Chapter 3

Matrices

Operations on Matrices

Algebra of matrix involves the operation of matrices, such as Addition, subtraction, multiplication etc.

Let us understand the operation of the matrix in a much better way-

1. Addition of Matrices :

Let A and B be two matrices of same order (i.e. comparable matrices). Then A + B is defined to be

$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij} \forall$ i & j.

2. Substraction of Matrices :

Let A & B be two matrices of same order. Then A – B is defined as A + (–B) where – B is (–1) B.

3. Multiplication of Matrix By Scalar :

Let λ be a scalar (real or complex number) & $A = [a_{ij}]_{m \times n}$ be a matrix. Thus the product λA is defined as $\lambda A = [b_{ij}]_{m \times n}$ where $b_{ij} = \lambda a_{ij}$ for all i & j.
Note : If A is a scalar matrix, then A = λI, where λ is the diagonal element.

Properties of Addition & Scalar Multiplication :

Consider all matrices of order m × n, whose elements are from a set F (F denote Q, R or C).

Let $M_{m\times n}$ (F) denote the set of all such matrices. Then

(a) $A \in M_{m \times n}(F)$ & $B \in M_{m \times n}(F) \Rightarrow A + B \in M_{m \times n}(F)$
(b) $A + B = B + A$
(c) $(A + B) + C = A + (B + C)$
(d) $O = [o]_{m \times n}$ is the additive identity.
(e) For every $A \in M_{n \times m}(F)$, $-A$ is the additive inverse.
(f) $\lambda(A + B) = \lambda A + \lambda B$
(g) $\lambda A = A\lambda$
(h) $(\lambda_1 + \lambda_2) A = \lambda_1 A + \lambda_2 A$

4. Multiplication of Matrices :

Let A and B be two matrices such that the number of columns of A is same as number of rows of B. i.e., A $= [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$. Then $AB = [c_{ij}]_{m \times n}$ where $c_{ij} = \sum_{k=1}^{p} a_{ik} a_{kj}$ which is the dot product of i^{th} row vector of A and j^{th} column vector of B.

Note :

1. The product AB is defined iff the number of columns of A is equal to the number of rows of B. A is called as premultiplier & B is called as post multiplier. AB is defined $\Rightarrow$ BA is defined.

2. In general $AB \neq BA$, even when both the products are defined.
3. $A(BC) = (AB)C$, whenever it is defined.

Properties of Matrix Multiplication :

Consider all square matrices of order 'n'. Let $M_n(F)$ denote the set of all square matrices of order n, (where F is Q, R or C). Then
(a) $A, B \in M_n(F) \Rightarrow AB \in M_n(F)$

(b) In general $AB \neq BA$

(c) $(AB)C = A(BC)$

(d) I_n, the identity matrix of order n, is the multiplicative identity. $AI_n = A = I_nA$

(e) For every non singular matrix A(i.e., $|A| \neq 0$) of Mn (F) there exist a unique (particular) matrix $B \in Mn(F)$ so that $AB = I_n = BA$. In this case we say that A & B are multiplicative inverse of one another. In notations, we write $B = A^{-1}$ or $A = B^{-1}$.

(f) If λ is a scalar $(\lambda A)B = \lambda(AB) = A(\lambda B)$.

(g) $A(B + C) = AB + AC \quad \forall\ A, B, C \in M_n(F)$
(h) $(A + B)C = AC + BC \quad \forall\ A, B, C \in M_n(F)$

Note :

1. Let $A = [a_{ij}]_{m \times n}$. Then $AI_n = A$ & $I_m A = A$, where I_n & I_m are identity matrices of order n & m respectively.

2. For a square matrix A, A^2 denotes AA, A^3 denotes AAA etc.

Solved Examples:

Ex.1 For the following pairs of matrices, determine the sum and difference, if they exist.

(a) $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1.5 & 6 \\ -3 & 2+i & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 0 \\ 3 & -4 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \end{pmatrix}$

Sol. (a) Matrices A and B are 2×3 and confirmable for addition and subtraction.

$$A + B = \begin{pmatrix} 1+2 & -1+1.5 & 2+6 \\ 0-3 & 1+2+i & 3+0 \end{pmatrix} = \begin{pmatrix} 3 & 0.5 & 8 \\ -3 & 3+i & 3 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 1-2 & -1-1.5 & 2-6 \\ 0-(-3) & 1-(2+i) & 3-0 \end{pmatrix} = \begin{pmatrix} -1 & -2.5 & -4 \\ 3 & -1-i & 3 \end{pmatrix}$$

(b) Matrix A is 2×2, and B is 2×3. Since A and B are not the same size, they are not confirmable for addition or subtraction.

Ex.2 Find the additive inverse of the matrix $A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 3 & -1 & 2 & 2 \\ 1 & 2 & 8 & 7 \end{bmatrix}$.

Sol. The additive inverse of the 3×4 matrix A is the 3×4 matrix each of whose elements is the negative of the corresponding element of A. Therefore if we denote the additive inverse of A by – A, we have $-A = \begin{bmatrix} -2 & -3 & 1 & -1 \\ -3 & 1 & -2 & -2 \\ -1 & -2 & -8 & -7 \end{bmatrix}$. Obviously $A + (-A) = (-A) + A = O$, where O is the null matrix of the type 3×4.

Ex.3 If $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 2 & 5 \end{bmatrix}, B = \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix}$, **find the matrix D such that $A + B - D = 0$.**

Sol. We have $A + B - D = 0 \Rightarrow (A + B) + (-D) = 0 \Rightarrow A + B = (-D) = D$

$\therefore\ D = A + B = \begin{bmatrix} 0 & 2 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$.

Ex.4 If $A = \begin{bmatrix} 3 & 9 & 0 \\ 1 & 8 & -2 \\ 7 & 5 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 & 2 \\ 7 & 1 & 4 \\ 2 & 2 & 6 \end{bmatrix}$, **verify that $3(A + B) = 3A + 3B$.**

Sol.

We have $A + B = \begin{bmatrix} 3+4 & 9+0 & 0+2 \\ 1+7 & 8+1 & -2+4 \\ 7+2 & 5+2 & 4+6 \end{bmatrix} = \begin{bmatrix} 7 & 9 & 2 \\ 8 & 9 & 2 \\ 9 & 7 & 10 \end{bmatrix}$

$\therefore 3(A + B) = \begin{bmatrix} 3\times7 & 3\times9 & 3\times2 \\ 3\times8 & 3\times9 & 3\times2 \\ 3\times9 & 3\times7 & 3\times10 \end{bmatrix} = \begin{bmatrix} 21 & 27 & 6 \\ 24 & 27 & 6 \\ 27 & 21 & 30 \end{bmatrix}$

Again $3A = 3\begin{bmatrix} 3 & 9 & 0 \\ 1 & 8 & -2 \\ 7 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 3\times3 & 3\times9 & 3\times0 \\ 3\times1 & 3\times8 & 3\times-2 \\ 3\times7 & 3\times5 & 3\times4 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 6 \\ 21 & 3 & 12 \\ 6 & 6 & 18 \end{bmatrix}$

Also $3B = 3\begin{bmatrix} 4 & 0 & 2 \\ 7 & 1 & 4 \\ 2 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 3\times4 & 3\times0 & 3\times2 \\ 3\times7 & 3\times1 & 3\times4 \\ 3\times2 & 3\times2 & 3\times6 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 6 \\ 21 & 3 & 12 \\ 6 & 6 & 18 \end{bmatrix}$

$\therefore\ 3A + 3B = \begin{bmatrix} 9 & 27 & 0 \\ 3 & 24 & -6 \\ 21 & 15 & 12 \end{bmatrix} + \begin{bmatrix} 12 & 0 & 6 \\ 21 & 3 & 12 \\ 6 & 6 & 18 \end{bmatrix}$

$\Rightarrow\ \begin{bmatrix} 9+12 & 27+0 & 0+6 \\ 3+21 & 24+3 & -6+12 \\ 21+6 & 15+6 & 12+18 \end{bmatrix} = \begin{bmatrix} 21 & 27 & 6 \\ 24 & 27 & 6 \\ 27 & 21 & 30 \end{bmatrix}$

$\therefore 3(A + B) = 3A + 3B$, i.e. the scalar multiplication of matrices distributes over the addition of matrices.

Ex.5 The set of natural numbers N is partitioned into arrays of rows and columns in the form of matrices as $M_1 = (1), M_2 = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}, M_3 \begin{pmatrix} 6 & 7 & 8 \\ 9 & 10 & 11 \\ 12 & 13 & 14 \end{pmatrix}, \ldots\ldots, M_n = (\ )$

and so on. Find the sum of the elements of the diagonal in M_n.

Sol. Let $M_n = (a_{ij})$ where $i, j = 1, 2, 3, \ldots\ldots, n$.
We first find out a_{11} for the nth matrix; which is the nth term in the series ; 1, 2, 6,......
Let $S = 1 + 2 + 6 + 15 + \ldots + T_{n-1} + T_n$.

Again writing $\quad S = 1 + 2 + 6 + ... + T_{n-1} + T_n$

$\Rightarrow 0 = 1 + 1 + 4 + 9 + ... + (T_n - T_{n-1}) - T_n \Rightarrow T_n = 1 + (1 + 4 + 9 + ...$ upto $(n - 1)$ terms)

$= 1 + (1^2 + 2^2 + 3^2 + 4^2 + ... + (n - 1)^2)$

$= 1 + \frac{n(n-1)(2n-1)}{6}$

Ex.6 If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$, $\quad B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, find AB and BA and show that $AB \neq BA$.

Sol.

We have $AB = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2.1 - 3.1 + 4.0 & 2.3 + 3.2 + 4.0 & 2.0 + 3.1 + 4.2 \\ 1.1 - 2.1 + 3.0 & 1.3 + 2.2 + 3.0 & 1.0 + 2.1 + 3.2 \\ -1.1 - 1.1 + 2.0 & -1.3 + 1.2 + 2.0 & -1.0 + 1.1 + 2.2 \end{bmatrix}$

$= \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix}$

Similarly, $BA = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1.2 + 3.1 - 0.1 & 1.3 + 3.2 + 0.1 & 1.4 + 3.3 + 0.2 \\ -1.2 + 2.1 - 1.1 & -1.3 + 2.2 + 1.1 & -1.4 + 2.3 + 1.2 \\ 0.2 + 0.1 - 2.1 & 0.3 + 0.2 + 2.1 & 0.4 + 0.3 + 2.2 \end{bmatrix}$

$= \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix}$

The matrix AB is of the type 3×3 and the matrix BA is also of the type 3×3. But the corresponding elements of these matrices are not equal. Hence $AB \neq BA$.

Ex.7 Show that for all values of p, q, r, s the matrices, $P = \begin{bmatrix} p & q \\ -q & p \end{bmatrix}$, and $Q = \begin{bmatrix} r & s \\ -s & r \end{bmatrix}$, PQ = QP.

Sol. We have $PQ = \begin{bmatrix} pr - qs & ps + qr \\ -qr - ps & -qs + pr \end{bmatrix}$.

