JEE ADVANCED

Single Correct Answer Type

1. Let E be the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and C be the circle $x^2 + y^2 = 9$. Let P and Q be the points $(1, 2)$ and $(2, 1)$, respectively. Then,
   a. Q lies inside C but outside E
   b. Q lies outside both C and E
   c. P lies inside both C and E
   d. P lies inside C but outside E **(IIT-JEE 1994)**

2. The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9}$ and having its center $(0, 3)$ is
   a. 4
   b. 3
   c. $\sqrt{12}$
   d. 7/2 **(IIT-JEE 1995)**

3. The number of values of c such that the straight line $y = 4x + c$ touches the curve $\frac{x^2}{4} + \frac{y^2}{1} = 1$ is
   a. 0
   b. 1
   c. 2
   d. infinite **(IIT-JEE 1998)**

4. If $P = (x, y)$, $F_1 = (3, 0)$, $F_2 = (-3, 0)$, and $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ equals
   a. 8
   b. 6
   c. 10
   d. 12 **(IIT-JEE 1998)**

5. The area of the quadrilateral formed by the tangents at the endpoint of the latus rectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is
   a. 27/4 sq. units
   b. 9 sq. units
   c. 27/2 sq. units
   d. 27 sq. units **(IIT-JEE 2003)**

6. If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$, then the locus of the midpoint of the intercept made by the tangents between the coordinate axes is
   a. $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$
   b. $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$
   c. $\frac{x^2}{2} + \frac{y^2}{4} = 1$
   d. $\frac{x^2}{4} + \frac{y^2}{2} = 1$ **(IIT-JEE 2004)**

7. The minimum area of the triangle formed by the tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the coordinate axes is
   a. ab sq. units
   b. $\frac{a^2 + b^2}{2}$ sq. units
   c. $\frac{(a + b)^2}{2}$ sq. units
   d. $\frac{a^2 + ab + b^2}{3}$ sq. units **(IIT-JEE 2004)**

8. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M, and O (the origin) is
   a. 31/10
   b. 29/10
   c. 21/10
   d. 27/10 **(IIT-JEE 2009)**

9. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the midpoint of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at points
   a. $(\pm(3\sqrt{5})/2, \pm2/7)$
   b. $(\pm(3\sqrt{5})/2, \pm\sqrt{19}/7)$
   c. $(\pm2\sqrt{3}, \pm1/7)$
   d. $(\pm2\sqrt{3}, \pm4\sqrt{3}/7)$ **(IIT-JEE 2009)**

10. The ellipse E_1: $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point $(0, 4)$ circumscribes the rectangle R. The eccentricity of the ellipse E_2 is
    a. $\sqrt{2}/2$
    b. $\sqrt{3}/2$
    c. 1/2
    d. 3/4 **(IIT-JEE 2012)**

Multiple Correct Answers Type

1. On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are
   a. $(2/5, 1/5)$
   b. $(-2/5, 1/5)$
   c. $(-2/5, -1/5)$
   d. $(2/5, -1/5)$ **(IIT-JEE 1999)**

2. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the endpoints of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are
   a. $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$
   b. $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$
   c. $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$
   d. $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$ **(IIT-JEE 2008)**

3. In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4 \sin^2 \frac{A}{2}$. If a, b, and c denote the lengths of the sides of the triangle opposite to the angles A, B, and C, respectively, then
   a. $b + c = 4a$
   b. $b + c = 2a$
   c. the locus of point A is an ellipse
   d. the locus of point A is a pair of straight lines **(IIT-JEE 2009)**

4. Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis, respectively. Let S be the circle $x^2 + (y - 1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 and E_2 at P, Q and R, respectively. Suppose that PQ

$= PR = \dfrac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is (are)

a. $e_1^2 + e_2^2 = \dfrac{43}{40}$ **b.** $e_1 e_2 = \dfrac{\sqrt{7}}{2\sqrt{10}}$

c. $|e_1^2 - e_2^2| = \dfrac{5}{8}$ **d.** $e_1 e_2 = \dfrac{\sqrt{3}}{4}$

(JEE Advanced 2015)

Linked Comprehension Type

For Problems 1-3

Tangents are drawn from the point $P(3, 4)$ to the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ touching the ellipse at points A and B.

(IIT-JEE 2010)

1. The coordinates of A and B are, respectively,
   a. $(3, 0)$ and $(0, 2)$
   b. $(-8/5, 2\sqrt{161}/15)$ and $(-9/5, 8/5)$
   c. $(-8/5, 2\sqrt{161}/15)$ and $(0, 2)$
   d. $(3, 0)$ and $(-9/5, 8/5)$

2. The orthocenter of triangle PAB is
   a. $(5, 8/7)$ **b.** $(7/5, 25/8)$
   c. $(11/5, 8/5)$ **d.** $(8/25, 7/5)$

3. The equation of the locus of the point whose distances from the point P and the line AB are equal is
   a. $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$
   b. $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$
   c. $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$
   d. $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

Matching Column Type

1. Match the conics in Column I with the statements/expressions in Column II.

Column I	Column II
(a) Circle	(p) The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
(b) Parabola	(q) Points z in the complex plane satisfying $|z + 2| - |z - 2| = \pm 3$
(c) Ellipse	(r) Points of the conic have parametric representation $x = \sqrt{3}\left(\dfrac{1-t^2}{1+t^2}\right)$, $y = \dfrac{2t}{1+t^2}$
(d) Hyperbola	(s) The eccentricity of the conic lies in the interval $1 \le x < \infty$
	(t) Points z in the complex plane satisfying $\operatorname{Re}(z + 1)^2 = |z|^2 + 1$

(IIT-JEE 2009)

2. Match the following:

List–I	List–II
(p) Let $y(x) = \cos(3\cos^{-1}x)$, $x \in [-1, 1], x \ne \pm \dfrac{\sqrt{3}}{2}$ Then $\dfrac{1}{y(x)}\left\{(x^2 - 1)\dfrac{d^2 y(x)}{dx^2} + x\dfrac{dy(x)}{dx}\right\}$ equals	(1) 1
(q) Let $A_1, A_2, \ldots, A_n$ ($n > 2$) be the vertices of a regular polygon of n sides with its centre at the origin. Let $\bar{a}_k$ be the position vector of the point A_k, $k = 1, 2, \ldots n$. If $\left|\sum_{k=1}^{n-1}(\bar{a}_k \times \bar{a}_{k+1})\right| = \left|\sum_{k=1}^{n-1}(\bar{a}_k \cdot \bar{a}_{k+1})\right|$, then the minimum value of n is	(2) 2
(r) If the normal from the point $P(h, 1)$ on the ellipse $\dfrac{x^2}{6} + \dfrac{y^2}{3} = 1$ is perpendicular to the line $x + y = 8$, then the value of h is	(3) 8
(s) Number of positive solutions satisfying the equation $\tan^{-1}\left(\dfrac{1}{2x+1}\right) + \tan^{-1}\left(\dfrac{1}{4x+1}\right) = \tan^{-1}\left(\dfrac{2}{x^2}\right)$ is	(4) 9

Codes:

	(p)	(q)	(r)	(s)
a.	(4)	(3)	(2)	(1)
b.	(2)	(4)	(3)	(1)
c.	(4)	(3)	(1)	(2)
d.	(2)	(4)	(1)	(3)

(JEE Advanced 2014)

Integer Answer Type

1. A vertical line passing through the point $(h, 0)$ intersects the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ at points P and Q. Let the tangents to the ellipse at P and Q meet at point R. If $\Delta(h) = $ area of triangle PQR, $\Delta_1 = \max\limits_{1/2 \le h \le 1} \Delta(h)$, and $\Delta_2 = \min\limits_{1/2 \le h \le 1} \Delta(h)$, then $\dfrac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = $

(JEE Advanced 2013)

2. Suppose that the foci of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$ are $(f_1, 0)$ and $(f_2, 0)$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at $(0, 0)$ and with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2 which passes through $(f_1, 0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\dfrac{1}{m_1^2} + m_2^2 \right)$ is

(JEE Advanced 2015)

Fill in the Blanks Type

1. An ellipse has OB as the semi-minor axis; F, F' as its foci; and $\angle FBF'$ is a right angle. Then, the eccentricity of the ellipse is _____.　　**(IIT-JEE 1997)**

Subjective Type

1. Let d be the perpendicular distance from the center of the ellipse to any tangent to the ellipse. If F_1 and F_2 are the two foci of the ellipse, then show that $(PF_1 - PF_2)^2 = 4a^2 \left(1 - \dfrac{b^2}{d^2} \right)$.　　**(IIT-JEE 1995)**

2. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

(IIT-JEE 1997)

3. Consider the family of circles $x^2 + y^2 = r^2$, $2 < r < 5$. If in the first quadrant, the common tangent to a circle of this family and the ellipse $4x^2 + 25y^2 = 100$ meets the coordinate axes at A and B, then find the equation of the locus of the midpoint of AB.　　**(IIT-JEE 1999)**

4. Find the coordinates of all the points P on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ for which the area of triangle PON is maximum, where O denotes the origin and N is the foot of the perpendicular from O to the tangent at P.

