CHAPTER 7

Indefinite Integration

- Integration as Reverse Process of Differentiation
- Elementary Integration
- Integration by Substitutions
- Integration by Parts
- Integration by Cancellation
- Integration by Partial Fractions
- Integrations of Irrational Functions

INTEGRATION AS REVERSE PROCESS OF DIFFERENTIATION

The concept of integration originated during the course of finding the area of a plane figure. It is based on the limit of the sum of the series whose each term tends to zero and the number of terms that tends to infinity. In fact, it is called integration because of the process of summation as integration means summation. But, later it was observed that integration is just the inverse process of differentiation.

In integration, we find the function whose differential coefficient is given. For example, consider the function $5x^4$, we want to know the function whose differential coefficient w.r.t. x is $5x^4$. One such function is x^5. Again since the differential coefficient of $x^5 + c$ is $5x^4$, where c is an arbitrary constant, therefore the general form of the function whose differential coefficient is $5x^4$ is $x^5 + c$.

If the differential coefficient of a function $F(x)$ is $f(x)$, i.e., if $\frac{d}{dx}(F(x)) = f(x)$, then we will say that one integral or primitive of $f(x)$ is $F(x)$, and in symbols we write $\int f(x)dx = F(x)$.

The process of finding the integral of a function is called integration, and the function which is integrated is called the integrand.

If $\frac{d}{dx}F(x) = f(x)$, then also $\frac{d}{dx}(F(x)+c) = f(x)$, where c is an arbitrary constant. Thus, here the general value of $\int f(x)dx$ is $F(x) + c$ and c is called the constant of integration.

Clearly, the integral will change if c changes. Thus, the integral of a function is not unique. Thus, $\int f(x)dx$ will have infinite number of values, and hence it is called the indefinite integral of $f(x)$.

ELEMENTARY INTEGRATION

Fundamental Integration Formulae

$$\frac{d}{dx}\{g(x)\} = f(x) \Leftrightarrow \int f(x)\, dx = g(x) + c$$

Based upon this definition and various standard differentiation formulae, we obtain the following integration formulae:

1. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$
2. $\int \frac{1}{x}\, dx = \log|x| + c$, when $x \neq 0$
3. $\int e^x\, dx = e^x + c$
4. $\int a^x\, dx = \frac{a^x}{\log_e a} + c$
5. $\int \sin x\, dx = -\cos x + c$
6. $\int \cos x\, dx = \sin x + c$
7. $\int \sec^2 x\, dx = \tan x + c$
8. $\int \text{cosec}^2 x\, dx = -\cot x + c$
9. $\int \sec x \tan x\, dx = \sec x + c$
10. $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + c$
11. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$ or $-\cos^{-1} x + C$
12. $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$ or $-\cot^{-1} x + C$
13. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$ or $-\text{cosec}^{-1} x + C$

Example 7.1 Evaluate $\int \frac{(1+x)^3}{\sqrt{x}} dx$.

Sol. $\int \frac{(1+x)^3}{\sqrt{x}} dx$

$$= \int \frac{1+3x+3x^2+x^3}{\sqrt{x}} dx$$

$$= \int x^{-1/2}dx + 3\int x^{1/2}dx + 3\int x^{3/2}dx + \int x^{5/2}dx$$

$$= \frac{x^{1/2}}{1/2} + 3\frac{x^{3/2}}{3/2} + 3\cdot\frac{x^{5/2}}{5/2} + \frac{x^{7/2}}{7/2} + c$$

$$= 2\sqrt{x} + 2x^{3/2} + \frac{6}{5}x^{5/2} + \frac{2}{7}x^{7/2} + c$$

Example 7.2 Evaluate $\int \frac{2^{x+1} - 5^{x-1}}{10^x} dx$.

Sol. $I = \int \frac{2^{x+1} - 5^{x-1}}{10^x} dx$

$$= \int \left[2\left(\frac{1}{5}\right)^x - \frac{1}{5}\left(\frac{1}{2}\right)^x\right] dx$$

$$= \frac{2\left(\frac{1}{5}\right)^x}{\log\left(\frac{1}{5}\right)} - \frac{1}{5}\frac{\left(\frac{1}{2}\right)^x}{\log\left(\frac{1}{2}\right)} + c$$

Example 7.3 Evaluate $\int \sec^2 x \,\text{cosec}^2 x dx$

Sol. $I = \int \sec^2 x\, \text{cosec}^2 x dx$

$$= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x \sin^2 x} dx$$

$$= \int \left(\sec^2 x + \text{cosec}^2 x\right) dx$$

$$= \tan x - \cot x + c$$

Example 7.4 Evaluate $\int \frac{x^2+\cos^2 x}{1+x^2}\operatorname{cosec}^2 x\,dx$.