Also $QP = \begin{bmatrix} p & q \\ -q & p \end{bmatrix} \begin{bmatrix} p & q \\ -q & p \end{bmatrix} = \begin{bmatrix} rp - sq & rq + sp \\ -sp - rq & -sq + rp \end{bmatrix}$

$= \begin{bmatrix} pr - qs & ps + pq \\ -qr - ps & -qs + pr \end{bmatrix}$ for all values of p, q, r,s. Hence PQ = QP, for all values of p, q, r, s.

Ex.8 If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$, **where k is any positive integer.**

Sol. We shall prove the result by induction on k.

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We have $A_1 = A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1+2.1 & -4.1 \\ 1 & 1-2.1 \end{bmatrix}$. Thus the result is true when k = 1.

Now suppose that the result is true for any positive integer k.

i.e., $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$ where k is any positive integer.

Now we shall show that the result is true for k + 1 if it is true for k. We have

$$A^{K+1} = AA^k = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$$

$$= \begin{bmatrix} 3+6k-4k & -12k-4+8k \\ 1+2k-k & -4k-1+2k \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+2k & -4-4k \\ 1+k & -2k-1 \end{bmatrix} = \begin{bmatrix} 1+2(k+1) & -4(1+k) \\ 1+k & 1-2(1+k) \end{bmatrix}.$$

Thus the result is true for k + 1 if it is true for k. But it is true for k = 1. Hence by induction it is true for all positive integral value of k.

Ex.9 If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, prove that $(aI + bA)^n = a^n I + na^{n-1} bA$. for " a, b $\in$ R **where I is the two rowed unit matrix n is a positive integer.**

Sol.

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0 \Rightarrow A^3 = A^2 . A = 0 \Rightarrow A^2 = A^3 = A^4 = \ldots\ldots$

$A^n = 0$

Now by binomial theorem

$(a I + b A)^n = (a I)^n + {}^nC_1 (a I)^{n-1} b A + {}^nC_2 (a I)^{n-2} (b A)^2 + \ldots\ldots + {}^nC_n (b A)^n$

$= a^n I + {}^nC_1 a^{n-1} b I A + {}^nC_2 an - 2 b^2 I A^2 + \ldots\ldots + {}^nC_n b_n A_n$

$= a^n I + n a^{n-1} b A + 0\ldots\ldots$

$(\because A^n = 0) \Rightarrow (a I + b A)^n = a^n I + n a^{n-1} b A.$

Ex.10 If $\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 18 & 2007 \\ 0 & 1 & 36 \\ 0 & 0 & 1 \end{bmatrix}$ **then find the value of (n + a).**

Sol. Consider

$$\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2a+8 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 3a+24 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 2n & na + 8\sum_{k=0}^{n-1} k \\ 0 & 1 & 4n \\ 0 & 0 & 1 \end{bmatrix}^n$$

Hence n = 9 and 2007 = $9a + 8\sum_{k=0}^{8} k = 9a + 8\left(\frac{8 \cdot 9}{2}\right)$

$\Rightarrow 2007 = 9a + 32 \cdot 9 = 9(a + 32)$

$\Rightarrow a + 32 = 223 \Rightarrow a = 191$

hence a + n = 200

Ex.11 Find the matrices of transformations T_1T_2 and T_2T_1, when T_1 is rotation through an angle 60º and T_2 is the reflection in the y–axis. Also verify that $T_1T_2 \neq T_2T_1$.

Sol.

$$T_1 = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

and $$T_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T_1T_2 = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -1+0 & 0-\sqrt{3} \\ -\sqrt{3}+0 & 0+1 \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \quad ...(1)$$

$$T_2T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1+0 & \sqrt{3}+0 \\ 0+\sqrt{3} & 0+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \quad .... (2)$$

It is clear from (1) and (2), $T_1T_2 \neq T_2T_1$

Ex.12 Find the possible square roots of the two rowed unit matrix I.

Sol.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be square root of the matrix = $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $A^2 = I$.

i.e. $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $$\begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & cb+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since the above matrices are equal, therefore

$a^2 + bc = 1$ ...(i)

$ab + bd = 0$ ...(ii)

$ac + cd = 0$,,,(iii)

$cb + d^2 = 0$ ....(iv)

must hold simultaneously.
If $a + d = 0$, the above four equations hold simultaneously if $d = -a$ and $a^2 + bc = 1$

Hence one possible square root of I is

$A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ where α, β, γ are any three numbers related by the condition $\alpha^2 + \beta\gamma = 1$.

If $a + d \neq 0$, the above four equations hold simultaneously if $b = 0, c = 0, a = 1, d = 1$ or if $b = 0, c = 0, a = -1, d = -1$. Hence $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ i.e. $\pm$ I are other possible square roots of I.

Ex.13 If $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$ and $B = \begin{bmatrix} x & -x \\ -x & x \end{bmatrix}$, **then prove that** $x\, e^A = \frac{1}{2}\,(A \,.\, e^{2x} + B)$. (where $e^A = I + A + \frac{A^2}{2!} + \ldots\ldots$)

Sol.

We have $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix} = x\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = x\,E$ ...(i)

$A^2 = A \,.\, A = x\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} . x\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = x^2\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = 2\,x^2\,E$ ...(ii)

$A^3 = A^2 \,.\, A = 2\,x^2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} . x\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2^2\,x^3\,E$ ...(iii)

Similarly it can be shown that $A^4 = 2^3\,x^4\,E$, $A^5 = 2^4\,x^5\,E$ ...

Now, $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$

$$= I + x.E + \frac{2x^2E}{2!} + \frac{2^2x^3E}{3!} + \ldots \text{ [by (1), (2), (3)]}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{2x^2}{2!}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \ldots$$

$$= \begin{bmatrix} 1 + x + \frac{2x^2}{2!} + \frac{2^2x^3}{3!} + \ldots & x + \frac{2x^2}{2!} + \frac{2^2x^3}{3!} + \ldots \\ x + \frac{2x^2}{2!} + \frac{2^2x^3}{3!} + \ldots & 1 + x + \frac{2x^2}{2!} + \frac{2^2x^3}{3!} + \ldots \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\left(1 + 2x + \frac{2x^2x^2}{2!} + \frac{2^3x^3}{3!} + \ldots\right) + \frac{1}{2} & \frac{1}{2}\left(1 + 2x + \frac{2x^2x^2}{2!} + \frac{2^3x^3}{3!} + \ldots\right) - \frac{1}{2} \\ \frac{1}{2}\left(1 + 2x + \frac{2x^2x^2}{2!} + \frac{2^3x^3}{3!} + \ldots\right) - \frac{1}{2} & \frac{1}{2}\left(1 + 2x + \frac{2x^2x^2}{2!} + \frac{2^3x^3}{3!} + \ldots\right) + \frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} (e^{2x}+1) & (e^{2x}-1) \\ (e^{2x}-1) & (e^{2x}+1) \end{bmatrix} = \frac{1}{2}e^{2x}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow e^A = \frac{1}{2}\left(e^{2x}\frac{A}{x} + \frac{B}{x}\right) \Rightarrow x.\ e^A = \frac{1}{2}(e^{2x}.A + B)$$

D. FURTHER TYPES OF MATRICES

(a) Nilpotent matrix : A square matrix A is said to be nilpotent (of order 2) if, $A^2 = O$. A square matrix is said to be nilpotent of order p, if p is the least positive integer such that $A^p = O$

(b) Idempotent matrix : A square matrix A is said to be idempotent if, $A^2 = A$.

eg. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an idempotent matrix.

(c) Involutory matrix : A square matrix A is said to be involutory if $A^2 = I$,I being the identity matrix. eg. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an involutory matrix.

(d) Orthogonal matrix : A square matrix A is said to be an orthogonal matrix if $A'A = I = A'A$

(e) Unitary matrix : A square matrix A is said to be unitary if $A(\overline{A})' = I$, where $\overline{A}$ is the complex conjugate of A.

Ex.14 Find the number of idempotent diagonal matrices of order n.

Sol. Let A = diag ($d_1, d_2, \ldots, d_n$) be any diagonal matrix of order n.

now $A^2 = A \cdot A = \begin{bmatrix} d_1 & 0 & 0 & \ldots & 0 \\ 0 & d_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & d_n \end{bmatrix} \times \begin{bmatrix} d_1 & 0 & 0 & \ldots & 0 \\ 0 & d_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & d_n \end{bmatrix} = \begin{bmatrix} d_1^2 & 0 & 0 & \ldots & 0 \\ 0 & d_2^2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & d_n^2 \end{bmatrix}$

But A is idempotent, so $A^2 = A$ and hence corresponding elements of A^2 and A should be equal

$\therefore$ $d_1^2 = d_1, d_2^2 = d_2, \ldots\ldots, d_n^2 = d_n$ or $d_1 = 0, 1; d_2 = 0, 1; \ldots\ldots\ldots; d_n = 0, 1$

$\Rightarrow$ each of $d_1, d_2 \ldots\ldots$, dn can be filled by 0 or 1 in two ways.

$\Rightarrow$ Total number of ways of selecting $d_1, d_2, \ldots\ldots, d_n = 2^n$

Hence total number of such matrices = 2^n.

Ex.15 Show that the matrix A = $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent and find its index.

Sol.

We have $A^2 = AA \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$

Again $A^3 = AA^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$

Thus 3 is the least positive integer such that A3 = 0. Hence the matrix A is nilpotent of index 3.

Ex.16 If AB = A and BA = B then B'A' = A' and A'B' = B' and hence prove that A' and B' are idempotent.

Sol. We have AB = A $\Rightarrow$ (AB)' = A' $\Rightarrow$ B'A' = A'. Also BA = B $\Rightarrow$ (BA)' = B' $\Rightarrow$ A'B' = B'.

Now A' is idempotent if A'2 = A'. We have A'2 =A'A' = A' (B'A') = (A'B') A' = B'A' =A'.

$\therefore$ A' is idempotent.

Again $B'^2 = B'B' = B'(A'B') = (B'A')B' = A'B' = B'$. $\therefore$ B' is idempotent.

E. TRANSPOSE OF MATRIX

Let $A = [a_{ij}]_{m \times n}$. Then the transpose of A is denoted by A'(or A^T) and is defined as $A' = [b_{ij}]_{n \times m}$ where $b_{ij} = a_{ji}$ for all i & j

i.e. A' is obtained by rewriting all the rows of A as columns (or by rewriting all the columns of A as rows).

(i) For any matrix $A = [aij]_{m \times n}$, $(A')' = A$

(ii) Let λ be a scalar & A be a matrix. Then $(\lambda A)' = \lambda A'$

(iii) $(A + B)' = A' + B'$ & $(A - B)' = A' - B'$ for two comparable matrices A and B.

(iv) $(A_1 \pm A_2 \pm .... \pm A_n)' = A_1' \pm A_2' \pm ..... \pm A_n'$, where A_j are comparable.

(v) Let $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$, then $(AB)' = B'A'$

(vi) $(A_1 A_2 .... A_n)' = An'. A_{n-1}' .... \pm A_2'. A_1'$, provided the product is defined.

