Z-Transform

6

Introduction

- Z - transform is a discrete-time counterpart of Laplace transform.
- For a discrete-time LTI system with impulse response h[n], the response y[n] of the system to a complex exponential input of the form z^n is

$$y[n] = z^n H(z)$$

where, $H(z)$ = Transform function of the system

Z-transform

Z-transform of a general discrete time signal

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

Note:

- The range of values of 'z' for which above equation is defined give ROC of Z-transform.
- Z-transform are used for digital filter design, sampled data control system.

Region of Convergence (ROC)

ROC is the region of range of values for which the summation

$\sum_{n=-\infty}^{+\infty} x[n]z^{-n}$ converges.

Properties of ROC

- The ROC of X(z) consists of a circle in the z-plane centered about the origin.
- ROC does not contain any poles, it is bounded by the poles.
- If x[n] is of finite duration then ROC is entire z-plane except possibly $z = 0$ and/or $z \to \infty$.
- If x[n] is a right sided sequence and if circle $|z| = a$ is in the ROC then all finite values of z for which $|z| > a$ will also be in ROC.
- If x[n] is a left sided sequence and if circle $|z| = a$ is the ROC then all finite values of z for which $|z| < a$ will also be in ROC.
- If x[n] is two sided and if the circle $|z| = a$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = a$.

Properties of Z-transform

Linearity

$x_1[n] \xrightarrow{Z.T.} X_1(z)$, ROC $= R_1$

$x_2[n] \xrightarrow{Z.T.} X_2(z)$, ROC $= R_2$

$$\boxed{ax_1[n] + bx_2[n] \xrightarrow{Z.T.} aX_1(z) + bX_2(z)}\ ;\ \text{ROC} = R_1 \cap R_2$$

Time shift

$x[n] \xrightarrow{Z.T.} x(z)$, ROC $= R$

$$\boxed{x[n - n_0] \xrightarrow{Z.T.} z^{-n_0}X(z)}\ ;\ \text{ROC}=R$$

Exponential multiplication or scaling in z-domain

$x[n] \xrightarrow{Z.T.} X(z)$ with ROC $= R$

$$\boxed{a^n x[n] \xrightarrow{Z.T.} X(z/a)}\ ;\ \text{ROC} = |a|\,R$$

Time reversal

$x[n] \xrightarrow{Z.T.} X(z)$, then

$$\boxed{x[-n] \xrightarrow{Z.T.} X(z^{-1})}\ ;\ \text{ROC} = 1/R$$

Differential in z-domain

$$\boxed{n\,x[n] \xrightarrow{Z.T.} -z\frac{d}{dz}X(z)}\ ;\ \text{ROC} = R$$

Convolution in Time

$x[n] \xrightarrow{Z.T.} X(z)$ with ROC $= R_1$

$h[n] \xrightarrow{Z.T.} H(Z)$ with ROC $= R_2$

$$\boxed{x[n] * h[n] \xrightarrow{Z.T.} X(z)H(z)}\ ;\ \text{ROC} = R_1 \cap R_2$$

Accumulation

$x[n] \xrightarrow{Z.T.} X(z)$, ROC $= R$

$$\boxed{\sum_{k=-\infty}^{n} x[k] \xrightarrow{Z.T} \frac{X(z)}{1 - z^{-1}}}\ ;\ \text{ROC} = R \cap |z| > 1$$

Unilateral Z-transform

$$\boxed{x[z] \xrightarrow{u.Z.T.} \sum_{n=0}^{\infty} x[n]z^{-n}}$$

Left shift

$$\boxed{x[n + 1] \xrightarrow{Z.T.} Z\,x(z) - Z\,x(0)}$$

Right shift

$$\boxed{\begin{aligned} x[n-1] &\xrightarrow{Z.T.} Z^{-1}x(z) + x(-1) \\ x[n-2] &\xrightarrow{Z.T.} Z^{-2}x(z) + x(-1) + x(-2) \end{aligned}}$$

First difference

$$\boxed{x[n] - x[n-1] = (1 - Z^{-1})\,x(z)}$$

Conjugation

$$\boxed{x^*[n] = x^*(z^*)}$$

Initial value theorem

If $\boxed{x[n] = 0}$ for $n < 0$

$$\boxed{x[0] = \lim_{Z\to\infty} x(z)}$$

Final value theorem

$$\boxed{x[\infty] = \lim_{Z\to 1}(1 - z^{-1})\,x(z)} \quad \text{or} \quad \boxed{x[\infty] = \lim_{Z\to 1}(z - 1)\,x(z)}$$

Characterization of LTI Systems Using Z-Transform

Causality

- A discrete-time LTI system is causal if and only if the ROC of its system function is the exterior of a circle, including infinity.
- A discrete-time LTI system with rational system function H(z) is causal if and only if.
  - (a) the ROC is the exterior of a circle outside the outermost pole; and
  - (b) with H(z) expressed as a ratio of polynomials in z, the order of the numerator cannot be greater than the order of the denominator.

Stability

- An LTI system is stable if and only if the ROC of its system function H(z) includes the unit circle, $|z| = 1$.

- A causal LTI system with rational system function H(z) is stable if and only if all of the poles of H(z) lie inside the unit circle i.e., they must all have magnitude smaller than 1.

Inverse Z-transform

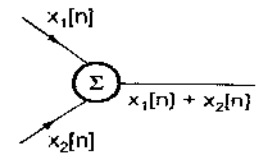
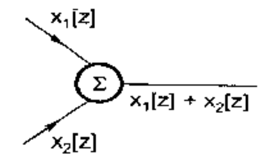
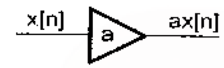
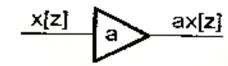
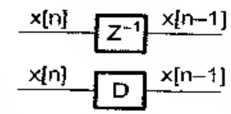
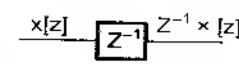
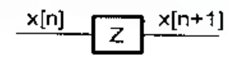
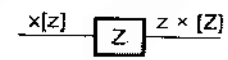
$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

- Where the symbol $\oint$ denotes integration around a counter clockwise circular contour centred at the origin with radius a.

Some Common Z - transform Pairs

	Signal	Transform	ROC
1.	$\delta[n]$	1	All z
2.	$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3.	$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4.	$\delta[n-m]$	z^{-m}	All z, except 0 (if m > 0) or ∞(if m < 0)
5.	$\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6.	$-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7.	$n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8.	$-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9.	$[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_o]z^{-1}}{1-[2\cos\omega_o]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10.	$[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11.	$[a^n \cos\omega_0 n]u[n]$	$\frac{1-[a\cos\omega_0]z^{-1}}{1-[2a\cos\omega_0]z^{-1}+a^2 z^{-2}}$	$\lvert z\rvert > a$
12.	$[a^n \sin\omega_0 n]u[n]$	$\frac{[a\sin\omega_o]z^{-1}}{1-[2a\cos\omega_o]z^{-}+a^2 z^{-2}}$	$\lvert z\rvert > a$

Basic Realization Structure

Element	Time domain representation	Z-domain representation
1. Adder		
2. Multiplier		
3. Delay element		
4. Unit advance element		

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