Chapter 3

ANALYTICAL GEOMETRY

Learning Objectives

After studying this chapter, the students will be able to understand

- the locus
- the angle between two lines.
- the concept of concurrent lines.
- the pair of straight lines
- the general equation and parametric equation of a circle.
- the centre and radius of the general equation of a circle.
- the equation of a circle when the extremities of a diameter are given.
- the equation of a tangent to the circle at a given point.
- identification of conics.
- the standard equation of a parabola, its focus, directrix, latus rectum and commercial applications

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Rene Descartes (1596-1650)

Introduction

The word "Geometry" is derived from the word "geo" meaning "earth" and "metron" meaning "measuring". The need of measuring land is the origin of geometry.

"Geometry" is the study of points, lines, curves, surface etc., and their properties. The importance of analytical geometry is that it establishes a correspondence between geometric curves and algebraic equations.

A systematic study of geometry by the use of algebra was first carried out by celebrated French Philosopher and mathematician Rene Descartes (1596-1650), in his book' La Geometry', published in 1637. The resulting combination of analysis and geometry is referred now as analytical geometry. He is known as the father of Analytical geometry

Analytical geometry is extremely useful in the aircraft industry, especially when dealing with the shape of an airplane fuselage.

3.1 Locus

Definition 3.1

The path traced by a moving point under some specified geometrical condition is called its locus.

3.1.1 Equation of a locus

Any relation in x and y which is satisfied by every point on the locus is called the equation of the locus.

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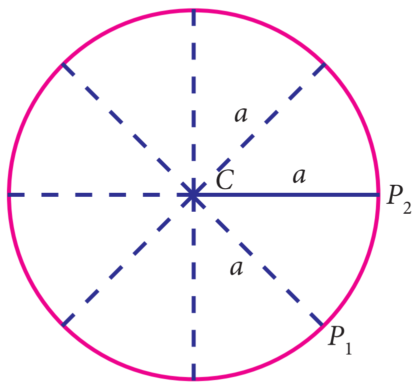

Fig. 3.1

For example,

(i) The locus of a point $P(x_1, y_1)$ whose distance from a fixed point $C(h, k)$ is constant, is a circle. The fixed point 'C' is called the centre.

(ii) The locus of a point whose distances from two points A and B are equal is the perpendicular bisector of the line segment AB.

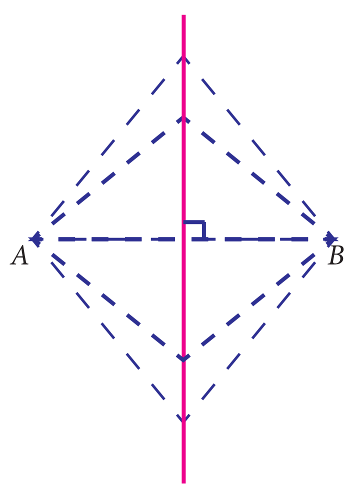

Fig. 3.2

DO YOU KNOW?

Straight line is the locus of a point which moves in the same direction.

Example 3.1

A point in the plane moves so that its distance from the origin is thrice its distance from the y- axis. Find its locus.

Solution

Let $P(x_1, y_1)$ be any point on the locus and A be the foot of the perpendicular from $P(x_1, y_1)$ to the y-axis.

$$\text{Given that } OP = 3\,AP$$

$$OP^2 = 9AP^2$$

$$(x_1 - 0)^2 + (y_1 - 0)^2 = 9x_1^2$$

$$x_1^2 + y_1^2 = 9x_1^2$$

$$8x_1^2 - y_1^2 = 0$$

$\therefore$ The locus of $P(x_1, y_1)$ is

$$8x^2 - y^2 = 0$$

Example 3.2

Find the locus of the point which is equidistant from (2, –3) and (3, –4).

Solution

Let $A(2, -3)$ and $B\ (3, -4)$ be the given points

Let $P(x_1, y_1)$ be any point on the locus.

$$\text{Given that } PA = PB.$$

$$PA^2 = PB^2$$

$$(x_1 - 2)^2 + (y_1 + 3)^2 = (x_1 - 3)^2 + (y_1 + 4)^2$$

$$x_1^2 - 4x_1 + 4 + y_1^2 + 6y_1 + 9 = x_1^2 - 6x_1 + 9 + y_1^2 + 8y_1 + 16$$

i.e., $2x_1 - 2y_1 - 12 = 0$

i.e., $x_1 - y_1 - 6 = 0$

The locus of $P(x_1, y_1)$ is $x - y - 6 = 0$.

Example 3.3

Find the locus of a point, so that the join of (–5, 1) and (3, 2) subtends a right angle at the moving point.

Solution

Let $A(-5, 1)$ and $B(3, 2)$ be the given points

Let $P(x_1, y_1)$ be any point on the locus.

Given that $\angle APB = 90^o$.

Triangle APB is a right angle triangle.

$$BA^2 = PA^2+PB^2$$

$$(3+5)^2 + (2-1)^2 = (x_1+5)^2 + (y_1-1)^2 + (x_1-3)^2 + (y_1-2)^2$$

$$65 = x_1^2 + 10x_1 + 25 + y_1^2 - 2y_1 + 1 + x_1^2 - 6x_1 + 9 + y_1^2 - 4y_1 + 4$$

i.e., $2x_1^2 + 2y_1^2 + 4x_1 - 6y_1 + 39 - 65 = 0$

i.e., $x_1^2 + y_1^2 + 2x_1 - 3y_1 - 13 = 0$

The locus of $P(x_1, y_1)$ is

$$x^2 + y^2 + 2x - 3y - 13 = 0$$

Exercise 3.1

1. Find the locus of a point which is equidistant from (1, 3) and x axis.
2. A point moves so that it is always at a distance of 4 units from the point (3, –2)
3. If the distance of a point from the points (2, 1) and (1, 2) are in the ratio 2 :1, then find the locus of the point.
4. Find a point on x axis which is equidistant from the points (7, –6) and (3, 4).
5. If $A(-1, 1)$ and $B(2, 3)$ are two fixed points, then find the locus of a point P so that the area of triangle APB = 8 sq.units.

3.2 System of Straight Lines

3.2.1 Recall

In lower classes, we studied the basic concept of coordinate geometry, like distance formula, section - formula, area of triangle and slope of a straight lines etc.

We also studied various form of equations of lines in X std. Let us recall the equations of straight lines. Which will help us for better understanding the new concept and definitions of XI std co-ordinate geometry.

Various forms of straight lines:

(i) Slope-intercept form

Equation of straight line having slope m and y-intercept 'c' is $y = mx+c$

(ii) Point- slope form

Equation of Straight line passing through the given point $P(x_1,y_1)$ and having a slope m is

$$y - y_1 = m(x - x_1)$$

(iii) Two-Point form

Equation of a straight line joining the given points $A(x_1,y_1), B(x_2,y_2)$ is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

In determinant form, equation of straight line joining two given points $A(x_1,y_1)$ and $B(x_2,y_2)$ is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(iv) Intercept form

Equation of a straight line whose x and y intercepts are a and b, is $\frac{x}{a}+\frac{y}{b}=1$.

(v) General form

Equation of straight line in general form is $ax+by+c=0$ where a, b and c are constants and a, b are not simultaneously zero.

3.2.2 Angle between two straight lines

Let l_1 and l_2 be two straight lines represented by the equations $l_1 : y=m_1x+c_1$ and $l_2 : y=m_2x+c_2$ intersecting at P.

If θ_1 and θ_2 are two angles made by l_1 and l_2 with x-axis then slope of the lines are $m_1=\tan\theta_1$ and $m_2=\tan\theta_2$.

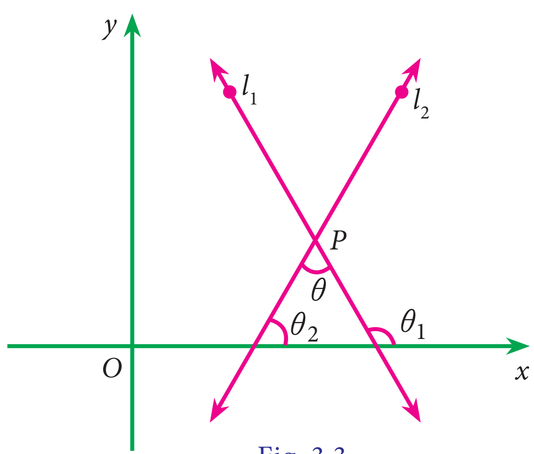

Fig. 3.3

From fig 3.3, if θ is angle between the lines l_1 & l_2 then

$$\theta=\theta_1-\theta_2$$

$$\therefore \tan\theta = \left|\tan(\theta_1-\theta_2)\right|$$

$$= \left|\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}\right|$$

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$\theta = \tan^{-1}\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

NOTE

(i) If $\frac{m_1-m_2}{1+m_1m_2}$ is positive, then θ, the angle between l_1 and l_2 is acute and if it is negative, then, θ is the obtuse.