(vii) Symmetric & Skew–Symmetric Matrix : A square matrix A is said to be symmetric if $A' = A$

i.e. Let $A = [a_{ij}]n$. A is symmetric iff $a_{ij} = a_{ij}$ for all i & j.
A square matrix A is said to be skew–symmetric if $A' = - A$

i.e. Let $A = [a_{ij}]_n$. A is skew–symmetric iff $a_{ij} = -a_{ji}$ for all i & j.

e.g. $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix

& $B = \begin{bmatrix} o & x & y \\ -x & o & z \\ -y & -z & 0 \end{bmatrix}$ is a skew–symmetric matrix.

Note :

1. In skew–symmetric matrix all the diagonal elements are zero. ($a_{ij} = - a_{ij} \Rightarrow a_{ij} = 0$)

2. For any square matrix A, A + A' is symmetric & A -A' is skew - symmetric.

3. Every square matrix can be uniquely expressed as a sum of two square matrices of which one is symmetric and the other is skew–symmetric.

$A = B + C$, where $B = \frac{1}{2}(A + A')$ & $C = \frac{1}{2}(A - A')$

System of Linear Equations

I. SYSTEM OF LINEAR EQUATIONS

System Of Linear Equation (In Two Variables) :

(i) Consistent Equations : Definite & unique solution . [intersecting lines]
(ii) Inconsistent Equation : No solution . [Parallel line]
(iii) Dependent equation : Infinite solutions . [Identical lines]

Let $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ then

$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \Rightarrow$ Given equations are inconsistent & $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow$ Given equations are dependent

Cramer's Rule :[Simultaneous Equations Involving Three Unknowns]

Let a1x + b1y + c1z = d1 ..... (I) ;

a2x + b2y + c2z = d2 ..... (II) ;

a3x + b3y + c3z = d3 ..... (III)

Then , $x = \frac{D_1}{D}, Y = \frac{D_2}{D}, Z = \frac{D_3}{D}$.

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$; $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ & $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Note :

(a) If $D \neq 0$ and atleast one of $D_1, D_2, D_3 \neq 0$, then the given system of equations are consistent and have unique non trivial solution.

(b) If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have trivial solution only.

(c) If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have infinite solutions.

(d) If $D = 0$ but atleast one of D_1, D_2, D_3 is not zero then the equations are inconsistent and have no solution.

(e) If x, y, z are not all zero, the condition for $a_1x + b_1y + c_1z = 0$; $a_2x + b_2y + c_2z = 0$ & $a_3x + b_3y + c_3z = 0$ to be consistent in x, y, z is that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Remember that if a given system of linear equations have **Only Zero** Solution for all its variables then
the given equations are said to have **Trivial Solution.**

Solving System of Linear Equations Using Matrices :

Consider the system $a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_n.$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$ & $B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_n \end{bmatrix}$

Then the above system can be expressed in the matrix form as AX = B.

The system is said to be consistent if it has atleast one solution.

(i) System of Linear Equations And Matrix Inverse :

If the above system consist of n equations in n unknowns, then we have AX = B where A is a
square matrix. If A is non–singular, solution is given by $X = A^{-1}B$.
If A is singular, (adj A) B = 0 and all the columns of A are not proportional, then the system has
infinitely many solutions.

If A is singular and (adj A) B $\neq$ 0, then the system has no solution (we say it is inconsistent).

(ii) Homogeneous System and Matrix Inverse :

If the above system is homogeneous, n equations in n unknowns, then in the matrix form it is

$AX = 0$. ($\because$ in this case $b_1 = b_2 = \ldots .. b_n = 0$), where A is a square matrix.
If A is non–singular, the system has only the trivial solution (zero solution) $X = 0$
If A is singular, then the system has infinitely many solutions (including the trivial solution) and hence it has non–trivial solutions.

(iii) Elementary Row Transformation of Matrix :
The following operations on a matrix are called as elementary row transformations.

(a) Interchanging two rows.
(b) Multiplications of all the elements of row by a nonzero scalar.
(c) Addition of constant multiple of a row to another row.

Note : Similar to above we have elementary column transformations also.

Remark : Two matrices A & B are said to be equivalent if one is obtained from other using elementary
transformations. We write $A \sim B$.

(iv) Echelon Form of A Matrix : A matrix is said to be in Echelon form if it satisfies the following
(a) The first non-zero element in each row is 1 & all the other elements in the corresponding
column (i.e. the column where 1 appears) are zeroes.
(b) The number of zeros before the first non zero element in any non zero row is less than
the number of such zeroes in succeeding non zero rows.

(v) System of Linear Equations : Let the system be $AX = B$ where A is an $m \times n$ matrix, X is
the n–column vector & B is the m–column vector. Let [AB] denote the augmented matrix (i.e.
matrix obtained by accepting elements of B as $n + 1^{th}$ column & first n columns are that of A).

Ex.25 Solve the equations

$\lambda x + 2y - 2z - 1 = 0$,
$4x + 2\lambda y - z - 2 = 0$,
$6x + 6y + \lambda z - 3 = 0$, considering specially the case when $\lambda = 2$.

Sol.

The matrix form of the given system is $\begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ .....(i)

The given system of equations will have a unique solution if and only if the coefficient matrix is non-singular, i.e., iff $\begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix} \neq 0$ i.e., iff $\lambda^3 + 11\lambda - 30 \neq 0$ i.e., iff $(\lambda - 2)(\lambda^2 + 2\lambda + 15) \neq 0$.

Now the only real root of the equation $(\lambda - 2)(\lambda^2 + 2\lambda + 15) \neq 2 = 0$ is $\lambda = 2$

Therefore if $\lambda \neq 2$, the given system of equations will have a unique solution given by

$$\frac{x}{\begin{vmatrix} 1 & 2 & -2 \\ 2 & 2\lambda & -1 \\ 3 & 6 & \lambda \end{vmatrix}} = \frac{y}{\begin{vmatrix} \lambda & 1 & -2 \\ 4 & 2 & -1 \\ 6 & 3 & \lambda \end{vmatrix}} = \frac{z}{\begin{vmatrix} \lambda & 2 & 1 \\ 4 & 2\lambda & 2 \\ 6 & 6 & 3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix}}$$

In case $\lambda = 2$, the equation (i) becomes $\begin{bmatrix} 2 & 2 & -2 \\ 4 & 4 & -1 \\ 6 & 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Performing $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, we get $\begin{bmatrix} 2 & 2 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

The above system of equations is equivalent to $8z = 0$, $3z = 0$, $2x + 2y - 2z = 1$.

$\therefore x = \frac{1}{2} - c$, $y = c$, $z = 0$ constitute the general solution of the given system of equations in case $\lambda = 2$.

Ex.26 Solve

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 4x_1 + 5x_2 + 6x_3 &= 7 \\ 7x_1 + 8x_2 + 9x_3 &= 10 \end{aligned}$$

Sol.

$$\begin{matrix} x_1 + 2x_2 + 3x_3 = 4 \\ 4x_1 + 5x_2 + 6x_3 = 7 \\ 7x_1 + 8x_2 + 9x_3 = 10 \end{matrix} \xrightarrow[-7E1+E3]{04E1+E2} \begin{matrix} x_1 + 2x_2 + 3x_3 = 4 \\ -3x_2 - 6x_3 = -9 \\ -6x_2 - 12x_3 = -18 \end{matrix}$$

$$\xrightarrow{-2E_2+E3} \begin{matrix} x_1 + 2x_2 + 3x_3 = 4 \\ -3x_2 - 6x_3 = -9 \\ 0 = 0 \end{matrix} \xrightarrow{-\frac{1}{3}E_2} \begin{matrix} x_1 + 2x_2 + 3x_3 = 4 \\ x_2 - 2x_3 = 3 \\ 0 = 0 \end{matrix}$$

Now we have only two equations in three unknowns. In the second equation, we can let $x_3 = k$, where k is any complex number. Then $x_2 = 3 - 2k$. Substituting $s_3 = k$ and $x_2 = 3 - 2k$ into the first equation, we have $x_1 = 4 - 2x_2 - 3x_3 = 4 - 2(3 - 2k) - 3(k) = -2 + k$

Thus the general solution is $(-2 + k, 3 - 2k, k)$ or $\begin{aligned} x_1 &= -2 + k \\ x_2 &= 3 - 2k \\ x_3 &= k \end{aligned}$

And we see that the system has an infinite number of solutions. Specific solutions can be generated by
choosing specific values for k.

Ex.27 Number of triplets of a, b & c for which the system of equations ax - by = 2a - b and (c + 1)x + cy = 10 - a + 3 b has infinitely many solutions and x = 1, y = 3 is one of the solutions is

Sol.

put x = 1 & y = 3 in 1st equation $\Rightarrow a = -2b$ & from 2nd equation

$c = \dfrac{9+5b}{4}$; Now use $\dfrac{a}{c+1} = -\dfrac{b}{c} = \dfrac{2a-b}{10-a+3b}$; from first two b = 0 or c = 1 ;

if $b = 0 \Rightarrow a = 0$ & $c = 9/4$; if c = 1 ; b = −1 ; a = 2

Ex.28 Solve

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 4x_1 + 5x_2 + 6x_3 &= 7 \\ 7x_1 + 8x_2 + 9x_3 &= 12 \end{aligned}$$

Sol.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 4x_1 + 5x_2 + 6x_3 &= 7 \\ 7x_1 + 8x_2 + 9x_3 &= 12 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 12 \end{array}\right) \xrightarrow[-7E2+E3]{-4E1+E2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -6 & -12 & -16 \end{array}\right)$$

$$\xrightarrow{-2E2+E3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & 0 & 0 & 2 \end{array}\right) \quad \begin{aligned} x_1 + 2x_2 + 3x_2 &= 4 \\ 0x_1 - 3x_2 - 6x_3 &= -9 \\ 0x_1 + 0x_2 + 0x_3 &= 2 \end{aligned}$$

The last equation, 0 = 2, can never hold regardless of the values assigned to x_1, x_2 and x_3. Because the last (equivalent) system has no solution, the original system of equations has no solution.

Ex.29 Solve

$$\begin{aligned} x_2 - x_3 &= -9 \\ 2x_1 - x_2 + 4x_3 &= 29 \\ x_1 + x_2 - 3x_3 &= -20 \end{aligned}$$

by reducing the augmented matrix of the system to reduced row echelon form.

Sol.

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & -9 \\ 2 & -1 & 4 & 29 \\ 1 & 1 & -3 & -20 \end{array}\right) \xrightarrow{R1 \leftrightarrow R3} \left(\begin{array}{ccc|c} 1 & 1 & -3 & -20 \\ 2 & -1 & 4 & 29 \\ 0 & 1 & -1 & -9 \end{array}\right) \xrightarrow{-2R1+R2} \left(\begin{array}{ccc|c} 1 & 1 & -3 & -20 \\ 0 & -3 & 10 & 69 \\ 0 & 1 & -1 & -9 \end{array}\right) \xrightarrow{-\frac{1}{3}R2}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & -20 \\ 0 & 1 & -\frac{10}{3} & -23 \\ 0 & 1 & -1 & -9 \end{array}\right) \xrightarrow{-1R2+R3} \left(\begin{array}{ccc|c} 1 & 1 & -3 & -20 \\ 0 & 1 & -\frac{10}{3} & -23 \\ 0 & 0 & \frac{7}{3} & 14 \end{array}\right) \xrightarrow{\frac{3}{7}R3} \left(\begin{array}{ccc|c} 1 & 1 & -3 & -20 \\ 0 & 1 & -\frac{10}{3} & -23 \\ 0 & 0 & 1 & 6 \end{array}\right) \xrightarrow[\substack{3R3+R1 \\ -1R2+R1}]{\frac{10}{3}R3+R2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 6 \end{array}\right).$$

It is easy to see that x1 = 1, x2 = −3, x3 = 6. The process of solving a system by reducin the augmented matrix to reduced row echelon form is called Gauss–Jordan elimination.