Sol. $I=\int \frac{(x^2+\cos^2 x)}{(1+x^2)}\operatorname{cosec}^2 x\,dx$

$=\int \frac{(1+x^2-\sin^2 x)}{(1+x^2)}\operatorname{cosec}^2 x\,dx$

$=\int \operatorname{cosec}^2 x\,dx-\int \frac{dx}{1+x^2}$

$=-\cot x-\tan^{-1}x+C$

Example 7.5 Evaluate $\int \frac{1}{1+\sin x}dx$

Sol. $\int \frac{1}{1+\sin x}dx$

$=\int \frac{1}{(1+\sin x)}\cdot\frac{(1-\sin x)}{(1-\sin x)}dx$

$=\int \frac{1-\sin x}{1-\sin^2 x}dx$

$=\int \frac{1-\sin x}{\cos^2 x}dx$

$=\int \frac{1}{\cos^2 x}dx-\int \frac{\sin x}{\cos^2 x}dx$

$=\int \sec^2 x dx-\int \tan x\sec x dx$

$=\tan x-\sec x+C$

Example 7.6 Evaluate $\int \tan^{-1}\left\{\sqrt{\left(\frac{1-\cos 2x}{1+\cos 2x}\right)}\right\}dx,$ $0<x<\pi/2$

Sol. $\int \tan^{-1}\left\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right\}dx$

$=\int \tan^{-1}\left\{\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}\right\}dx$

$=\int \tan^{-1}(\tan x)dx=\int x dx=\frac{x^2}{2}+C$

Example 7.7 Evaluate $\int \frac{\sec x}{\sec x+\tan x}dx$

Sol. $\int \frac{\sec x}{\sec x+\tan x}dx$

$=\int \frac{\sec x(\sec x-\tan x)}{(\sec x+\tan x)(\sec x-\tan x)}dx$

$=\int \frac{\sec^2 x-\sec x\tan x}{\sec^2 x-\tan^2 x}dx$

$=\int \sec^2 x dx-\int \sec x\tan x dx$

$=\tan x-\sec x+C$

Concept Application Exercise 7.1

Evaluate the following

1. $\int (\sec x+\tan x)^2\,dx$
2. $\int (1-\cos x)\ \operatorname{cosec}^2 x\,dx$
3. $\int a^{mx}b^{nx}\,dx$
4. $\int \frac{\tan x}{\sec x+\tan x}\,dx$
5. If $\int \frac{1}{x+x^5}\,dx=f(x)+c$, then evaluate $\int \frac{x^4}{x+x^5}dx$.
6. Evaluate $\int \frac{(x^3+8)(x-1)}{x^2-2x+4}dx$.
7. Find $\int \frac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}dx$.
8. Evaluate $\int \tan^{-1}(\sec x+\tan x)dx,\ -\pi/2<x<\pi/2$

Properties of Indefinite Integration

1. $\int kf(x)\,dx=k\int f(x)$, where k is a constant
2. $\int \{f_1(x)\pm f_2(x)\pm\ldots\pm f_n(x)\}\,dx$
   $=\int f_1(x)\,dx\pm\int f_2(x)\,dx\pm\ldots\pm\int f_n(x)\,dx$
3. $\int f(x)dx=F(x)+C$
   then $\int f(ax+b)dx=\frac{F(ax+b)}{a}+c$

Proof: Let $ax+b=t$, then $adx=dt$

$\Rightarrow\quad I=\int f(ax+b)dx=\frac{1}{a}\int f(t)dt$

$=\frac{1}{a}F(t)+c=\frac{1}{a}F(ax+b)+C$

Example 7.8 Evaluate $\int \frac{x+2}{(x+1)^2}dx$.

Sol. $\int \frac{x+2}{(x+1)^2}dx$

$$= \int \frac{x+1+1}{(x+1)^2}dx$$

$$= \int \frac{x+1}{(x+1)^2} + \frac{1}{(x+1)^2}dx$$

$$= \int \frac{1}{x+1}dx + \int (x+1)^{-2}dx$$

$$= \log|x+1| + \frac{(x+1)^{-1}}{(-1)} + c = \log|x+1| - \frac{1}{x+1} + C$$

Example 7.9 Evaluate $\int \frac{8x+13}{\sqrt{4x+7}}dx$.

