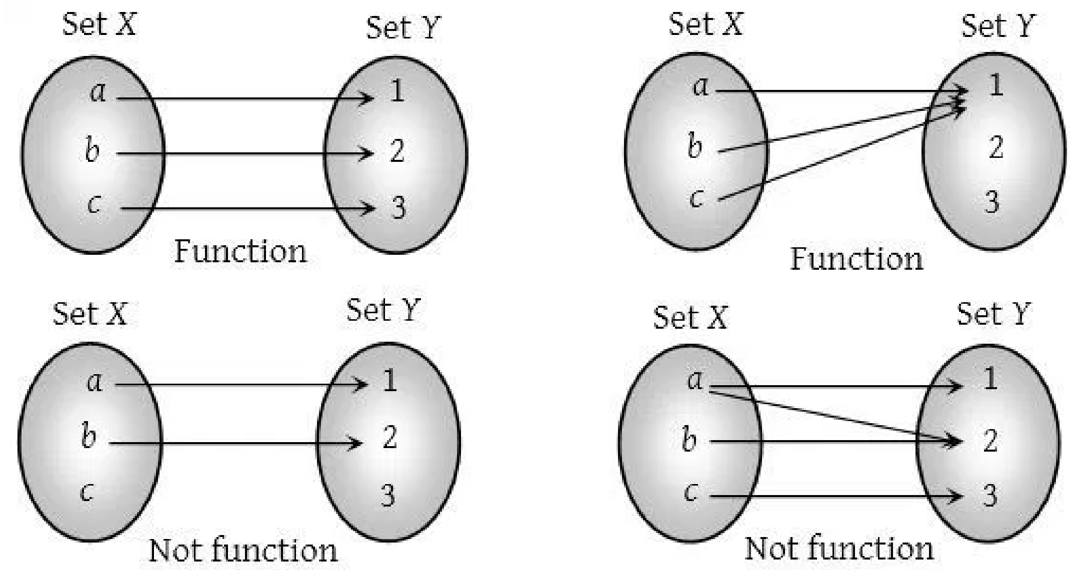
# 8. Functions

## What is a Function?

### Definition of function

**Function** can be easily defined with the help of the concept of mapping. Let X and Y be any two non-empty sets. "A function from X to Y is a rule or correspondence that assigns to each element of set X, one and only one element of set Y". Let the correspondence be 'f' then mathematically we write f : X ⟶ Y where y = f(x), x ∈ X and y ∈ Y. We say that 'y' is the image of 'x' under f (or x is the pre image of y). Two things should always be kept in mind:

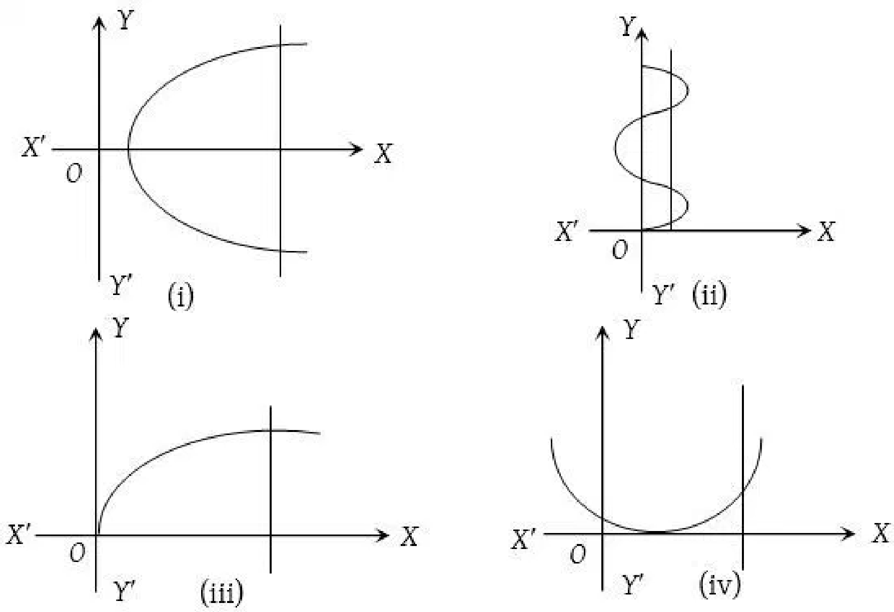
1. A mapping f : X ⟶ Y is said to be a function if each element in the set X has its image in set Y. It is also possible that there are few elements in set Y which are not the images of any element in set X.
2. Every element in set X should have one and only one image. That means it is impossible to have more than one image for a specific element in set X. Functions can not be multi-valued (A mapping that is multi-valued is called a relation from X and Y) e.g.




3.

## Testing for a function by vertical line test

A relation f : A ⟶ B is a function or not it can be checked by a graph of the relation. If it is possible to draw a vertical line which cuts the given curve at more than one point then the given relation is not a function and when this vertical line means line parallel to Y-axis cuts the curve at only one point then it is a function. Figure (iii) and (iv) represents a function.



## Number of functions

Let X and Y be two finite sets having m and n elements respectively. Then each element of set X can be associated to any one of n elements of set Y. So, total number of functions from set X to set Y is $n^m$.

## Value of the function

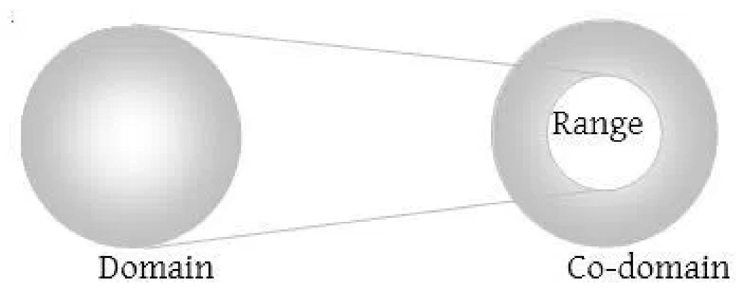
If y = f(x) is a function then to find its values at some value of x, say x = a we directly substitute x = a in its given rule f(x) and it is denoted by f(a).
e.g. If $f(x) = x^2 + 1$, then $f(1) = 1^2 + 1 = 2$, $f(2) = 2^2 + 1 = 5$, $f(0) = 0^2 + 1 = 1$, etc.

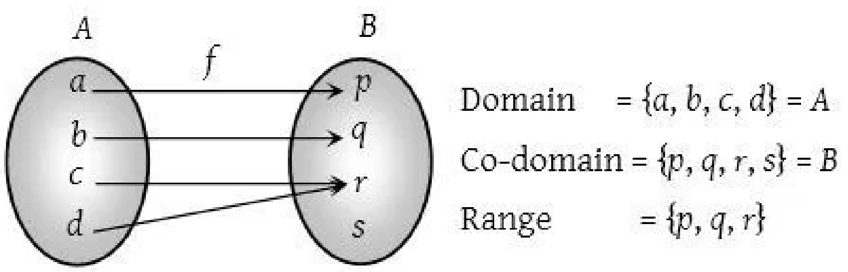
## Algebra of functions

1. **Scalar multiplication of a function:**
   (c f)x = c f(x) where c is a scalar. The new function has the domain $X_f$.
2. **Addition/subtraction of functions:**
   (f ± g)(x) = f(x) ± g(x). The new function has the domain X.
3. **Multiplication of functions:**
   (f.g)(x) = (g.f)(x) = f(x)g(x). The product function has the domain X.
4. **Division of functions:**
   (i) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$. The new function has the domain X, except for the values of x for which g(x) = 0.
   (ii) $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$. The new function has the domain X, except for the values of x for which f(x) = 0.
5. **Equal functions:** Two function f and g are said to be equal functions, if and only if
   (i) Domain of f = Domain of g.
   (ii) Co-domain of f = Co-domain of g.
   (iii) f(x) = g(x) ∀ x ∈ their common domain.
6. **Real valued function:** If R, be the set of real numbers and A, B are subsets of R, then the function f : A ⟶ B is called a real function or real –valued function.

## Domain, co-domain and range of function

If a function f is defined from a set A to set B then for f : A ⟶ B set A is called the domain of function f and set B is called the co-domain of function f. The set of all f-images of the elements of A is called the range of function f.
In other words, we can say
Domain = All possible values of x for which f(x) exists.
Range = For all values of x, all possible values of f(x).



## Methods for finding domain and range of function

**(i) Domain**

Expression under even root (i.e., square root, fourth root etc.) ≥ 0. Denominator ≠ 0.
If domain of y = f(x) and y = g(x) are $D_1$ and $D_2$ respectively then the domain of f(x) ± g(x) or f(x) . g(x) is $D_1 \cap D_2$.
While domain of $\frac{f(x)}{g(x)}$ is $D_1 \cap D_2$ – {g(x) = 0}.
Domain of (√f(x)) = $D_1 \cap$ {x : f(x) ≥ 0}

**(ii) Range:**

Range of y = f(x) is collection of all outputs f(x) corresponding to each real number in the domain.

1. If domain ∈ finite number of points ⇒ range ∈ set of corresponding f(x) values.
2. If domain ∈ R or R – [some finite points]. Then express x in terms of y. From this find y for x to be defined (i.e., find the values of y for which x exists).
3. If domain ∈ a finite interval, find the least and greatest value for range using monotonicity.

## Domain and Range of Some Standard Functions

| Function | Domain | Range |
|---|---|---|
| Polynomial function | $R$ | $R$ |
| Identity function $x$ | $R$ | $R$ |
| Constant function $K$ | $R$ | $\{K\}$ |
| Reciprocal function $\frac{1}{x}$ | $R_0$ | $R_0$ |
| $x^2, \lvert x \rvert$ | $R$ | $R^+ \cup \{0\}$ |
| $x^3, x\lvert x \rvert$ | $R$ | $R$ |
| Signum function | $R$ | $\{-1, 0, 1\}$ |
| $x + \lvert x \rvert$ | $R$ | $R^+ \cup \{0\}$ |
| $x - \lvert x \rvert$ | $R$ | $R^- \cup \{0\}$ |
| $[x]$ | $R$ | $I$ |
| $x - [x]$ | $R$ | $[0, 1)$ |
| $\sqrt{x}$ | $[0, \infty)$ | $[0, \infty]$ |
| $a^x$ | $R$ | $R^+$ |
| $\log x$ | $R^+$ | $R$ |
| $\sin x$ | $R$ | $[-1, 1]$ |
| $\cos x$ | $R$ | $[-1, 1]$ |

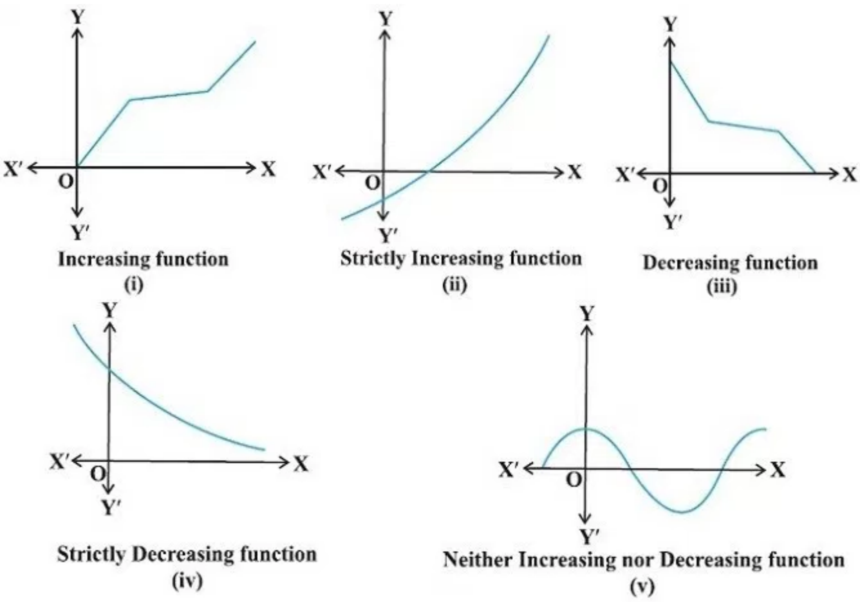
| | | |
|---|---|---|
| $\tan x$ | $R-\left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\right\}$ | $R$ |
| $\cot x$ | $R-\{0, \pm\pi, \pm 2\pi, \ldots\}$ | $R$ |
| $\sec x$ | $R-\left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\right\}$ | $R-(-1, 1)$ |
| $\text{cosec} x$ | $R-\{0, \pm\pi, \pm 2\pi, \ldots\}$ | $R-(-1, 1)$ |
| $\sin^{-1} x$ | $[-1, 1]$ | $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ |
| $\cos^{-1} x$ | $[-1, 1]$ | $[0, \pi]$ |
| $\tan^{-1} x$ | $R$ | $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ |
| $\cot^{-1} x$ | $R$ | $(0, \pi)$ |
| $\sec^{-1} x$ | $R-(-1, 1)$ | $[0, \pi]-\left\{\frac{\pi}{2}\right\}$ |
| $\text{cosec}^{-1} x$ | $R-(-1,1)$ | $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$ |

## How do you know if a Function is Increasing or Decreasing?

## Increasing and decreasing functions

**Definition:**

(1) A function f is said to be an increasing function in ]a,b[, if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in$ ]a,b[.
(2) A function f is said to be a decreasing function in ]a,b[, if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2), \forall x_1, x_2 \in$ ]a,b[.
f(x) is known as non-decreasing if $f'(x) \geq 0$ and non-increasing if $f'(x) \leq 0$.