Ex.30 Determine conditions on a, b and c so that $\begin{aligned} x_1 + 2x_2 + 3x_3 &= a \\ 4x_1 + 5x_2 + 6x_3 &= b \\ 7x_1 + 8x_2 + 9x_3 &= c \end{aligned}$ **will have no solutions or have an infinite number of solution.**

Sol.

$\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & -3 & -6 & b-4a \\ 0 & 0 & 0 & c-2b+a \end{array}\right)$. If $c - 2b + a \neq 0$, then no solution exists. If $c - 2b + a = 0$, we have two equations in three unknowns and we can set x_3 arbitrarily and then solve for x_1 and x_2.

J. INVERSE OF A MATRIX

(i) Singular & Non Singular Matrix : A square matrix A is said to be singular or non–singular according as $|A|$ is zero or non–zero respectively.

Ex.31 Show that every skew-symmetric matrix of odd order is singular.

Sol.

Since $|A| = |A'| = (-1)^n |A| \Rightarrow \quad |A| (1 - (-1)^n) = 0.$

As n is odd $\Rightarrow \quad |A| = 0.$ Hence A is singular.

(ii) Cofactor Matrix & Adjoint Matrix : Let $A = [a_{ij}]_n$ be a square matrix. The matrix obtained by
replacing each element of A by corresponding cofactor is called as cofactor matrix of A,
denoted as cofactor A. The transpose of cofactor matrix of A is called as adjoint of A, denoted
as adj A. i.e. If $A = [a_{ij}]_n$ then cofactor $A = [c_{ij}]_n$ when c_{ij} is the cofactor of $a_{ij} \ \forall$ i & j.
Adj $A = [d_{ij}]_n$ where $d_{ij} = c_{ji} \ \forall$ i & j.

(iii) Properties of Cofactor A and adj A :

(a) $A \cdot \text{adj } A = |A| I_n = (\text{adj } A) A$ where $A = [a_{ij}]_n$.
(b) $|\text{adj } A| = |A|^{n-1}$, where n is order of A. In particular, for 3 × 3 matrix, $|\text{adj } A| = |A|^2$
(c) If A is a symmetric matrix, then adj A are also symmetric matrices.
(d) If A is singular, then adj A is also singular.

(iv) Inverse of A Matrix (Reciprocal Matrix) : Let A be a non–singular matrix. Then the matrix

$\frac{1}{|A|}$ adj A is the multiplicative inverse of A(we call it inverse of A) and is denoted by A^{-1}.

When have A (adj A) = |A| I_n = (adj A) A

$\Rightarrow A\left(\frac{1}{|A|}\text{adj } A\right) = I_n = \left(\frac{1}{|A|}\text{adj } A\right) A$, for A is non–singular $\Rightarrow A^{-1} = \frac{1}{|A|}$ adj A.

Remarks :

1. The necessary and sufficient condition for existence of inverse of A is that A is non–singular.
2. A^{-1} is always non–singular.
3. If A = diag $(a_{11}, a_{22}, \ldots, a_{nn})$ where $a_{ij} \neq 0 \ \forall i$, then A^{-1} = diag $(a_{11}^{-1}, a_{22}^{-1}, \ldots, a_{nn}^{-1})$.
4. $(A^{-1})' = (A')^{-1}$ for any non–singular matrix A. Also adj (A′) = (adj A)′.
5. $(A^{-1})^{-1} = A$ if A is non–singular.
6. Let k be non–zero scalar & A be a non–singular matrix. Then $(kA)^{-1} = \frac{1}{k} A^{-1}$.
7. $|A^{-1}| = \frac{1}{|A|}$ for $|A| \neq 0$
8. Let A be a non–singular matrix. Then AB = AC ⇒ B = C & BA = CA ⇒ B = C.
9. A is non–singular and symmetric ⇒ A^{-1} is symmetric.
10. In general AB = **0** does not imply A = **0** or B = **0**. But if **A** is non–singular and **AB = 0,** then **B = 0**. Similarly B is non–singular and **AB = 0** ⇒ **A = 0**. Therefore, **AB = 0** ⇒ either both are singular or one of them is **0**.

Characteristic Polynomial & Characteristic Equation : Let A be a square matrix. Then the
polynomial |A – xI| is called as characteristic polynomial of A & the equation |A – xI| = 0 is called as
characteristic equation A.

Remark : Every square matrix A satisfies its characteristic equation (Cayley – Hamilton Theorem).
i.e.

$a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0$ is the characteristic equation of A, then
$a_0 A^n + a_1 A^{n-1} + \ldots + a_{n-1} A + a_n I = 0$

Ex.32 Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$.

Sol.

We have $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{vmatrix}$. The cofactors of the elements of the first row of | A | are

$\begin{vmatrix} 5 & 0 \\ 4 & 3 \end{vmatrix}, -\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix}$ i.e., are 15, 0, – 10 respectively.

The cofactors of the elements of the second row of |A| are $-\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$ i.e. are 6, –3, 0 respectively.

The cofactors of the elements of the third row of |A| are $\begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix}$ i.e., are – 15, 0, 5 respectively.

Therefore the adj. A = the transpose of the matrix B where $B = \begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}$ $\therefore$ adj $A = \begin{bmatrix} 15 & 0 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$

Ex.33 If A and B are square matrices of the same order, then adj (AB) = adj B. adj A.

Sol.

We have AB adj (AB) = $|AB| I_n$ = (adj AB) AB. ...(1)

Also AB (adj B. adj A) = A(B adj B) adj A

= A $|B| I_n$ adj A = |B| (A adj A) = | B | | A | I_n = | BA | I_n = | AB | I_n ...(2)

Similarly, we have (adj B adj A) AB = adj B [(adj A [(adj A) A] B

= adj B. $|A| I_n$ B = |A|. (adj B) B = |A|. $|B| I_n$ = | AB| I_n. ...(3)

From (1), (2) and (3), the required result follows, provided AB is non–singular.

Note : The result adj (AB) = adj B adj A holds goods even if A or B is singular. However the proof given above is applicable only if A and B are non–singular.

Ex.34 If A be an n-square matrix and B be its adjoint, then show that Det $(AB + KI_n)$ = $[\text{Det}(A) + K]_n$, where K is a scalar quantity.

Sol.

We have, A B = A (adj A) = Det (A). I_n $\Rightarrow$ $AB + KI_n = \text{Det}(A) I_n + KI_n$

$\Rightarrow \text{Det}(AB + KI_n) = \text{Det}(\text{Det}(A) I_n + KI_n) = (\text{Det}(A) + K)^n$ $(\because \text{Det}(\alpha I_n) = \alpha^n)$

$\Rightarrow$ $\text{Det}(AB + KI_n) = [\text{Det}(A) + K]^n$.

Ex.35 If (ℓ_r, m_r, n_r), r = 1, 2, 3 be the direction cosines of three mutually perpendicular lines referred to an orthogonal Cartesian co–ordinate system, then prove that $\begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix}$ is an orthogonal matrix.

Sol.

Let $A = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix}$. Then $A' = \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$. We have $AA' = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$.

$$= \begin{bmatrix} \ell_1^2 + m_1^2 + n_1^2 & \ell_1\ell_2 + m_1m_2 + n_1n_2 & \ell_1\ell_3 + m_1m_2 + n_1n_3 \\ \ell_2\ell_1 + m_2m_1 + n_2n_1 & \ell_2^2 + m_2^2 + n_2^2 & \ell_2\ell_3 + m_2m_3 + n_2n_3 \\ \ell_3\ell_1 + m_3m_1 + n_3n_1 & \ell_3\ell_2 + m_3m_2 + n_3n_2 & \ell_3^2 + m_3^2 + n_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

$\begin{bmatrix} \therefore & \ell_1^2 + m_1^2 + n_1^2 = 1 \text{ etc.} \\ \text{and} & \ell_1\ell_2 + m_1m_2 + n_2n_3 = 0 \text{ etc.} \end{bmatrix}$ Hence the matrix A is orthogonal.

Ex.36 Obtain the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and verify that it is satisfied y A and hence find its inverse.

Sol.

We have $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix}$

$= (1 - \lambda)(2 - \lambda)(3 - \lambda) + 2[0 - 2(2 - \lambda)] = (2 - \lambda)[(1 - \lambda)(3 - \lambda) - 4]$

$= (2 - \lambda)[\lambda^2 - 4\lambda - 1] = -(\lambda^3 - 6\lambda^2 + 7\lambda + 2).$

$\therefore$ the characteristic equation of A is $\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$ ......(i)

By the Cayley–Hamilton theorem $A^3 - 6A^2 + 7A + 2I = O.$ .....(ii)

Verification of (ii). We have $A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$.

Also $A^3 = A \,.\, A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$.

Now $A^2 - 6A^2 + 7A + 2I = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}$.

Hence Cayley–Hamilton theorem is verified. Now we shall compute A^{-1}.
Multiplying (ii) by A^{-1}, we get $A^2 - 6A + 7I + 2A^{-1} = \mathbf{O}$.

$$\therefore \quad A^{-1} = -\frac{1}{2}(A^2 - 6A + 7I) = -\frac{1}{2}\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 3 & 13 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - \frac{7}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{bmatrix}.$$

Ex.37 Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Sol.

We have $|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 3 & 1 & -1 \end{vmatrix}$, applying $C_3 \to C_3 \to 2C_2$ $= -1 \begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix}$, expanding the determinant along the first row $= -2$. Since $|A| \neq 0$, therefore A^{-1} exists.

Now the cofactors of the elements of the first row of $|A|$ are $\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$

i.e., are $-1, 8, -5$ respectively.

The cofactors of the elements of the second row of $|A|$ are $-\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$

i.e. are $1, -6, 3$ respectively.

The cofactors of the elements of the third row of $|A|$ are $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$

i.e. are $-1, 2, -1$ respectively.

Therefore the Adj. A = the transpose of the matrix B where $B = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$ $\therefore$ Adj. $A = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ 5 & 3 & -1 \end{bmatrix}$.

Now $A^{-1} = \frac{1}{|A|}$ Adj. A and here $|A| = -2$. $\quad \therefore A^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$.

Ex.38 If a non-singular matrix A is symmetric, show that A^{-1} is also symmetric.

Sol.