(ii) We know that two straight lines are parallel if and only if their slopes are equal.

(iii) We know that two lines are perpendicular if and only if the product of their slopes is –1. (Here the slopes m_1 and m_2 are finite.)

DO YOU KNOW?

The straight lines x-axis and y-axis are perpendicular to each other. But, the condition $m_1m_2=-1$ is not true because the slope of the x-axis is zero and the slope of the y-axis is not defined.

Example 3.4

Find the acute angle between the lines $2x-y+3=0$ and $x+y+2=0$

Solution

Let m_1 and m_2 be the slopes of $2x-y+3=0$ and $x+y+2=0$

Now $m_1=2$, $m_2=-1$

Let θ be the angle between the given lines

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$= \left|\frac{2+1}{1+2(-1)}\right| = 3$$

$$\theta = \tan^{-1}(3)$$

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3.2.3 Distance of a point from a line

(i) The length of the perpendicular from a point $P(l, m)$ to the line $ax+by+c=0$ is $d=\left|\frac{al+bm+c}{\sqrt{a^2+b^2}}\right|$

(ii) The length of the perpendicular form the origin (0,0) to the line $ax+by+c=0$ is $d=\left|\frac{c}{\sqrt{a^2+b^2}}\right|$

Example 3.5

Show that perpendicular distances of the line $x-y+5=0$ from origin and from the point $P(2, 2)$ are equal.

Solution

Given line is $x-y+5=0$

Perpendicular distance of the given line from $P(2, 2)$ is $=\left|\frac{2-2+5}{\sqrt{1^2+1^2}}\right|$

$$=\left|\frac{5}{\sqrt{2}}\right| = \frac{5}{\sqrt{2}}$$

Distance of (0,0) from the given line

$$=\left|\frac{5}{\sqrt{1^2+1^2}}\right|=\left|\frac{5}{\sqrt{2}}\right| = \frac{5}{\sqrt{2}}$$

The given line is equidistance from origin and (2, 2)

Example 3.6

If the angle between the two lines is $\frac{\pi}{4}$ and slope of one of the lines is 3, then find the slope of the other line.

Solution

We know that the acute angle θ between two lines with slopes m_1 and m_2 is given by

$$\tan\,\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

Given $m_1=3$ and $\theta=\frac{\pi}{4}$

$$\therefore \tan\frac{\pi}{4} = \left|\frac{3-m_2}{1+3m_2}\right|$$

$$1 = \left(\frac{3-m_2}{1+3m_2}\right)$$

$$1+3m_2 = 3-m_2$$

$$\Rightarrow \quad m_2 = \frac{1}{2}$$

Hence the slope of the other line is $\frac{1}{2}$.

3.2.4 Concurrence of three lines

If two lines l_1 and l_2 meet at a common point P, then that P is called point of intersection of l_1 and l_2. This point of intersection is obtained by solving the equations of l_1 and l_2.

If three or more straight lines will have a point in common then they are said to be concurrent.

The lines passing through the common point are called concurrent lines and the common point is called concurrent point.

Conditions for three given straight lines to be concurrent

Let $a_1x+b_1y+c_1 = 0 \quad \rightarrow (1)$

$a_2x+b_2y+c_2 = 0 \quad \rightarrow (2)$

$a_3x+b_3y+c_3 = 0 \quad \rightarrow (3)$

be the equations of three straight lines, then the condition that these lines to be concurrent is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Example 3.7

Show that the given lines $3x-4y-13=0$, $8x-11y=33$ and $2x-3y-7=0$ are concurrent and find the concurrent point.

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Solution

Given lines $3x - 4y - 13 = 0 \quad \ldots (1)$

$8x - 11y = 33 \quad \ldots (2)$

$2x - 3y - 7 = 0 \quad \ldots (3)$

Conditon for concurrent lines is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

i.e., $\begin{vmatrix} 3 & -4 & -13 \\ 8 & -11 & -33 \\ 2 & -3 & -7 \end{vmatrix} =$

$3(77-99) + 4(-56+66) - 13(-24+22)$

$= -66 + 40 + 26 = 0$

$\Rightarrow$ Given lines are concurrent.

To get the point of concurrency solve the equations (1) and (3)

Equation (1) $\times 2 \Rightarrow 6x-8y = 26$

Equation (3) $\times 3 \Rightarrow 6x-9y = 21$

$y = 5$

When $y=5$ from (2) $8x = 88$

$x = 11$

Point of concurrency is $(11, 5)$

Example 3.8

If the lines $3x - 5y - 11 = 0$, $5x + 3y - 7 = 0$ and $x + ky = 0$ are concurrent, find the value of k.

Solution

Given the lines are concurrent.

Therefore $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

$\begin{vmatrix} 3 & -5 & -11 \\ 5 & 3 & -7 \\ 1 & k & 0 \end{vmatrix} = 0$

$1(35+33) - k(-21+55) = 0$

$\Rightarrow 34k = 68. \quad \therefore k = 2.$

Example 3.9

A private company appointed a clerk in the year 2012, his salary was fixed as ₹20,000. In 2017 his salary raised to ₹25,000.

(i) Express the above information as a linear function in x and y where y represent the salary of the clerk and x-represent the year

(ii) What will be his salary in 2020?

Solution

Let y represent the salary (in Rs) and x represent the year

year (x)	salary (y)
2012(x_1)	20,000(y_1)
2017(x_2)	25,000(y_2)
2020	?

The equation of straight line expressing the given information as a linear equation in x and y is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-20,000}{25,000-20,000} = \frac{x-2012}{2017-2012}$$

$$\frac{y-20,000}{5000} = \frac{x-2012}{5}$$

$y = 1000x - 2012000+20,000$

$y = 1000x - 19,92,000$

In 2020 his salary will be

$y = 1000(2020)-19,92,000$

$y = ₹28000$

Exercise 3.2

1. Find the angle between the lines whose slopes are $\frac{1}{2}$ and 3
2. Find the distance of the point (4,1) from the line $3x - 4y + 12 = 0$
3. Show that the straight lines $x + y - 4 = 0$, $3x + 2 = 0$ and $3x - 3y + 16 = 0$ are concurrent.
4. Find the value of 'a' for which the straight lines $3x + 4y = 13; 2x - 7y = -1$ and $ax - y - 14 = 0$ are concurrent.
5. A manufacturer produces 80 TV sets at a cost of ₹2,20,000 and 125 TV sets at a cost of ₹2,87,500. Assuming the cost curve to be linear, find the linear expression of the given information. Also estimate the cost of 95 TV sets.

3.3 Pair of Straight Lines

3.3.1 Combined equation of the pair of straight lines

Let us consider the two individual equations of straight lines

$$l_1 x + m_1 y + n_1 = 0 \text{ and}$$

$$l_2 x + m_2 y + n_2 = 0$$

Then their combined equation is

$$(l_1 x + m_1 y + n_1)\ (l_2 x + m_2 y + n_2) = 0$$

$$l_1 l_2 x^2 + (l_1 m_2 + l_2 m_1)xy + m_1 m_2 y^2 + (l_1 n_2 + l_2 n_1)x + (m_1 n_2 + m_2 n_1)y + n_1 n_2 = 0$$

Hence the general equation of pair of straight lines can be taken as

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

where a, b, c, f, g and h are all constants.

3.3.2 Pair of straight lines passing through the origin

The homogeneous equation

$$ax^2 + 2hxy + by^2 = 0 \quad \ldots (1)$$

of second degree in x and y represents a pair of straight lines passing through the origin.

Let $y = m_1 x$ and $y = m_2 x$ be two straight lines passing through the origin.

Then their combined equation is

$$(y - m_1 x)(y - m_2 x) = 0$$

$$\Rightarrow \quad m_1 m_2 x^2 - (m_1 + m_2)xy + y^2 = 0 \quad \ldots (2)$$

(1) and (2) represent the same pair of straight lines

$$\therefore \frac{a}{m_1 m_2} = \frac{2h}{-(m_1 + m_2)} = \frac{b}{1}$$

$$\Rightarrow m_1 m_2 = \frac{a}{b} \text{ and } m_1 + m_2 = -\frac{2h}{b}.$$

i.e., product of the slopes = $\frac{a}{b}$ and sum of the slopes $= -\frac{2h}{b}$

3.3.3 Angle between pair of straight lines passing through the origin

The equation of the pair of straight lines passing through the origin is

$$ax^2 + 2hxy + by^2 = 0$$

Let m_1 and m_2 be the slopes of above lines.

Here $m_1 + m_2 = -\frac{2h}{b}$ and $m_1 m_2 = \frac{a}{b}$.

Let θ be the angle between the pair of straight lines.