**Monotonic function:** A function f is said to be monotonic in an interval if it is either increasing or decreasing in that interval.
We summarize the results in the table below :

| $f'(a_1)$ | $f''(a_1)$ | $f'''(a_1)$ | Behaviour of f at $a_1$ |
|---|---|---|---|
| + | | | Increasing |
| – | | | Decreasing |
| 0 | + | | Minimum |
| 0 | – | | Maximum |
| 0 | 0 | | ? |
| 0 | 0 | ± | Inflection |
| | 0 | 0 | ? |

- Blank space indicates that the function may have any value at $a_1$.
- Question mark indicates that the behaviour of f cannot be inferred from the data.

## Properties of monotonic functions

1. If f(x) is a strictly increasing function on an interval [*a, b*], then $f^{-1}$ exists and it is also a strictly increasing function.
2. If f(x) is a strictly increasing function on an interval [*a, b*] such that it is continuous, then $f^{-1}$ is continuous on [f(a), f(b)].
3. If f(x) is continuous on [*a, b*] such that f′(c) ≥ 0 [f′(c) > 0] for each c ∈ (*a*, *b*), then f(x) is monotonically (strictly) increasing function on [*a*, *b*].
4. If f(x) is continuous on  such that f′(c) ≤ 0 [f′(c) > 0] for each c ∈ (*a*, *b*), then f(x) is monotonically (strictly) decreasing function on [*a*, *b*].
5. If f(x) and *g*(*x*) are monotonically (or strictly) increasing (or decreasing) functions on [*a*, *b*], then  is a monotonically (or strictly) increasing function on [*a*, *b*].
6. If one of the two functions f(x), g(x) is strictly (or monotonically) increasing and other a strictly (monotonically) decreasing, then *gof*(*x*) is strictly (monotonically) decreasing on [*a*, *b*].

## How do you find the Minimum and Maximum Values of a Function?

## Maxima and Minima

(1) A function f(x) is said to attain a maximum at x = a if there exists a neighbourhood (a – δ, a + δ) such that f(x) < f(a) for all x ∈ (a – δ, a + δ), x ≠ a
⇒ f(x) – f(a) < 0 for all x ∈ (a – δ, a + δ), x ≠ a
In such a case, *f* (a) is said to be the maximum value of f(x) at x = a.

(2) A function f(x) is said to attain a minimum at x = a if there exists a nbd(a – δ, a + δ) such that f(x) > f(a) for all x ∈ (a – δ, a + δ), x ≠ a
⇒ f(x) – f(a) > 0 for all x ∈ (a – δ, a + δ), x ≠ a

In such a case, f(a) is said to be the minimum value of f(x) at x = a. The points at which a function attains either the maximum values or the minimum values are known as the extreme points or turning points and both maximum and minimum values of f(x) are called extreme or extreme values.

Thus a function attains an extreme value at x = a if f(a) is either  a maximum or a minimum value. Consequently at an extreme point a, f(x) – f(a) keeps the same sign for all values of x in a deleted nbd of a.

## Necessary condition for extreme values

A necessary condition for f(a) to be an extreme value of a function f(x) is that f′(a) = 0, in case it exists.

**Note :**

1. This result states that if the derivative exists, it must be zero at the extreme points. A function may however attain an extreme value at a point without being derivable there at.
   For example, the function f(x) = |x| attains the minimum value at the origin even though it is not differentiable at *x* = 0.
2. This condition is only a necessary condition for the point x = a to be an extreme point. It is not sufficient *i.e.,* f′(a) = 0 does not necessarily imply that x = a is an extreme point. There are functions for which the derivatives vanish at a point but do not have an extreme value there at *e.g*. f(x) = $x^3$ at x = 0 does not attain an extreme value at x = 0 and f′(0) = 0.
3. Geometrically, the above condition means that the tangent to the curve y = f(x) at a point where the ordinate is maximum or minimum is parallel to the *x*-axis.
4. The values of *x* for which f′(x) = 0 are called stationary values or critical values of *x* and the corresponding values of f(x) are called stationary or turning values of f(x).
5. The points where a function attains a maximum (or minimum) are also known as points of local maximum (or local minimum) and the corresponding values of f(x) are called local maximum (or local minimum) values.


## Sufficient criteria for extreme values ($1^{st}$ derivative test)

Let f(x) be a function differentiable at x = a.
Then (a) x = a is a point of local maximum of f(x) if

1. f′(a) = 0 and
2. f′(a) changes sign from positive to negative as *x* passes through *a i.e.,* f′(x) > 0 at every point in the left neighbourhood (a – δ, a) of *a* and f′(x) < 0 at every point in the right neighbourhood (a, a + δ) of *a*.

(b) x = a is a point of local minimum of f(x) if

1. f′(a) = 0 and
2. f′(a) changes sign from negative to positive as *x* passes through *a, i.e*., f′(x) < 0 at every point in the left neighbourhood (a – δ, a) of *a* and f′(x) > 0 at every point in the right neighbourhood (a, a + δ) of *a*.

(c) If f′(a) = 0 but f′(a) does not change sign, that is, has the same sign in the complete neighbourhood of *a*, then *a* is neither a point of local maximum nor a point of local minimum.

**Working rule for determining extreme values of a function f(x)**

**Step I :** Put $y = f(x)$

**Step II :** Find $\frac{dy}{dx}$

**Step III :** Put $\frac{dy}{dx} = 0$ and solve this equation for $x$. Let $x = c_1, c_2, \ldots, c_n$ be values of $x$ obtained by putting $\frac{dy}{dx} = 0$.

$c_1, c_2, \ldots\ldots c_n$ are the stationary values of $x$.

**Step IV :** Consider $x = c_1$.

If dy/dx changes its sign from positive to negative as *x* passes through $c_1$, then the function attains a local maximum at $x = c_1$. If dy/dxchanges its sign from negative to positive as *x* passes through $c_1$, then the function attains a local minimum at $x = c_1$. In case there is no change of sign, then $x = c_1$ is neither a point of local maximum nor a point of local minimum.

## Higher order derivative test

1. Find $f'(x)$ and equate it to zero. Solve $f'(x) = 0$ let its roots are $x = a_1, a_2\ldots$.
2. Find $f''(x)$ and at $x = a_1$;
   1. If $f''(a_1)$ is positive, then f(x) is minimum at $x = a_1$.
   2. If $f''(a_1)$ is negative, then f(x) is maximum at $x = a_1$.
   3. If $f''(a_1) = 0$, go to step 3.
3. If at $x = a_1$, $f''(a_1) = 0$, then find $f'''(x)$. If $f'''(a_1) \neq 0$, then f(x) is neither maximum nor minimum at $x = a_1$.

   If $f'''(a_1) = 0$, then find $f^{iv}(x)$.

   If $f^{iv}(x)$ is $+ve$ (Minimum value)

   $f^{iv}(x)$ is $-ve$ (Maximum value)
4. If at $x = a_1$, $f^{iv}(a_1) = 0$ then find $f^{v}(x)$ and proceed similarly.

## Properties of maxima and minima

1. Maxima and minima occur alternately, that is between two maxima there is one minimum and vice-versa.

2. If $f(x) \to \infty$ as $x \to a$ or $b$ and $f'(x) = 0$ only for one value of x (say c) between a and b, then f(c) is necessarily the minimum and the least value.
   If $f(x) \to -\infty$ as $x \to a$ or $b$, then f(c) is necessarily the maximum and the greatest value.

## Greatest and least values of a function defined on an interval [a, b]

By maximum (or minimum) or local maximum (or local minimum) value of a function f(x) at a point c ∈ [a, b] we mean the greatest (or the least) value in the immediate neighbourhood of x = c. It does not mean the greatest or absolute maximum (or the least or absolute minimum) of f(x) in the interval [a, b].

A function may have a number of local maxima or local minima in a given interval and even a local minimum may be greater than a relative maximum.

Thus a local maximum value may not be the greatest (absolute maximum) and a local minimum value may not be the least (absolute minimum) value of the function in any given interval.

However, if a function f(x) is continuous on a closed interval [a, b], then it attains the absolute maximum (absolute minimum) at critical points, or at the end points of the interval [a, b]. Thus, to find the absolute maximum (absolute minimum) value of the function, we choose the largest and smallest amongst the numbers f(a), $f(c_1)$, $f(c_2)$,….$f(c_n)$, f(b) where $x = c_1, c_2, \ldots, c_n$ are the critical points.

## Maxima and Minima Problems with Solutions

**1.**

The value of the function $(x-1)(x-2)^2$ at its maxima is

(a) 1 (b) 2

(c) 0 (d) $\frac{4}{27}$

**Solution:**

(d) Given $f(x) = (x-1)(x-2)^2$

$f(x) = (x-1)(x^2+4-4x)$; $f(x) = (x^3-5x^2+8x-4)$

Now $f'(x) = 3x^2-10x+8$, $f'(x) = 0$

$\Rightarrow 3x^2-10x+8=0 \Rightarrow (3x-4)(x-2)=0 \Rightarrow x = \frac{4}{3}, 2$

Now $f''(x) = 6x-10$

$f''(4/3) = 6\times 4/3 - 10 < 0$

$f''(2) = 12-10 > 0$

Hence at $x = \frac{4}{3}$ the function will occupy maximum value.

$\therefore$ Maximum value $= f(4/3) = 4/27$.

**2.**

The minimum value of $|x| + |x+\frac{1}{2}| + |x-3| + |x-\frac{5}{2}|$ is

(a) 0 (b) 2

(c) 4 (d) 6

**Solution:**

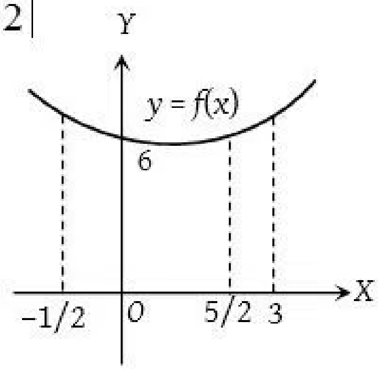
(d) $f(x) = |x| + \left|x + \frac{1}{2}\right| + |x-3| + \left|x - \frac{5}{2}\right|$

$= -4x + 5$, for $x \le -\frac{1}{2}$

$= -2x + 6$, for $-\frac{1}{2} \le x \le 0$

$= 6$, for $0 \le x \le \frac{5}{2}$

$= 2x + 1$, for $\frac{5}{2} \le x \le 3$ $= 4x - 5$, for $x \ge 3$




From the graph, minimum value of the function is 6.