Since A is symmetric, $A' = A \quad \Rightarrow \quad A'A^{-1} = AA^{-1} = I$

$\Rightarrow \quad (A^{-1})A'A^{-1} = (A^{-1})'I = (A^{-1})'I' \quad \Rightarrow \quad (AA^{-1})' A^{-1} = (A^{-1}I)' = (A^{-1})'$

$\Rightarrow \quad I'A^{-1} = (A^{-1})' \Rightarrow A^{-1} = (A^{-1})'$. Hence A^{-1} is also symmetric.

Matrices: Overview

Definition

- A set of numbers or objects or symbols represented in the form of a rectangular array is called a matrix.

- The order of the matrix is defined by the number of rows and number of columns present in the rectangular array of representation.
- Unlike determinants, it has no value.
- A matrix of order m × n, **i.e.** m rows and n columns, is represented below:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

<u>Abbreviated as:</u> $A = [a_{ij}]_{m \times n}$, where $1 \le i \le m$; $1 \le j \le n$, i denotes the row and j denotes the column.

- The number a_{11}, a_{12}, ... etc., are known as the elements of matrix A, where a_{ij} belongs to the i^{th} row and j^{th} column and is called the $(i, j)^{th}$ element of the matrix $A = [a_{ij}]$.

Type of Matrices

1. **Row Matrix:** Matrix having one row **i.e.** matrix of order 1 × n. They are also known as row vectors.
   Example: $A = [a_{11}, a_{12}, \dots\dots a_{1n}]$
2. **Column Matrix:** Matrix having one column **i.e.** matrix of order m × 1. They are also known as column vectors.
   Example: $$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$
3. **Zero or Null Matrix:** An m × n matrix all whose entries are zero. It is denoted as $O_{m\times n}$.
   Example: $$A = O_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ or } B = O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
4. **Horizontal Matrix:** A matrix of order m × n is a horizontal matrix if n > m.
   Example: $$A = \begin{bmatrix} 1 & 8 & 2 \\ 3 & 4 & 7 \end{bmatrix}$$

5. **Verical Matrix:** A matrix of order m × n is a vertical matrix if m > n.

Example:

$$A = \begin{bmatrix} 5 & 2 \\ 6 & 8 \\ 9 & 4 \end{bmatrix}$$

6. **Square Matrix:** If number of rows is equal to number of column, then the matrix is a square matrix.

   Example:

$$A = \begin{bmatrix} 1 & 6 & 3 \\ 3 & 2 & 4 \\ 9 & 5 & 7 \end{bmatrix}$$

Note:

In a square matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

(a) The pair of elements a_{ij} & a_{ji} are called Conjugate Elements.

Example: a_{12} and a_{21}

(b) The elements a_{11}, a_{22}, a_{33}, ... a_{nn} are called **Diagonal Elements**. The line along which the diagonal elements lie is called the "**Principal or Leading**" diagonal.

(c) Sum of all the diagonal elements, **i.e.** Σa_{ii} is known as trace of the matrix. It is denoted as $t_r(A)$.

1. **Diagonal Matrix:** A square matrix in which all the elements are zero except the diagonal element. It is denoted as dia(d_1, d_2, ..., d_n).

Example:

$$A = \text{dia}(3, -2, 9) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Note:

Min. number of zeros in a diagonal matrix of order n = n(n – 1)

2. **Scalar Matrix:** A square matrix in which every non-diagonal element is zero and all diagonal elements are equal, is called a scalar matrix.
   i.e. in scalar matrix, $a_{ij} = 0$, for $i \neq j$ and $a_{ij} = k$, for $i = j$

   Example:
$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

3. **Unit/Identity Matrix:** A square matrix, in which every non-diagonal element is zero and every diagonal element is 1, is called, unit matrix or an identity matrix.
   i.e. in scalar matrix, $a_{ij} = 0$, for $i \neq j$ and $a_{ij} = 1$, for $i = j$

   Example:
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equality of Matrices

$A = [a_{ij}]$ & $B = [b_{ij}]$ will be equal, only if

- Both have the same order
- $a_{ij} = b_{ij}$ for each pair of i & j

Operations on Matrices

For the Matrices,
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \; B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

- **Addition of Matrices:** $A + B = [a_{ij} + b_{ij}]$, where A & B are of the same order.

  i.e.
$$A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{23} \end{bmatrix}$$

  Properties:
  (a) Addition of matrices is commutative **i.e.** $A + B = B + A$.
  (b) Matrix addition is associative **i.e.** $(A + B) + C = A + (B + C)$

(c) Additive inverse. If A + B = O = B + A , then A and B are additive inverse of each other.

- **Multiplication of a Matrix by a Scalar:** If a matrix is multiplied by a scalar quantity, then each element is multiplied by that quantity for the resulting matrix.

  i.e.
  $$kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$$
  , where k is a scalar quantity.

- **Multiplication of Matrices:** Two matrices A, B can be multiplied to give resulting matrix AB, only if, no. of a column of A(prefactor) is equal to the no. of rows of B (post factor)

  i.e. A is a matrix of order n x m and B is a matrix of order p x q, then AB exists only if m = p.

  If m=p, order of AB = n x q

  $$(AB)_{ij} = \sum_{r=1}^{n} a_{ir} \cdot b_{rj}$$

  i.e.

  $$AB = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}) & (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}) & (a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}) \\ (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}) & (a_{21}b_{22} + a_{22}b_{22} + a_{23}b_{32}) & (a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}) \\ (a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31}) & (a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32}) & (a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33}) \end{bmatrix}$$

  Properties:

  (a) Matrix multiplication may or maynot be commutative **i.e.** AB may or may not be equal to BA.

  (b) Matrix multiplication is Associative **i.e.** (A . B) . C = A . (B . C)

  (c) Matrix multiplication is Distributive over Matrix Addition **i.e.** A(B+C) = AB + AC

  (d) Cancellation Laws not necessary hold in case of matrix multiplication **i.e.** if AB = AC ⇒ B = C even if A ≠ 0.

  (e) AB = 0 i.e., Null Matrix, does not necessarily imply that either A or B is a null matrix.

  (f) Positive Integral Powers of a Square Matrix **i.e.** For a square matrix A , $A^2A = (AA)\ A = A\ (AA) = A^3$

Note:

- If A and B are two non-zero matrices such that AB = O then A and B are called the divisors of zero.

- <u>If A and B are two matrices such that:</u>
  (i) AB = BA ⇒ A and B commute each other
  (ii) AB = – BA ⇒ A and B anti commute each other
- For a unit matrix I of any order , $I^m = I$ for all $m \in N$.

Illustration 1. Find the value of x and y, if

$$2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix} + \begin{bmatrix}y & 0\\1 & 2\end{bmatrix} = \begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$$

Ans.

$$2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix} + \begin{bmatrix}y & 0\\1 & 2\end{bmatrix} = \begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2 & 6\\0 & 2x\end{bmatrix} + \begin{bmatrix}y & 0\\1 & 2\end{bmatrix} = \begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$$

$$\therefore \begin{bmatrix}2+y & 6+0\\0+1 & 2x+2\end{bmatrix} = \begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$$

On equating the corresponding elements of L.H.S. and R.H.S.

A2 + y = 5 ⇒ y = 3
2x + 2 = 8 ⇒ 2x = 6 ⇒ x = 3

Thus, x = 3 and y = 3.

Illustration 2. If $A = \begin{bmatrix}2 & 1 & 3\\3 & -2 & 1\\-1 & 0 & 1\end{bmatrix} and B = \begin{bmatrix}1 & -2\\2 & 1\\4 & -2\end{bmatrix}$**, find AB and BA if possible.**

Ans. A is a 3 × 3 matrix and B is a 3 × 2 matrix, therefore, A and B are conformable for the product AB and it is of the order 3 × 2.

$$AB = \begin{bmatrix} [2\times 1 + 1\times 2 + 3\times 4] & [2\times(-2) + 1\times 1 + 3\times 3] \\ [3\times 1 + (-2)\times 2 + 1\times 4] & [3\times(-2) + (-2)\times 1 + 1\times 3] \\ [(-1)\times 1 + 0\times 2 + 1\times 4] & [(-1)\times(-2) + 0\times 1 + 1\times 3] \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} 16 & -12 \\ 3 & -10 \\ 3 & 0 \end{bmatrix}$$

BA is not possible since the number of columns of B $\neq$ number of rows of A.

Matrix Polynomial

- If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_nx^0$, then we define a matrix polynomial $f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots + a_nI^n$ where A is the given square matrix.
- If f (A) is the null matrix then A is called the zero or root of the polynomial f (x).

Special Matrices

(a) Idempotent Matrix: A square matrix is idempotent provided $A^2 = A$

(b) Nilpotent Matrix: A square matrix is said to be nilpotent matrix of order m, $m \in N$, if $A^m = O$, $A^{m-1} \neq O$

(c) Periodic Matrix: A square matrix is periodic when it satisfies the relation $A^{K+1} = A$, for some positive integer K. The period of the matrix is the least value of K for which this holds true.

(d) Involutary Matrix: If $A^2 = I$, the matrix is said to be an involutary matrix.

Note:

- For an idempotent matrix, $A^n = A \ \forall\ n > 2, n \in N$.
- Period of an idempotent matrix is 1.
- For an involuntary Matrix, $A = A^{-1}$.

Transpose of a Matrix

- The transpose of a matrix is obtained by changing its rows & columns.

- It is denoted as A^T or A'.
- If a matrix be $A = [a_{ij}]$ of order m x n, then A^T or $A' = [a_{ji}]$ for $1 \le i \le n$ & $1 \le j \le m$ of order $n \times m$

  i.e.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow A' = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

- **Example:**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 3 & 2 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 3 \\ 3 & 6 & 2 \end{bmatrix}$$

- **Reversal law for transpose:** $(A_1, A_2, \ldots\ldots A_n)^T = A_n^T, \ldots\ldots, A_2^T, A_1^T$

Properties of Transpose:

If A^T & B^T denote the transpose of A and B.

(a) $(A \pm B)^T = A^T \pm B^T$; note that A & B have the same order.
(b) $(A B)^T = B^T A^T$ A & B are conformable for matrix product AB.
(c) $(A^T)^T = A$
(d) $(kA)^T = k A^T$ k is a scalar.

Symmetric & Skew-Symmetric Matrix

- A square matrix $A = [a_{ij}]$ is said to be, symmetric if, $a_{ij} = a_{ji}$ $\forall$ i & j (conjugate elements are equal)
- A square matrix $A = [a_{ij}]$ is said to be, Skew-symmetric if, $a_{ij} = -a_{ji}$ $\forall$ i & j (the pair of conjugate elements are additive inverse of each other)

Note:

(i) For symmetric matrix, $A = A^T$

(ii) Max. number of distinct entries in a symmetric matrix of order n is $\frac{n(n+1)}{2}$.

(iii) The digaonal elements of a skew symmetric matrix are all zero, but not the converse **i.e.** if digaonal elements are 0 doesn't mean matrix is skew symmetric.

Properties Of Symmetric & Skew Matrix

(a) A is symmetric, if $A^T = A$ and A is skew-symmetric, if $A^T = -A$.

(b) $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew symmetric matrix.
Consider $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$. Thus, $A + A^T$ is symmetric.
Similarly, we can prove that $A - A^T$ is skew-symmetric.