(IIT-JEE 1999)

5. Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Suppose perpendiculars from A, B, and C to the major axis of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $(a > b)$, meet the ellipse, respectively, at P, Q, and R so that P, Q, and R lie on the same side of the major axis as A, B, and C, respectively. Prove that the normals to the ellipse drawn at the points P, Q, and R are concurrent.

(IIT-JEE 2000)

6. Let P be a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $0 < b < a$. Let the line parallel to the y-axis passing through P meets the circle $x^2 + y^2 = a^2$ at point Q such that P and Q are on the same side of the x-axis. For two positive real numbers r and s, find the locus of the point R on PQ such that $PR : RQ = r : s$ as P varies over the ellipse.　　**(IIT-JEE 2001)**

7. Prove that in an ellipse, the perpendicular from a focus upon any tangent and the line joining the center of the ellipse to the point of contact meet at the corresponding directrix.　　**(IIT-JEE 2002)**

8. From a point, common tangents are drawn to the curves $x^2 + y^2 = 16$ and $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$. Find the slope of the common tangent in the first quadrant and also find the length of the intercept between the coordinate axes.

(IIT-JEE 2005)

Answer Type

JEE Advanced

Single Correct Answer Type

1. d.　　2. a.　　3. c.　　4. c.　　5. d.
6. a.　　7. a.　　8. d.　　9. c.　　10. c.

Multiple Correct Answers Type

1. b., d.　　2. b., c.　　3. b., c.　　4. a., b

Linked Comprehension Type

1. d.　　2. c.　　3. a.

Matching Column Type

1. (r) – (c)　　2. a.

Integer Answer Type

1. 9　　　　2. 4

Fill in the Blanks Type

1. $\dfrac{1}{\sqrt{2}}$

Subjective Type

3. $\dfrac{25}{x^2} + \dfrac{4}{y^2} = 4$　　4. $\left(\pm \dfrac{a^2}{\sqrt{a^2 + b^2}}, \pm \dfrac{b^2}{\sqrt{a^2 + b^2}} \right)$

6. $\dfrac{x^2}{a^2} + \dfrac{y^2 (r + s)^2}{(bs + ar)^2} = 1$　　8. $m = -\dfrac{2}{\sqrt{3}}, \dfrac{14}{\sqrt{3}}$

Hints and Solutions

JEE Advanced

Single Correct Answer Type

1. **d.** Since $1^2 + 2^2 = 5 < 9$ and $2^2 + 1^2 = 5 < 9$, both P and Q lie inside C. Also,

$$\frac{1^2}{9} + \frac{2^2}{4} = \frac{1}{9} + 1 > 1 \text{ and } \frac{2^2}{9} + \frac{1}{4} = \frac{25}{36} < 1$$

Hence, P lies outside E and Q lies inside E. Thus, P lies inside C but outside E.

2. **a.** The given ellipse is

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

Here, $a^2 = 16$ and $b^2 = 9$

$\therefore b^2 = a^2 (1 - e^2)$

or $9 = 16(1 - e^2)$

or $e = \dfrac{\sqrt{7}}{4}$

Hence, the foci are $(\pm\sqrt{7}, 0)$.

Radius of circle = Distance between $(\pm\sqrt{7}, 0)$ and $(0, 3)$

$$= \sqrt{7 + 9} = 4$$

3. **c.** For given slope, there exist two parallel tangents to the ellipse. Hence, there are two values of c.

4. **c.** The ellipse can be written as

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

Here, $a^2 = 25$, $b^2 = 16$.

Now, $b^2 = a^2(1 - e^2)$

or $\dfrac{16}{25} = 1 - e^2$

or $e^2 = 1 - \dfrac{16}{25} = \dfrac{9}{25}$

or $e = \dfrac{3}{5}$

The foci of the ellipse are $(\pm ae, 0) \equiv (\pm 3, 0)$, i.e., F_1 and F_2 are the foci of the ellipse.

Therefore, we have $PF_1 + PF_2 = 2a = 10$ for every point P on the ellipse.

5. d. The given ellipse is

$\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$

Then, $a^2 = 9$, $b^2 = 5$. Therefore,

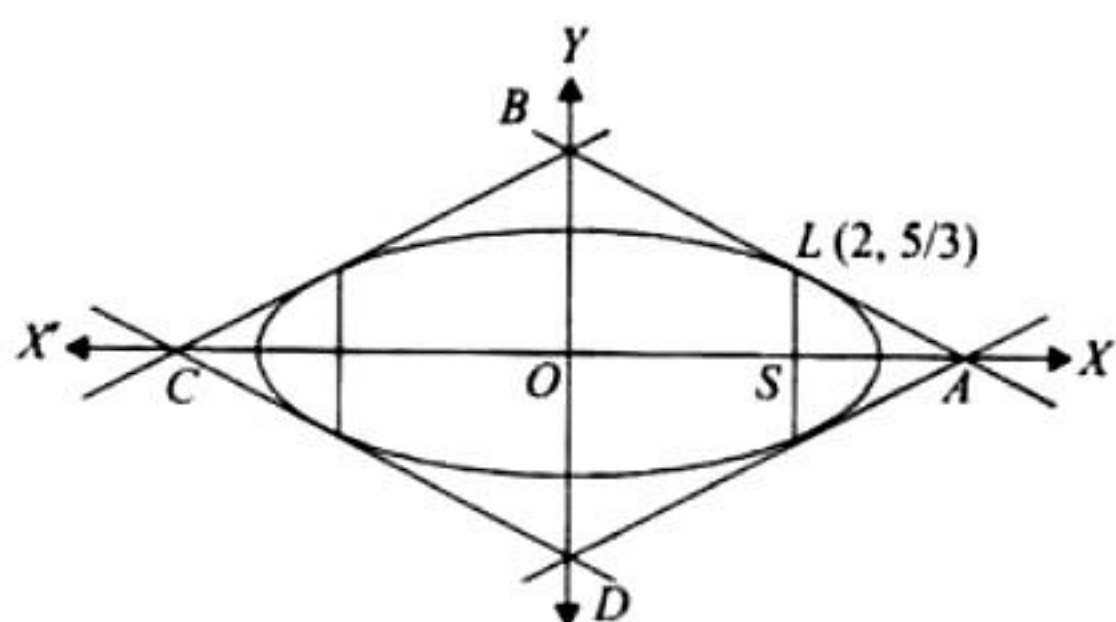$e = \sqrt{1 - \dfrac{5}{9}} = \dfrac{2}{3}$

Hence, the endpoint of latus rectum in first quadrant is $L(2, 5/3)$.

The equation of tangent at L is

$\dfrac{2x}{9} + \dfrac{y}{3} = 1$

The tangent meets the x-axis at $A(9/2, 0)$ and the y-axis at $B(0, 3)$. Therefore,

Area of $\triangle OAB = \dfrac{1}{2} \times \dfrac{9}{2} \times 3 = \dfrac{27}{4}$

By symmetry,

Area of quadrilateral = 4 × (Area of $\triangle OAB$)

$= 4 \times \dfrac{27}{4} = 27$ sq. units

6. a. Any tangent to ellipse

$\dfrac{x^2}{2} + \dfrac{y^2}{1} = 1$ is given by

$\dfrac{x \cos \theta}{\sqrt{2}} + y \sin \theta = 1$

Let it meet axes at A and B.

∴ $A \equiv (\sqrt{2} \sec \theta, 0)$ and $B \equiv (0, \csc \theta)$.
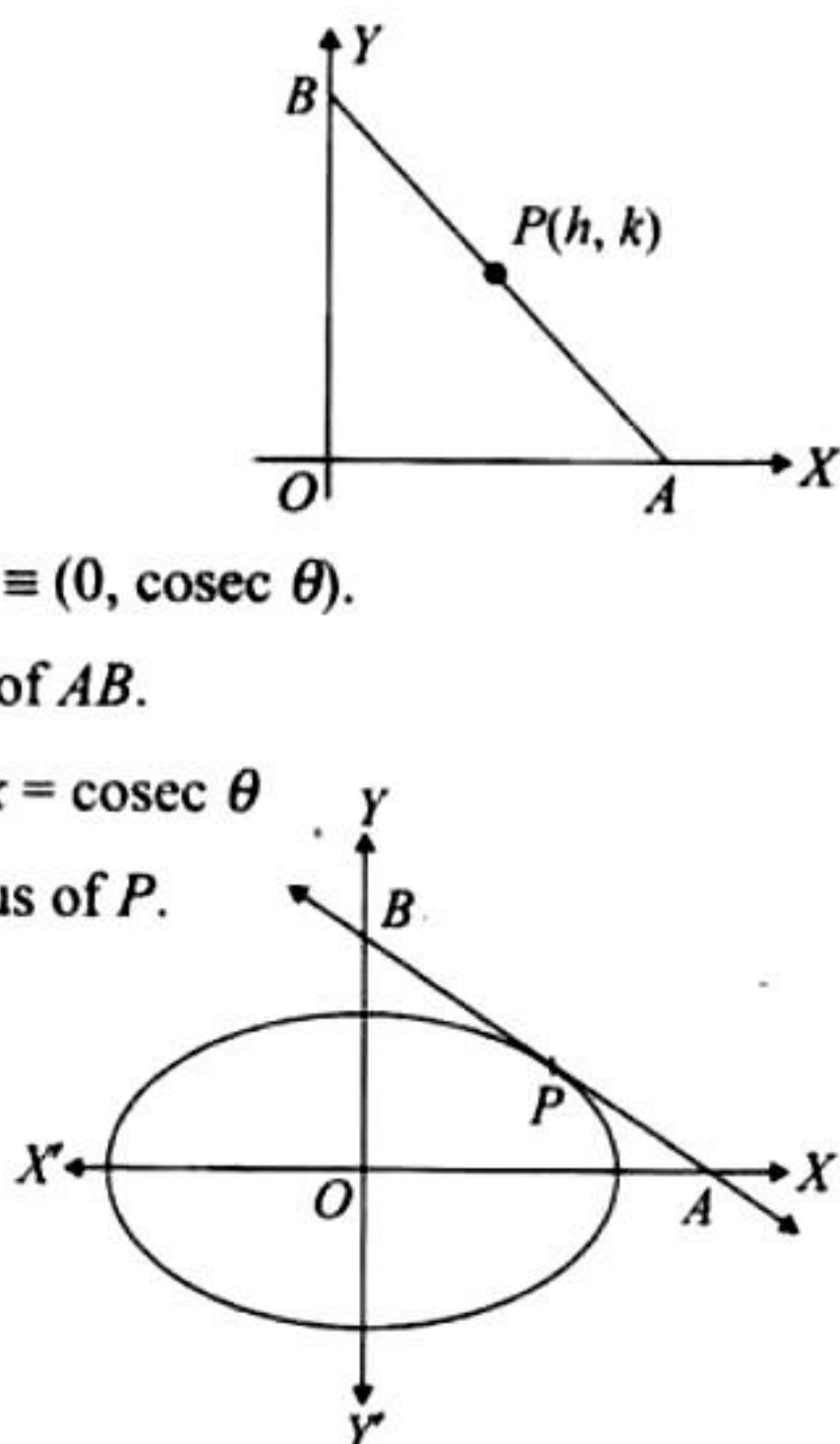
Let $p(h, k)$ be the mid point of AB.