**3.**

Local maximum value of the function $\frac{\log x}{x}$ is

(a) $e$ (b) 1

(c) $\frac{1}{e}$ (d) $2e$

**Solution:**

(c) Let $f(x) = \frac{\log x}{x} \Rightarrow f'(x) = \frac{1}{x^2} - \frac{\log x}{x^2}$

For maximum or minimum value of $f(x)$, $f'(x) = 0$

$\Rightarrow f'(x) = \frac{1-\log_e x}{x^2} = 0$ or $\frac{1-\log_e x}{x^2} = 0$

$\therefore \log_e x = 1$ or $x = e$, which lie in $(0, \infty)$.

For $x = e$, $\frac{d^2y}{dx^2} = -\frac{1}{e^3}$, which is $-ve$.

Hence $y$ is maximum at $x = e$ and its maximum value $= \frac{\log e}{e} = \frac{1}{e}$.

**4.**

The function $\sin x(1 + \cos x)$ at $x = \frac{\pi}{3}$, is

(a) Maximum

(b) Minimum

(c) Neither maximum nor minimum

(d) None of these

**Solution:**

(a) Let $f(x) = \sin x(1+\cos x)$

$\Rightarrow f'(x) = \cos 2x + \cos x$

and $f''(x) = -2\sin 2x - \sin x = -(2\sin 2x + \sin x)$

For maximum or minimum value of $f(x)$, $f'(x) = 0$

$\cos 2x + \cos x = 0 \Rightarrow \cos x = -\cos 2x$

$\Rightarrow \cos x = \cos(\pi \pm 2x)$

$\therefore x = \pi \pm 2x$ or $x = \frac{\pi}{3}, -\pi$

Now $f''\left(\frac{\pi}{3}\right) = -2\sin\frac{2\pi}{3} - \sin\frac{\pi}{3} = -2\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2} = -ve$

Hence $f(x)$ is maximum at $x = \frac{\pi}{3}$.

**5.**

$x^x$ has a stationary point at

(a) $x = e$ (b) $x = \frac{1}{e}$

(c) $x = 1$ (d) $x = \sqrt{e}$

**OR**

When $x$ is positive, the minimum value of $x^x$ is

(a) $e^{-1}$ (b) $e^{-1/e}$

(c) $e^{1/e}$ (d) $e^e$

**Solution:**

(b) Let $y = x^x \Rightarrow \log y = x.\log x$, $(x > 0)$

Differentiating $\frac{dy}{dx} = x^x(1+\log x)$; $\therefore \frac{dy}{dx} = 0$

$\Rightarrow \log x = -1 \Rightarrow x = e^{-1} = \frac{1}{e}$

$\therefore$ Stationary point is $x = \frac{1}{e}$

$\frac{d^2y}{dx^2} = x^x(1+\log x)^2 + x^x.\frac{1}{x}$

When $x = \frac{1}{e}$, $\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{(1/e)-1} > 0$

Therefore $y$ is minimum at $x = \frac{1}{e}$ and minimum value

$= \left(\frac{1}{e}\right)^{1/e} = e^{-1/e}$.

**6.**

The largest term in the sequence $a_n = \dfrac{n^2}{n^3 + 200}$ is given by

(a) $\dfrac{529}{49}$ (b) $\dfrac{8}{89}$

(c) $\dfrac{49}{543}$ (d) None of these

**Solution:**

(c) Consider the function

$$f(x) = \frac{x^2}{(x^3 + 200)} \qquad \text{.....(i)}$$

$$f'(x) = x\frac{(400 - x^3)}{(x^3 + 200)^2} = 0$$

When $x = (400)^{1/3}$, $(\because x \neq 0)$

$x = (400)^{1/3} - h \Rightarrow f'(x) > 0$

$x = (400)^{1/3} + h \Rightarrow f'(x) < 0$

$\therefore f(x)$ has maxima at $x = (400)^{1/3}$

Since $7 < (400)^{1/3} < 8$, either $a_7$ or $a_8$ is the greatest term of the sequence.

$\because a_7 = \dfrac{49}{543}$ and $a_8 = \dfrac{8}{89}$ and $\dfrac{49}{543} > \dfrac{8}{89}$

$\therefore a_7 = \dfrac{49}{543}$ is the greatest term.

**7.**

The minimum value of the expression $7 - 20x + 11x^2$ is

(a) $\dfrac{177}{11}$ (b) $-\dfrac{177}{11}$

(c) $-\dfrac{23}{11}$ (d) $\dfrac{23}{11}$

**Solution:**

(c) Given $f(x) = 7 - 20x + 11x^2$

$f'(x) = -20 + 22x$

Put $f'(x) = 0$ *i.e.,* $-20 + 22x = 0$

$\Rightarrow x = 10/11$ and $f''(x) = 22 > 0$

Hence at $x = 10/11$, $f(x)$ will have minimum value,

$$\therefore f\left(\frac{10}{11}\right) = 7 - \frac{200}{11} + \frac{100 \times 11}{121} = 7 - \frac{200}{11} + \frac{100}{11} = -\frac{23}{11}.$$

**8.**

If $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$, for every real number *x*, then the minimum value of *f*

(a) Does not exist because *f* is unbounded

(b) Is not attained even though *f* is bounded

(c) Is equal to 1

(d) Is equal to –1

**Solution:**

(d) $f(x) = \dfrac{x^2 - 1}{x^2 + 1} = \dfrac{x^2 + 1 - 2}{x^2 + 1} = 1 - \dfrac{2}{x^2 + 1}$

$\therefore f(x) < 1 \forall x$ and $\geq -1$ as $\dfrac{2}{x^2 + 1} \leq 2$

$\therefore -1 \leq f(x) < 1$

Hence $f(x)$ has minimum value –1 and also there is no maximum value.

**Aliter :** $f'(x) = \dfrac{(x^2 + 1)2x - (x^2 - 1)2x}{(x^2 + 1)^2} = \dfrac{4x}{(x^2 + 1)^2}$

$f'(x) = 0 \Rightarrow x = 0$

$$f''(x) = \frac{(x^2 + 1)^2 4 - 4x.2(x^2 + 1)2x}{(x^2 + 1)^4}$$

$$= \frac{(x^2 + 1)4 - 16x(x)}{(x^2 + 1)^3} = \frac{-12x^2 + 4}{(x^2 + 1)^3}$$

$\therefore f''(0) > 0$

$\therefore$ There is only one critical point having minima.

Hence $f(x)$ has least value at $x = 0$.

$$f_{\min} = f(0) = \frac{-1}{1} = -1.$$

**9.**

The maximum value of $\sin x\ (1+\cos x)$ will be at the

(a) $x=\frac{\pi}{2}$ (b) $x=\frac{\pi}{6}$

(c) $x=\frac{\pi}{3}$ (d) $x=\pi$

**Solution:**

(c) $y=\sin x(1+\cos x)=\sin x+\frac{1}{2}\sin 2x$

$\therefore\ \frac{dy}{dx}=\cos x+\cos 2x$ and $\frac{d^2y}{dx^2}=-\sin x-2\sin 2x$

On putting $\frac{dy}{dx}=0$ , $\cos x+\cos 2x=0$

$\Rightarrow\ \cos x=-\cos 2x=\cos(\pi-2x)\Rightarrow\ x=\pi-2x$

$\therefore\ x=\frac{\pi}{3};\ \therefore\ \left(\frac{d^2y}{dx^2}\right)_{x=\pi/3}=-\sin\left(\frac{1}{3}\pi\right)-2\sin\left(\frac{2}{3}\pi\right)$

$=\frac{-\sqrt{3}}{2}-2.\frac{\sqrt{3}}{2}=\frac{-3\sqrt{3}}{2}$, which is negative.

$\therefore$ At $x=\frac{\pi}{3}$ the function is maximum.

**10.**

If $x$ is real, then greatest and least values of $\frac{x^2-x+1}{x^2+x+1}$ are

(a) $3,-\frac{1}{2}$ (b) $3,\frac{1}{3}$

(c) $-3,-\frac{1}{3}$ (d) None of these

**Solution:**

(b) Let $y = \frac{x^2 - x + 1}{x^2 + x + 1}$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2 + x + 1)(2x - 1) - (x^2 - x + 1)(2x + 1)}{(x^2 + x + 1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x^2 - 2}{(x^2 + x + 1)^2} = 0 \Rightarrow 2x^2 - 2 = 0 \Rightarrow x = -1, +1$$

$$\frac{d^2y}{dx^2} = \frac{4(-x^3 + 3x + 1)}{x^2 + x + 1}$$

At $x = -1$, $\frac{d^2y}{dx^2} < 0$, the function will occupy maximum value, $\therefore f(-1) = 3$ and at $x = 1$ $\frac{d^2y}{dx^2} > 0$, the function will occupy minimum value $\therefore f(1) = \frac{1}{3}$.

**11.**

The minimum value of $\frac{\log x}{x}$ in the interval $[2, \infty)$ is

(a) $\frac{\log 2}{2}$

(b) Zero

(c) $\frac{1}{e}$

(d) Does not exist

**Solution:**

(d) Let $y = \frac{\log x}{x} \Rightarrow \frac{dy}{dx} = \frac{x.\frac{1}{x} - \log x}{x^2} = \frac{1 - \log x}{x^2}$

Put $\frac{dy}{dx} = 0 \Rightarrow \frac{1 - \log x}{x^2} = 0$

$\Rightarrow 1 - \log x = 0 \Rightarrow x = e$ and $\frac{d^2y}{dx^2} = \frac{-3x + 2x \log x}{x^4}$

At $x = e$, $\frac{d^2y}{dx^2} = \frac{1}{-e^3} < 0$

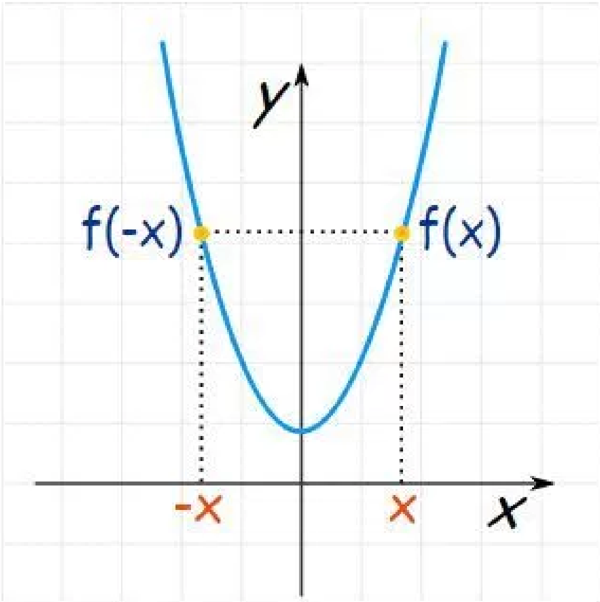
$\therefore$ In $[2, \infty)$ the function $\frac{\log x}{x}$ will be maximum and minimum value does not exist.

## Even and Odd Functions

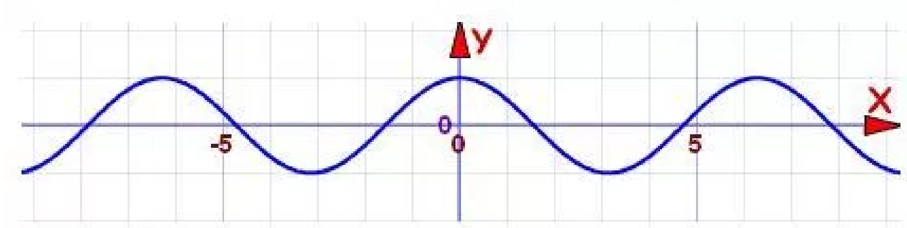
**(1) Even function:** If we put (–x) in place of x in the given function and if f(–x) = f(x), ∀ x ∈ domain then function f(x) is called even function. e.g. $f(x) = e^x + e^{-x}$ , $f(x) = x^2$, f(x) = x sin x, f(x) = cos x, $f(x) = x^2 \cos x$ all are even functions.