Sol. $\int \frac{8x+13}{\sqrt{4x+7}}dx$

$$= \int \frac{8x+14-1}{\sqrt{4x+7}}dx$$

$$= \int \frac{2(4x+7)-1}{\sqrt{4x+7}}dx$$

$$= 2\int \sqrt{4x+7}dx - \int \frac{1}{\sqrt{4x+7}}dx$$

$$= 2\left\{\frac{(4x+7)^{3/2}}{4\times\frac{3}{2}}\right\} - \left\{\frac{(4x+7)^{1/2}}{4\times\frac{1}{2}}\right\} + C$$

$$= \frac{1}{3}(4x+7)^{3/2} - \frac{1}{2}(4x+7)^{1/2} + C$$

Example 7.10 Evaluate $\int \sin^3 x\,dx$.

Sol. $\sin 3x = 3\sin x - 4\sin^3 x$

$$\Rightarrow \sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

$$\Rightarrow \int \sin^3 dx = \int \frac{3\sin x - \sin 3x}{4}dx$$

$$= \frac{3}{4}\int \sin x dx - \frac{1}{4}\int \sin 3x dx$$

$$= \frac{3}{4}(-\cos x) - \frac{1}{4}\left(-\frac{\cos 3x}{3}\right) + C$$

$$= -\frac{3}{4}\cos x + \frac{\cos 3x}{12} + C$$

Example 7.11 Evaluate $\int \sin 2x \sin 3x dx$.

Sol. $\int \sin 2x \sin 3x dx$

$$= \frac{1}{2}\int 2\sin 3x \sin 2x dx$$

$$= \frac{1}{2}\int [\cos(3x-2x) - \cos(3x+2x)]dx$$

$[\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$

$$= \frac{1}{2}\int [\cos x - \cos 5x]dx$$

$$= \frac{1}{2}\int \cos x dx - \frac{1}{2}\int \cos 5x dx$$

$$= \frac{1}{2}\sin x - \frac{1}{2}\frac{\sin 5x}{5} + C$$

$$= \frac{\sin x}{2} - \frac{\sin 5x}{10} + C$$

Example 7.12 Evaluate $\int \frac{dx}{(2x-7)\sqrt{(x-3)(x-4)}}$.

Sol. $I = \int \frac{dx}{(2x-7)\sqrt{(x-3)(x-4)}}$

$$= \int \frac{dx}{(2x-7)\sqrt{x^2-7x+12}}$$

$$= \int \frac{2dx}{(2x-7)\sqrt{(2x-7)^2-1}}$$

$$= \frac{1}{2}\sec^{-1}(2x-7) + c$$

Example 7.13 Evaluate $\int \frac{1}{\sqrt{3x+4}-\sqrt{3x+1}}dx$.

Sol. $\int \frac{1}{\sqrt{3x+4}-\sqrt{3x+1}}dx$

$$= \int \frac{\sqrt{(3x+4)}+\sqrt{(3x+1)}}{(\sqrt{3x+4}+\sqrt{3x+1})(\sqrt{3x+4}-\sqrt{3x+1})}dx$$

$$= \int \frac{\sqrt{3x+4}+\sqrt{3x+1}}{(3x+4)-(3x+1)}dx$$

$$= \frac{1}{3}\int \sqrt{3x+4}+\sqrt{3x+1}\,dx$$

$$= \frac{1}{3}\int \sqrt{3x+4}dx + \frac{1}{3}\int \sqrt{3x+1}\,dx$$

$$= \frac{1}{3}\left\{\frac{(3x+4)^{3/2}}{3\times\frac{3}{2}}\right\} + \frac{1}{3}\left\{\frac{(3x+1)^{3/2}}{3\times\frac{3}{2}}\right\} + C$$

$$= \frac{2}{27}\{(3x+4)^{3/2} + (3x+1)^{3/2}\} + C$$

Example 7.14 Find the values of a and b such that

$$\int \frac{dx}{1+\sin x} = \tan\left(\frac{x}{2}+a\right) + b.$$

Sol. $\int \frac{dx}{1+\sin x} = \int \frac{dx}{1+\cos(\pi/2-x)}$

$$= \int \frac{dx}{2\cos^2(\pi/4-x/2)}$$

$$= \frac{1}{2}\int \sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$$

$$= \frac{1}{2}\frac{\tan(\pi/4 - x/2)}{-\frac{1}{2}} + c$$

$$= \tan\left(\frac{x}{2} - \frac{\pi}{4}\right) + c$$

Given, $\int \frac{dx}{1+\sin x} = \tan\left(\frac{x}{2} + a\right) + b$

$$\Rightarrow \quad \tan\left(\frac{x}{2} - \frac{\pi}{4}\right) + c = \tan\left(\frac{x}{2} + a\right) + b$$

$\therefore a = -\frac{\pi}{4}$ and $b = c =$ an arbitrary constant.