Hence, $2h = \sqrt{2} \sec \theta$ and $2k = \csc \theta$

Eliminating 'θ', which is locus of P.

$\left(\dfrac{1}{\sqrt{2}h}\right)^2 + \left(\dfrac{1}{2k}\right)^2 = 1$

or $\dfrac{1}{2h^2} + \dfrac{1}{4k^2} = 1$

or $\dfrac{1}{2x^2} + \dfrac{1}{4y^2} = 1$

7. a. Tangent to the ellipse

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

at $P(a \cos \theta, b \sin \theta)$ is given by

$\dfrac{x \cos \theta}{a} + \dfrac{y \sin \theta}{b} = 1$

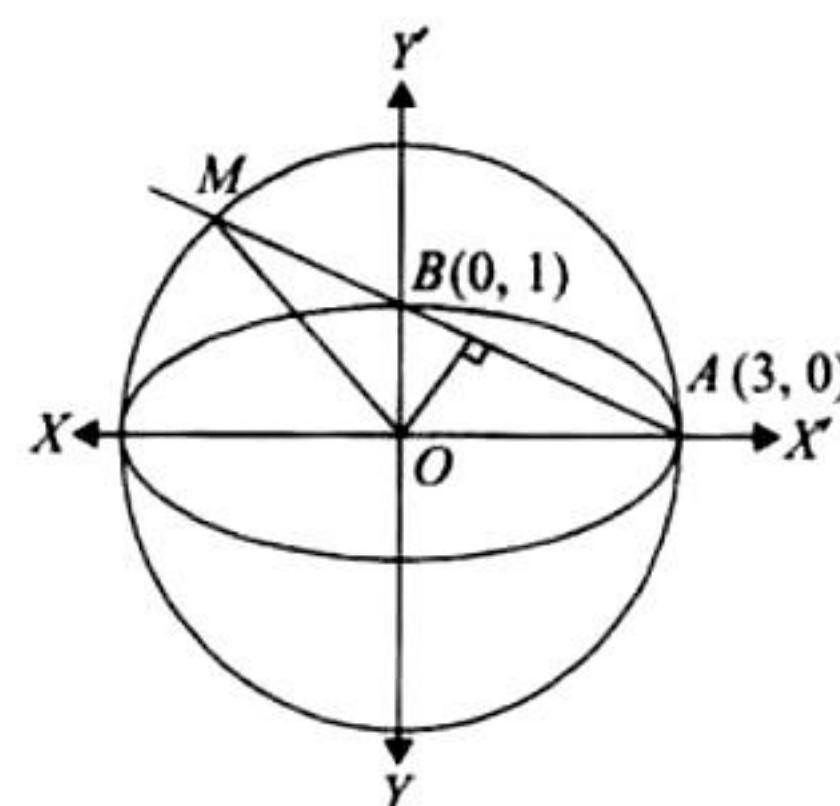It meets the coordinate axes at $A(a \sec \theta, 0)$ and $B(0, b \csc \theta)$. Therefore,

Area of $\triangle OAB = \dfrac{1}{2} \times a \sec \theta \times b \csc \theta$

or $\Delta = \dfrac{ab}{\sin 2\theta}$

For area to be minimum, $\sin 2\theta$ should be maximum and we know that the maximum value of $\sin 2\theta$ is 1. Therefore,

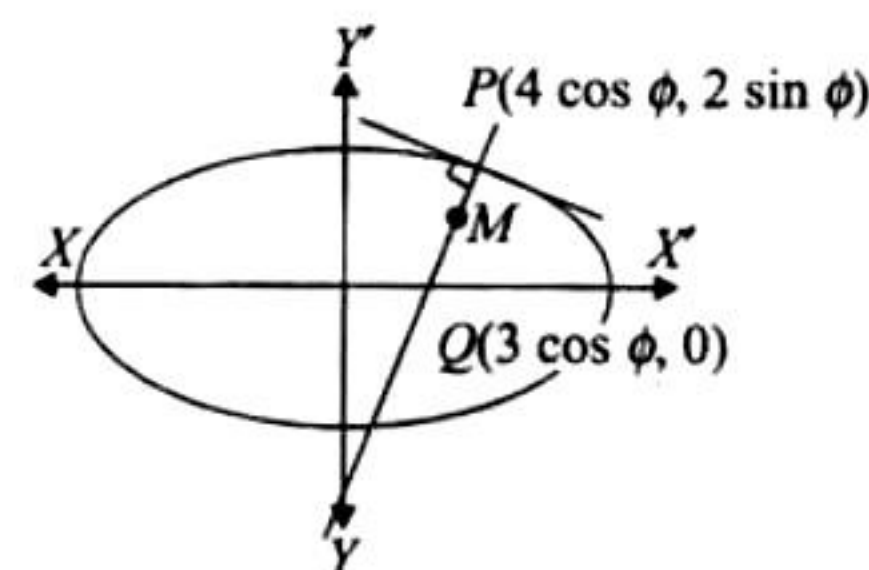$\Delta_{\min} = ab$

8. d.

The equation of line AM is

$x + 3y - 3 = 0$

Perpendicular distance of line from the origin $= \dfrac{3}{\sqrt{10}}$

Length of $AM = 2\sqrt{9 - \dfrac{9}{10}} = 2 \times \dfrac{9}{\sqrt{10}}$

∴ Area of $\triangle OAM = \dfrac{1}{2} \times 2 \times \dfrac{9}{\sqrt{10}} \times \dfrac{3}{\sqrt{10}} = \dfrac{27}{10}$ sq. units.

9. c. Given ellipse is $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$. Normal at $P(4 \cos \phi, 2 \sin \phi)$ is given by $4x \sec \phi - 2y \csc \phi = 12$

It meets x-axis at

$Q \equiv (3 \cos \phi, 0)$

Let $M \equiv (\alpha, \beta)$ be the mid point of PQ.

∴ $\alpha = \dfrac{3 \cos \phi + 4 \cos \phi}{2} = \dfrac{7}{2} \cos \phi$

or $\cos \phi = \dfrac{2}{7} \alpha$

and $\beta = \sin \phi$

Using $\cos^2 \phi + \sin^2 \phi = 1$, we have

$$\frac{4}{49} \alpha^2 + \beta^2 = 1$$

or $\frac{4}{49} x^2 + y^2 = 1$ (i)

Now, the latus rectum is

$x = \pm 2\sqrt{3}$ (ii)

Solving (i) and (ii), we have

$$\frac{48}{49} + y^2 = 1$$

or $y = \pm \frac{1}{7}$

The points of intersection are $(\pm 2\sqrt{3}, \pm 1/7)$.