Then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\pm 2\sqrt{h^2 - ab}}{a+b}\right|$$

Let us take 'θ' as acute angle

$$\therefore \qquad \theta = \tan^{-1}\left[\left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|\right]$$

NOTE

(i) If θ is the angle between the pair of straight lines

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$,

then $\theta = \tan^{-1}\left[\left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|\right]$

(ii) If the straight lines are parallel, then $h^2 = ab$.

(iii) If the straight lines are perpendicular, then $a + b = 0$

i.e., *coefficient of* x^2 + *coefficient of* $y^2 = 0$

3.3.4 The condition for general second degree equation to represent the pair of straight lines

The condition for a general second degree equation in x, y namely

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

to represent a pair of straight lines is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

NOTE

The condition in determinant form is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Example 3.10

Find the combined equation of the given straight lines whose separate equations are $2x + y - 1 = 0$ and $x + 2y - 5 = 0$.

Solution

The combined equation of the given straight lines is

$$(2x + y - 1)(x + 2y - 5) = 0$$

i.e., $2x^2 + xy - x + 4xy + 2y^2 - 2y - 10x - 5y + 5 = 0$

i.e., $2x^2 + 5xy + 2y^2 - 11x - 7y + 5 = 0$

Example 3.11

Show that the equation $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ represents a pair of straight lines. Also find the angle between them.

Solution

Compare the equation

$2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$

with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$,

we get

$a = 2, b = 3, h = \frac{5}{2}, g = 3, f = \frac{7}{2}$ and $c = 4$

Condition for the given equation to represent a pair of straight lines is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 24 + \frac{105}{2} - \frac{49}{2} - 27 - 25 = 0$$

Hence the given equation represents a pair of straight lines.

Let θ be the angle between the lines.

Then $\theta = \tan^{-1}\left[\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|\right]$

$= \tan^{-1}\left[\frac{2\sqrt{\frac{25}{4}-6}}{5}\right]$

$\therefore\ \theta = \tan^{-1}\left(\frac{1}{5}\right)$

Example 3.12

The slope of one of the straight lines $ax^2+2hxy+by^2=0$ is twice that of the other, show that $8h^2=9ab$.

Solution

Let m_1 and m_2 be the slopes of the pair of straight lines

$$ax^2+2hxy+by^2 = 0$$

$$\therefore\ m_1+m_2 = -\frac{2h}{b} \text{ and } m_1m_2=\frac{a}{b}$$

It is given that one slope is twice the other, so let $m_2=2m_1$

$$\therefore m_1+2m_1=-\frac{2h}{b} \text{ and } m_1\cdot 2m_1=\frac{a}{b}$$

$$\therefore\ m_1 = -\frac{2h}{3b} \text{ and } 2m_1^2=\frac{a}{b}$$

$$\Rightarrow\ 2\left(-\frac{2h}{3b}\right)^2 = \frac{a}{b}$$

$$\Rightarrow\ \frac{8h^2}{9b^2} = \frac{a}{b}$$

$$\Rightarrow\ 8h^2 = 9ab$$

Example 3.13

Show that the equation $2x^2+7xy+3y^2+5x+5y+2=0$ represent two straight lines and find their separate equations.

Solution

Compare the equation

$$2x^2+7xy+3y^2+5x+5y+2=0$$

with

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

we get,

$$a=2,\ b=3,\ h=\frac{7}{2}, g=\frac{5}{2},\ f=\frac{5}{2},\ c=2$$

Now $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 2 & \frac{7}{2} & \frac{5}{2} \\ \frac{7}{2} & 3 & \frac{5}{2} \\ \frac{5}{2} & \frac{5}{2} & 2 \end{vmatrix}$

$$= 2\left(6-\frac{25}{4}\right)-\frac{7}{2}\left(7-\frac{25}{4}\right) + \frac{5}{2}\left(\frac{35}{4}-\frac{15}{2}\right)$$

$$= 2\left(-\frac{1}{4}\right)-\frac{7}{2}\cdot\frac{3}{4}+\frac{5}{2}\cdot\frac{5}{4}$$

$$= -\frac{1}{2}-\frac{21}{8}+\frac{25}{8} = 0$$

Hence the given equation represents a pair of straight lines.

Now consider,

$$2x^2+7xy+3y^2 = 2x^2+6xy+xy+3y^2$$

$$= 2x(x+3y)+y(x+3y)$$

$$= (x+3y)(2x+y)$$

Let $2x^2+7xy+3y^2+5x+5y+2 = (x+3y+l)(2x+y+m)$

Comparing the coefficient of x,

$2l+m=5$ (1)

Comparing the coefficient of y,

$l+3m=5$ (2)

Solving (1) and (2), we get $m=1$ and $l=2$

$\therefore$ The separate equations are

$$x+3y+2=0 \text{ and } 2x+y+1=0.$$

Example 3.14

Show that the pair of straight lines $4x^2-12xy+9y^2+18x-27y+8=0$ represents a pair of parallel straight lines and find their separate equations.

Solution

The given equation is $4x^2-12xy+9y^2+18x-27y+8=0$

Here $a=4$, $b=9$ and $h=-6$

$$h^2-ab = 36-36=0$$

Hence the given equation represents a pair of parallel straight lines.

$$\text{Now } 4x^2 - 12xy + 9y^2 = (2x-3y)^2$$

Consider,

$$4x^2 - 12xy + 9y^2 + 18x - 27y + 8 = 0$$

$$\Rightarrow \quad (2x-3y)^2 + 9(2x-3y) + 8 = 0$$

$$\text{Put} \quad 2x-3y = z$$

$$z^2 + 9z + 8 = 0$$

$$(z+1)(z+8) = 0$$

$$z+1 = 0 \qquad z+8 = 0$$

$$2x-3y+1 = 0 \quad 2x-3y+8 = 0$$

Hence the separate equations are

$2x-3y+1 = 0$ and $2x-3y+8 = 0$

Example 3.15

Find the angle between the straight lines $x^2 + 4xy + y^2 = 0$

Solution

The given equation is $x^2 + 4xy + y^2 = 0$

Here $a = 1$, $b = 1$ and $h = 2$.

If θ is the angle between the given straight lines, then

$$\theta = \tan^{-1}\left[\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|\right]$$

$$= \tan^{-1}\left[\left|\frac{2\sqrt{4-1}}{2}\right|\right]$$

$$= \tan^{-1}(\sqrt{3})$$

$$\theta = \frac{\pi}{3}$$

Example 3.16

For what value of k does $2x^2 + 5xy + 2y^2 + 15x + 18y + k = 0$ represent a pair of straight lines.

Solution

Here $a = 2$, $b = 2$, $h = \frac{5}{2}$, $g = \frac{15}{2}$, $f = 9$, $c = k$.

The given line represents a pair of straight lines if,

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\text{i.e.,} \quad 4k + \frac{675}{2} - 162 - \frac{225}{2} - \frac{25}{4}k = 0$$

$$\Rightarrow \quad 16k + 1350 - 648 - 450 - 25k = 0$$

$$\Rightarrow \quad 9k = 252 \qquad \therefore k = 28$$

Exercise 3.3

1. If the equation $ax^2 + 5xy - 6y^2 + 12x + 5y + c = 0$ represents a pair of perpendicular straight lines, find a and c.
2. Show that the equation $12x^2 - 10xy + 2y^2 + 14x - 5y + 2 = 0$ represents a pair of straight lines and also find the separate equations of the straight lines.
3. Show that the pair of straight lines $4x^2 + 12xy + 9y^2 - 6x - 9y + 2 = 0$ represents two parallel straight lines and also find the separate equations of the straight lines.
4. Find the angle between the pair of straight lines $3x^2 - 5xy - 2y^2 + 17x + y + 10 = 0$

3.4 Circles

Definition 3.2

A circle is the locus of a point which moves in such a way that its distance from a fixed point is always constant. The fixed point is called the centre of the circle and the constant distance is the radius of the circle.

3.4.1 The equation of a circle when the centre and radius are given

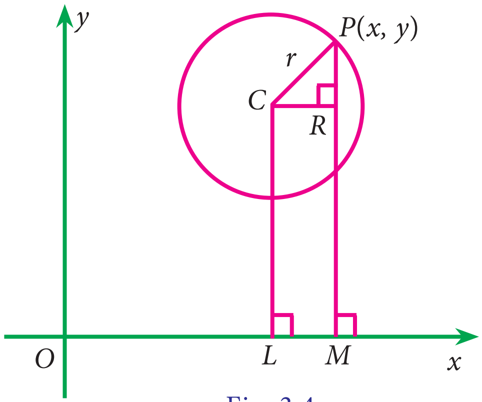

Fig. 3.4

Let $C(h, k)$ be the centre and 'r' be the radius of the circle

Let $P(x, y)$ be any point on the circle

$$CP = r$$

$$CP^2 = r^2$$

$$(x-h)^2 + (y-k)^2 = r^2$$

is the equation of the circle.