Example 7.15 Evaluate $\int \left(x + \frac{1}{x}\right)^{3/2}\left(\frac{x^2-1}{x^2}\right)dx.$

Sol. $I = \int \left(x + \frac{1}{x}\right)^{3/2}\left(\frac{x^2-1}{x^2}\right)dx$

$$= \int \left(x + \frac{1}{x}\right)^{3/2}\left(1 - \frac{1}{x^2}\right)dx$$

Let $t = x + \frac{1}{x} \Rightarrow dt = \left(1 - \frac{1}{x^2}\right)dx$

$$\Rightarrow \quad I = \int t^{3/2}dt = \frac{2}{5}t^{5/2} + c = \frac{2}{5}\left(x + \frac{1}{x}\right)^{5/2} + C$$

Example 7.16 Evaluate $\int \frac{x}{\sqrt{x+2}}\,dx.$

Sol. $\int \frac{x}{\sqrt{x+2}}\,dx$

$$= \frac{x+2-2}{\sqrt{x+2}}\,dx$$

$$= \int \frac{x+2}{\sqrt{x+2}}\,dx - \int \frac{2}{\sqrt{x+2}}\,dx$$

$$= \int \sqrt{x+2}\,dx - 2\int (x+2)^{-1/2}\,dx$$

$$= \frac{(x+2)^{3/2}}{3/2} - 2\,\frac{(x+2)^{1/2}}{1/2} + C$$

$$= 2/3(x+2)^{3/2} - 4\sqrt{x+2} + C$$

FORM 1: $\int \frac{f'(x)}{f(x)}dx = \log_e|f(x)| + C$

Proof: Let $f(x) = t \;\Rightarrow\; dt = f'(x)dx$

$$\Rightarrow \int \frac{f'(x)}{f(x)}dx = \int \frac{dt}{t} = \log_e|t| + C = \log_e|f(x)| + C$$

Example 7.17 Evaluate $\int \frac{\sec^2 x}{3+\tan x}\,dx.$

Sol. $\int \frac{\sec^2 x}{3+\tan x}\,dx = \int \frac{\frac{d}{dx}(3+\tan x)}{3+\tan x}\,dx = \log|3+\tan x| + C$

Example 7.18 Evaluate $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx.$

Sol. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{\frac{d}{dx}(e^x + e^{-x})}{e^x + e^{-x}}\,dx = \log|e^x + e^{-x}| + C$

Example 7.19 Evaluate $\int \frac{1-\tan x}{1+\tan x}\,dx.$

Sol. $\int \frac{1-\tan x}{1+\tan x}\,dx = \int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$

$$= \int \frac{\frac{d}{dx}(\cos x + \sin x)}{\cos x + \sin x}\,dx = \log|\cos x + \sin x| + C$$

Example 7.20 Evaluate $\int \frac{1}{1+e^{-x}}\,dx.$

Sol. $\int \frac{1}{1+e^{-x}}\,dx = \int \frac{e^x}{e^x+1}\,dx$

$$= \int \frac{\frac{d}{dx}(e^x+1)}{e^x+1}\,dx = \log(1+e^x) + C$$

Example 7.21 Evaluate $\int \frac{\sin 2x}{a^2\sin^2 x + b^2\cos^2 x}\,dx.$

Sol. $\frac{d}{dx}\left(a^2\sin^2 x + b^2\cos^2 x\right) = (a^2 - b^2)\sin 2x$

Now, $\int \frac{\sin 2x}{a^2\sin^2 x + b^2\cos^2 x}\,dx$

$$= \frac{1}{(a^2-b^2)}\int \frac{(a^2-b^2)\sin 2x}{a^2\sin^2 x + b^2\cos^2 x}\,dx$$

$$= \frac{1}{(a^2-b^2)}\int \frac{\frac{d}{dx}(a^2\sin^2 x + b^2\cos^2 x)}{a^2\sin^2 x + b^2\cos^2 x}\,dx$$

$$= \frac{1}{(a^2-b^2)}\log|a^2\sin^2 x + b^2\cos^2 x| + C$$

FORM 2: $\int (f(x))^n f'(x)dx = \frac{(f(x))^{n+1}}{n+1} + C$

Proof: Let $f(x) = t \Rightarrow dt = f'(x)dx$

$$\Rightarrow \int (f(x))^n f'(x)dx = \int t^n dt = \frac{t^{n+1}}{n+1} + C = \frac{(f(x))^{n+1}}{n+1} + C$$