10. c. Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
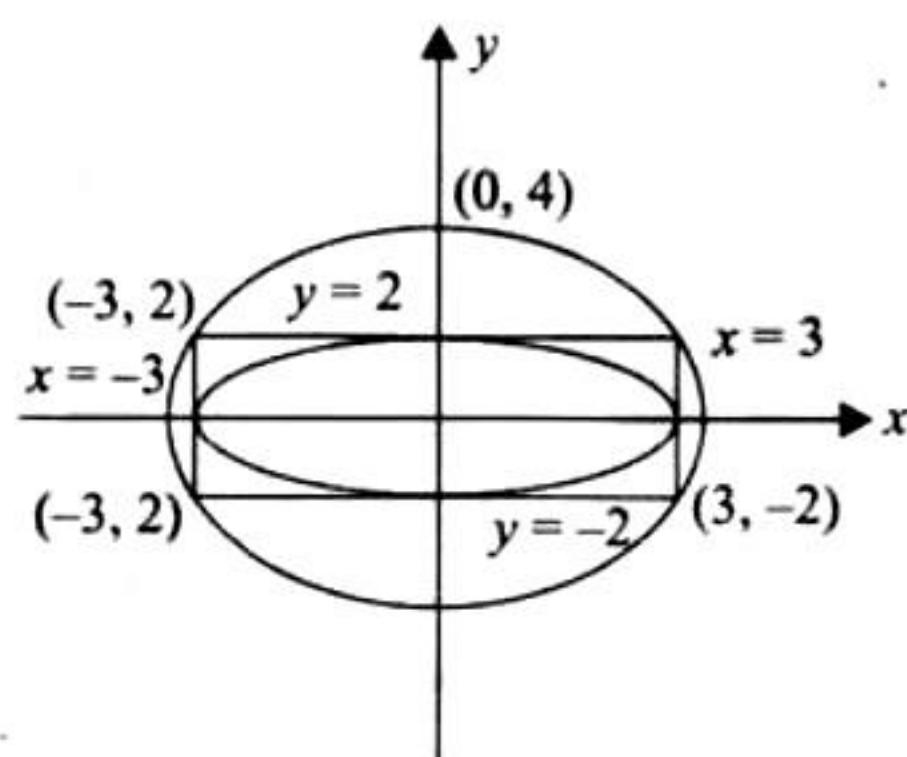
as it is passing through

(0, 4) and (3, 2). So,

$b^2 = 16$ and $\frac{9}{a^2} + \frac{4}{16} = 1$

or $a^2 = 12$

So, $12 = 16(1 - e^2)$

or $e = \frac{1}{2}$.

Multiple Correct Answers Type

1. b., d. Let (x_1, y_1) be the point at which tangents to the ellipse $4x^2 + 9y^2 = 1$ are parallel to $8x = 9y$.

Then the slope of the tangent is 8/9, i.e.,

$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{8}{9}$ (i)

Differentiating the equation of ellipse w.r.t. x, we get

$$8x + 18y \frac{dx}{dy} = 0$$

or $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-8x_1}{18y_1} = \frac{-4x_1}{9y_1}$

Substituting in (i), we get

$$\frac{-4x_1}{9y_1} = \frac{8}{9}$$

or $-x_1 = 2y_1$ (ii)

Also, (x_1, y_1) is the point of contact which must be on the curve. Hence,

$4x_1^2 + 9y_1^2 = 1$

or $4 \times 4y_1^2 + 9y_1^2 = 1$ [Using (ii)]

or $y_1^2 = \frac{1}{25}$

or $y_1 = \pm \frac{1}{5}$

$\therefore x_1 = \mp \frac{2}{5}$

Thus, the required points are (–2/5, 1/5) and (2/5, –1/5).

Alternative Method:

Let $y = \frac{8}{9} x + c$

be the tangent to

$$\frac{x^2}{1/4} + \frac{y^2}{1/9} = 1$$

where $c = \pm \sqrt{a^2 m^2 + b^2} = \pm \sqrt{\frac{1}{4} \times \frac{64}{81} + \frac{1}{9}} = \pm \frac{5}{9}$

So, the points of contact are $(-a^2 m/c, \ b^2/c) \equiv (2/5, -1/5)$ or (–2/5 , 1/5).

2. b., c. The given ellipse is

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

$b^2 = a^2(1 - e^2)$

or $e = \frac{\sqrt{3}}{2}$

Hence, the endpoints P and Q of the latus rectum are given by

$P \equiv \left(\sqrt{3}, -\frac{1}{2}\right)$

and $Q \equiv \left(-\sqrt{3}, -\frac{1}{2}\right)$ (Given $y_1, y_2 < 0$)

The coordinates of the midpoint of PQ are

$R \equiv \left(0, -\frac{1}{2}\right)$

Length of latus rectum, $PQ = 2\sqrt{3}$

Hence, two parabolas are possible whose vertices are

$S\left(0, +\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$ and $T\left(0, -\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$

The equations of the parabolas are

$x^2 = 2\sqrt{3} \left(y + \frac{\sqrt{3}}{2} + \frac{1}{2}\right)$

and $x^2 = -2\sqrt{3} \left(y - \frac{\sqrt{3}}{2} + \frac{1}{2}\right)$

or $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$

and $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$

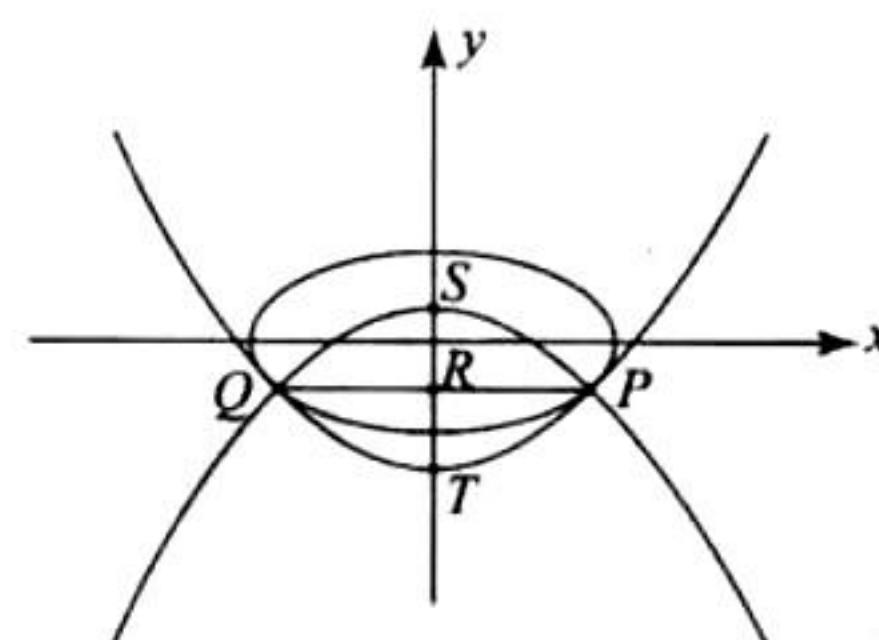**3. b., c.**

Given $\cos B + \cos C = 4 \sin^2 \frac{A}{2}$

or $2 \cos\left(\frac{B+C}{2}\right) \cos\left(\frac{B-C}{2}\right) = 4 \sin^2 \frac{A}{2}$

or $\cos\left(\frac{B-C}{2}\right) = 2 \sin\left(\frac{A}{2}\right)$

or $2 \sin \frac{B+C}{2} \cos \frac{B-C}{2} = 4 \sin \frac{A}{2} \cos \frac{A}{2}$

or $\sin B + \sin C = 2 \sin A$
(using sine rule)
Thus sum of two variable sides is constant.
or $b + c = 2a$ (constant)
Hence, the locus of vertex A is an ellipse with B and C as foci.

4. a., b.

Let ellipses be $E_1 : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and $E_2 = \dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} = 1$

Since $x + y = 3$ is a tangent,
$a^2 + b^2 = A^2 + B^2 = 9$ (using condition $c^2 = a^2m^2 + b^2$ etc.)
Point P lies on $x^2 + (y - 1)^2 = 2$.
Equation of normal to circle having slope 1 is $y - 1 = 1(x - 0)$
or $x - y + 1 = 0$
Solving this normal with tangent line we get point $P(1, 2)$.

Now $PQ = PR = \dfrac{2\sqrt{2}}{3}$

So, points on line $x + y - 3 = 0$ at distance $\dfrac{2\sqrt{2}}{3}$ from point P

are $\left(1 \mp \dfrac{1}{\sqrt{2}} \dfrac{2\sqrt{2}}{3}, 2 \mp \dfrac{1}{\sqrt{2}} \dfrac{2\sqrt{2}}{3}\right)$

or $Q\left(\dfrac{5}{3}, \dfrac{4}{3}\right)$ and $Q\left(\dfrac{1}{3}, \dfrac{8}{3}\right)$

Now $Q\left(\dfrac{5}{3}, \dfrac{4}{3}\right)$ lies on E_1

So, $\dfrac{25}{9a^2} + \dfrac{16}{9(9 - a^2)} = 1$

$\Rightarrow \quad 225 - 25a^2 + 16a^2 = 9a^2(9 - a^2)$
$\Rightarrow \quad a^4 - 10a^2 + 25 = 0$
$\Rightarrow \quad a^2 = 5$ so $b^2 = 4$

$\therefore \quad e_1^2 = 1 - \dfrac{b^2}{a^2} = 1 - \dfrac{4}{5} = \dfrac{1}{5}$

Now $\left(\dfrac{1}{3}, \dfrac{8}{3}\right)$ lies on E_2

So, $\dfrac{1}{A^2} + \dfrac{64}{(9 - A^2)} = 9$

$\Rightarrow \quad 9 - A^2 + 64A^2 = 9A^2(9 - A^2)$
$\Rightarrow \quad A^4 - 2A^2 + 1 = 0$
$\Rightarrow \quad A^2 = 1 \qquad$ so, $B^2 = 8$

$\therefore \quad e_2^2 = 1 - \dfrac{A^2}{B^2} = 1 - \dfrac{1}{8} = \dfrac{7}{8}$

Linked Comprehension Type

1. d.