(c) The sum of two symmetric matrices is a symmetric matrix and the sum of two skew-symmetric matrices is a skew-symmetric matrix.

(d) If A & B are symmetric matrices then,
(AB + BA) is a symmetric matrix and (AB − BA) is a skew-symmetric matrix.

(e) Every square matrix can be uniquely expressed as a sum of a symmetric and a skew-symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\substack{P \\ \text{Symmetric}}} + \underbrace{\frac{1}{2}(A - A^T)}_{\substack{Q \\ \text{Skew Symmetric}}}$$

i.e.

Adjoint of A Square Matrix

Let $A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix and let the matrix formed by the cofactors of $[a_{ij}]$ in determinant A is $= \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$. Then (adj A) $= \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$.

Note: Co-factors of the elements of any matrix are obtained by eliminating all the elements of the same row and column and calculating the determinant of the remaining elements.

Theorem : A (adj. A) = (adj. A).A = $|A|\, I_n$, if A be a square matrix of order n.

Properties:

(i) $|\text{adj } A| = |A|^{n-1}$

(ii) adj (AB) = (adj B) (adj A)

(iii) adj(KA) = K^{n-1} (adj A), K is a scalar

Illustration 3. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$**, find adj A.**

Ans. Each element of cofactor matrix

$$A_{11} = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = 3 \times 3 - 4 \times 4 = -7$$

$$A_{12} = -\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1, A_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1; A_{21} = -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6, A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

$$A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -2, \quad A_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1; \quad A_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1, \quad A_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$$

Cofactor matrix =

$$\begin{bmatrix} -7 & 1 & 1 \\ 6 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

adj A = transpose of codaftor matrix =

$$\begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

Elementary Transformation

Any one of the following operations on a matrix is called an elementary transformation.

- Interchanging any two rows (or columns), denoted by $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix}$$

We apply $R_1 \leftrightarrow R_2$ and obtain:

$$A = \begin{bmatrix} 4 & -5 & 6 \\ 1 & 2 & -3 \end{bmatrix}$$

- Multiplication of the element of any row (or column) by a non-zero quantity and denoted by $Ri \rightarrow kRi$ or $Ci \rightarrow kCj$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix}$$

We apply $R_1 \leftrightarrow 3R_1$ and obtain:

$$A = \begin{bmatrix} 3 & 6 & -9 \\ 4 & -5 & 6 \end{bmatrix}$$

- Addition of constant multiple of the elements of any row to the corresponding element of any other row, denoted by $Ri \rightarrow Ri + kRj$ or $Ci \rightarrow Ci + kCj$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix}$$

We apply $R_2 \leftrightarrow R_2 + 4R_1$ and obtain:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 8 & 3 & -6 \end{bmatrix} <$$

Inverse Of A Matrix (Reciprocal Matrix)

- A square matrix A said to be invertible (non singular) if there exists a matrix B such that, $AB = I = BA$. B is called the inverse (reciprocal) of A and is denoted by A^{-1}.
- Thus $A^{-1} = B \Leftrightarrow AB = I = BA$.
- We have, $A.(\text{adj } A) = |A| \; I_n$
  $A^{-1} A \,(\text{adj } A) = A^{-1} I_n |A|$
  $I_n (\text{adj } A) = A^{-1} |A| \; I_n$
  $\therefore A^{-1} = \frac{(\text{adj } A)}{|A|}$

Imp. Theorem: If A & B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$. This is reversal law for inverse

Note:

- The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$.
- If A be an invertible matrix, then A^T is also invertible & $(A^T)^{-1} = (A^{-1})^T$.
- If A is invertible,
  (a) $(A^{-1})^{-1} = A$;
  (b) $(A^k)^{-1} = (A^{-1})^k = A^{-k}, k \in N$
- If A is an Orthogonal Matrix. $AA^T = I = A^TA$
- A square matrix is said to be orthogonal if, $A^{-1} = A^T$.
- $|A^{-1}| = \frac{1}{|A|}$

System of Equation & Criterian for Consistency Gauss - Jordan Method

$a1x + b1y + c1z = d1$
$a2x + b2y + c2z = d2$
$a3x + b3y + c3z = d3$

We can write it in the form of matrix:

$$\begin{bmatrix} a1x + b1y + c1z \\ a2x + b2y + c2z \\ a3x + b3y + c3z \end{bmatrix} = \begin{bmatrix} d1 \\ d2 \\ d3 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d1 \\ d2 \\ d3 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}\ and\ B = \begin{bmatrix} d1 \\ d2 \\ d3 \end{bmatrix}$$

$\Rightarrow AX = B$

$\Rightarrow A^{-1} A X = A^{-1} B$

$\Rightarrow X = A^{-1} B =$

$\frac{(\text{adj. } A).B}{|A|}$.

Note:

- If $|A| \neq 0$, system is consistent having unique solution
- If $|A| \neq 0$ & (adj A). B $\neq O$ (Null matrix), system is consistent having unique non − trivial solution.
- If $|A| \neq 0$ & (adj A) . B $= O$ (Null matrix) , system is consistent having trivial solution
- If $|A| = 0$, matrix method fails

If (adj A) . B = null matrix = **O** → Consistent (Infinite solutions)

If (adj A) . B $\neq$ **O** → Inconsistent (no solution)

Illustration 4. Solve the following equation,

2x+y+2z=0, 2x-y+z=10, x+3y-z=5.

Ans. The given equation can be written in matrix form:

$$\begin{bmatrix} 2x + y + 2z \\ 2x - y + z \\ x + 3y - z \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$$

$$\because X = A^{-1} B = \frac{(adj.\ A).B}{|A|}.$$

Therefore,

$$|A| = \begin{vmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = 2(1-3) - 1(-2-1) + 2(6+1) = 13$$

$$adj\ A = \begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{13} \begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 0+70+15 \\ 0-40+10 \\ 0-50-20 \end{bmatrix} = \begin{bmatrix} 85/13 \\ -30/13 \\ -70/13 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 85/13 \\ -30/13 \\ -70/13 \end{bmatrix} \Rightarrow x = \frac{85}{13}, y = \frac{-30}{13}, z = \frac{-70}{13}$$

Elementary Operation of Matrix

A matrix is an array of numbers arranged in the form of rows and columns. The number of rows and columns of a matrix are known as its dimensions which is given by **m [latex]\times[/latex] n**, where m and n represent the number of rows and columns respectively. Apart from basic mathematical operations there are certain elementary operations that can be performed on matrix namely transformations. It is a special type of matrix that can be illustrate 2d and 3d transformations. Let's have a look on different types of elementary operations.

Types of Elementary Operations

There are two types of elementary operations of a matrix:

- **Elementary row operations:** when they are performed on rows of a matrix.
- **Elementary column operations:** when they are performed on columns of a matrix.

Elementary Operations of a Matrix

- Any 2 columns (or rows) of a matrix can be exchanged. If the i^{th} and j^{th} rows are exchanged, it is shown by $R_i \leftrightarrow R_j$ and if the i^{th} and j^{th} columns are exchanged, it is shown by $C_i \leftrightarrow C_j$.

For example, given the matrix A below:
[latex]A =

$$\begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix}$$

[/latex]

We apply [latex]R_{1}\leftrightarrow R_{2}[/latex] and obtain:
[latex]A =

$$\begin{bmatrix} 4 & -5 & 6 \\ 1 & 2 & -3 \end{bmatrix}$$

[/latex]

- The elements of any row (or column) of a matrix can be multiplied with a non-zero number. So if we multiply the i^{th} row of a matrix by a non-zero number k, symbolically it can be denoted by $R_i \leftrightarrow kR_i$. Similarly, for column it is given by $C_i \leftrightarrow kC_i$.

For example, given the matrix A below:

[latex]A =

$$\begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix}$$

[/latex]

We apply [latex]R_{1}\leftrightarrow 3R_{1}[/latex] and obtain:
[latex]A =

$$\begin{bmatrix} 3 & 6 & -9 \\ 4 & -5 & 6 \end{bmatrix}$$

[/latex]

- The elements of any row (or column) can be added with the corresponding elements of another row (or column) which is multiplied by a non-zero number. So if we add the i^{th} row of a matrix to the j^{th} row which is multiplied by a non-zero number k, symbolically it can be denoted by $R_i \leftrightarrow R_i + kR_j$. Similarly, for column it is given by $C_i \leftrightarrow C_i + kC_j$.

For example, given the matrix A below:
[latex]A =

$$\begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix}$$

[/latex]
We apply [latex]R_{2}\leftrightarrow R_{2}+4R_{1}[/latex] and obtain:
[latex]A =

$$\begin{bmatrix} 1 & 2 & -3 \\ 8 & 3 & -6 \end{bmatrix}$$

[/latex]<

Invertible Matrices

A matrix is an array of numbers arranged in the form of rows and columns. The number of rows and columns of a matrix are known as its dimensions, which is given by m x n where m and n represent the number of rows and columns respectively. The basic mathematical operations like addition, subtraction, multiplication and division can be done on matrices. In this article, we will discuss the inverse of a matrix or the invertible vertices.

What is Invertible Matrix?

A matrix A of dimension n x n is called invertible if and only if there exists another matrix B of the same dimension, such that AB = BA = I, where I is the identity matrix of the same order. Matrix B is known as the inverse of matrix A. Inverse of matrix A is symbolically represented by A^{-1}. Invertible matrix is also known as a non-singular matrix or nondegenerate matrix.

For example, matrices A and B are given below:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

Now we multiply A with B and obtain an identity matrix:

$$AB = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly, on multiplying B with A, we obtain the same identity matrix:

$$BA = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It can be concluded here that AB = BA = I. Hence $A^{-1} = B$, and B is known as the inverse of A. Similarly, A can also be called an inverse of B, or $B^{-1} = A$.

A square matrix that is not invertible is called singular or degenerate. A square matrix is called singular if and only if the value of its determinant is equal to zero. Singular matrices are unique in the sense that if the entries of a square matrix are randomly selected from any finite region on the number line or complex plane, then the probability that the matrix is singular is 0, that means, it will “rarely” be singular.

Invertible Matrix Theorem

Theorem 1

If there exists an inverse of a square matrix, it is always unique.

Proof:

Let us take A to be a square matrix of order n x n. Let us assume matrices B and C to be inverses of matrix A.

Now AB = BA = I since B is the inverse of matrix A.

Similarly, AC = CA = I.

But, B = BI = B (AC) = (BA) C = IC = C

This proves B = C, or B and C are the same matrices.

Theorem 2:

If A and B are matrices of the same order and are invertible, then $(AB)^{-1} = B^{-1} A^{-1}$.