**Examples:**

1. $f(x) = x^2 + 1$




2. This is the curve $f(x) = x^2+1$
3. f(x) = cos x

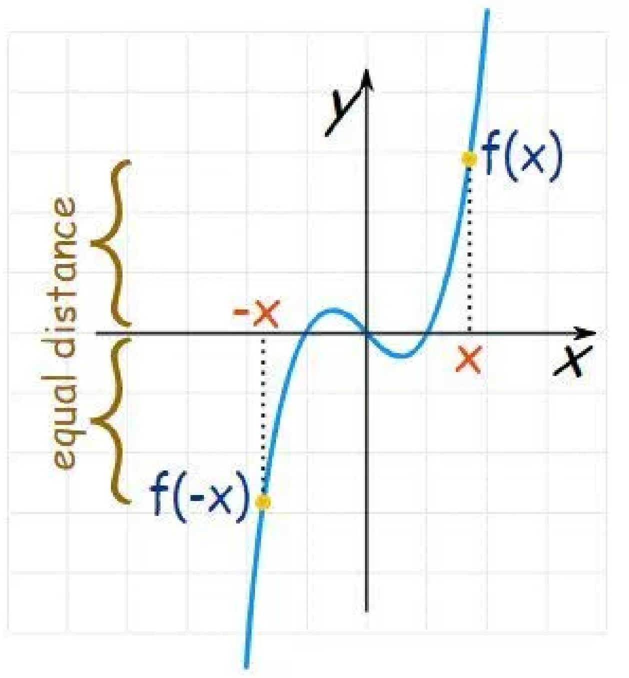



Cosine function: f(x) = cos(x)
It is an even function

**(2) Odd function:** If we put (–x) in place of x in the given function and if f(–x) = –f(x), ∀ x ∈ domain then f(x) is called odd function. e.g. $f(x) = e^x – e^{-x}$ , $f(x) = x^3$, f(x) = sin x, f(x) = x cos x, $f(x) = x^2 \sin x$ all are odd functions.

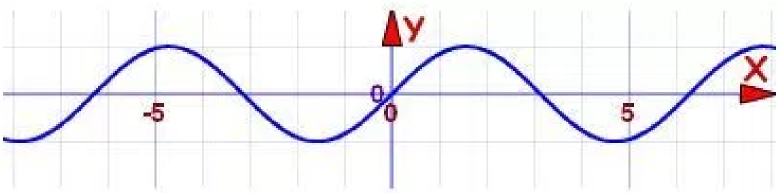
**Examples:**

1. $f(x) = x^3 - x$




This is the curve $f(x) = x^3 - x$

2. $f(x) = \sin x$



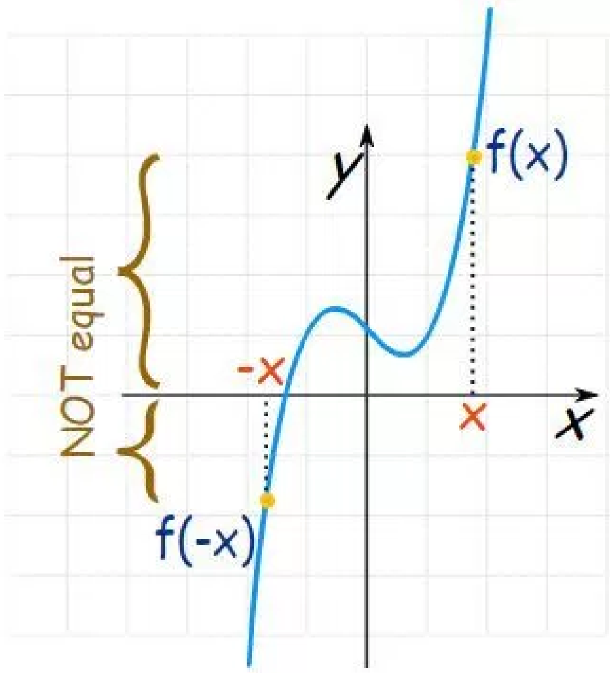

Sine function: $f(x) = \sin(x)$

It is an odd function

## Properties of even and odd function

- The graph of even function is always symmetric with respect to y-axis. The graph of odd function is always symmetric with respect to origin.
- The product of two even functions is an even function.
- The sum and difference of two even functions is an even function.
- The sum and difference of two odd functions is an odd function.
- The product of two odd functions is an even function.
- The product of an even and an odd function is an odd function. It is not essential that every function is even or odd. It is possible to have some functions which are neither even nor

- odd function. e.g. $f(x) = x^2 + x^3$, $f(x) = \log_e x$, $f(x) = e^x$.




This is the curve $f(x) = x^3 - x + \mathbf{1}$

It is **not an odd function**, and it is **not an even function** either.
It is neither odd nor even!

- The sum of even and odd function is neither even nor odd function.
- Zero function $f(x) = 0$ is the only function which is even and odd both.

## Periodic function

A function is said to be periodic function if its each value is repeated after a definite interval. So a function $f(x)$ will be periodic if a positive real number T exist such that, $f(x + T) = f(x)$, $\forall\ x \in$ domain. Here the least positive value of T is called the period of the function.

Clearly $f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = \ldots\ldots$ *e.g.*, sin x, cos x, tan x are periodic functions with period 2π, 2π and π respectively.

**Some standard results on periodic functions**

| Functions | Periods |
|---|---|
| $\sin^n x,\ \cos^n x,\ \sec^n x,\ \text{cosec}^n x$ | $\begin{cases} \pi; & \text{if } n \text{ is even} \\ 2\pi; & \text{if } n \text{ is odd or fraction} \end{cases}$ |
| $\tan^n x, \cot^n x$ | $\pi; n$ is even or odd. |
| $\sin(ax+b), \cos(ax+b)$<br>$\sin(ax+b), \cos(ax+b)$ | $2\pi / a$ |
| $\tan(ax+b), \cot(ax+b)$ | $\pi / a$ |
| $\lvert\sin x\rvert, \lvert\cos x\rvert, \lvert\tan x\rvert,$<br>$\lvert\cot x\rvert, \lvert\sec x\rvert, \lvert\text{cosec } x\rvert$ | $\pi$ |
| $\lvert\sin(ax+b)\rvert, \lvert\cos(ax+b)\rvert,$<br>$\sec\lvert ax+b\rvert, \lvert\text{cosec}(ax+b)\rvert$<br>$\lvert\tan(ax+b)\rvert, \lvert\cot(ax+b)\rvert$ | $\pi / a$ |
| $x - [x]$ | 1 |
| Algebraic functions *e.g.*,<br>$\sqrt{x}, x^2, x^3 + 5, \ldots.$etc | Period does not exist |

## Composite function

If f : A ⟶ B and g : B ⟶ C are two function then the composite function of f and g, gof A ⟶ C will be defined as gof(x) = g[f(x)], ∀ x ∈ A.

**Properties of composition of function:**

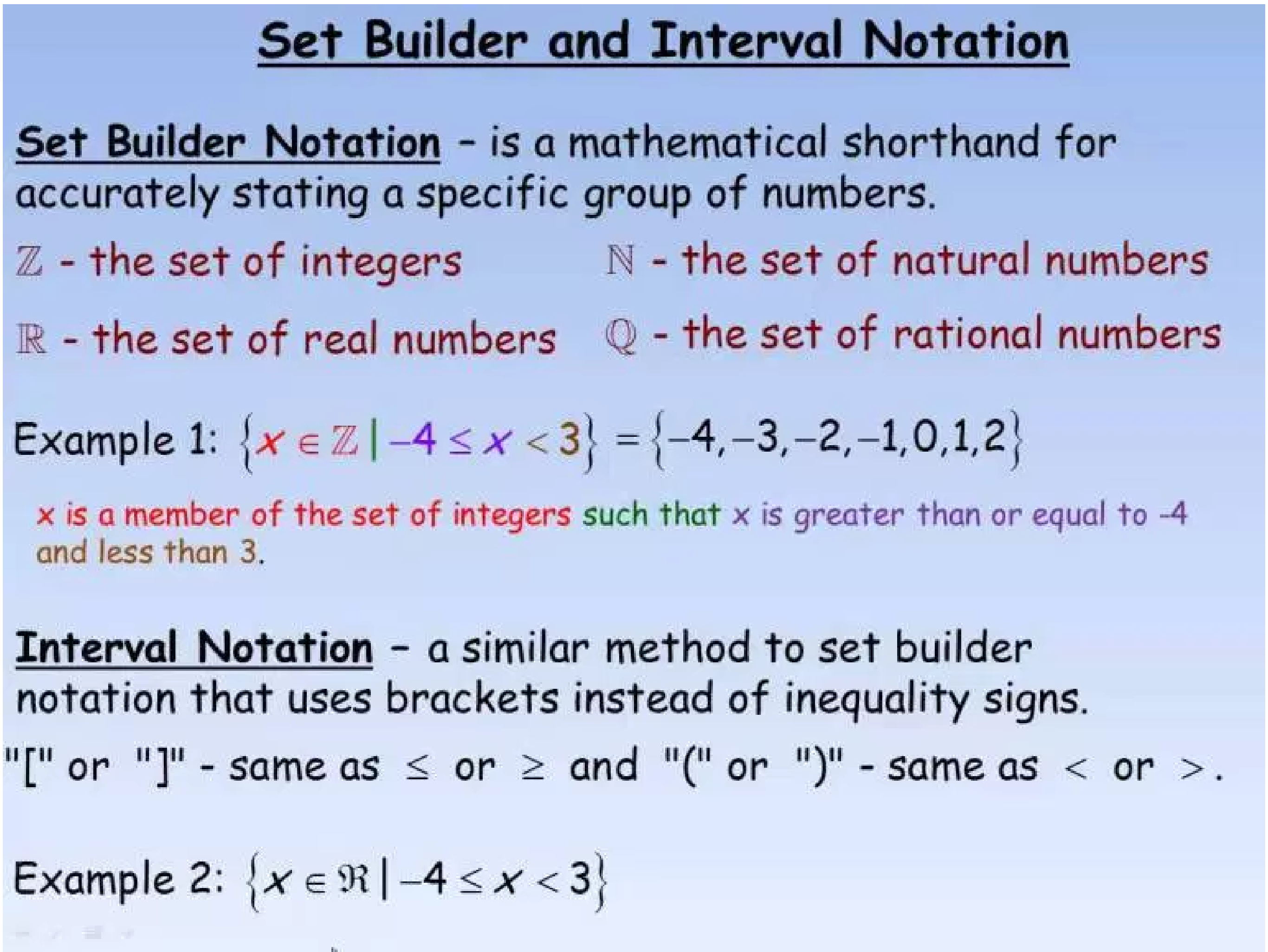
1. f is even, g is even ⇒ fog is even function.
2. f is odd, g is odd ⇒ fog is odd function.
3. f is even, g is odd ⇒ fog is even function.
4. f is odd, g is even ⇒ fog is even function.
5. Composite of functions is not commutative i.e., fog ≠ gof.
6. Composite of functions is associative i.e., (fog)oh = fo(goh)
7. If f : A ⟶ B is bijection and g : B ⟶ A is inverse of f. Then fog = $I_B$ and gof = $I_A$ where, $I_A$ and I are identity functions on the sets A and B respectively.
8. If f : A ⟶ B and g : B ⟶ C are two bijections, then gof A ⟶ C is bijection and $(gof)^{-1} = (f^{-1}og^{-1})$.
9. fog ≠ gof but if, fog = gof then either $f^{-1} = g$ or $g^{-1} = f$ also, (fog)(x) = (gof)(x) = (x).
10. gof(x) is simply the g-image of f(x), where f(x) is f-image of elements x ∈ A.
11. Function gof will exist only when range of f is the subset of domain of g.
12. fog does not exist if range of g is not a subset of domain of f.
13. fog and gof may not be always defined.
14. If both f and g are one-one, then fog and gof are also one-one.
15. If both f and g are onto, then gof is onto.