Given ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

The equation of tangent having slope m is

$$y = mx + \sqrt{9m^2 + 4}$$

The tangent passes through the point $(3, 4)$. Therefore,

$$4 - 3m = \sqrt{9m^2 + 4}$$

Squaring, we have

$$16 + 9m^2 - 24m = 9m^2 + 4$$

or $m = \dfrac{12}{24} = \dfrac{1}{2}$

Therefore, the equation of tangent is

$$y - 4 = \dfrac{1}{2}(x - 3) \text{ or } x - 2y + 5 = 0 \qquad \text{(i)}$$

Let the point of contact on the curve be $B(\alpha, \beta)$.
Equation of tangent at this point is

$$\dfrac{x\alpha}{9} + \dfrac{y\beta}{4} - 1 = 0 \qquad \text{(ii)}$$

Comparing equations (i) and (ii)

or $\dfrac{\alpha/9}{1} = \dfrac{\beta/4}{-2} = -\dfrac{1}{5}$

or $\alpha = -\dfrac{9}{5}, \beta = \dfrac{8}{5}$

$B \equiv \left(-\dfrac{9}{5}, \dfrac{8}{5}\right)$

Another slope of tangent is ∞. Then the equation of tangent is $x = 3$ and the corresponding point of contact is $A(3, 0)$.

2. c. Since the slope of PA is ∞, the slope of altitude through B must be 0, for which the orthocenter is $(11/5, 8/5)$.

3. a. The equation of chord of contact AB is

$\dfrac{3x}{9} + \dfrac{4y}{4} = 1$ or $\dfrac{x}{3} + y = 1$ or $x + 3y - 3 = 0$

Thus, equation of locus of the point whose distances from the point P and the line AB are equal is

$$(x - 3)^2 + (y - 4)^2 = \dfrac{(x + 3y - 3)^2}{10}$$

or $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

Matching Column Type

1. (r) – (c)

$$x = \sqrt{3}\left(\dfrac{1 - t^2}{1 + t^2}\right), y = \dfrac{2t}{1 + t^2}$$

Let $t = \tan \alpha$

$\therefore \cos 2\alpha = \dfrac{x}{\sqrt{3}}$ and $\sin 2\alpha = y$

$\therefore \dfrac{x^2}{3} + y^2 = \sin^2 2\alpha + \cos^2 2\alpha = 1$, which is an ellipse.

Note: Solutions of the remaining parts are given in their respective chapters.

2. a.

For statement (r):

Point $(h, 1)$ lies on ellipse $\dfrac{x^2}{6} + \dfrac{y^2}{3} = 1$

$\Rightarrow \quad \dfrac{h^2}{6} + \dfrac{1^2}{3} = 1$

$\therefore \quad h = \pm 2$

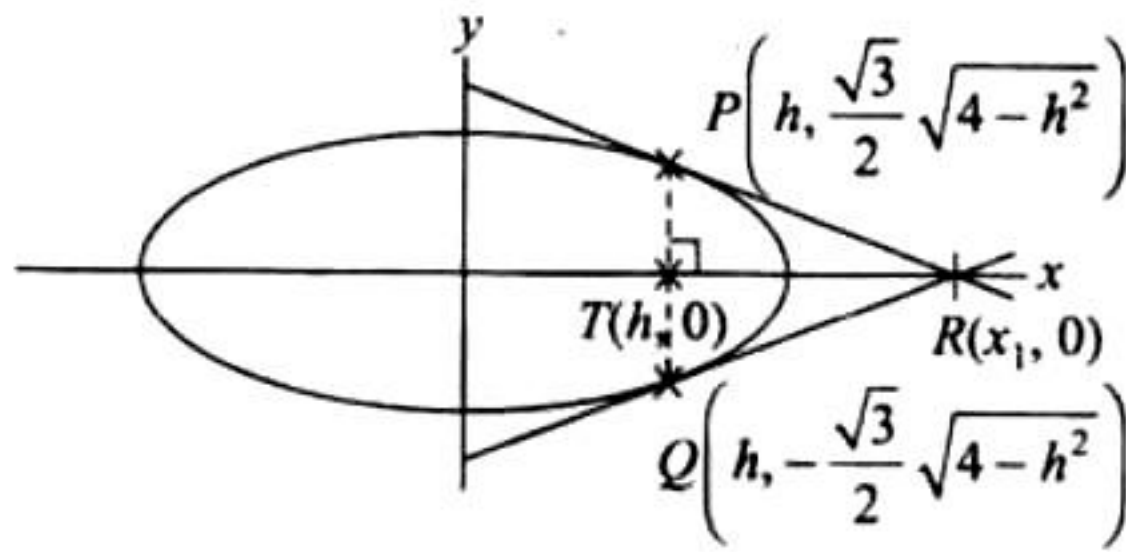Tangent at $(2, 1)$ is $\dfrac{2x}{6} + \dfrac{y}{3} = 1$

or $x + y = 3$ which is parallel to $x + y = 8$

$\therefore \quad h = 2$

Note: Solutions of the remaining parts are given in their respective chapters.

Integer Answer Type

1. (9)

$\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

$\therefore y = \dfrac{\sqrt{3}}{2}\sqrt{4 - h^2}$ at $x = h$

Let $R(x_1, 0)$.

PQ is the chord of contact. So,

$\dfrac{xx_1}{4} = 1$

or $x = \dfrac{4}{x_1}$

which is the equation of PQ. At $x = h$,

$\dfrac{4}{x_1} = h$ or $x_1 = \dfrac{4}{h}$

$\Delta(h) = \text{Area of } \Delta PQR = \dfrac{1}{2} \times PQ \times RT$

$= \dfrac{1}{2} \times \dfrac{2\sqrt{3}}{2}\sqrt{4 - h^2} \times (x_1 - h)$

$= \dfrac{\sqrt{3}}{2h}(4 - h^2)^{3/2}$

$\Delta'(h) = \dfrac{-\sqrt{3}(4 + 2h^2)}{2h^2}\sqrt{4 - h^2} < 0$

Therefore, $\Delta(h)$ is always decreasing. So,

$\Delta_1 = \text{Maximum of } \Delta(h) = \dfrac{45\sqrt{5}}{8}$ at $h = \dfrac{1}{2}$

and $\Delta_2 = \text{Minimum of } \Delta(h) = \dfrac{9}{2}$ at $h = 1$

So, $\dfrac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = \dfrac{8}{\sqrt{5}} \times \dfrac{45\sqrt{5}}{8} - 8 \times \dfrac{9}{2} = 45 - 36 = 9$

2.(4) Ellipse is $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$

$\therefore \quad a^2 e^2 = a^2 - b^2 = 9 - 5 = 4$

$\therefore \quad$ foci are $(2, 0)$ and $(-2, 0)$

$\therefore \quad f_1 = 2$ and $f_2 = -2$

Parabola with vertex $O(0, 0)$ and focus $(f_1, 0)$ or $(2, 0)$ is $y^2 = 8x$.

Equation of tangent having slope m_1 is

$y = m_1 x + \dfrac{2}{m_1}$

It passes through $(2f_2, 0)$ or $(-4, 0)$.

$\therefore \quad 0 = -4m_1 + \dfrac{2}{m_1}$

$\Rightarrow \quad m_1 = \pm \dfrac{1}{\sqrt{2}}$

Parabola with vertex $O(0, 0)$ and focus $(2f_2, 0)$ or $(-4, 0)$ is $y^2 = -16x$.

Equation of tangent having slope m_2 is

$y = m_2 x - \dfrac{4}{m_2}$

It passes through $(f_1, 0)$ or $(2, 0)$.

$\therefore \quad 0 = 2m_2 - \dfrac{4}{m_2}$

$\Rightarrow \quad m_2 = \pm \sqrt{2}$

$\Rightarrow \quad \dfrac{1}{m_1^2} + m_2^2 = 2 + 2 = 4$

Fill in the Blanks Type

1. Consider ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
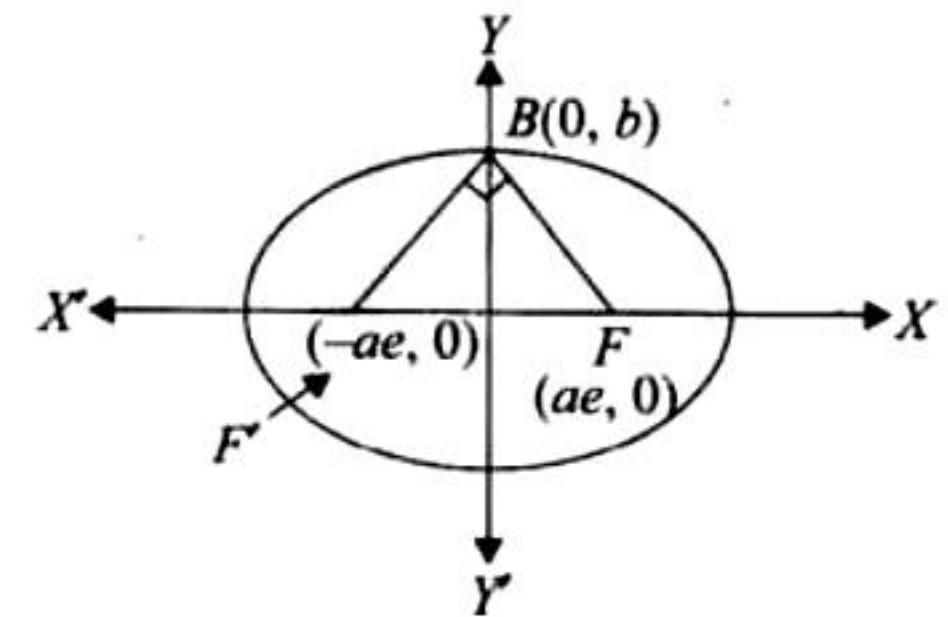
Given $m_{BF} \cdot m_{BF'} = -1$