Proof:

$(AB)(AB)^{-1} = I$ (From the definition of inverse of a matrix)
$A^{-1}(AB)(AB)^{-1} = A^{-1}I$ (Multiplying A^{-1} on both sides)
$(A^{-1}A)B(AB)^{-1} = A^{-1}$ ($A^{-1}I = A^{-1}$)
$IB(AB)^{-1} = A^{-1}$
$B(AB)^{-1} = A^{-1}$
$B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$
$I(AB)^{-1} = B^{-1}A^{-1}$
$(AB)^{-1} = B^{-1}A^{-1}$

Matrix Inversion Methods

Matrix inversion is the method of finding the other matrix, say B that satisfies the previous equation for the given invertible matrix, say A. Matrix inversion can be found using the following methods:

- Gaussian Elimination
- Newton's Method
- Cayley-Hamilton Method
- Eigen Decomposition Method

Applications of Invertible Matrix

For many practical applications, the solution for the system of the equation should be unique and it is necessary that the matrix involved should be invertible. Such applications are:

- Least-squares or Regression
- Simulations
- MIMO Wireless Communications

Invertible Matrix Example

Now, go through the solved example given below to understand the matrix which can be invertible and how to verify the relationship between matrix inverse and the identity matrix.

Example: If

$$A = \begin{bmatrix} -3 & 1 \\ 5 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & \frac{1}{5} \\ 1 & \frac{3}{5} \end{bmatrix},$$

then show that A is invertible matrix and B is its inverse.

Solution:

Given,

$$A = \begin{bmatrix} -3 & 1 \\ 5 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & \frac{1}{5} \\ 1 & \frac{3}{5} \end{bmatrix}$$

Now, finding the determinant of A,

$$|A| = \begin{vmatrix} -3 & 1 \\ 5 & 0 \end{vmatrix}$$

= -3(0) – 1(5)
= 0 – 5
= -5 $\neq$ 0
Thus, A is an invertible matrix.
We know that, if A is invertible and B is its inverse, then AB = BA = I, where I is an identity matrix.

$$AB = \begin{pmatrix} -3 & 1 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{5} \\ 1 & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} (-3)\cdot 0 + 1\cdot 1 & (-3)\frac{1}{5} + 1\cdot\frac{3}{5} \\ 5\cdot 0 + 0\cdot 1 & 5\cdot\frac{1}{5} + 0\cdot\frac{3}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & \frac{1}{5} \\ 1 & \frac{3}{5} \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 0\cdot(-3) + \frac{1}{5}\cdot 5 & 0\cdot 1 + \frac{1}{5}\cdot 0 \\ 1\cdot(-3) + \frac{3}{5}\cdot 5 & 1\cdot 1 + \frac{3}{5}\cdot 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

AB = BA = I

Therefore, the matrix A is invertible and the matrix B is its inverse.

Properties

Below are the following properties hold for an invertible matrix A:

- $(A^{-1})^{-1} = A$
- $(kA)^{-1} = k^{-1}A^{-1}$ for any nonzero scalar k
- $(Ax)^{+} = x^{+}A^{-1}$ if A has orthonormal columns, where + denotes the Moore–Penrose inverse and x is a vector
- $(A^{T})^{-1} = (A^{-1})^{T}$
- For any invertible n x n matrices A and B, $(AB)^{-1} = B^{-1}A^{-1}$. More specifically, if $A_1, A_2 \ldots, A_k$ are invertible n x n matrices, then $(A_1A_2 \cdots A_{k-1}A_k)^{-1} = A^{-1}{}_kA^{-1}{}_{k-1} \cdots A^{-1}{}_2A^{-1}{}_1$
- $\det A^{-1} = (\det A)^{-1}$

Adjoint and inverse of a matrix

The adjoint of a matrix (also called the adjugate of a matrix) is defined as the transpose of the cofactor matrix of that particular matrix. For a matrix A, the adjoint is denoted as **adj (A)**. On the other hand, the inverse of a matrix A is that matrix which when multiplied by the matrix A give an identity matrix. The inverse of a Matrix A is denoted by A^{-1}.

Adjoint of a Matrix

Let the determinant of a square matrix A be |A|

$$If A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad Then \quad |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The matrix formed by the cofactors of the elements is

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Where

$$A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}.a_{32}$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -a_{21}.a_{33} + a_{23}.a_{31}; A_{13} =$$

$$(-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31};$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = -a_{12}a_{33} + a_{13}.a_{32}; A_{22} =$$

$$(-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}a_{33} - a_{13}.a_{31};$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -a_{11}a_{32} + a_{12}.a_{31}; A_{31} =$$

$$(-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = a_{12}a_{23} - a_{13}.a_{22};$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = -a_{11}a_{23} + a_{13}.a_{21}; A_{33} =$$

$$(-1)^{3+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}.a_{21};$$

Then the transpose of the matrix of co-factors is called the adjoint of the matrix A and is written as adj A.

$$adj\ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

The product of a matrix A and its adjoint is equal to unit matrix multiplied by the determinant A.

Let A be a square matrix, then (Adjoint A). A = A. (Adjoint A) = | A |. I

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $and\ adj\ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

A. (adj. A) $= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} & a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} & a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} \\ a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} & a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} & a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} \\ a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} & a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} & a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\, I.$$

Example Problems on How to Find the Adjoint of a Matrix

Example 1: If $A^T = -A$ then the elements on the diagonal of the matrix are equal to

(a) 1
(b) -1
(c) 0
(d) none of these

Solution: (c) $A^T = -A$; A is skew-symmetric matrix; diagonal elements of A are zeros. so option (c) is the answer.

Example 2: If A and B are two skew-symmetric matrices of order n, then,

(a) AB is a skew-symmetric matrix
(b) AB is a symmetric matrix
(c) AB is a symmetric matrix if A and B commute
(d) None of these

Solution: (c) We are given A' = -A and B' = -B;
Now, (AB)' = B'A' = (-B) (-A) = BA = AB, if A and B commute.

Example 3: Let A and B be two matrices such that AB' + BA' = 0. If A is skew symmetric, then BA
(a) Symmetric
(b) Skew symmetric
(c) Invertible
(d) None of these

Solution: (c) we have, (BA)' = A'B' = -AB' [A is skew symmetric]; = BA' = B(-A) = -BA
BA is skew symmetric.

Example 4: Let A

$$= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix},$$

then adj A is given by –

Solution: Co-factors of the elements of any matrix are obtained by eliminating all the elements of the same row and column and calculating the determinant of the remaining elements.

$A_{11} = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = 3 \times 3 - 4 \times 4 = -7$

$A_{12} = -\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1, A_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1; A_{21} = -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6, A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$

$A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -2, \quad A_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1; \quad A_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1, \quad A_{33} =$
$\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$

adj A = transpose of cofactor matrix.

$$\therefore \quad Adj\ A = \begin{vmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{vmatrix}$$

Example 5: Which of the following statements are false –

(a) If | A | = 0, then | adj A | = 0;
(b) Adjoint of a diagonal matrix of order 3 × 3 is a diagonal matrix;
(c) Product of two upper triangular matrices is an upper triangular matrix;
(d) adj (AB) = adj (A) adj (B);

Solution: (d) We have, adj (AB) = adj (B) adj (A) and not adj (AB) = adj (A) adj (B)

Inverse of a Matrix

If A and B are two square matrices of the same order, such that AB = BA = I (I = unit matrix)
Then B is called the inverse of A, i.e. B = A^{-1} and A is the inverse of B. Condition for a square matrix A to possess an inverse is that the matrix A is non-singular, i.e., | A | ≠ 0. If A is a square matrix and B is its inverse then AB = I. Taking determinant of both sides | AB | = | I | or | A | | B | = I. From this relation it is clear that | A | ≠ 0, i.e. the matrix A is non-singular.

How to find the inverse of a matrix by using the adjoint matrix?

We know that,

$$A.(Adj\ A) = |A|\, I \quad or \quad \frac{A.(Adj\ A)}{|A|} = I \ \ (Provided\, |A| \neq 0)$$

And

$$A.A^{-1} = I; \ A^{-1} = \frac{1}{|A|}(Adj.A)$$

Properties of Inverse and Adjoint of a Matrix

- **Property 1:** For a square matrix A of order n, A adj(A) = adj(A) A = |A|I, where I is the identitiy matrix of order n.
- **Property 2:** A square matrix A is invertible if and only if A is a non-singular matrix.

Problems on Finding the Inverse of a Matrix

Illustration : Let A

$$= \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$$

What is inverse of A?

Solution: By using the formula

$A^{-1} = \frac{adj\ A}{|A|}$ *we can obtain the value of* A^{-1}

We have

$$A_{11} = \begin{bmatrix} 4 & 5 \\ -6 & -7 \end{bmatrix} = 2 \ A_{12} = -\begin{bmatrix} 3 & 5 \\ 0 & -7 \end{bmatrix} = 21$$

Similarly

A_{13} = -18 A31 =4, $A_{32} = -8$, $A_{33} = 4$, $A_{21} = +6$, $A_{22} = -7$, $A_{23} = 6$

cofactor matrix of A

$$= \begin{bmatrix} 2 & 21 & -18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{bmatrix}$$

adj A = transpose of cofactor matrix

adj A $= \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$

Also $|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix} = \{4 \times (-7) - (-6) \times 5 - 3 \times (-6)\}$

= -28 + 30 + 18

= 20

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{20} \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

Types of Matrices

Matrix Types: Overview

The different types of matrices are:

Type of Matrix	Details
Row Matrix	A = $[a_{ij}]_{1\times n}$
Column Matrix	A = $[aij]_{m\times 1}$
Zero or Null Matrix	A = $[a_{ij}]_{mxn}$ where, a_{ij} = 0
Singleton Matrix	A = $[a_{ij}]_{mxn}$ where, m = n =1
Horizontal Matrix	$[a_{ij}]_{mxn}$ where, n > m
Vertical Matrix	$[a_{ij}]_{mxn}$ where, m > n
Square Matrix	$[a_{ij}]_{mxn}$ where, m = n
Diagonal Matrix	A = $[a_{ij}]$ when i ≠ j
Scalar Matrix	A = $[a_{ij}]_{mxn}$ where, a_{ij} = $\begin{Bmatrix} 0, & i \neq j \\ k, & i = j \end{Bmatrix}$ where k is a constant.
Identity (Unit) Matrix	A = $[a_{ij}]_{m\times n}$ where, $a_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$
Equal Matrix	A = $[a_{ij}]_{mxn}$ and B = $[b_{ij}]_{rxs}$ where, a_{ij} = b_{ij}, m = r, and n = s
Triangular Matrices	Can be either upper triangular (a_{ij} = 0, when i > j) or lower triangular (a_{ij} = 0 when i < j)

Singular Matrix	$\|A\| = 0$
Non-Singular Matrix	$\|A\| \neq 0$
Symmetric Matrices	$A = [a_{ij}]$ where, $a_{ij} = a_{ji}$
Skew-Symmetric Matrices	$A = [a_{ij}]$ where, $a_{ij} = a_{ji}$
Hermitian Matrix	$A = A^{\theta}$
Skew – Hermitian Matrix	$A^{\theta} = -A$
Orthogonal Matrix	$A A^T = I_n = A^T A$
Idempotent Matrix	$A^2 = A$
Involuntary Matrix	$A^2 = I$, $A^{-1} = A$
Nilpotent Matrix	$\exists\, p \in N$ such that $A^P = 0$

Types of Matrices: Explanations

Row Matrix

A matrix having only one row is called a row matrix. Thus $A = [a_{ij}]_{mxn}$ is a row matrix if m = 1. So, a row matrix can be represented as $A = [a_{ij}]_{1\times n}$. It is called so because it has only one row and the order of a row matrix will hence be 1 × n. For example, A = [1 2 4 5] is row matrix of order 1 x 4. Another example of the row matrix is P = [-4 -21 -17] which is of the order 1 × 3.