Example 7.22 Evaluate $\int \frac{\log\left(\tan\frac{x}{2}\right)}{\sin x}\,dx.$

Sol. $\frac{d}{dx}\left[\log\left(\tan\frac{x}{2}\right)\right] = \frac{\frac{1}{2}\sec^2\frac{x}{2}}{\tan\frac{x}{2}} = \frac{1}{\sin x}$

Now, $\int \frac{\log\left(\tan\frac{x}{2}\right)}{\sin x}\,dx = \int \log\left(\tan\frac{x}{2}\right)\frac{d}{dx}\left[\log\left(\tan\frac{x}{2}\right)\right]dx$

$$= \frac{\left[\log\left(\tan\frac{x}{2}\right)\right]^2}{2} + C$$

Example 7.23 Evaluate $\int \frac{\sqrt{2+\log x}}{x} dx$.

Sol. $\int \frac{\sqrt{2+\log x}}{x} dx = \int (2+\log x)^{\frac{1}{2}} \frac{d}{dx}(2+\log x) dx$

$$= \frac{(2+\log x)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2(2+\log x)^{\frac{3}{2}}}{3} + C$$

Example 7.24 Evaluate $\int \tan^4 x dx$.

Sol. $\int \tan^4 x\, dx = \int \tan^2 x \tan^2 x dx = \int \tan^2 x(\sec^2 x - 1) dx$

$$= \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx = \frac{\sec^3 x}{3} - \int (\sec^2 x - 1) dx$$

$$= \frac{\sec^3 x}{3} - \tan x + x + C$$

Example 7.25 Evaluate $\int (\tan x - x)\tan^2 x dx$.

Sol. $\int (\tan x - x)\tan^2 x dx = \int (\tan x - x)(\sec^2 x - 1) dx$

$$= \int (\tan x - x)\frac{d}{dx}(\tan x - x) dx = \frac{(\tan x - x)^2}{2} + C$$

Example 7.26 Evaluate $\int \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}} dx$.

Sol. $\int \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}} dx$

$$= \int (\sin^{-1} x)^3 \frac{d}{dx}(\sin^{-1} x) dx = \frac{(\sin^{-1} x)^4}{4} + C$$

Example 7.27 Evaluate $\int \left(\frac{x+1}{x}\right)(x+\log x)^2 dx$.

Sol. $\int (x+\log x)^2 \left(\frac{x+1}{x}\right) dx = \int (x+\log x)^2 \left(1+\frac{1}{x}\right) dx$

$$= \int (x+\log x)^2 \frac{d}{dx}(x+\log x) dx = \frac{(x+\log x)^3}{3} + C$$

Example 7.28 Evaluate $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$.

Sol. $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx = \int \frac{\sqrt{\tan x}\,\frac{1}{\cos^2 x}}{\frac{\sin x \cos x}{\cos^2 x}} dx$

$$= \int \frac{\sqrt{\tan x}\sec^2 x}{\tan x} dx = \int (\tan x)^{-1/2} \sec^2 x dx$$

$$= \frac{(\tan x)^{1/2}}{1/2} + C = 2\sqrt{\tan x} + C$$

Example 7.29 Evaluate $\int \frac{\cot x}{\sqrt{\sin x}} dx$.

Sol. $\int \frac{\cot x}{\sqrt{\sin x}} dx = \int \frac{\cos x}{\sin x \sqrt{\sin x}} dx$

$$= \int (\sin x)^{-3/2} \cos x dx = \frac{(\sin x)^{-1/2}}{-1/2} + C = \frac{-2}{\sqrt{\sin x}} + C$$

Example 7.30 Evaluate $\int \frac{dx}{x^2(x^4+1)^{\frac{3}{4}}}$.

Sol. $I = \int \frac{dx}{x^2(x^4+1)^{\frac{3}{4}}} = \int \frac{dx}{x^2 x^3\left(1+\frac{1}{x^4}\right)^{\frac{3}{4}}}$

$$= \int \frac{\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}} dx}{x^5}$$

Let $1+\frac{1}{x^4} = t \Rightarrow \frac{-4}{x^5} dx = dt$

$$\Rightarrow I = -\frac{1}{4}\int t^{-\frac{3}{4}} dt = -\frac{1}{4}\frac{t^{\frac{1}{4}}}{\frac{1}{4}} + C$$

$$= -\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}} + C$$

Concept Application Exercise 7.2

Evaluate the following:

1. $\int \frac{dx}{\sqrt{2ax - x^2}}$
2. $\int \frac{e^{3x} + e^{5x}}{e^x + e^{-x}} dx$
3. $\int \tan^2 x \sin^2 x dx$
4. $\int \frac{\cos x - \sin x}{\cos x + \sin x}(2+2\sin 2x) dx$
5. $\int \operatorname{cosec}^4 x\, dx$
6. $\int \frac{\sin 2x}{(a+b\cos x)^2} dx$
7. $\int \sin x \cos x \cos 2x \cos 4x \cos 8x\, dx$
8. $\int \frac{(1+\ln x)^5}{x} dx$
9. $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos\theta} dx$
10. $\int \frac{x^3}{x+1} dx$
11. $\int \frac{dx}{\sqrt{x} + \sqrt{x-2}}$
12. $\int \left(1+2x+3x^2+4x^3+\cdots\right) dx \quad (0<|x|<1)$
13. $\int \frac{\ln(\ln x)}{x \ln x} dx$ is $(x>0)$

14. $\int \frac{dx}{x + x\log x}$

15. $\int \sec^p x \tan x dx$

16. $\int \frac{\sin^6 x}{\cos^8 x} dx$

17. $\int (\tan x - x)\tan^2 x dx$

Some More Standard Formulae

1. $\int \tan x\, dx = \ln|\sec x| + c$

Proof: $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{\frac{d}{dx}(\cos x)}{\cos x} dx$

$= \ln|\sec x| + c$

2. $\int \cot x\, dx = \ln|\sin x| + c$

Proof: $\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \int \frac{\frac{d}{dx}(\sin x)}{\sin x} dx$

$= \ln|\sin x| + c$

3. $\int \sec x\, dx = \ln|\sec x + \tan x| + c = \ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

$= \frac{1}{\sqrt{a^2 + b^2}}$

Proof: $\int \sec x\, dx$

$= \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{(\sec x + \tan x)} dx$

$= \ln|\sec x + \tan x| + c$

$= \ln\left|\frac{1 + \sin x}{\cos x}\right| + c$

$= \ln\left|\frac{1 - \cos\left(\frac{\pi}{2} + x\right)}{\sin\left(\frac{\pi}{2} + x\right)}\right| + c$

$= \ln\left|\frac{2\sin^2\left(\frac{\pi}{4} + \frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)}\right| + c$

$= \ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

4. $\int \operatorname{cosec} x\, dx = \ln|\operatorname{cosec} x - \cot x| + c = \ln\left|\tan\frac{x}{2}\right| + c$

FORM3:

$$\int \frac{1}{a \sin x + b \cos x} dx$$

Working Rule:

Substitute $a = r\cos\theta$, $b = r\sin\theta$ and so $r = \sqrt{a^2 + b^2}$,

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$\therefore a\sin x + b\cos x = r\sin(x + \theta)$

So, $\int \frac{1}{a \sin x + b\cos x} dx$

$= \frac{1}{r}\int \frac{1}{\sin(x + \theta)} dx$

$= \frac{1}{r}\int \operatorname{cosec}(x + \theta)\, dx$

$= \frac{1}{r}\log\left|\tan\left(\frac{x}{2} + \frac{\theta}{2}\right)\right| + c$

$\therefore \int \frac{1}{a \sin x + b\cos x} dx$

$\log\left|\tan\left(\frac{x}{2} + \frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right| + c$

Example 7.31 Evaluate $\int \sin 2x d(\tan x)$.

Sol $I = \int \sin 2x\, d(\tan x)$

$= \int \sin 2x \cdot \frac{d(\tan x)}{dx} dx$

$= \int \sin 2x \sec^2 x dx$

$= 2\int \tan x dx$

$= 2\ln|\sec x| + c$

Example 7.32 Evaluate $\int \tan x \tan 2x \tan 3x dx$.

Sol. $\tan 3x = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$

$\Rightarrow \tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$

$\therefore \int (\tan 3x - \tan 2x - \tan x)\, dx$

$= -\frac{1}{3}\log|\cos 3x| + \frac{1}{2}\log|\cos 2x| + \log|\cos x| + c$

Example 7.33 Evaluate $\int \frac{1}{\sqrt{3}\sin x + \cos x} dx$.