## Set-builder & Interval Notation

A **set** is a collection of unique elements. Elements in a set do not “repeat”.

## Methods of Describing Sets:

Sets may be described in many ways: by roster, by set-builder notation, by interval notation, by graphing on a number line, and/or by Venn diagrams. For graphing on a number line, see Linear Inequalities. For Venn diagrams, see Working with Sets and Venn Diagrams.



**By roster:** A roster is a list of the elements in a set, separated by commas and surrounded by French curly braces.

| | |
|---|---|
| $\{2,3,4,5,6\}$ | is a roster for the set of integers from 2 to 6, inclusive. |
| $\{1,2,3,4,\ \ldots\}$ | is a roster for the set of positive integers. The three dots indicate that the numbers continue in the same pattern indefinitely. (Those three dots are called an **ellipsis.**) |
| Rosters may be awkward to write for certain sets that contain an infinite number of entries. | |

**By set-builder notation:** Set-builder notation is a mathematical shorthand for precisely stating all numbers of a specific set that possess a specific property.

$\mathbb{R}$ = real numbers; $\mathbb{Z}$ = integer numbers; $\mathbb{N}$ = natural numbers.

| | |
|---|---|
| $\{x \in \mathbb{Z} \mid 2 \leq x \leq 6\}$ | is set-builder notation for the set of integers from 2 to 6, inclusive. $\in$ = "is an element of" Z = the set of integers \| = the words "such that" The statement is read, "all $x$ that are elements of the set of integers, such that, $x$ is between 2 and 6 inclusive." |
| $\{x \in \mathbb{Z} \mid x > 0\}$ | The statement is read, "all $x$ that are elements of the set of integers, such that, the $x$ values are greater than 0, positive." (The positive integers can also be indicated as the set $Z^+$.) |
| It is also possible to use a colon ( : ), instead of the \| , to represent the words "such that". $\{x \in \mathbb{Z} \mid 2 \leq x \leq 6\}$ is the same as $\{x \in \mathbb{Z} : 2 \leq x \leq 6\}$ | |

**By interval notation:** An **interval** is a connected subset of numbers. **Interval notation** is an alternative to expressing your answer as an inequality. Unless specified otherwise, we will be working with real numbers.

| When using interval notation, the symbol: |
|---|
| ( means "not included" or "open". |
| [ means "included" or "closed". |

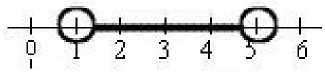
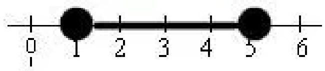
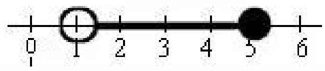
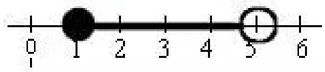
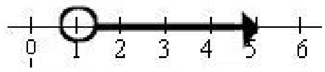
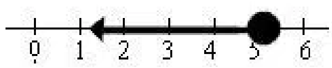
| | |
|---|---|
| $2 \leq x < 6$ | as an inequality. |
| $[2,6)$ | in interval notation. |

**Intervals**

**There are four types of interval:**

1. **Open interval:** Let a and b be two real numbers such that a<b, then the set of all real numbers lying strictly between a and b is called an open interval and is denoted by ]a, b[ or (a, b). Thus, ]a, b[ or (a, b)= { x ∈ R: a < x < b}.
2. **Closed interval:** Let a and b be two real numbers such that a<b, then the set of all real numbers lying between a and b including a and b is called a closed interval and is denoted by [a, b]. Thus, [a, b] = { x ∈ R: a ≤ x ≤ b}.
3. **Open-Closed interval:** It is denoted by ]a, b] or (a, b] and ]a, b] or (a, b] = { x ∈ R: a < x ≤ b}.
4. **Closed-Open interval:** It is denoted by [a, b[ or [a, b) and [a, b[ or [a, b) = { x ∈ R: a ≤ x < b}.

The chart below will show you all of the possible ways of utilizing interval notation.

| **Interval Notation:** (description) | (diagram) |
|---|---|
| Open Interval: $(a, b)$ is interpreted as $a < x < b$ where the endpoints are NOT included. (While this notation resembles an ordered pair, in this context it refers to the interval upon which you are working.) | $(1, 5)$  |
| Closed Interval: $[a, b]$ is interpreted as $a \le x \le b$ where the endpoints are included. | $[1, 5]$  |
| Half-Open Interval: $(a, b]$ is interpreted as $a < x \le b$ where a is not included, but b is included. | $(1, 5]$  |
| Half-Open Interval: $[a, b)$ is interpreted as $a \le x < b$ where a is included, but b is not included. | $[1, 5)$  |
| Non-ending Interval: $(a, \infty)$ is interpreted as $x > a$ where $a$ is not included and infinity is always expressed as being "open" (not included). | $(1, \infty)$  |
| Non-ending Interval: $(-\infty, b]$ is interpreted as $x \le b$ where $b$ is included and again, infinity is always expressed as being "open" (not included). | $(-\infty, 5]$  |

For some intervals it is necessary to use combinations of interval notations to achieve the desired set of numbers. Consider how you would express the interval **"all numbers except 13".**

| | |
|---|---|
| As an inequality: | $x < 13$ or $x > 13$ |
| In interval notation: | $(-\infty, 13) \cup (13, \infty)$ |
| **Notice that the word "or" has been replaced with the symbol "U", which stands for "union".** | |

Consider expressing in interval notation, the set of numbers which contains all numbers less than 0 and also all numbers greater than 2 but less than or equal to 10.

| | |
|---|---|
| As an inequality: | $x < 0$ or $2 < x \le 10$ |
| In interval notation: | $(-\infty, 0) \cup (2, 10]$ |

As you have seen, there are many ways of representing the same interval of values. These ways may include word descriptions or mathematical symbols.

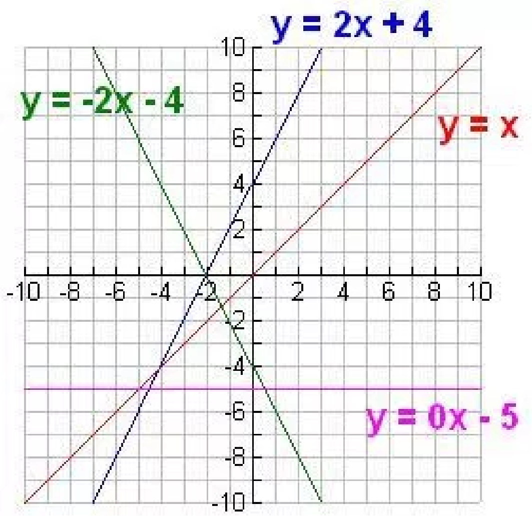
| The following statements and symbols ALL represent the same interval: | |
|---|---|
| **WORDS:** | **SYMBOLS:** |
| "all numbers between positive one and positive five, including the one and the five." | $1 \leq x \leq 5$ |
| "x is less than or equal to 5 and greater than or equal to 1" | $\{x \in \mathbb{R} \mid 1 \leq x \leq 5\}$ |
| "x is between 1 and 5, inclusive" | $[1,5]$ |

## Graphing Functions and Examining Coefficients

## Behavior of Specific Functions

### Linear Functions:

Linear functions are straight lines. Form being examined is $y = mx + b$.

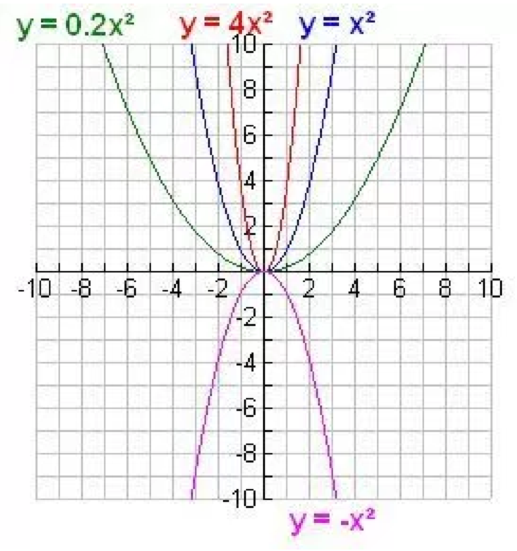



**Some Observations:**

1. If the slope, m, is positive, the line slants uphill. As the slope gets larger, the uphill slant of the line gets steeper. As the slope gets extremely large (a very big number), the line becomes nearly vertical. If the line is vertical, the slope is undefined (because it has no horizontal change).
2. As the slope gets smaller (closer to zero), the line loses steepness and starts to flatten. If the slope is zero, the line is horizontal (flat),
3. If the slope is negative the line slants downhill. As the slope decreases (remember -2 is > -3), the downhill slant of the line gets steeper.

### Quadratic Functions:

Quadratic functions are parabolas.
Form being examined is $y = ax^2$.

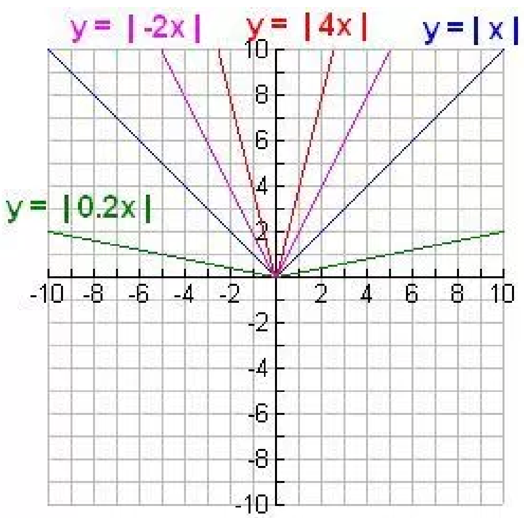



**Some Observations:**

1. If the coefficient of $x^2$ gets larger, the parabola becomes thinner (narrower), closer to its line of symmetry.
2. If the coefficient of $x^2$ gets smaller, the parabola becomes thicker (wider), further from its line of symmetry.
3. If the coefficient of $x^2$ is negative, the parabola opens downward.

## Absolute Value Functions:

The graph of the absolute value function takes the shape of a V.
Form being examined is $y = | x |$.



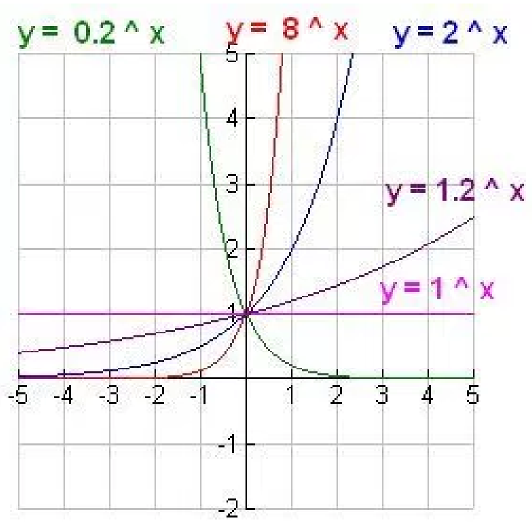

Notice that the graphs of these absolute value functions are on or above the x-axis. Absolute value always yields answers which are positive or zero.

**Some Observations:**

1. As the coefficient of x gets larger, the graph becomes thinner, closer to its line of symmetry.
2. As the coefficient of x gets smaller, the graph becomes thicker, further from its line of symmetry.
3. If the coefficient of x is negative, the graph is the same as if that coefficient were positive. It is acted upon by the absolute value property.

## Exponential Functions:

Exponential functions have the variable x as an exponent.
Form being examined is $y = a^x$.



