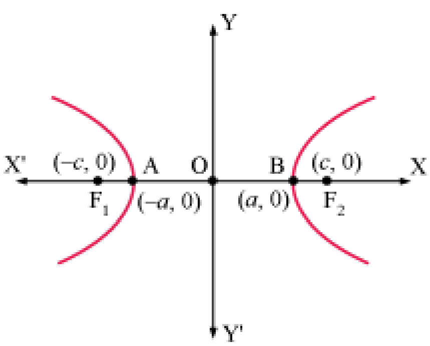
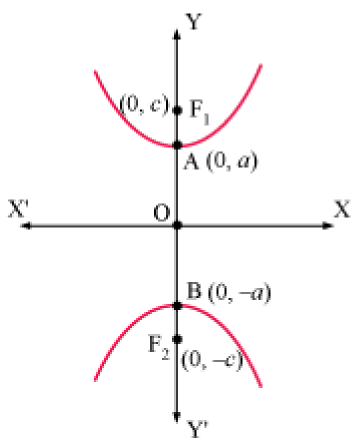
<tr><td rowspan="4">

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$</td><td>Standard Equation</td><td>$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$</td></tr>
<tr><td>Centre</td><td>(0, 0)</td></tr>
<tr><td>Vertices</td><td>(±a, 0)</td></tr>
<tr><td>Foci</td><td>(±c, 0)</td></tr>
</table>

	Conjugate axis	*y*-axis
	Transverse axis	*x*-axis
	Length of conjugate axis	$2b$
	Length of transverse axis	$2a$
	Length between foci	$2c$
	Relation between *a*, *b* and *c*	$c^2 = a^2 + b^2$
	Length of latus rectum	$\frac{2b^2}{a}$
	Eccentricity (*e*> 1)	$\frac{c}{a}$
	Standard Equation	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

<table>
<tr><td rowspan="10">

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$</td><td>Centre</td><td>(0, 0)</td></tr>
<tr><td>Vertices</td><td>(0, ±a)</td></tr>
<tr><td>Foci</td><td>(0, ±c)</td></tr>
<tr><td>Conjugate axis</td><td>x-axis</td></tr>
<tr><td>Transverse axis</td><td>y-axis</td></tr>
<tr><td>Length of conjugate axis</td><td>2b</td></tr>
<tr><td>Length of transverse axis</td><td>2a</td></tr>
<tr><td>Length between foci</td><td>2c</td></tr>
<tr><td>Relation between a, b and c</td><td>$c^2 = a^2 + b^2$</td></tr>
</table>

	Length of latus rectum	$\frac{2b^2}{a}$
	Eccentricity (*e*> 1)	$\frac{c}{a}$

- The hyperbola whose transverse axis and conjugate axis have the same length (*a* = *b*) is called an **equilateral hyperbola**.
- In such a case, the equation of an equilateral hyperbola will be $\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$ or $\frac{y^2}{a^2}-\frac{x^2}{a^2}=1$, which reduces to $x^2-y^2=a^2$ or $y^2-x^2=a^2$

Solved Examples

Example 1:
The ratio of the lengths of the major axis and minor axis of a hyperbola is 4: 3. The centre of the hyperbola is at the origin and its transverse axis is on the *y*-axis. If the total length of the latus recta of the hyperbola is 45 units, then find:

1. Lengths of the transverse axis and the conjugate axis
2. Distance between the foci
3. Coordinates of the foci
4. Coordinates of the vertices
5. Equation
6. Eccentricity

Solution:

It is given that the ratio of the lengths of the major axis and the minor axis of a hyperbola is 4: 3.

$$\therefore \frac{2a}{2b} = \frac{a}{b} = \frac{4}{3}$$

Let $a = 4x$ and $b = 3x$

It is also given that the total length of the latus recta is 45 units. Since there are 2 latus recta, the length of each is $\frac{45}{2}$ units.

$$\therefore \frac{2b^2}{a} = \frac{45}{2}$$
$$\Rightarrow \frac{2(3x)^2}{4x} = \frac{9x}{2} = \frac{45}{2}$$
$$\Rightarrow x = 5$$
$$\therefore a = 4a = 4 \times 5 = 20$$
$$b = 3a = 3 \times 5 = 15$$

1. Length of the transverse axis is $2a = 2 \times 20 = 40$ units

   Length of the conjugate axis is $2b = 2 \times 15 = 30$ units

2. We know that

   $$c^2 = a^2 + b^2 = 20^2 + 15^2 = 625 = 25^2$$
   $$\therefore c = 25$$

   Hence, the distance between the foci = $2c = 2 \times 25 = 50$ units

3. Since the centre of the hyperbola is at the origin and its transverse axis is on the *y*-axis, its focus lies on the *y*-axis. Hence, the coordinates of the foci of the hyperbola are $(0, \pm c) = (0, \pm 25)$.