In particular, if the centre is at the origin, the equation of circle is $x^2 + y^2 = r^2$

Example 3.17

Find the equation of the circle with centre at (3, –1) and radius is 4 units.

Solution

Equation of circle is

$$(x-h)^2 + (y-k)^2 = r^2$$

Here $(h, k) = (3, -1)$ and $r = 4$

Equation of circle is

$$(x-3)^2 + (y+1)^2 = 16$$

$$x^2 - 6x + 9 + y^2 + 2y + 1 = 16$$

$$x^2 + y^2 - 6x + 2y - 6 = 0$$

Example 3.18

Find the equation of the circle with centre at origin and radius is 3 units.

Solution

Equation of circle is $x^2 + y^2 = r^2$

Here $r = 3$

i.e equation of circle is $x^2 + y^2 = 9$

3.4.2 Equation of a circle when the end points of a diameter are given

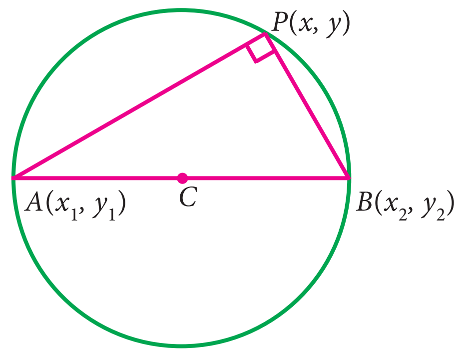

Fig. 3.5

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter of a circle and

$P(x, y)$ be any point on the circle.

We know that angle in the semi circle is 90°

$$\therefore \angle APB = 90^\circ$$

$$\therefore \text{(Slope of AP) (Slope of BP)} = -1$$

$$\left(\frac{y-y_1}{x-x_1}\right) \times \left(\frac{y-y_2}{x-x_2}\right) = -1$$

$$(y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$$

$\Rightarrow (x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$ is the required equation of circle.

Example 3.19

Find the equation of the circle when the end points of the diameter are (2, 4) and (3, –2).

Solution

Equation of a circle when the end points of the diameter are given is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$$

Here (x_1,y_1) = (2, 4) and

(x_2,y_2) = (3, –2)

$$\therefore\ (x-2)(x-3)+(y-4)(y+2) = 0$$

$$x^2+y^2-5x-2y-2 = 0$$

3.4.3 General equation of a circle

The general equation of circle is $x^2+y^2+2gx+2fy+c = 0$ where g, f and c are constants.

i.e $$x^2+y^2+2gx+2fy = -c$$

$$x^2+2gx+g^2-g^2+y^2+2fy+f^2-f^2 = -c$$

$$(x+g)^2-g^2+(y+f)^2-f^2 = -c$$

$$(x+g)^2+(y+f)^2 = g^2+f^2-c$$

$$[x-(-g)]^2+[y-(-f)]^2 = \left[\sqrt{g^2+f^2-c}\right]^2$$

Comparing this with the circle $(x-h)^2+(y-k)^2=r^2$

We get, centre is $(-g,-f)$ and radius is $\sqrt{g^2+f^2-c}$

NOTE

The general second degree equation $ax^2+by^2+2hxy+2gx+2fy+c=0$ represents a circle if

(i) $a = b$ i.e., co-efficient of x^2 = co-efficient of y^2.

(ii) $h = 0$ i.e., no xy term.

Example 3.20

Find the centre and radius of the circle $x^2+y^2-8x+6y-24=0$

Solution

Equation of circle is

$$x^2+y^2-8x+6y-24 = 0$$

Here $g = -4$, $f = 3$ and $c = -24$

Centre = $C(-g,-f) = C(4,-3)$ and

Radius: $r = \sqrt{g^2+f^2-c}$

$= \sqrt{16+9+24} = 7$ unit.

Example 3.21

For what values of a and b does the equation

$(a-2)x^2+by^2+(b-2)xy+4x+4y-1=0$ represents a circle? Write down the resulting equation of the circle.

Solution

The given equation is

$$(a-2)x^2+by^2+(b-2)xy+4x+4y-1=0$$

As per conditions noted above,

(i) coefficient of $xy = 0 \Rightarrow b-2 = 0$

$\therefore\ b = 2$

(ii) coefficient of x^2 = Coefficient of y^2

$\Rightarrow a - 2 = b$

$a - 2 = 2 \Rightarrow a = 4$

$\therefore$ Resulting equation of circle is $2x^2+2y^2+4x+4y-1=0$

Example 3.22

If the equation of a circle $x^2+y^2+ax+by=0$ passing through the points (1, 2) and (1, 1), find the values of a and b

Solution

The circle $x^2+y^2+ax+by = 0$ passing through (1, 2) and (1, 1)

$\therefore$ We have $1 + 4 + a + 2b = 0$ and $1 + 1 + a + b = 0$

$\Rightarrow \quad a+2b = -5 \quad (1)$

and $\quad a + b = -2 \quad (2)$

Solving (1) and (2), we get $a = 1$, b, = –3.

Example 3.23

If the centre of the circle $x^2+y^2+2x-6y+1=0$ lies on a straight line $ax+2y+2=0$, then find the value of 'a'

Solution

Centre C(–1, 3)

It lies on $ax+2y+2=0$

$$-a+6+2=0$$

$$a=8$$

Example 3.24

Show that the point (7, –5) lies on the circle $x^2+y^2-6x+4y-12=0$ and find the coordinates of the other end of the diameter through this point.

Solution

Let $A(7, -5)$

Equation of circle is

$$x^2+y^2-6x+4y-12=0$$

Substitute (7, –5) for (x, y), we get

$$x^2+y^2-6x+4y-12=$$

$$7^2+(-5)^2-6(7)+4(-5)-12$$

$$= 49 + 25 - 42 - 20 - 12 = 0$$

∴ (7, –5) lies on the circle

Here $g=-3$ and $f=2$

∴ Centre = $C(3, -2)$

Let the other end of the diameter be $B(x, y)$

Midpoint of AB =

$$\left(\frac{x+7}{2}, \frac{y-5}{2}\right)=C(3,-2)$$

$$\frac{x+7}{2}=3 \qquad \frac{y-5}{2}=-2$$

$$x=-1 \qquad y=1$$

Other end of the diameter is (–1 , 1).

Example 3.25

Find the equation of the circle passing through the points (0,0), (1, 2) and (2,0).

Solution

Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$

The circle passes through the point (0, 0)

$$c=0 \qquad \text{... (1)}$$

The circle passes through the point (1, 2)

$$1^2+2^2+2g(1)+2f(2)+c=0$$

$$2g+4f+c=-5 \qquad \text{... (2)}$$

The circle passes through the point (2, 0)

$$2^2+0+2g(2)+2f(0)+c=0$$

$$4g+c=-4 \qquad \text{... (3)}$$

Solving (1), (2) and (3), we get

$$g=-1,\ f=-\frac{3}{4} \text{ and } c=0$$

∴ The equation of the circle is

$$x^2+y^2+2(-1)x+2\left(\frac{-3}{4}\right)y+0=0$$

ie., $$2x^2+2y^2-4x-3y=0$$

3.4.4 Parametric form of a circle

Consider a circle with radius r and centre at the origin. Let $P(x, y)$ be any point on the circle. Assume that OP makes an angle θ with the positive direction of x-axis.

Draw PM perpendicular to x-axis.

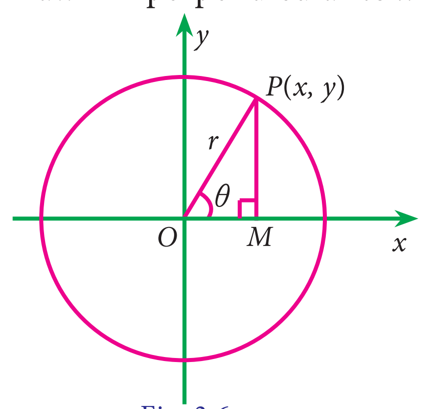

Fig. 3.6

From the figure,

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

The equations $x = r\cos\theta$, $y = r\sin\theta$ are called the parametric equations of the circle $x^2 + y^2 = r^2$. Here 'θ' is called the parameter and $0 \le \theta \le 2\pi$.