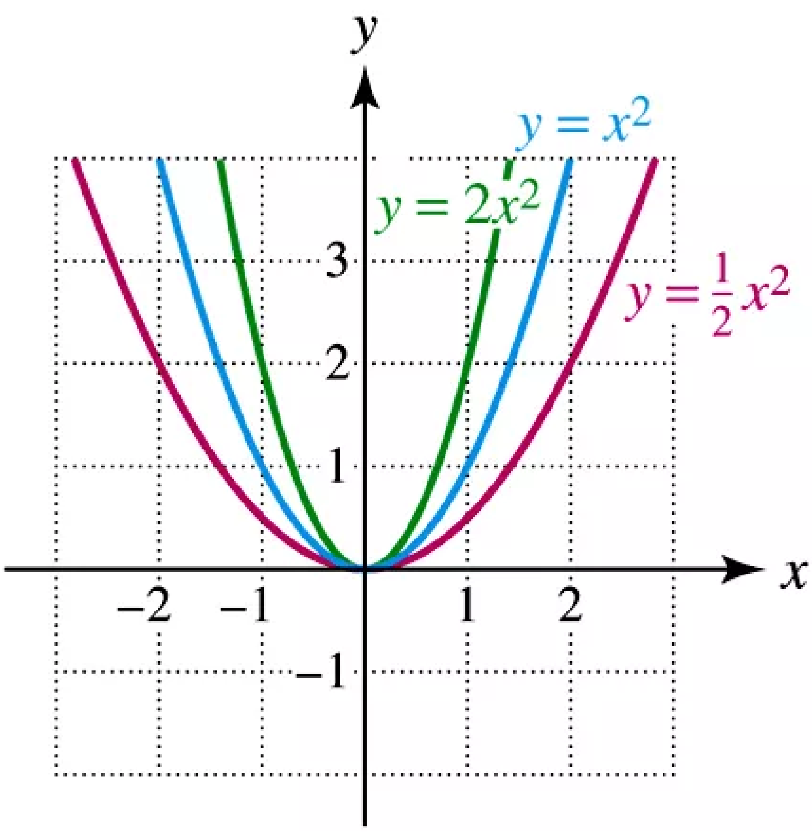

Notice that all of the exponential graphs pass through the point (0,1). This occurs because values raised to the zero power equal 1.

**Some Observations:**

1. As the base value ( a ) gets larger, the graph becomes steeper faster – it appears to stretch upward more quickly.
2. As the base value ( a ) gets closer to 1, the graph flattens. If the base were to become one, the graph would be a horizontal (flat) straight line (not an exponential graph).
3. If the base value ( a ) is between 0 and 1, the graph appears to have reflected itself over the y-axis. The graph has just turned from exponential growth to exponential decay.

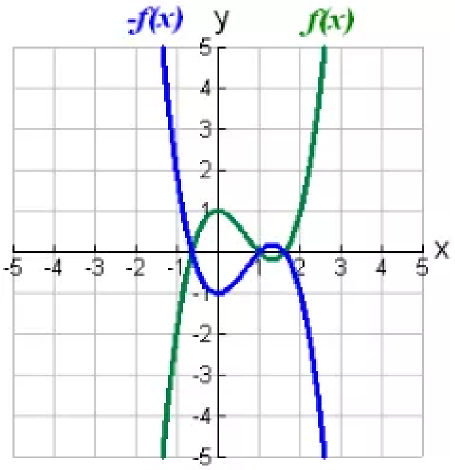
## Function Transformations

There are numerous ways to apply transformations to functions to create new functions.




Let's look at some of the possibilities. Remember to utilize your graphing calculator to compare the graphs of your functions and their transformations.

**Reflections and Functions: Examining -f (x) and f (-x)**
**Reflection over the x-axis**
-f (x) reflects f (x) over the x-axis.

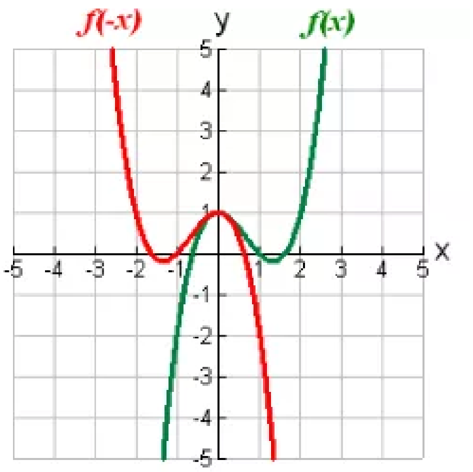



A reflection is a mirror image. Placing the edge of a mirror on the x-axis will form a reflection in the x-axis. This can also be thought of as "folding" over the x-axis.

If the original (parent) function is y = f (x), the reflection over the x-axis is function -f (x).

**Reflection over the y-axis**

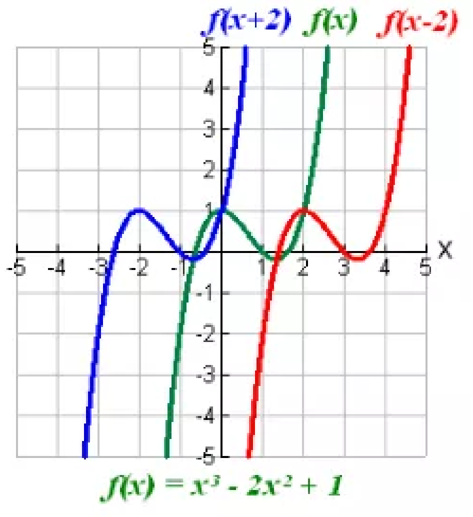
f (-x) reflects f (x) over the y-axis.




Placing the edge of a mirror on the y-axis will form a reflection in the y-axis. This can also be thought of as "folding" over the y-axis.
If the original (parent) function is y = f (x), the reflection over the y-axis is function f (-x).

**Translations and Functions: Examining f (x + a) and f (x) + a**

**Slide to the right or left**
f (x + a) translates f (x) horizontally

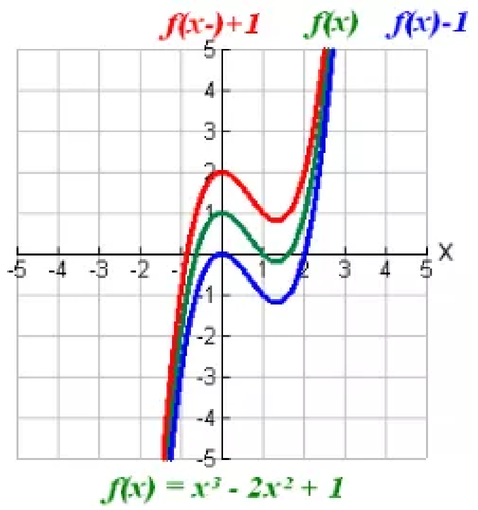



If the original (parent) function is y = f (x), the translation (sliding) of the function horizontally to the left or right is given by the function f (x – a).
if a > 0, the graph translates (slides) to the right.
if a < 0, the graph translates (slides) to the left.

Remember that you are "subtracting" the value of a from x. Thus f (x + 2) is really f (x – (-2)) and the graph moves to the left.

**Slide upward or downward**

f (x)+ a translates f (x) vertically
If the original (parent) function is y = f (x), the translation (sliding) of the function vertically upward or downward is the function f (x) + a.



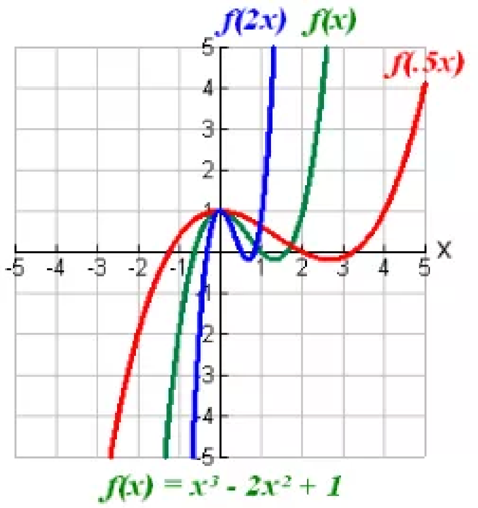

if a > 0, the graph translates (slides) upward.
if a < 0, the graph translates (slides) downward.
Remember that you are adding the value of a to the y-values of the function.

**Stretch or Compress Functions: Examining f (ax) and a f (x)**
**Horizontal Stretch or Compress**

f (ax) stretches/compresses f (x) horizontally

A horizontal stretching is the stretching of the graph away from the y-axis.
A horizontal compression is the squeezing of the graph towards the y-axis.
If the original (parent) function is y = f (x), the horizontal stretching or compressing of the function is the function f (ax).

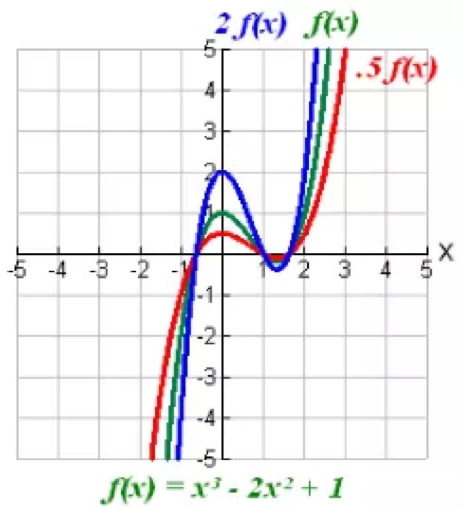
if 0 < a < 1 (a fraction), the graph is stretched horizontally by a factor of a units.




if a > 1, the graph is compressed horizontally by a factor of a units.
if a should be negative, the horizontal compression or horizontal stretching of the graph is followed by a reflection of the graph across the y-axis.

**Vertical Stretch or Compress**
a f (x) stretches/compresses f (x) vertically

A vertical stretching is the stretching of the graph away from the x-axis.
A vertical compression is the squeezing of the graph towards the x-axis.
If the original (parent) function is y = f (x), the vertical stretching or compressing of the function is the function a f(x).




if 0 < a < 1 (a fraction), the graph is compressed vertically by a factor of a units.
if a > 1, the graph is stretched vertically by a factor of a units.

If a should be negative, then the vertical compression or vertical stretching of the graph is followed by a reflection across the x-axis.

## Composition of Functions (f o g)(x)

The term "composition of functions" (or "composite function") refers to the combining of functions in a manner where the output from one function becomes the input for the next function.