$\therefore \dfrac{b - 0}{0 - ae} \times \dfrac{b - 0}{0 + ae} = -1$

$\therefore \dfrac{b^2}{a^2 e^2} = 1$

$\therefore e^2 = \dfrac{b^2}{a^2} = 1 - e^2$

$\therefore e^2 = \dfrac{1}{2}$

$\therefore e = \dfrac{1}{\sqrt{2}}$

Subjective Type

1. The equation of the tangent at the point $P(a\cos\theta, b\sin\theta)$ on

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

is $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$ \hfill (i)

The perpendicular distance of (i) from the center $(0, 0)$ of the ellipse is given by

$d = \dfrac{1}{\sqrt{\dfrac{1}{a^2}\cos^2\theta + \dfrac{1}{b^2}\cos^2\theta}} = \dfrac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$

$$\therefore 4a^2\left(1 - \frac{b^2}{a^2}\right) = 4a^2\left\{1 - \frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2}\right\}$$
$$= 4(a^2 - b^2)\cos^2\theta = 4a^2e^2\cos^2\theta \qquad \text{(ii)}$$

The foci are $F_1 \equiv (ae, 0)$ and $F_2 \equiv (-ae, 0)$. Therefore,
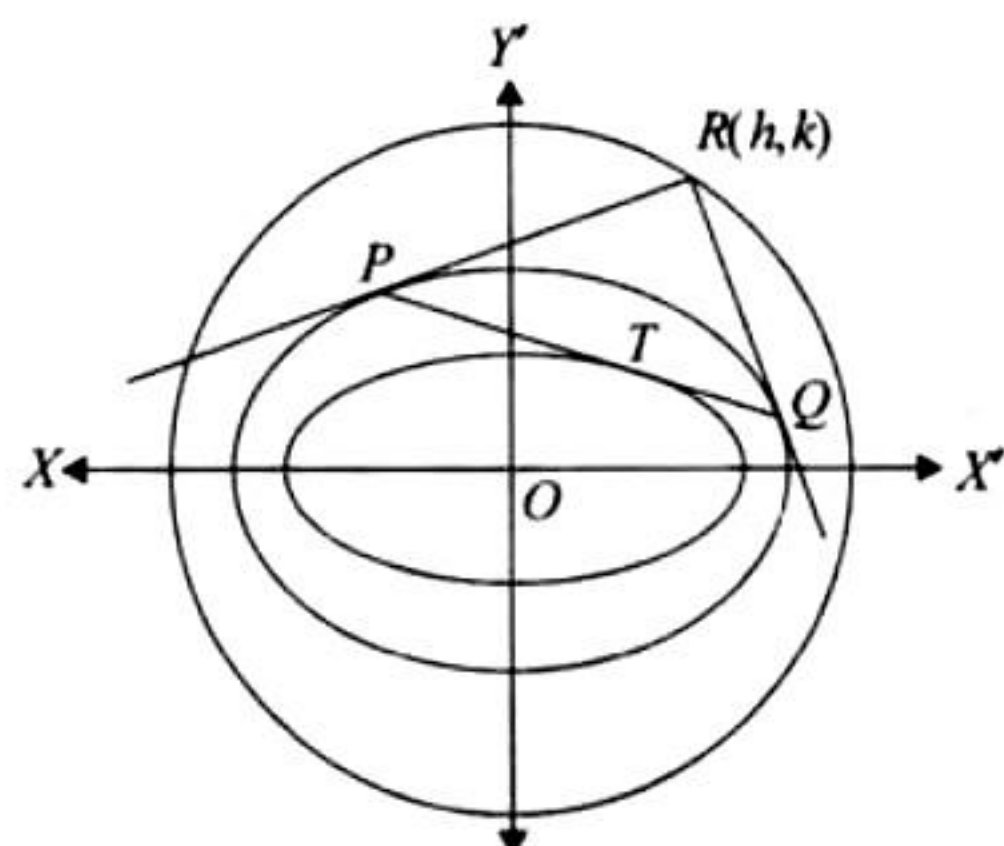
$PF_1 = e(1 - e\cos\theta)$

and $PF_2 = a(1 + e\cos\theta)$

$\therefore (PF_1 - PF_2)^2 = 4a^2e^2\cos^2\theta \qquad \text{(iii)}$

Hence, from (ii) and (iii), we have

$$(PF_1 - PF_2)^2 = 4a^2\left(1 - \frac{b^2}{a^2}\right)$$

2.

The given ellipses are

$$\frac{x^2}{4} + \frac{y^2}{1} = 1 \qquad \text{(i)}$$

and $\dfrac{x^2}{6} + \dfrac{y^2}{3} = 1 \qquad \text{(ii)}$

Then the equation of tangent to (i) at any point $T(2\cos\theta, \sin\theta)$ is given by

$$\frac{x\cos\theta}{2} + \frac{y\sin\theta}{1} = 1 \qquad \text{(iii)}$$

Let this tangent meets the ellipse (ii) at P and Q. Let the tangents drawn to ellipse (ii) at P and Q meet each other at $R(h, k)$.

Then PQ is the chord of contact of ellipse (ii) with respect to the point $R(h, k)$ and is given by

$$\frac{xh}{6} + \frac{yk}{3} = 1 \qquad \text{(iv)}$$

Clearly, (iii) and (iv) represent the same line and, hence, should be identical.

Therefore, comparing the ratio of coefficients, we get

$$\frac{(\cos\theta)/2}{h/6} = \frac{\sin\theta}{k/3} = \frac{1}{1}$$

or $h = 3\cos\theta$, $k = 3\sin\theta$

or $h^2 + k^2 = 9$

Therefore, the locus of (h, k) is

$x^2 + y^2 = 9$

which is the director circle of the ellipse

$$\frac{x^2}{6} + \frac{y^2}{3} = 1$$

and we know that the director circle is the locus of the point of intersection of the tangents which are at right angle.

Thus, tangents at P and Q are perpendicular.

3. Let the midpoint of AB be (h, k). Then the coordinates of A and B are $(2h, 0)$ and $(0, 2k)$, respectively.

Then the equation of line AB is

$$\frac{x}{2h} + \frac{y}{2k} = 1 \text{ or } y = -\frac{k}{h}x + 2k$$

It touches the ellipse

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$
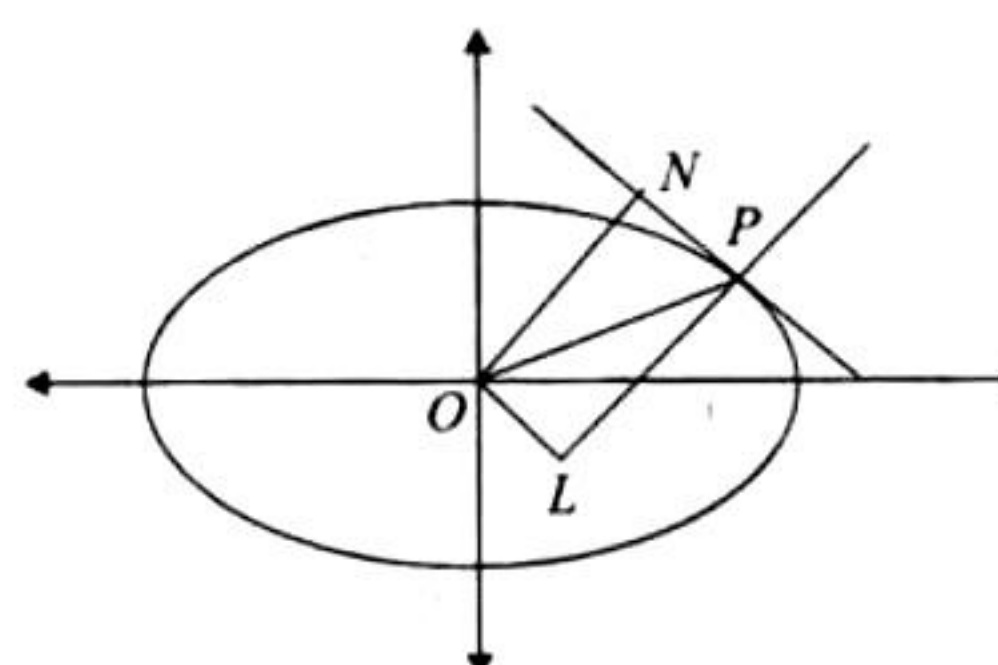
if $4k^2 = 25\left(-\dfrac{k}{h}\right)^2 + 4$

or $\dfrac{25}{h^2} + \dfrac{4}{k^2} = 4$

Therefore, the locus of (h, k) is

$$\frac{25}{x^2} + \frac{4}{y^2} = 4$$

(For the given tangent to the ellipse, the radius of the circle is automatically fixed.)