Example 3.26

Find the parametric equations of the circle $x^2 + y^2 = 25$

Solution

Here $r^2 = 25 \Rightarrow r = 5$

Parametric equations are $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow x = 5\cos\theta,\ y = 5\sin\theta,\ 0 \le \theta \le 2\pi$$

Exercise 3.4

1. Find the equation of the following circles having
   (i) the centre (3,5) and radius 5 units
   (ii) the centre (0,0) and radius 2 units
2. Find the centre and radius of the circle
   (i) $x^2 + y^2 = 16$
   (ii) $x^2 + y^2 - 22x - 4y + 25 = 0$
   (iii) $5x^2 + 5y^2 + 4x - 8y - 16 = 0$
   (iv) $(x+2)(x-5) + (y-2)(y-1) = 0$
3. Find the equation of the circle whose centre is (–3, –2) and having circumference 16π
4. Find the equation of the circle whose centre is (2,3) and which passes through (1,4)
5. Find the equation of the circle passing through the points (0, 1),(4, 3) and (1, –1).
6. Find the equation of the circle on the line joining the points (1,0), (0,1) and having its centre on the line $x + y = 1$
7. If the lines $x + y = 6$ and $x + 2y = 4$ are diameters of the circle, and the circle passes through the point (2, 6) then find its equation.
8. Find the equation of the circle having (4, 7) and (–2, 5) as the extremities of a diameter.
9. Find the Cartesian equation of the circle whose parametric equations are $x = 3\cos\theta, y = 3\sin\theta,\ 0 \le \theta \le 2\pi$.

3.4.5 Tangents

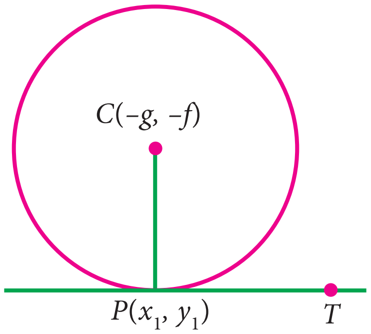

Fig. 3.7

The equation of the tangent to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

Corollary:

The equation of the tangent at (x_1, y_1) to the circle

$x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$

NOTE

To get the equation of the tangent at (x_1, y_1) to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots (1)$$

replace x^2 by xx_1, y^2 by yy_1, x by $\frac{x+x_1}{2}$ and y by $\frac{y+y_1}{2}$ in equation (1).

Example 3.27

Find the equation of tangent at the point (–2, 5) on the circle

$$x^2 + y^2 + 3x - 8y + 17 = 0.$$

Solution

The equation of the tangent at (x_1, y_1) to the given circle $x^2 + y^2 + 3x - 8y + 17 = 0$ is

$$xx_1 + yy_1 + 3 \times \frac{1}{2}(x + x_1) - 8 \times \frac{1}{2}(y + y_1) + 17 = 0$$

Here $(x_1, y_1) = (-2, 5)$

$$-2x + 5y + \frac{3}{2}(x-2) - 4(y+5) + 17 = 0$$

$$-2x + 5y + \frac{3}{2}x - 3 - 4y - 20 + 17 = 0$$

$$-4x + 10y + 3x - 6 - 8y - 40 + 34 = 0$$

$x - 2y + 12 = 0$ is the required equation.

Length of the tangent to the circle

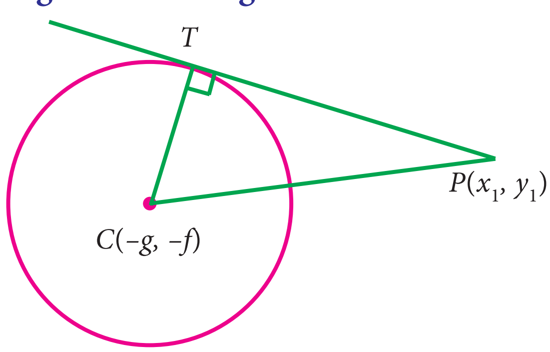

Fig. 3.8

Length of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from a point $P(x_1, y_1)$ is $PT = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

NOTE

(i) If the point P is on the circle then $PT^2 = 0$

(ii) If the point P is outside the circle then $PT^2 > 0$

(iii) If the point P is inside the circle then $PT^2 < 0$

Example 3.28

Find the length of the tangent from the point (2, 3) to the circle

$$x^2 + y^2 + 8x + 4y + 8 = 0$$

Solution

The length of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from a point (x_1, y_1) is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

Length of the tangent

$$= \sqrt{x_1^2 + y_1^2 + 8x_1 + 4y_1 + 8}$$

$$= \sqrt{2^2 + 3^2 + 8(2) + 4(3) + 8}$$

[Here $(x_1, y_1) = (2, 3)$]

$$= \sqrt{49}$$

Length of the tangent = 7 units

Example 3.29

Determine whether the points $P(0,1)$, $Q(5,9)$, $R(-2, 3)$ and $S(2, 2)$ lie outside the circle, on the circle or inside the circle $x^2 + y^2 - 4x + 4y - 8 = 0$

Solution

The equation of the circle is $x^2 + y^2 - 4x + 4y - 8 = 0$

$$PT^2 = x_1^2 + y_1^2 - 4x_1 + 4y_1 - 8$$

At $P(0, 1)$

$PT^2 = 0 + 1 + 0 + 4 - 8 = -3 < 0$

At $Q(5, 9)$

$QT^2 = 25 + 81 - 20 + 36 - 8 = 114 > 0$

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At $R(-2, 3)$

$RT^2 = 4 + 9 + 8 + 12 - 8 = 25 > 0$

At $S(2, 2)$

$ST^2 = 4 + 4 - 8 + 8 - 8 = 0$

$\therefore$ The point P lies inside the circle. The points Q and R lie outside the circle and the point S lies on the circle.

Result	Condition for the straight line $y = mx + c$ to be a tangent to the circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1 + m^2)$

Example 3.30

Find the value of k so that the line $3x + 4y - k = 0$ is a tangent to $x^2 + y^2 - 64 = 0$

Solution

The given equations are $x^2 + y^2 - 64 = 0$ and $3x + 4y - k = 0$

The condition for the tangency is $c^2 = a^2(1 + m^2)$

Here $a^2 = 64$, $m = \frac{-3}{4}$ and $c = \frac{k}{4}$

$$c^2 = a^2(1 + m^2) \Rightarrow \frac{k^2}{16} = 64\left(1 + \frac{9}{16}\right)$$

$$k^2 = 64 \times 25$$

$$k = \pm 40$$

Exercise 3.5

1. Find the equation of the tangent to the circle $x^2 + y^2 - 4x + 4y - 8 = 0$ at $(-2, -2)$
2. Determine whether the points $P(1, 0)$, $Q(2, 1)$ and $R(2, 3)$ lie outside the circle, on the circle or inside the circle $x^2 + y^2 - 4x - 6y + 9 = 0$
3. Find the length of the tangent from $(1, 2)$ to the circle $x^2 + y^2 - 2x + 4y + 9 = 0$
4. Find the value of P if the line $3x + 4y - P = 0$ is a tangent to the circle $x^2 + y^2 = 16$

3.5 Conics

Definition 3.3

If a point moves in a plane such that its distance from a fixed point bears a constant ratio to its perpendicular distance from a fixed straight line, then the path described by the moving point is called a conic.

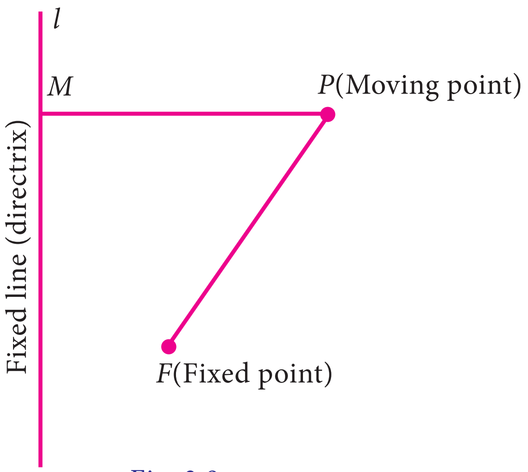

Fig. 3.9

In figure, the fixed point F is called focus, the fixed straight line l is called directrix and P is the moving point such that $\frac{FP}{PM} = e$, a constant. Here the locus of P is called a conic and the constant 'e' is called the eccentricity of the conic.

Based on the value of eccentricity we can classify the conics namely,

a) If $e = 1$, then, the conic is called a parabola

b) If $e < 1$, then, the conic is called an ellipse

c) If $e > 1$, then, the conic is called a hyperbola.

The general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents,

(i) a pair of straight lines if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

(ii) a circle if $a = b$ and $h = 0$

If the above two conditions are not satisfied, then $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents,

(i) a parabola if $h^2 - ab = 0$

(ii) an ellipse if $h^2 - ab < 0$

(iii) a hyperbola if $h^2 - ab > 0$

In this chapter, we study about parabola only.

3.5.1 Parabola

Definition 3.4

The locus of a point whose distance from a fixed point is equal to its distance from a fixed line is called a parabola.

$y^2 = 4ax$ is the standard equation of the parabola. It is open rightward.