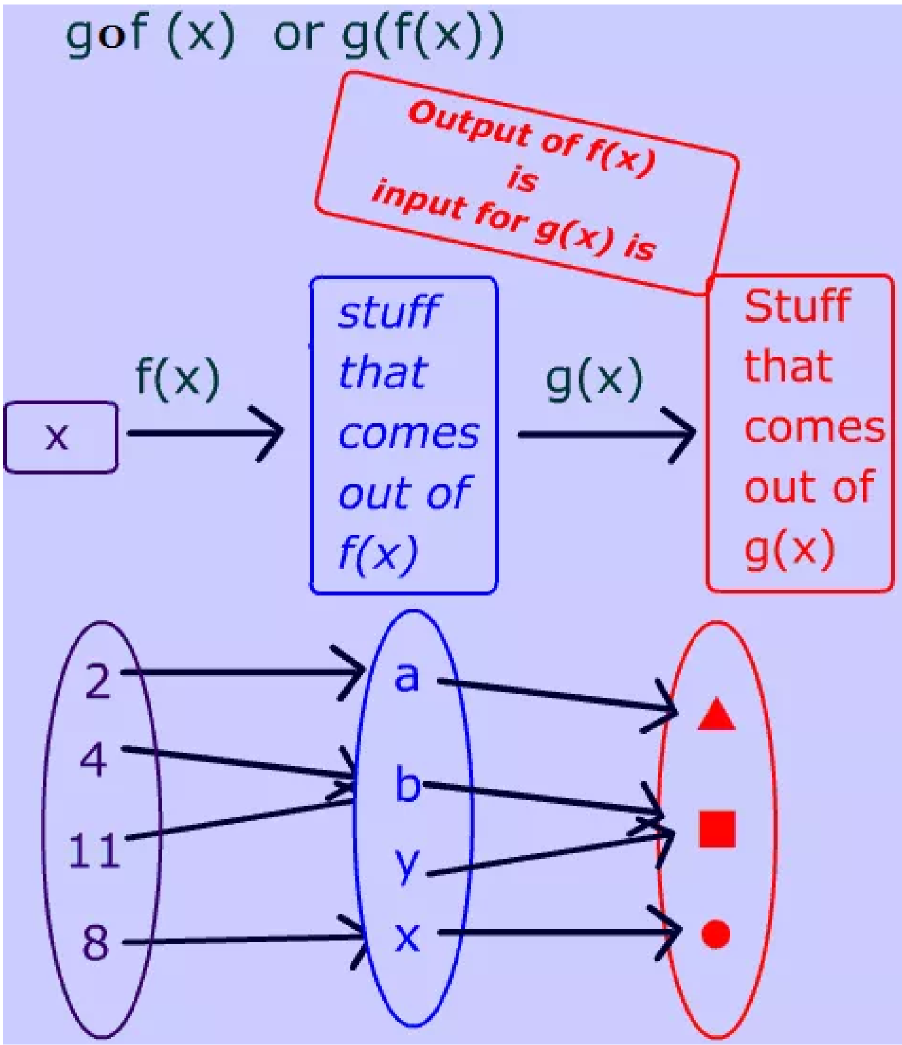
## Composition of Functions and Domain

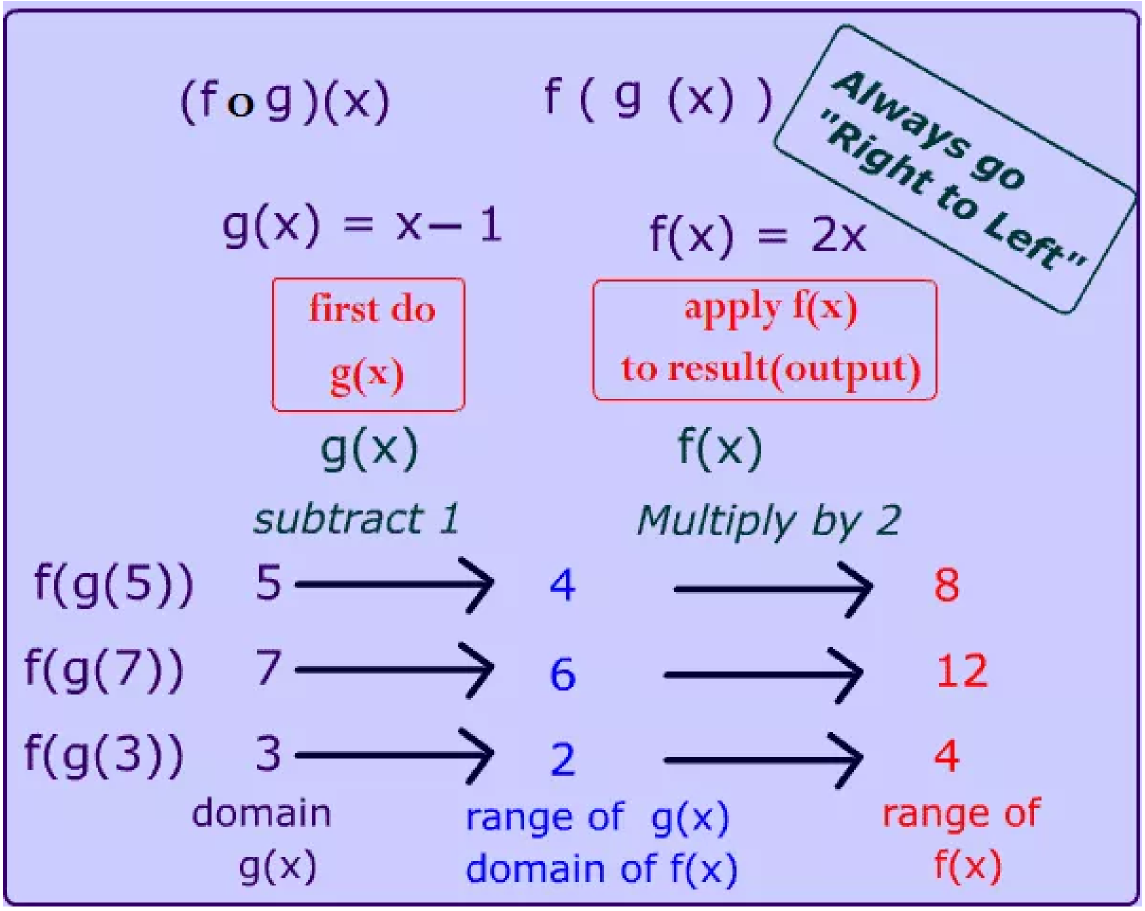
If $f$ and $g$ are functions, then the **composite function**, or **composition**, of $g$ and $f$ is defined by $(g \circ f)(x) = g(f(x))$.

The **domain of** $g \circ f$ is the set of all numbers $x$ in the domain of $f$ such that $f(x)$ is in the domain of $g$.

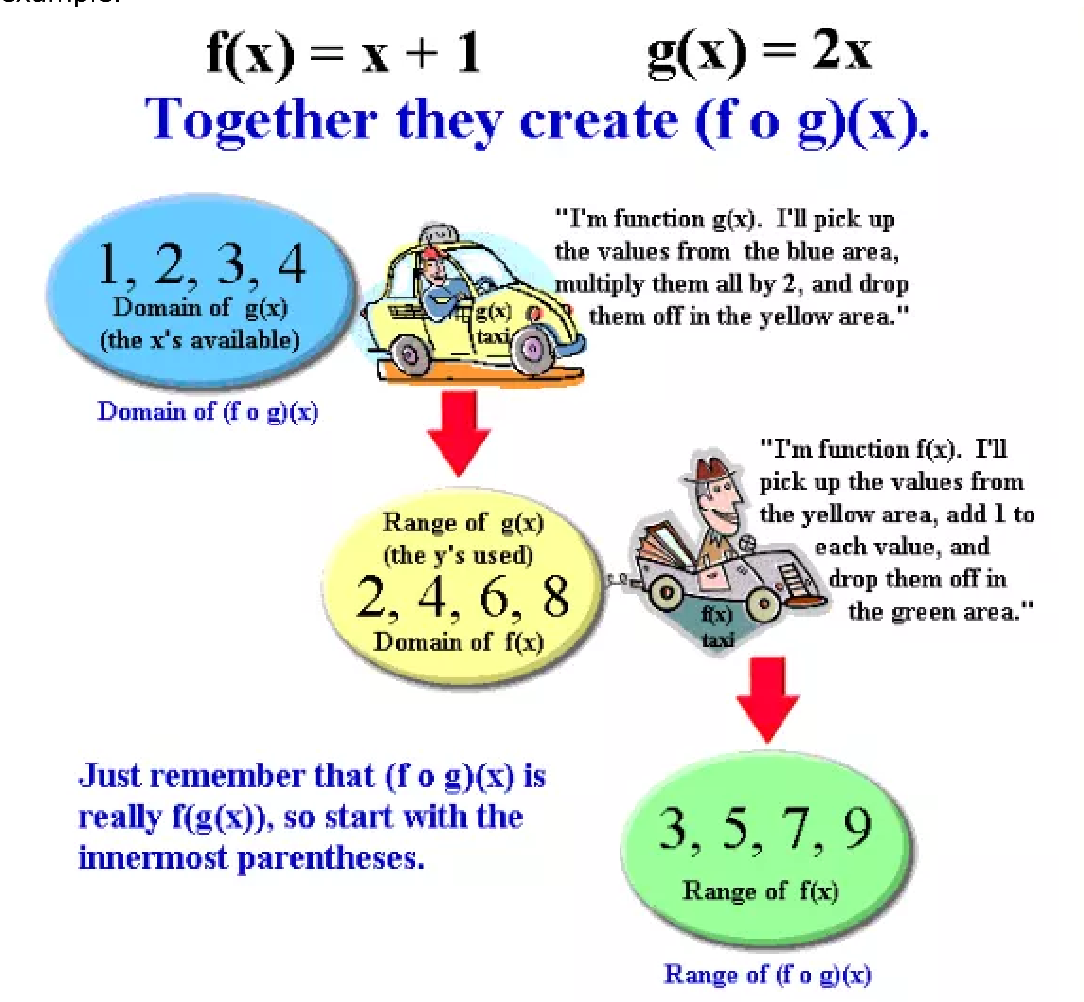
In math terms, the range (the y-value answers) of one function becomes the domain (the x-values) of the next function.

The notation used for composition is:
(f o g)(x) = f(g(x)) and is read "f composed with g of x" or "f of g of x".






Notice how the letters stay in the same order in each expression for the composition.
f (g(x)) clearly tells you to start with function g (innermost parentheses are done first).
Composition of functions can be thought of as a series of taxicab rides for your values.
The example below shows functions f and g working together to create the composition .
Note: The starting domain for function g is being limited to the four values 1, 2, 3 and 4 for this example.



In the example above, you can see what is happening to the individual elements throughout the composition. Now, suppose that we wish to write our composition as an algebraic expression.

**Examples** :

**1.** Given the functions $f(x) = 5x$ and $g(x) = x^2 + 1$, find **a.)** $(f \circ g)(x)$ and **b.)** $(g \circ f)(x)$

Answer: **a.)** $(f \circ g)(x) = f(g(x)) = f(x^2 + 1) = 5(x^2 + 1) = 5x^2 + 5$

**b.)** $(g \circ f)(x) = g(f(x)) = g(5x) = (5x)^2 + 1 = 25x^2 + 1$

Notice that $(g \circ f)(x)$ and $(f \circ g)(x)$ do not necessarily yield the same answer. Composition of functions is not commutative.

**2.** Given the functions $p(x) = x + 2$ and $h(x) = x^2$, find **a.)** $(h \circ p)(3)$ and **b.)** $(h \circ p)(x)$

Answer: **a.)** $(h \circ p)(3) = h(p(3))$ where $p(3)$ gives an answer of 5 and $h(5)$ then gives an answer of 25. The answer is 25.

**b.)** $(h \circ p)(x) = h(p(x)) = h(x + 2) = (x + 2)^2 = x^2 + 4x + 4$

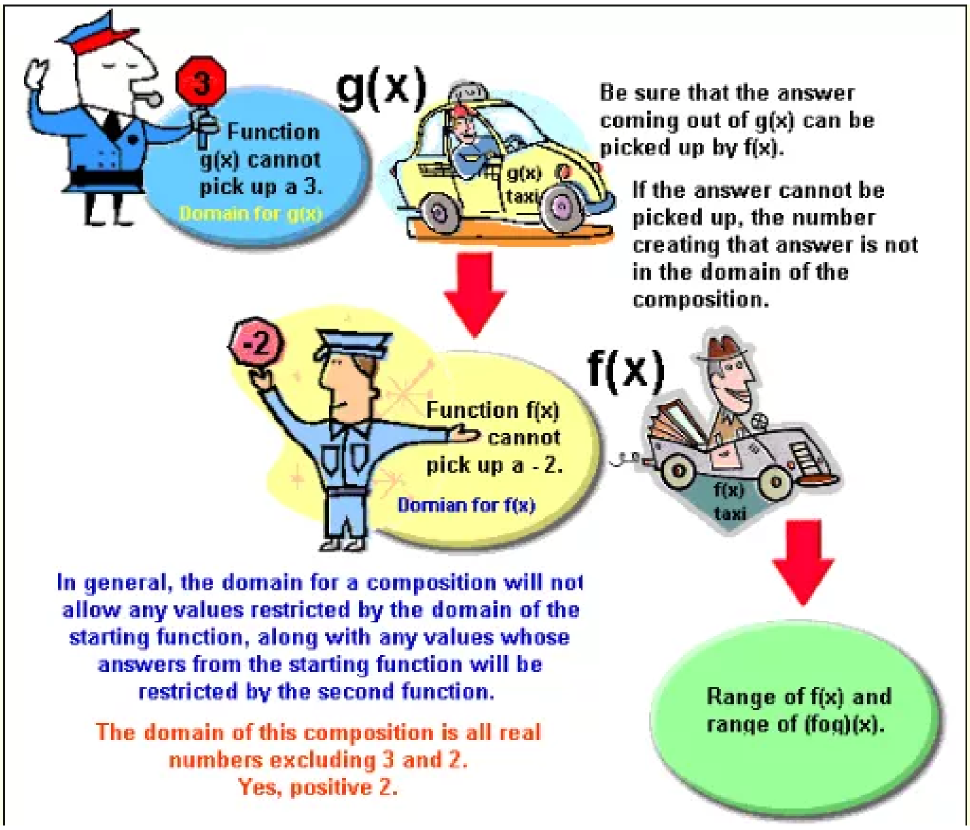
## Finding Domains of Composite Functions

At times, the domain of a composite function can be a bit confusing. Let's examine what happens to values as they "travel" through a composition of functions.