Column Matrix

A matrix having only one column is called a **column matrix**. Thus, $A = [a_{ij}]_{mxn}$ is a column matrix if n = 1. Thus, the value of for a column matrix will be 1. Hence, the order is m × 1.
An example of a column matrix is:

$$A = \begin{bmatrix} -1 \\ 2 \\ -4 \\ 5 \end{bmatrix}$$ is column matrix of order 4 x 1.

Just like the row matrices had only one row, column matrices have only one column. Thus, the value of for a column matrix will be 1. Hence, the order is m × 1. The general form of a column matrix is given by $\mathbf{A} = [\mathbf{a}_{ij}]_{m\times1}$. Other examples of a column matrix include:

$$P = \begin{bmatrix} 2 \\ 7 \\ -17 \end{bmatrix} \qquad Q = \begin{bmatrix} -1 \\ -18 \\ -19 \\ 9 \\ 13 \end{bmatrix}$$

In the above example , P and Q are 3×1 and 5×1 order matrices respectively.

Zero or Null Matrix

If in a matrix all the elements are zero then it is called a zero matrix and it is generally denoted by 0. Thus, A = $[a_{ij}]_{mxn}$ is a zero-matrix if $a_{ij} = 0$ for all i and j; E.g. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of order 2 x 3.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is a 3 x 2 null matrix & B = $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 3 x 3 null matrix.

Singleton Matrix

If in a matrix there is only element then it is called **singleton matrix**. Thus, A = $[a_{ij}]_{mxn}$ is a singleton matrix if m = n = 1. E.g. [2], [3], [a], [] are singleton matrices.

Horizontal Matrix

A matrix of order m x n is a horizontal matrix if n > m; E.g.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$$

Vertical Matrix

A matrix of order m x n is a **vertical matrix** if m > n; E.g.

$$\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$$

Square Matrix

If the number of rows and the number of columns in a matrix are equal, then it is called a square matrix.
Thus, $A = [a_{ij}]_{mxn}$ is a square matrix if m = n; E.g.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is a square matrix of order 3 × 3.
The sum of the diagonal elements in a square matrix A is called the trace of matrix A, and which is denoted by

tr(A); tr(A) = $\sum_{i=1}^{n} a_{ij} = a_{11} + a_{22} + \ldots. + a_{mn}.$

Another example of a square matrix is:

$$P = \begin{bmatrix} 4 & 7 \\ 9 & 13 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2 & 1 & 13 \\ -5 & -8 & 0 \\ 14 & -7 & 9 \end{bmatrix}$$

The order of P and Q is 2×2 and 3×3 respectively.

Diagonal Matrix

If all the elements, except the principal diagonal, in a square matrix, are zero, it is called a **diagonal matrix**. Thus, a square matrix $A = [a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$; E.g.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

is a diagonal matrix of order 3 x 3, which can also be denoted by diagonal [2 3 4]. The special thing is, all the non-diagonal elements of this matrix are zero. That means only the diagonal has non-zero elements. There are two important things to note here which are

(i) A diagonal matrix is always a square matrix

(ii) The diagonal elements are characterized by this general form: a_{ij} where $i = j$. This means that a matrix can have only one diagonal.
Few more example of diagonal matrix are:

$P = [9]$

$$Q = \begin{bmatrix} 9 & 0 \\ 0 & 13 \end{bmatrix} \quad R = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

In the above examples, P, Q, and R are diagonal matrices with order 1×1, 2×2 and 3×3 respectively. When all the diagonal elements of a diagonal matrix are the same, it goes by a different name- scalar matrix which is discussed below.

Scalar Matrix

If all the elements in the diagonal of a diagonal matrix are equal, it is called a scalar matrix. Thus, a square matrix A = $[a_{ij}]_{mx}$ is a scalar matrix if

a_{ij} = $\begin{Bmatrix} 0, & i \neq j \\ k, & i = j \end{Bmatrix}$ where k is a constant.

$$E.g. \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

is a scalar Matrix.
More examples of scalar matrix are:

$$P = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad Q = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$$

Now, what if all the diagonal elements are equal to 1? That will still be a scalar matrix and obviously a diagonal matrix. It has got a special name which is known as the identity matrix.

Unit Matrix or Identity Matrix

If all the elements of a principal diagonal in a diagonal matrix are 1, then it is called a **unit matrix**. A unit matrix of order n is denoted by I_n. Thus, a square matrix A = $[a_{ij}]_{m\times n}$ is an identity matrix if

$$a_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$\text{E.g. } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Conclusions:

- All identity matrices are scalar matrices
- All scalar matrices are diagonal matrices
- All diagonal matrices are square matrices

It should be noted that the converse of the above statements is not true for any of the cases.

Equal Matrices

Equal matrices are those matrices which are equal in terms of their elements. The conditions for matrix equality are discussed below.

Equality of Matrices Conditions

Two matrices A and B are said to be equal if they are of the same order and their corresponding elements are equal, i.e. Two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{r\times s}$ are equal if:
(a) $m = r$ i.e. the number of rows in A = the number of rows in B.
(b) $n = s$, i.e. the number of columns in A = the number of columns in B
(c) $a_{ij} = b_{ij}$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, i.e. the corresponding elements are equal;

For example, Matrices $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} and \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are not equal because their orders are not the same.
But, If

$$A = \begin{bmatrix} 1 & 6 & 3 \\ 5 & 2 & 1 \end{bmatrix} and \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$ are equal matrices then,

$a_1 = 1, a_2 = 6, a_3 = 3, b_1 = 5, b_2 = 2, b_3 = 1.$

Triangular Matrix

A square matrix is said to be a **triangular matrix** if the elements above or below the principal diagonal are zero. There are two types:

Upper Triangular Matrix

A square matrix $[a_{ij}]$ is called an **upper triangular matrix**, if $a_{ij} = 0$, when $i > j$.

$$E.g. \begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \end{bmatrix}$$ is an upper uriangular matrix of order 3 x 3.

Lower Triangular Matrix

A square matrix is called a **lower triangular matrix**, if $a_{ij} = 0$ when $i < j$.

$$E.g. \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 2 \end{bmatrix}$$ is a lower triangular matrix of order 3 x 3.

Singular Matrix and Non-Singular Matrix

Matrix A is said to be a singular matrix if its determinant $|A| = 0$, otherwise a non-singular matrix, i.e. If for det $|A| = 0$, it is singular matrix and for det $|A| \neq 0$, it is non-singular.

Symmetric and Skew Symmetric Matrices

- **Symmetric matrix:** A square matrix $A = [a_{ij}]$ is called a symmetric matrix if $a_{ij} = a_{ji}$, for all i,j values;

  $$\text{Eg. } A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 2 \end{pmatrix}$$ is symmetric, because $a_{12} = 2 = a_{21}$, $a_{31} = 3 = a_{13}$ etc.

Note: A is symmetric if A' = A (where 'A' is the transpose of matrix)

- **Skew-Symmetric Matrix:** A square matrix $A = [a_{ij}]$ is a skew-symmetric matrix if $a_{ij} = a_{ji}$, for all values of i,j. [putting j = i] $a_{ii} = 0$

Thus, in a skew-symmetric matrix all diagonal elements are zero; E.g.

$$A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

are skew-symmetric matrices.
Note: A square matrix A is a skew-symmetric matrix A' = -A.

Some important Conclusions on Symmetric and Skew-Symmetric Matrices:

- If A is any square matrix, then A + A' is a symmetric matrix and A – A' is a skew-symmetric matrix.
- Every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix. A = 1/2 (A + A') + 1/2 (A - A') = 1/2 (B + C), where B is symmetric and C is a skew symmetric matrix.
- If a and B are symmetric matrices, then AB is symmetric AB = BA, i.e. A & B commute.
- The matrix B'AB is symmetric or skew-symmetric in correspondence if A is symmetric or skew-symmetric.
- All positive integral powers of a symmetric matrix are symmetric.
- Positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric matrix are symmetric.

Hermitian and Skew-Hermitian Matrices

A square matrix A = [a_{ij}] is said to be a Hermitian matrix if

$a_{ij} = \overline{a}_{ji} \,\forall\, i, j;\ i.e.\ A = A^{\theta}$

$$E.g. \begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix} \cdot \begin{bmatrix} 3 & 3-4i & 5+2i \\ 3+4i & 5 & -2+i \\ 5-2i & -2-i & 2 \end{bmatrix}$$ are Hermitian matrices

Important Notes:

- If A is a Hermitian matrix then $a_{ii} = \overline{a}_{ii} \Rightarrow a_{ii}\, is\, real \,\forall\, i,$ thus every diagonal element of a Hermitian Matrix must be real.
- If a Hermitian matrix over the set of real numbers is actually a real symmetric matrix; and A a square matrix,

A = [a_{ij}] is said to be a skew-Hermitian if $a_{ij} = -\overline{a}_{ji},\ \forall\, i, j;$ i.e. $A^{\theta} = -A$;

$$\text{E.g.} \begin{bmatrix} 0 & -2+i \\ 2-i & 0 \end{bmatrix} \begin{bmatrix} 3i & -3+2i & -1-i \\ 3-2i & -2i & -2-4i \\ 1+i & 2+4i & 0 \end{bmatrix}$$

are skew-Hermitin matrices.

- If A is a skew-Hermitian matrix then =\frac{1}{2}(A+A')+\frac{1}{2}(A-A')=\frac{1}{2}(B+C),

$$a_{ii} = -\overline{a}_{ii} \Rightarrow a_{ii} + \overline{a_{ii}} = 0$$

i.e. a_{ii} must be purely imaginary or zero.

- A skew-Hermitian matrix over the set of real numbers is actually is a real skew-symmetric matrix.

Special Matrices

A^2, □A□^{2} = □A□)..

(a) Idempotent Matrix: A square matrix is idempotent, provided $A^2 = A$. For an idempotent matrix

$$A, A^n = A \,\forall\, n > 2, n \in$$
$$N \Rightarrow A^n = A, n \geq 2.$$

For an idempotent matrix A, det A = 0 or x.

(b) Nilpotent Matrix: A nilpotent matrix is said to be nilpotent of index p, ($p \in N$), if $A^p = O$, $A^{p-1} \neq O$, i.e. if p is the least positive integer for which $A^p = O$, then A is said to be nilpotent of index p.

(c) Periodic Matrix:

A square matrix which satisfies the relation $A^{k+1} = A$, for some positive integer K, then A is periodic with period K, i.e. if K is the least positive integer for which $A^{k+1} = A$, nd A is said to be periodic with period K. If K =1 then A is called idempotent.

E.g. the matrix $\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$

has the period 1.

Notes:

(i) Period of a square null matrix is not defined.
(ii) Period of an idempotent matrix is 1.

(d) Involutary Matrix:

If $A^2 = I$, the matrix is said to be an involutary matrix. An involutary matrix its own inverse

E.g. (i) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$