Sol. Let $\sqrt{3} = r\sin\theta$ and $1 = r\cos\theta$

Then $r=\sqrt{(\sqrt{3})^2+1^2}=2$ and $\tan\theta=\frac{\sqrt{3}}{1}\Rightarrow\theta=\frac{\pi}{3}$

$$\therefore \int\frac{1}{\sqrt{3}\sin x+\cos x}dx$$

$$=\int\frac{1}{r\sin\theta\sin x+r\cos\theta\cos x}dx$$

$$=\frac{1}{r}\int\frac{1}{\cos(x-\theta)}dx=\frac{1}{r}\int\sec(x-\theta)dx$$

$$=\frac{1}{r}\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}-\frac{\theta}{2}\right)\right|+c$$

$$=\frac{1}{2}\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}-\frac{\pi}{6}\right)\right|+c$$

$$=\frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)\right|+c$$

Example 7.34 Evaluate $\int\frac{1}{\sin(x-a)\sin(x-b)}dx$.

Sol. $\int\frac{1}{\sin(x-a)\sin(x-b)}dx$

$$=\frac{1}{\sin(a-b)}\int\frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(a-b)}\times$$

$$\int\frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(a-b)}\int\{\cot(x-a)-\cot(x-b)\}dx$$

$$=\frac{1}{\sin(a-b)}\{\log|\sin(x-a)|-\log|\sin(x-b)|\}+c$$

$$=\operatorname{cosec}(a-b)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|+c$$

Example 7.35 Evaluate $\int\{1+2\tan x(\tan x+\sec x)\}^{1/2}dx$.

Sol. $\int\left(1+2\tan^2x+2\tan x\sec x\right)^{1/2}dx$

$$=\int\left(\sec^2x+\tan^2x+2\tan x\sec x\right)^{1/2}dx$$

$$=\int(\sec x+\tan x)dx$$

$$=\log(\sec x+\tan x)+\log\sec x+c$$

$$=\log\sec x(\sec x+\tan x)+c$$

Concept Application Exercise 7.3

Evaluate the following:

1. $\int\frac{dx}{(1+\sin x)^{1/2}}$
2. $\int\frac{dx}{\cos x-\sin x}$
3. $\int\frac{\sin x}{\sin(x-a)}dx$
4. $\int\tan^3 x dx$

INTEGRATION BY SUBSTITUTIONS

If $g(x)$ is a continuously differentiable function, then to evaluate the integrals of the form $I=\int f(g(x))g'(x)\,dx$, we substitute $g(x)=t$ and $g'(x)\,dx=dt$.

The substitution reduces the integral to $\int f(t)\,dt$.

After evaluating this integral we substitute the value of t.

Example 7.36 Evaluate $\int\sin(e^x)d(e^x)$.

Sol. $I=\int\sin(e^x)d(e^x)$

Let $e^x=t$

$\Rightarrow\quad I=\int\sin(e^x)d(e^x)$

$$=\int\sin t\,dt=-\cos t+C=-\cos(e^x)+c$$

Example 7.37 Evaluate $\int\cos^3x\sqrt{\sin x}\,dx$.

Sol. [Here, the power of cos x is 3 which is an odd positive integer, therefore, put $z=\sin x$]

Let $z=\sin x$, then $dz=\cos x\,dx$

Now $\int\cos^3x\sqrt{\sin x}\,dx$

$$=\int\cos^2x\sqrt{\sin x}\cos x\,dx$$

$$=\int(1-\sin^2x)\sqrt{\sin x}\cos x\,dx$$

$$=\int(1-z^2)\sqrt{z}\,dz$$

$$=\int(\sqrt{z}-z^{5/2})dz$$

$$=\frac{z^{3/2}}{3/2}-\frac{z^{7/2}}{7/2}+c=\frac{2}{3}z^{3/2}-\frac{2}{7}z^{7/2}+c$$

$$=\frac{2}{3}\sin^{3/2}x-\frac{2}{7}\sin^{7/2}x+c$$

Example 7.38 Evaluate $\int \frac{\sin 2x}{a^2+b^2\sin^2 x}dx$.

Sol. Let $z = a^2 + b^2 \sin^2 x$,

$\Rightarrow \quad dz = 2b^2 \sin x \cos x\, dx = b^2 \sin 2x\, dx$

$$\Rightarrow \quad I = \int \frac{\sin 2x}{a^2+b^2\sin^2 x}dx$$

$$= \frac{1}{b^2}\int \frac{dz}{z}$$

$$= \frac{1}{b^2}\log|z| + c$$

$$= \frac{1}{b^2}\log|a^2+b^2\sin^2 x| + c$$

Example 7.39 Evaluate $\int 2^{2^{2^x}} 2^{2^x} 2^x dx$.