3.5.2 Definitions regarding a parabola: $y^2 = 4ax$

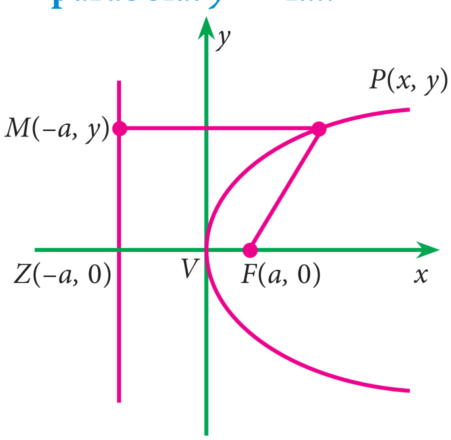

Fig. 3.10

Focus	The fixed point used to draw the parabola is called the focus (F). Here, the focus is $F(a, 0)$.
Directrix	The fixed line used to draw a parabola is called the directrix of the parabola. Here, the equation of the directrix is $x = -a$.
Axis	The axis of the parabola is the axis of symmetry. The curve $y^2 = 4ax$ is symmetrical about x-axis and hence x-axis or $y = 0$ is the axis of the parabola $y^2 = 4ax$. Note that the axis of the parabola passes through the focus and perpendicular to the directrix.
Vertex	The point of intersection of the parabola with its axis is called its vertex. Here, the vertex is $V(0, 0)$.
Focal distance	The distance between a point on the parabola and its focus is called a focal distance
Focal chord	A chord which passes through the focus of the parabola is called the focal chord of the parabola.
Latus rectum	It is a focal chord perpendicular to the axis of the parabola. Here, the equation of the latus rectum is $x = a$. Length of the latus rectum is $4a$.

3.5.3 Other standard parabolas

1. Open leftward : $y^2 = -4ax [a > 0]$

If $x > 0$, then y become imaginary. i.e., the curve exist for $x \leq 0$.

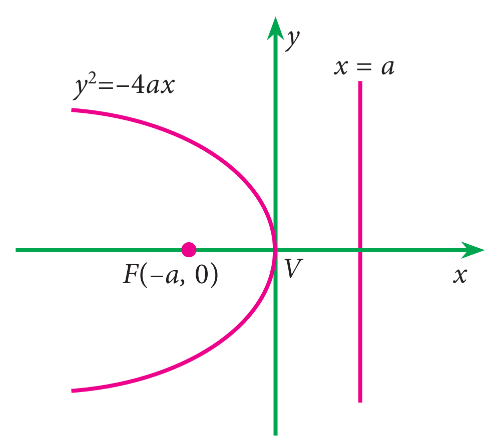

Fig. 3.11

2. Open upward : $x^2 = 4ay[a > 0]$

If $y < 0$, then x becomes imaginary. i.e., the curve exist for $y \geq 0$.

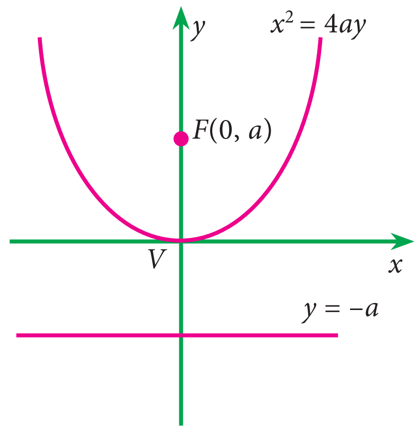

Fig. 3.12

3. Open downward : $x^2 = -4ay[a > 0]$

If $y > 0$, then x becomes imaginary. i.e., the curve exist for $y \leq 0$.

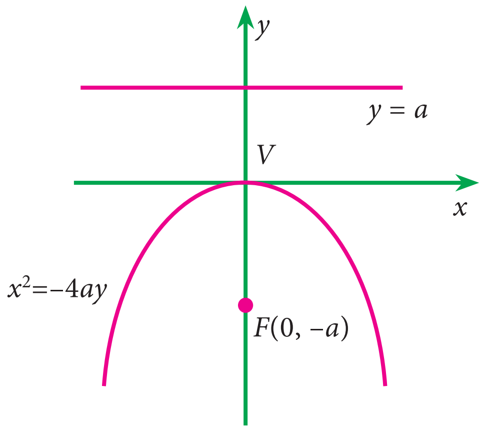

Fig. 3.13

Equations	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Vertex	$V(0, 0)$	$V(0, 0)$	$V(0, 0)$	$V(0, 0)$
Focus	$F(a, 0)$	$F(-a, 0)$	$F(0, a)$	$F(0, -a)$
Equation of directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Length of Latus rectum	$4a$	$4a$	$4a$	$4a$
Equation of Latus rectum	$x = a$	$x = -a$	$y = a$	$y = -a$

The process of shifting the origin or translation of axes.

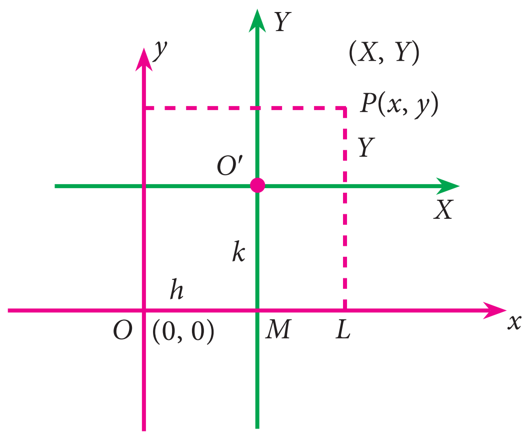

Fig. 3.14

Consider the xoy system. Draw a line parallel to x -axis (say X axis) and draw a line parallel to y – axis (say Y axis). Let $P(x,y)$ be a point with respect to xoy system and $P(X,Y)$ be the same point with respect to $XO'Y$ system.

Let the co-ordinates of O' with respect to xoy system be (h, k)

The co-ordinate of P with respect to xoy system:

$$OL = OM + ML$$

$$= h + X$$

i.e) $x = X + h$

similarly $y = Y + k$

$\therefore$ The new co-ordinates of P with respect to $XO'Y$ system.

$$X = x - h \text{ and } Y = y - k$$

3.5.4 General form of the standard equation of a parabola, which is open rightward (i.e., the vertex other than origin):

Consider a parabola with vertex V whose co-ordinates with respect to XOY system is (0, 0) and with respect to xoy system is (h, k). Since it is open rightward, the equation of the parabola w.r.t. XOY system is $Y^2 = 4aX$.

By shifting the origin $X = x - h$ and $Y = y - k$, the equation of the parabola with respect to old xoy system is

$$(y-k)^2 = 4a(x-h)$$

This is the general form of the standard equation of the parabola, which is open rightward.

Similarly the other general forms are

$(y-k)^2 = -4a(x-h)$ (open leftwards)

$(x-h)^2 = 4a(y-k)$ (open upwards)

$(x-h)^2 = -4a(y-k)$ (open downwards)

Example 3.31

Find the equation of the parabola whose focus is (1,3) and whose directrix is $x - y + 2 = 0$.

Solution

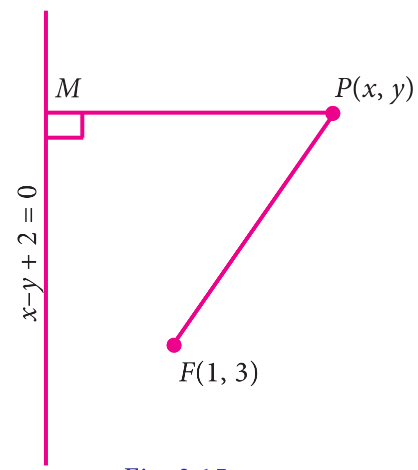

Fig. 3.15

F is (1, 3) and directrix is $x - y + 2 = 0$

$$x - y + 2 = 0$$

Let $P(x,y)$ be any point on the parabola.

For parabola $\frac{FP}{PM} = 1$

$FP = PM$

$$FP^2 = (x-1)^2 + (y-3)^2$$
$$= x^2 - 2x + 1 + y^2 - 6y + 9$$
$$= x^2 + y^2 - 2x - 6y + 10$$

$$PM = \pm\frac{(x-y+2)}{\sqrt{1+1}}$$
$$= \pm\frac{(x-y+2)}{\sqrt{2}}$$

$$PM^2 = \frac{(x-y+2)^2}{2}$$
$$= \frac{x^2 + y^2 + 4 - 2xy - 4y + 4x}{2}$$

$$FP^2 = PM^2$$

$$2x^2 + 2y^2 - 4x - 12y + 20 = x^2 + y^2 - 2xy - 4y + 4x + 4$$

The required equation of the parabola is

$$\Rightarrow x^2 + y^2 + 2xy - 8x - 8y + 16 = 0$$

Example 3.32

Find the focus, the vertex, the equation of the directrix, the axis and the length of the latus rectum of the parabola $y^2 = -12x$

Solution

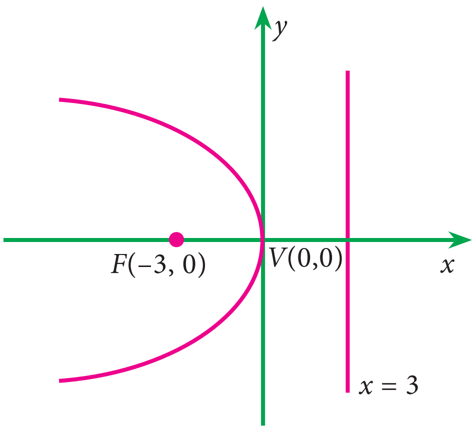

Fig. 3.16

The given equation is of the form $y^2 = -4ax$

where $4a = 12$, $a = 3$.