Consider the following example:

$$f(x) = \frac{1}{x+2} \quad \text{and} \quad g(x) = \frac{x}{x-3}$$

What is the domain of $(f \circ g)(x)$?



In this problem, function cannot pick up the value x = 3, and function cannot pick up the value x = -2.

The domain of will be the values from the domain of g(x) which can "move through" to the end of the composition. This means that
the answers created by these values from function must be "picked up" by function f(x).

Let's follow this process algebraically:

**1**. Function cannot pick up the value 3. Consequently, the composition also cannot pick up the value 3.

**2**. The answers coming out of function come out in the form x/x-3. Since function f(x) cannot pick up -2, we must lookout for any values of x that cause
x/x-3 =-2 since these values create an answer that cannot progress through the composition (cannot be picked up by function f(x) ).

**3**. When does x-x-3= -2 ? Solve algebraically ....

$$\frac{x}{x-3} = -2$$
$$x = -2x + 6$$
$$3x = 6$$
$$x = 2$$

**4**. The domain of will be all real numbers with the exclusion of 3 and 2.
(Notice that one of the excluded values is 2, not -2. The value x = -2 makes it through the composition very nicely because its answer from function g(x) is 2/5 which is then picked up by function f(x) )

Is there an easier way to find the domain of a composition?

If you are finding the algebraic expression for the composition of two functions, you can examine your answer to determine any additional restrictions on the domain of the composition. Let's continue with our problem....

$$f(x) = \frac{1}{x+2}, g(x) = \frac{x}{x-3}$$
$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x}{x-3}\right) = \frac{1}{\frac{x}{x-3}+2}$$
$$= \frac{1}{\frac{x+2(x-3)}{x-3}} = \frac{1}{\frac{3x-6}{x-3}} = \frac{x-3}{3x-6} = \frac{x-3}{3(x-2)}$$

The algebraic expression for this composition (the final answer)"SHOWS" us that x = 2 would not be an acceptable domain element since it creates a zero denominator problem in the answer.

Just remember that you must also specify any restrictions on the domain of the starting function. In this problem, x = 3 is not allowed since it is a restriction on g(x).

Answer: The domain of the composition is all real numbers with the exclusion of 3 and 2.

## Definition of Inverse Function

If f : A ⟶ B be a one-one onto (bijection) function, then the mapping $f^{-1}$ : B ⟶ A which associates each element a ∊ A with element such that f(a) = b is called the inverse function of the function f : A ⟶ B.

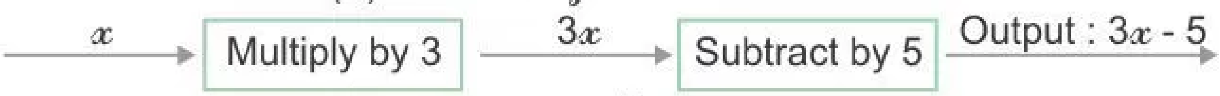
$f^{-1}$ : B ⟶ A, $f^{-1}(b)$ = a ⇒ f(a) = b
In terms of ordered pairs inverse function is defined as $f^{-1}(b, a)$ if (*a, b*) ∊ f.
For the existence of inverse function, it should be one-one and onto.

## Properties of Inverse function:

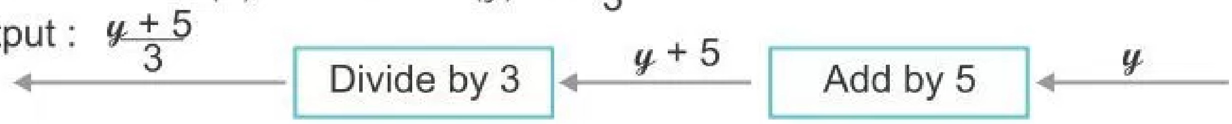
1. Inverse of a bijection is also a bijection function.
2. Inverse of a bijection is unique.
3. $(f^{-1})^{-1} = f$
4. If f and g are two bijections such that (gof) exists then $(gof)^{-1} = f^{-1}og^{-1}$.
5. If f : A ⟶ B is a bijection then $f^{-1}$. B ⟶ A is an inverse function of f. $f^{-1}of = I_A$ and $fof^{-1} = I_B$. Here $I_A$, is an identity function on set A, and $I_B$, is an identity function on set B.

The **inverse function** $f^{-1}$ of a function f is the function such that, for every value of $x$ that f maps to $f(x)$, $f^{-1}$ maps $f(x)$ back to $x$.

Consider the function: $f(x) = 3x - 5 = y$




The inverse of $f(x)$ would be $f^{-1}(y) = \frac{y+5}{3}$




Some common inverse functions:

| Function | Inverse | Exceptions |
|---|---|---|
| + | - | |
| × | ÷ | Division by zero |
| $x^n$ | $y^{\frac{1}{n}}$ or $\sqrt[n]{y}$ or $\frac{1}{y^n}$ | n = zero |
| $a^x$ | $\log_a(y)$ | '$y$' and 'a' equal to 0 or negative number |

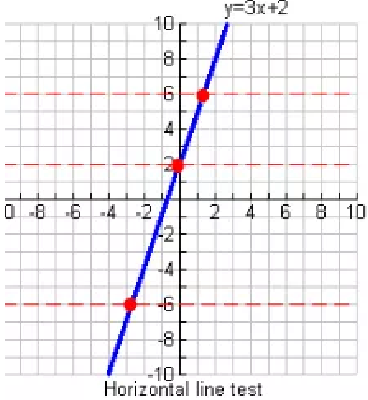
A function and its inverse function can be described as the "DO" and the "UNDO" functions. A function takes a starting value, performs some operation on this value, and creates an output answer. The inverse function takes the output answer, performs some operation on it, and arrives back at the original function's starting value.

This "DO" and "UNDO" process can be stated as a composition of functions. If functions f and g are inverse functions, f(g(x)) = g(f(x)) . A function composed with its inverse function yields the original starting value. Think of them as "undoing" one another and leaving you right where you started.

So how do we find the inverse of a function?
Basically speaking, the process of finding an inverse is simply the swapping of the x and y coordinates. This newly formed inverse will be a relation, but may not necessarily be a function.

Consider this subtle difference in terminology:
Definition: INVERSE OF A FUNCTION: The relation formed when the independent variable is exchanged with the dependent variable in a given relation. (This inverse may NOT be a function.)

**Definition**: INVERSE FUNCTION: If the above mentioned inverse of a function is itself a function, it is then called an inverse function.



Horizontal line test

**Remember**:

The inverse of a function may not always be a function!
The original function must be a one-to-one function to guarantee that its inverse will also be a function.

**Definition**: A function is a one-to-one function if and only if each second element corresponds to one and only one first element. (each x and y value is used only once)
Use the horizontal line test to determine if a function is a one-to-one function.
If ANY horizontal line intersects your original function in ONLY ONE location, your function will be a one-to-one function and its inverse will also be a function.

The function y = 3x + 2, shown at the right, IS a one-to-one function and its inverse will also be a function.

(Remember that the vertical line test is used to show that a relation is a function.)

**Definition**: The inverse of a function is the set of ordered pairs obtained by interchanging the first and second elements of each pair in the original function.

Should the inverse of function f (x) also be a function, this inverse function is denoted by $f^{-1}$ (x).
**Note**: If the original function is a one-to-one function, the inverse will be a function.

[The notation $f^{-1}$ (x) refers to "inverse function". It does not algebraically mean 1/f (x).]

**Swap ordered pairs**: If your function is defined as a list of ordered pairs, simply swap the x and y values. Remember, the inverse relation will be a function only if the original function is one-to-one.

**Examples**:

**a.** Given function *f*, find the inverse relation. Is the inverse relation also a *function*?
$f(x) = \{(3,4),(1,-2),(5,-1),(0,2)\}$
**Answer:**
Function *f* is a one-to-one function since the *x* and *y* values are used only once. Since function *f* is a one-to-one function, the inverse relation is also a function.
Therefore, the inverse function is:
$f^{-1}(x) = \{(4,3),(-2,1),(-1,5),(2,0)\}$

**b.** Determine the inverse of this function. Is the inverse also a *function*?

| $x$ | 1 | -2 | -1 | 0 | 2 | 3 | 4 | -3 |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 2 | 0 | 3 | -1 | 1 | -2 | 5 | 1 |

**Answer:** Swap the *x* and *y* variables to create the inverse relation. The inverse relation will be the set of ordered pairs:
{(2,1), (0,-2), (3,-1), (-1,0), **(1,2)**, (-2,3), (5,4),**(1,-3)**}
Since function *f* was **not** a one-to-one function (the *y* value of 1 was used twice), the inverse relation will **NOT** be a function (because the *x* value of 1 now gets mapped to two separate *y* values which is not possible for functions).

**Solve algebraically:** Solving for an inverse relation algebraically is a three step process:

1. Set the function = y
2. Swap the x and y variables
3. Solve for y

**Examples:**

**a.** Find the inverse of the function $f(x) = x - 4$

**Answer:**

| | |
|---|---|
| $y = x - 4$ | *Remember:* Set = y. |
| $x = y - 4$ | Swap the variables. |
| $x + 4 = y$ | Solve for y. |
| $f^{-1}(x) = x + 4$ | Use the inverse function notation since $f(x)$ is a one-to-one function. |

**b.** Find the inverse of the function $f(x) = \frac{x+1}{x}$ (given that $x$ is not equal to 0).

**Answer:**

| | |
|---|---|
| $y = \frac{x+1}{x}$ | *Remember:* Set = y. |
| $x = \frac{y+1}{y}$ | Swap the variables. |
| $xy = y + 1$ | Eliminate the fraction by multiplying each side by y. |
| $xy - y = 1$ | Get the y's on one side of the equal sign by subtracting y from each side. |
| $y(x-1) = 1$ | Isolate the y by factoring out the y. |
| $y = \frac{1}{x-1}$ | Solve for y. |
| $f^{-1}(x) = \frac{1}{x-1}$ | Use the inverse function notation since $f(x)$ is a one-to-one function. |

**Graph:** The graph of an inverse relation is the reflection of the original graph over the identity line, y = x. It may be necessary to restrict the domain on certain functions to guarantee that the inverse relation is also a function. (Read more about graphing inverses.)

**Example:**

Consider the straight line, y = 2x + 3, as the original function. It is drawn in blue.
If reflected over the identity line, y = x, the original function becomes the red dotted graph. The new red graph is also a straight line and passes the vertical line test for functions. The inverse relation of y = 2x + 3 is also a function.

Not all graphs produce an inverse relation which is also a function.