4.

The ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since this ellipse is symmetrical in all four quadrants, either there exists no such point P or there are four points, one in each quadrant. Without loss of generality, we can assume that $a > b$ and P lies in the first quadrant.

Let P be $(a\cos\theta, b\sin\theta)$. Then the equation of tangent is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

$$\therefore ON = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

The equation of ON is

$$\frac{x}{b}\sin\theta - \frac{y}{a}\cos\theta = 0$$

The equation of normal at P is

$ax\sec\theta - by\csc\theta = a^2 - b^2$

$$\therefore OL = \frac{a^2 - b^2}{\sqrt{a^2\sec^2\theta + b^2\csc^2\theta}}$$

$$= \frac{(a^2 - b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

Now, $NP = OL$

$$\therefore NP = \frac{(a^2 - b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$\therefore \Delta = $ Area of triangle OPN

$$= \frac{1}{2} \times ON \times NP$$

$$= \frac{1}{2} ab(a^2 - b^2) \frac{\sin\theta\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= \frac{1}{2} ab(a^2 - b^2) \frac{1}{a^2\tan\theta + b^2\cot\theta}$$

$$= \frac{1}{2} ab(a^2 - b^2) \frac{1}{(a\sqrt{\tan\theta} - b\sqrt{\cot\theta})^2 + 2ab}$$

Now, Δ is maximum when

$a\sqrt{\tan\theta} - b\sqrt{\cot\theta} = 0$ or $\tan\theta = \dfrac{b}{a}$

Therefore, P has coordinates $\left(a^2/\sqrt{a^2 + b^2},\, b^2/\sqrt{a^2 + b^2}\right)$.

By symmetry, we have four such points, i.e.,

$$\left(\pm \frac{a^2}{\sqrt{a^2 + b^2}},\, \pm \frac{b^2}{\sqrt{a^2 + b^2}}\right).$$

5. Let A, B, and C be the points on the circle whose coordinates are

$A(a\cos\theta,\, a\sin\theta)$

$B\left(a\cos\left(\theta + \dfrac{2\pi}{3}\right),\, a\sin\left(\theta + \dfrac{2\pi}{3}\right)\right)$

$C\left(a\cos\left(\theta + \dfrac{4\pi}{3}\right),\, a\sin\left(\theta + \dfrac{4\pi}{3}\right)\right)$

Hence, $P \equiv (a\cos\theta,\, b\sin\theta)$ [Given]

$Q \equiv \left(a\cos\left(\theta + \dfrac{2\pi}{3}\right),\, b\sin\left(\theta + \dfrac{2\pi}{3}\right)\right)$

$R \equiv \left(a\cos\left(\theta + \dfrac{4\pi}{3}\right),\, b\sin\left(\theta + \dfrac{4\pi}{3}\right)\right)$
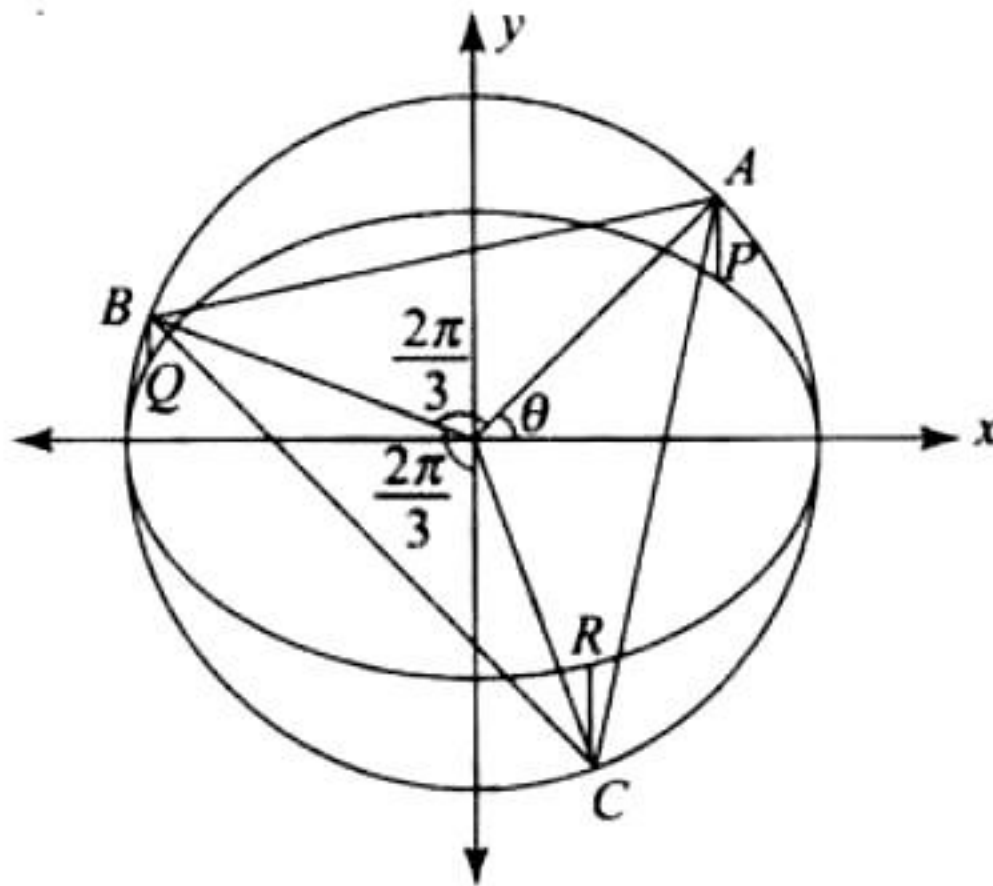
It is given that P, Q, and R are on the same side of the x-axis as A, B, and C.

So, the required normals to the ellipse at P, Q and R are

$ax\sec\theta - by\csc\theta = a^2 - b^2$ (i)

$ax\sec\left(\theta + \dfrac{2\pi}{3}\right) - by\csc\left(\theta + \dfrac{2\pi}{3}\right) = a^2 - b^2$ (ii)

$ax\sec\left(\theta + \dfrac{4\pi}{3}\right) - by\csc\left(\theta + \dfrac{4\pi}{3}\right) = a^2 - b^2$ (iii)

Now, $\quad \Delta = \begin{vmatrix} \sec\theta & \csc\theta & 1 \\ \sec\left(\theta + \dfrac{2\pi}{3}\right) & \csc\left(\theta + \dfrac{2\pi}{3}\right) & 1 \\ \sec\left(\theta + \dfrac{4\pi}{3}\right) & \csc\left(\theta + \dfrac{4\pi}{3}\right) & 1 \end{vmatrix}$

Multiplying R_1, R_2 and R_3 by $\sin\theta\cos\theta$, $\sin\left(\theta + \dfrac{2\pi}{3}\right)\cos\left(\theta + \dfrac{2\pi}{3}\right)$, and $\sin\left(\theta + \dfrac{4\pi}{3}\right)\cos\left(\theta + \dfrac{4\pi}{3}\right)$ respectively, we get

$$\Delta = \frac{1}{k}\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta + \dfrac{2\pi}{3}\right) & \cos\left(\theta + \dfrac{2\pi}{3}\right) & \sin\left(2\theta + \dfrac{4\pi}{3}\right) \\ \sin\left(\theta - \dfrac{2\pi}{3}\right) & \cos\left(\theta - \dfrac{2\pi}{3}\right) & \sin\left(2\theta - \dfrac{4\pi}{3}\right) \end{vmatrix}$$

where

$k = \sin\theta\cos\theta\,\sin\left(\theta + \dfrac{2\pi}{3}\right)\cos\left(\theta + \dfrac{2\pi}{3}\right)\sin\left(\theta + \dfrac{4\pi}{3}\right)$
$\cos\left(\theta + \dfrac{4\pi}{3}\right)$