4. Coordinates of the vertices = $(0, \pm a) = (0, \pm 20)$

5. Equation of the hyperbola is given by

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$
$$\Rightarrow \frac{y^2}{20^2} - \frac{x^2}{15^2} = 1$$
$$\Rightarrow \frac{y^2}{400} - \frac{x^2}{225} = 1$$

6. Eccentricity (*e*) of the hyperbola is given by

$$e = \frac{c}{a} = \frac{25}{20} = \frac{5}{4}$$

Example 2:

The eccentricity and the distance between the foci of a hyperbola are 3/2 and 24 units. The centre of the hyperbola is at the origin and its transverse axis is on the *x*-axis. Find:

1. Lengths of transverse axis and conjugate axis

2. Coordinates of the vertices

1. Equation

3. Length of the latus rectum

Also, check whether the point (−10, $-3\sqrt{5}$) lies on this hyperbola or not.

Solution:
It is given that the distance between the foci is 24.

$$\therefore 2c = 24$$
$$\Rightarrow c = 12$$

It is also given that the eccentricity of the hyperbola is 3/2.

$$\therefore \frac{c}{a} = \frac{3}{2}$$
$$\Rightarrow \frac{12}{a} = \frac{3}{2}$$
$$\Rightarrow a = 8$$

We know that

$$c^2 = a^2 + b^2$$
$$\Rightarrow b = \sqrt{c^2 - a^2} = \sqrt{12^2 - 8^2} = \sqrt{80} = 4\sqrt{5}$$

1. Length of the transverse axis = 2*a* = 2 × 8 = 16 units

   Length of the conjugate axis = 2*b* = 2 × $4\sqrt{5}$ = $8\sqrt{5}$ units

2. Since the centre of the hyperbola is at the origin and its transverse axis is on the *x*-axis, its vertices lie on the *x*-axis and the coordinates of these vertices are (±*a*, 0) = (±8, 0).

3. Equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\Rightarrow \frac{x^2}{8^2} - \frac{y^2}{(4\sqrt{5})^2} = 1$$
$$\Rightarrow \frac{x^2}{64} - \frac{y^2}{80} = 1$$

4. The length of the latus rectum is given by

$$\frac{2b^2}{a} = \frac{2 \times (4\sqrt{5})^2}{8} = 20$$ units

The point (−10, $-3\sqrt{5}$) lies on the hyperbola $\frac{x^2}{64} - \frac{y^2}{80} = 1$ if it satisfies this equation.

On substituting $x = -10$ and $y = -3\sqrt{5}$ in the LHS of the equation of the hyperbola, we obtain

$$\frac{(-10)^2}{64} - \frac{(-3\sqrt{5})^2}{80} = \frac{100}{64} - \frac{45}{80} = \frac{25}{16} - \frac{9}{16} = 1$$

This shows that the point (−10, $-3\sqrt{5}$) lies on the hyperbola $\frac{x^2}{64} - \frac{y^2}{80} = 1$.

Example 3:

A hyperbola has its centre at the origin and its conjugate axis is on the *y*-axis. It passes through the points (−10, $12\sqrt{3}$) and $\left(\frac{-25}{4}, 9\right)$. Find its equation, length of the latus rectum and eccentricity.

Solution:

Since the hyperbola has its centre at the origin and its conjugate axis is on the *y*-axis, let us assume that the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{... (1)}$$

Since the hyperbola passes through the point (−10, $12\sqrt{3}$),

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\Rightarrow \frac{(-10)^2}{a^2} - \frac{(12\sqrt{3})^2}{b^2} = 1$$
$$\Rightarrow \frac{100}{a^2} - \frac{432}{b^2} = 1 \qquad \text{... (2)}$$

The hyperbola also passes through the point $\left(\frac{-25}{4}, 9\right)$. So,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\Rightarrow \frac{\left(\frac{-25}{4}\right)^2}{a^2} - \frac{(9)^2}{b^2} = 1$$
$$\Rightarrow \frac{625}{16a^2} - \frac{81}{b^2} = 1 \qquad \text{... (3)}$$

On multiplying equation (2) by 25 and equation (3) by 64, and then subtracting these results, we obtain

$$25\left(\frac{100}{a^2}-\frac{432}{b^2}\right)-64\left(\frac{625}{16a^2}-\frac{81}{b^2}\right)=25-64$$
$$\Rightarrow \frac{5616}{b^2}=39$$
$$\Rightarrow b^2=\frac{5616}{39}=144$$
$$\Rightarrow b=12$$

On substituting the value of *b* in equation (2), we obtain

$$\frac{100}{a^2}-\frac{432}{12^2}=1$$
$$\Rightarrow \frac{100}{a^2}=1+3=4$$
$$\Rightarrow a^2=25$$
$$\Rightarrow a=5$$

On substituting the values of *a* and *b* in equation (1), we obtain the equation of the required hyperbola as

$$\frac{x^2}{25}-\frac{y^2}{144}=1$$

Length of the latus rectum is given by

$$\frac{2b^2}{a}=\frac{2\times(12)^2}{5}=\frac{288}{5}\text{ units}$$

We know that $c=\sqrt{a^2+b^2}=\sqrt{5^2+12^2}=13$

Eccentricity (*e*) of the hyperbola is given by

$$e=\frac{c}{a}=\frac{13}{5}$$