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The parabola is open left, its focus is

$$F(-a,o), F(-3,0)$$

Its vertex is $V(h,k) = V(0,0)$

The equation of the directrix is $x = a$

i.e., $x = 3$

Its axis is x – axis whose equation is $y = 0$.

Length of the latus rectum = $4a = 12$

Example 3.33

Show that the demand function $x = 10p - 20 - p^2$ is a parabola and price is maximum at its vertex.

Solution

$$x = 10p - 20 - p^2$$
$$= -p^2 - 20 + 10p + 5 - 5$$
$$x - 5 = -p^2 + 10p - 25$$
$$= -(p^2 - 10p + 25)$$
$$= -(p-5)^2$$

Put $X = x - 5$ and $P = p - 5$

$$X = -P^2 \quad \text{i.e } P^2 = -X$$

It is a parabola open downward.

$\therefore$ At the vertex $p = 5$ when $x = 5$. i.e., price is maximum at the vertex.

Example 3.34

Find the axis, vertex, focus, equation of directrix and the length of latus rectum for the parabola $x^2 + 6x - 4y + 21 = 0$

Solution

$$4y = x^2 + 6x + 21$$
$$4y = (x^2 + 6x + 9) + 12$$
$$4y - 12 = (x+3)^2$$
$$(x+3)^2 = 4(y-3)$$
$$X = x + 3, Y = y - 3$$
$$x = X - 3 \text{ and } y = Y + 3$$
$$X^2 = 4Y$$

Comparing with $X^2 = 4aY, 4a = 4$

$a = 1$

	Referred to (X, Y)	Referred to (x, y) $x = X - 3$, $y = Y + 3$
Axis $y = 0$	$Y = 0$	$y = 3$
Vertex $V(0,0)$	$V(0,0)$	$V(-3,3)$
Focus $F(0, a)$	$F(0, 1)$	$F(-3, 4)$
Equation of directrix $(y=-a)$	$Y = -1$	$y = 2$
Length of Latus rectum $(4a)$	$4(1) = 4$	4

Example 3.35

The supply of a commodity is related to the price by the relation $x = \sqrt{5P - 15}$. Show that the supply curve is a parabola.

Solution

The supply price relation is given by

$$x^2 = 5p - 15$$
$$= 5(p - 3)$$

$\Rightarrow$ $X^2 = 4aP$ where $X = x$ and $P = p - 3$

$\therefore$ the supply curve is a parabola whose vertex is $(X = 0, P = 0)$

i.e., The supply curve is a parabola whose vertex is (0, 3)

Exercise 3.6

1. Find the equation of the parabola whose focus is the point $F(-1, -2)$ and the directrix is the line $4x - 3y + 2 = 0$.

2. The parabola $y^2 = kx$ passes through the point (4,–2). Find its latus rectum and focus.

3. Find the vertex, focus, axis, directrix and the length of latus rectum of the parabola $y^2 - 8y - 8x + 24 = 0$

4. Find the co-ordinates of the focus, vertex, equation of the directrix, axis and the length of latus rectum of the parabola

   (a) $y^2 = 20x$

   (b) $x^2 = 8y$

   (c) $x^2 = -16y$

5. The average variable cost of a monthly output of x tonnes of a firm producing a valuable metal is ₹ $\frac{1}{5}x^2 - 6x + 100$. Show that the average variable cost curve is a parabola. Also find the output and the average cost at the vertex of the parabola.

6. The profit ₹y accumulated in thousand in x months is given by $y = -x^2 + 10x - 15$. Find the best time to end the project.

Exercise 3.7

Choose the correct answer

1. If m_1 and m_2 are the slopes of the pair of lines given by $ax^2 + 2hxy + by^2 = 0$, then the value of $m_1 + m_2$ is

   (a) $\frac{2h}{b}$ (b) $-\frac{2h}{b}$

   (c) $\frac{2h}{a}$ (d) $-\frac{2h}{a}$

2. The angle between the pair of straight lines $x^2 - 7xy + 4y^2 = 0$ is

   (a) $\tan^{-1}\left(\frac{1}{3}\right)$ (b) $\tan^{-1}\left(\frac{1}{2}\right)$

   (c) $\tan^{-1}\left(\frac{\sqrt{33}}{5}\right)$ (d) $\tan^{-1}\left(\frac{5}{\sqrt{33}}\right)$

3. If the lines $2x - 3y - 5 = 0$ and $3x - 4y - 7 = 0$ are the diameters of a circle, then its centre is

   (a) (–1, 1) (b) (1,1)

   (c) (1, –1) (d) (–1, –1)

4. The x–intercept of the straight line $3x + 2y - 1 = 0$ is

   (a) 3 (b) 2 (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

5. The slope of the line $7x + 5y - 8 = 0$ is

   (a) $\frac{7}{5}$ (b) $-\frac{7}{5}$ (c) $\frac{5}{7}$ (d) $-\frac{5}{7}$

6. The locus of the point P which moves such that P is at equidistance from their coordinate axes is

   (a) $y = \frac{1}{x}$ (b) $y = -x$

   (c) $y = x$ (d) $y = \frac{-1}{x}$

7. The locus of the point P which moves such that P is always at equidistance from the line $x + 2y + 7 = 0$ is

   (a) $x + 2y + 2 = 0$ (b) $x - 2y + 1 = 0$

   (c) $2x - y + 2 = 0$ (d) $3x + y + 1 = 0$

8. If $kx^2 + 3xy - 2y^2 = 0$ represent a pair of lines which are perpendicular then k is equal to

   (a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 2 (d) –2

9. (1, –2) is the centre of the circle $x^2 + y^2 + ax + by - 4 = 0$, then its radius

   (a) 3 (b) 2 (c) 4 (d) 1

10. The length of the tangent from (4,5) to the circle $x^2+y^2=16$ is

(a) 4 (b) 5 (c) 16 (d) 25

11. The focus of the parabola $x^2=16y$ is

(a) (4 ,0) (b) (–4 , 0)

(c) (0, 4) (d) (0, –4)

12. Length of the latus rectum of the parabola $y^2=-25x$.

(a) 25 (b) –5 (c) 5 (d) –25

13. The centre of the circle $x^2+y^2-2x+2y-9=0$ is

(a) (1 ,1) (b) (–1, –1)

(c) (–1,1) (d) (1, –1)