Operating $R_2 \to R_2 + R_3$

$$\Delta = \frac{1}{k}\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ 2\sin\theta\cdot\cos\dfrac{2\pi}{3} & 2\cos\theta\cdot\cos\dfrac{2\pi}{3} & 2\sin 2\theta\cdot\cos\dfrac{4\pi}{3} \\ \sin\left(\theta - \dfrac{2\pi}{3}\right) & \cos\left(\theta - \dfrac{2\pi}{3}\right) & \sin\left(2\theta - \dfrac{4\pi}{3}\right) \end{vmatrix}$$

$$= \frac{1}{k}\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ -\sin\theta & -\cos\theta & -\sin 2\theta \\ \sin\left(\theta - \dfrac{2\pi}{3}\right) & \cos\left(\theta - \dfrac{2\pi}{3}\right) & \sin\left(2\theta - \dfrac{4\pi}{3}\right) \end{vmatrix}$$
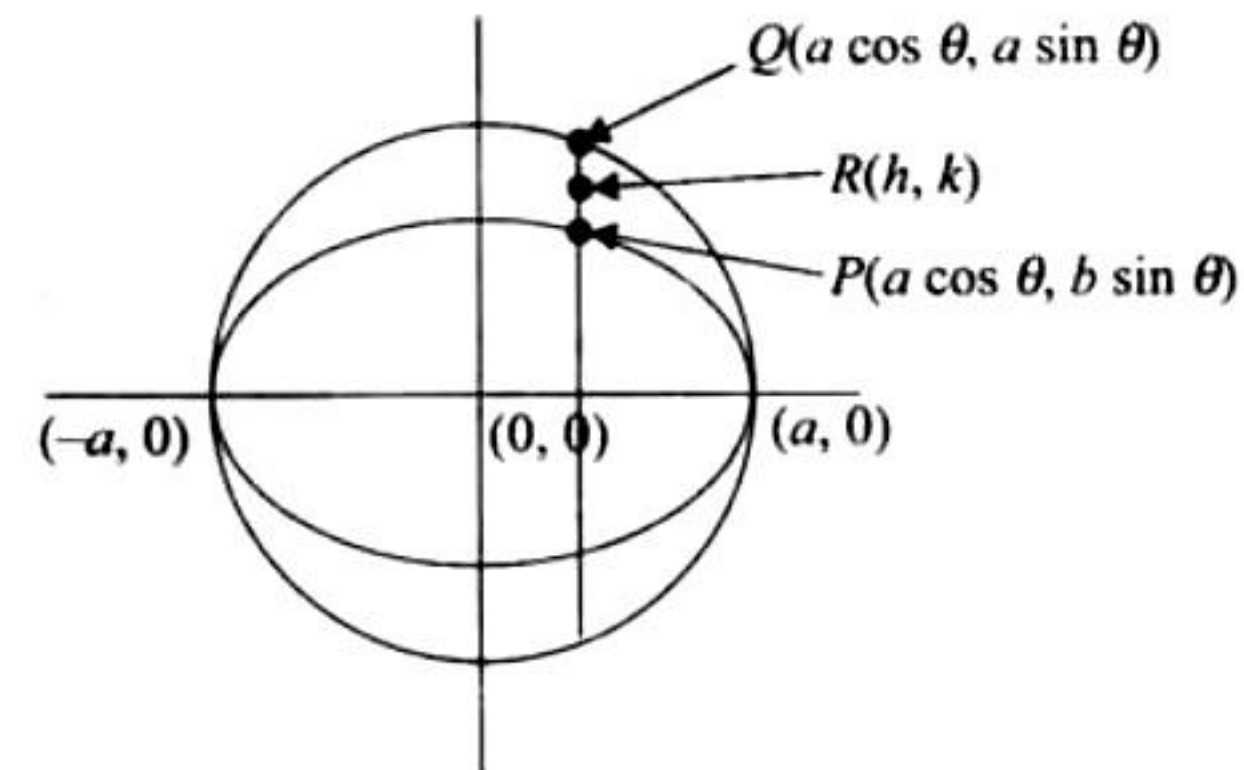
Hence, $\Delta = 0$.

6. Let the coordinates of P be $(a\cos\theta, b\sin\theta)$. Then the coordinates of Q are $(a\cos\theta, a\sin\theta)$.

Let $R(h, k)$ divides PQ in the ratio $r : s$. Then,

$h = \dfrac{s(a\cos\theta) + r(a\cos\theta)}{(r + s)} = a\cos\theta$

or $\cos\theta = \dfrac{h}{a}$

$k = \dfrac{s(b\sin\theta) + r(a\sin\theta)}{(r + s)}$

$= \dfrac{\sin\theta(bs + ar)}{(r + s)}$

or $\sin\theta = \dfrac{k(r + s)}{(bs + ar)}$

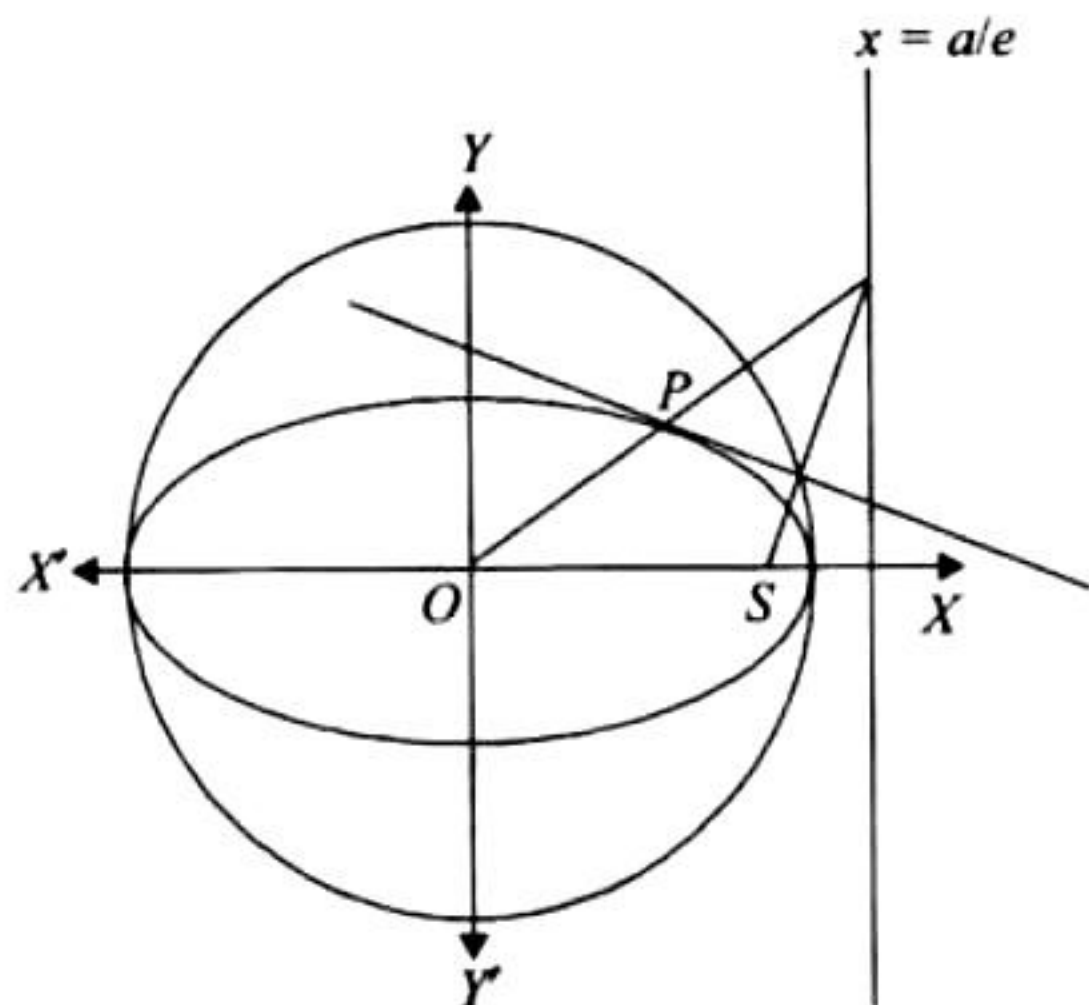We know that $\cos^2 \theta + \sin^2 \theta = 1$. Therefore,

$$\frac{h^2}{a^2} + \frac{k^2 (r+s)^2}{(bs+ar)^2} = 1$$

Hence, the locus of R is

$$\frac{x^2}{a^2} + \frac{y^2 (r+s)^2}{(bs+ar)^2} = 1$$

which is an ellipse.

7.

Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and O be the center.
The tangent at $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$$

whose slope is $-b^2 x_1 / a^2 y_1$.

The focus of the ellipse is $S(ae, 0)$.

The equation of the line through $S(ae, 0)$ perpendicular to the tangent at P is

$$y = \frac{a^2 y_1}{b^2 x_1}(x - ae) \qquad \text{(i)}$$

The equation of OP is

$$y = \frac{y_1}{x_1} x \qquad \text{(ii)}$$

Solving (i) and (ii), we get

$$\frac{y_1}{x_1} x = \frac{a^2 y_1}{b^2 x_1}(x - ae)$$

or $x(a^2 - b^2) = a^3 e$

or $x\, a^2 e^2 = a^3 e$

or $x = \dfrac{a}{e}$

This is the corresponding directrix.

8. Any tangent on the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$ is

$$y = mx \pm \sqrt{25m^2 + 4}$$

If this is also the tangent on the circle, $x^2 + y^2 = 16$ then distance of center of circle from tangent is equal to radius

$$\therefore \quad \left| \frac{0 - m \times 0 \pm \sqrt{25m^2 + 4}}{\sqrt{1 + m^2}} \right| = 4$$

or $m = \pm \dfrac{2}{\sqrt{3}}$

Since the common tangent is in the first quadrant,

$$m = -\frac{2}{\sqrt{3}}$$

Hence, the common tangent in the first quadrant is given by

$$y = \frac{-2}{\sqrt{3}} x + \sqrt{\frac{112}{3}}$$

$$\sqrt{3}\, y + 2x = 4\sqrt{7} \qquad \text{(i)}$$

The points of intersection of this tangent with the x- and the y-axis are $(2\sqrt{7}, 0)$ and $(0, 4\sqrt{7}/\sqrt{3})$, respectively.

Therefore, the length of intercept is

$$\sqrt{(2\sqrt{7} - 0)^2 + \left(0 - \frac{4\sqrt{7}}{\sqrt{3}}\right)^2} = \frac{14}{\sqrt{3}}$$