14. The equation of the circle with centre on the x axis and passing through the origin is

(a) $x^2-2ax+y^2=0$

(b) $y^2-2ay+x^2=0$

(c) $x^2+y^2=a^2$

(d) $x^2-2ay+y^2=0$

15. If the centre of the circle is $(-a, -b)$ and radius is $\sqrt{a^2-b^2}$, then the equation of circle is

(a) $x^2+y^2+2ax+2by+2b^2=0$

(b) $x^2+y^2+2ax+2by-2b^2=0$

(c) $x^2+y^2-2ax-2by-2b^2=0$

(d) $x^2+y^2-2ax-2by+2b^2=0$

16. Combined equation of co-ordinate axes is

(a) $x^2-y^2=0$ (b) $x^2+y^2=0$

(c) $xy=c$ (d) $xy=0$

17. $ax^2+4xy+2y^2=0$ represents a pair of parallel lines then 'a' is

(a) 2 (b) –2 (c) 4 (d) –4

18. In the equation of the circle $x^2+y^2=16$ then y intercept is (are)

(a) 4 (b)16 (c) ±4 (d) ±16

19. If the perimeter of the circle is 8π units and centre is (2,2) then the equation of the circle is

(a) $(x-2)^2+(y-2)^2=4$

(b) $(x-2)^2+(y-2)^2=16$

(c) $(x-4)^2+(y-4)^2=2$

(d) $x^2+y^2=4$

20. The equation of the circle with centre (3, –4) and touches the x – axis is

(a) $(x-3)^2+(y-4)^2=4$

(b) $(x-3)^2+(y+4)^2=16$

(c) $(x-3)^2+(y-4)^2=16$

(d) $x^2+y^2=16$

21. If the circle touches x axis, y axis and the line $x = 6$ then the length of the diameter of the circle is

(a) 6 (b)3 (c) 12 (d) 4

22. The eccentricity of the parabola is

(a) 3 (b) 2 (c) 0 (d) 1

23. The double ordinate passing through the focus is

(a) focal chord (b) latus rectum

(c) directrix (d) axis

24. The distance between directrix and focus of a parabola $y^2=4ax$ is

(a) a (b) $2a$ (c) $4a$ (d) $3a$

25. The equation of directrix of the parabola $y^2 = -x$ is

(a) $4x+1=0$ (b) $4x-1=0$
(c) $x-4=0$ (d) $x+4=0$

Miscellaneous Problems

1. A point P moves so that P and the points (2, 2) and (1, 5) are always collinear. Find the locus of P.
2. As the number of units produced increases from 500 to 1000 and the total cost of production increases from ₹ 6000 to ₹9000. Find the relationship between the cost (y) and the number of units produced (x) if the relationship is linear.
3. Prove that the lines $4x+3y=10$, $3x-4y=-5$ and $5x+y=7$ are concurrent.
4. Find the value of p for which the straight lines $8px+(2-3p)y+1=0$ and $px+8y-7=0$ are perpendicular to each other.
5. If the slope of one of the straight lines $ax^2+2hxy+by^2=0$ is thrice that of the other, then show that $3h^2=4ab$.
6. Find the values of a and b if the equation $(a-1)x^2+by^2+(b-8)xy+4x+4y-1=0$ represents a circle.
7. Find whether the points $(-1,-2)$, $(1,0)$ and $(-3,-4)$ lie above, below or on the line $3x+2y+7=0$.
8. If $(4,1)$ is one extremity of a diameter of the circle $x^2+y^2-2x+6y-15=0$, find the other extremity.
9. Find the equation of the parabola which is symmetrical about x-axis and passing through (–2, –3).
10. Find the axis, vertex, focus, equation of directrix and the length of latus rectum of the parabola $(y-2)^2=4(x-1)$

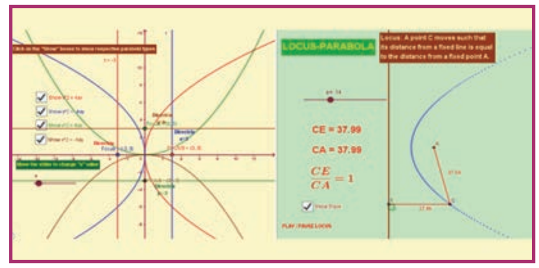
Summary

- Angle between the two intersecting lines $y=m_1x+c_1$ and $y=m_2x+c_2$ is $\theta=\tan^{-1}\left|\frac{m_1-m_2}{1+m_1m_2}\right|$.
- Condition for the three straight lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$, and $a_3x+b_3y+c_3=0$ to be concurrent is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$.
- The condition for a general second degree equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ to represent a pair of straight lines is $abc+2fgh-af^2-bg^2-ch^2=0$.
- Pair of straight lines passing through the origin is $ax^2+2hxy+by^2=0$.
- Let $ax^2+2hxy+by^2=0$ be the straight lines passing through the origin then the product of the slopes is $\frac{a}{b}$ and sum of the slopes is $\frac{-2h}{b}$.
- If θ is the angle between the pair of straight lines. $ax^2+2hxy+by^2+2gx+2fy+c=0$, then $\theta = \tan^{-1}\left|\frac{\pm 2\sqrt{h^2-ab}}{a+b}\right|$.

- The equation of a circle with centre (h, k) and the radius r is $(x-h)^2+(y-k)^2=r^2$.
- The equation of a circle with centre at origin and the radius r is $x^2+y^2=r^2$.
- The equation of the circle with (x_1, y_1) and (x_2, y_2) as end points of a diameter is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.
- The parametric equations of the circle $x^2+y^2=r^2$ are $x=r\cos\theta$, $y=r\sin\theta$ $0\le\theta\le 2\pi$
- The equation of the tangent to the circle $x^2+y^2+2gx+2fy+c=0$ at (x_1, y_1) is $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$.
- The equation of the tangent at (x_1, y_1) to the circle $x^2+y^2=a^2$ is $xx_1+yy_1=a^2$.
- Length of the tangent to the circle $x^2+y^2+2gx+2fy+c=0$ from a point (x_1, y_1) is $\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$.
- Condition for the straight line $y=mx+c$ to be a tangent to the circle $x^2+y^2=a^2$ is $c^2=a^2(1+m^2)$.
- The standard equation of the parabola is $y^2=4ax$.

GLOSSARY (கலைச்சொற்கள்)	
Centre	மையம்
Chord	நாண்
Circle	வட்டம்
Concurrent Line	ஒரு புள்ளி வழிக் கோடு
Conics	கூம்பு வெட்டிகள்
Diameter	விட்டம்
Directix	இயக்குவரை
Equation	சமன்பாடு
Focal distance	குவியத்தூரம்
Focus	குவியம்
Latus rectum	செவ்வகலம்
Length of the tangent	தொடுகோட்டின் நீளம்
Locus	நியமப்பாதை அல்லது இயங்குவரை
Origin	ஆதி
Pair of straight line	இரட்டை நேர்க்கோடு
Parabola	பரவளையம்
Parallel line	இணை கோடு
Parameter	துணையலகு
Perpendicular line	செங்குத்து கோடு
Point of concurrency	ஒருங்கிணைவுப் புள்ளி
Point of intersection	வெட்டும் புள்ளி
Radius	ஆரம்
Straight line	நேர்க்கோடு
Tangent	தொடுகோடு
Vertex	முனை

ICT Corner



Expected final outcomes

Step – 1

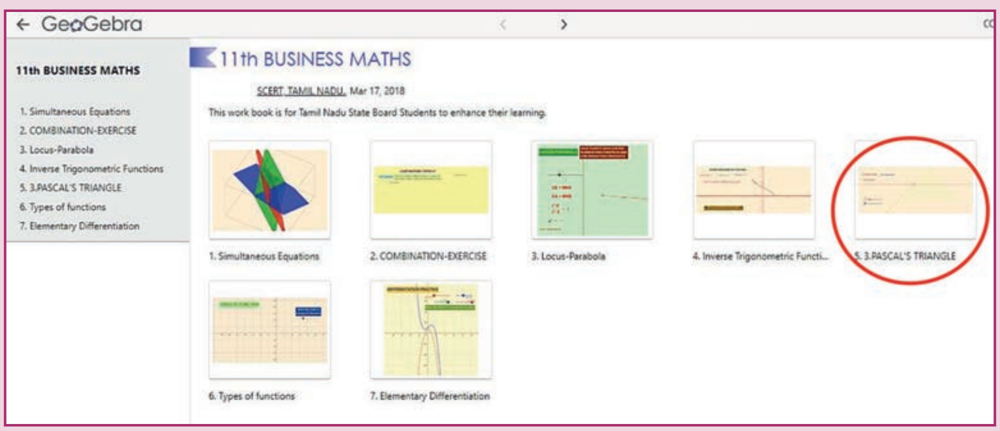
Open the Browser and type the URL given (or) Scan the QR Code.

GeoGebra Work book called "**11th BUSINESS MATHEMATICS and STATISTICS**" will appear. In this several work sheets for Business Maths are given, Open the worksheet named "**Locus-Parabola**"

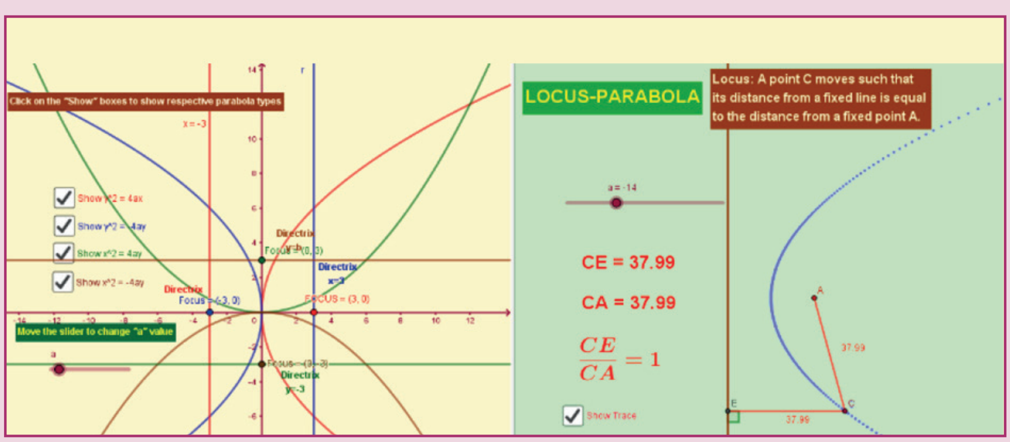
Step - 2

Locus-Parabola page will open. Click on the check boxes on the Left-hand side to see the respective parabola types with Directrix and Focus. On right-hand side Locus for parabola is given. You can play/pause for the path of the locus and observe the condition for the locus.

Step 1



Step 2



Browse in the link

11th Business Mathematics and Statistics: https://ggbm.at/qKj9gSTG (or) scan the QR Code

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