Sol. $I = \int 2^{2^{2^x}}\ 2^{2^x}\ 2^x\ dx$

Let $2^{2^{2^x}} = t \Rightarrow 2^{2^{2^x}}\ 2^{2^x}\ 2^x (\log 2)^3 dx = dt$

$$\Rightarrow \quad I = \int \frac{1}{(\log 2)^3}dt = \frac{1}{(\log 2)^3}\ t + c$$

$$= \frac{1}{(\log 2)^3}\ 2^{2^{2^x}} + C$$

Example 7.40 Evaluate $\int \frac{1}{x^{1/2}+x^{1/3}}dx$.

Sol. $\frac{1}{x^{1/2}+x^{1/3}} = \frac{1}{x^{1/3}\left(1+x^{1/6}\right)}$

Let $x = t^6 \Rightarrow dx = 6t^5\, dt$

$$\therefore \int \frac{1}{x^{1/2}+x^{1/3}}dx = \int \frac{1}{x^{1/3}\left(1+x^{1/6}\right)}dx = \int \frac{6t^5}{t^2(1+t)}dt$$

$$= 6\int \frac{t^3}{(1+t)}dt$$

On dividing, we obtain

$$\int \frac{1}{x^{1/2}+x^{1/3}}dx = 6\int\left\{(t^2-t+1) - \frac{1}{1+t}\right\}dt$$

$$= 6\left[\left(\frac{t^3}{3}\right) - \left(\frac{t^2}{2}\right) + t - \log|1+t|\right] + C$$

$$= 2x^{1/2} - 3x^{1/3} + 6x^{1/6} - 6\log\left(1+x^{1/6}\right) + C$$

Example 7.41 Evaluate $\int x^2 \frac{\tan^{-1} x^3}{1+x^6}dx$.

Sol. $I = \int x^2 \frac{\tan^{-1} x^3}{1+x^6}dx$

Let $\tan^{-1} x^3 = t$

$$\Rightarrow \quad \frac{1}{1+x^6}\ 3x^2\, dx = dt \Rightarrow dx = \frac{(1+x^6)}{3x^2}dt$$

$$\Rightarrow \quad I = \int x^2 \frac{t}{1+x^6} \times \frac{1+x^6}{3x^2}dt$$

$$= \frac{1}{3}\int t\ dt$$

$$= \frac{1}{6}\ t^2 + C$$

$$= \frac{1}{6}\ \{\tan^{-1} x^3\}^2 + C$$

Example 7.42 Evaluate $\int \frac{2x-\sqrt{\sin^{-1} x}}{\sqrt{1-x^2}}dx$.

Sol. $I = \int \frac{2x-\sqrt{\sin^{-1} x}}{\sqrt{1-x^2}}dx$

$$= -\int \frac{-2x}{\sqrt{1-x^2}}dx - \int \frac{\left(\sin^{-1} x\right)^{1/2}}{\sqrt{1-x^2}}dx$$

$$= -\int \left(1-x^2\right)^{-1/2}\left(1-x^2\right)' dx - \int \left(\sin^{-1} x\right)^{1/2}\left(\sin^{-1} x\right)' dx$$

$$= -\frac{\left(1-x^2\right)^{1/2}}{1/2} - \frac{\left(\sin^{-1} x\right)^{3/2}}{3/2} + c$$

Example 7.43 Evaluate $\int \frac{e^{\sqrt{x}}\cos(e^{\sqrt{x}})}{\sqrt{x}}dx$.

Sol. Put $e^{\sqrt{x}} = t \Rightarrow \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2dt$

$$\Rightarrow \int \frac{e^{\sqrt{x}}\cos(e^{\sqrt{x}})}{\sqrt{x}}dx$$

$$= 2\int \cos t\, dt$$

$$= 2\sin t + c$$

$$= 2\sin e^{\sqrt{x}} + c$$

Example 7.44 Evaluate $\int \frac{\tan x}{a+b\tan^2 x}dx$.

Sol. $\int \frac{\tan x}{a+b\tan^2 x}dx$

$$= \int \frac{(\sin x)/(\cos x)}{a+b\dfrac{\sin^2 x}{\cos^2 x}}dx$$

$$= \int \frac{\sin x \cos x}{a\cos^2 x + \sin^2 x}dx$$

$$= \frac{1}{2}\int \frac{\sin 2x}{a\cos^2 x + b\sin^2 x}dx$$

$$= \frac{1}{2(b-a)}\log|a\cos^2 x + b\sin^2 x| + C$$

Example 7.45 Find $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx.$

Sol. Let $z = xe^x$, then $dz = (1e^x + xe^x)\,dx = e^x(1+x)\,dx$

$$\Rightarrow \int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$$

$$= \int \frac{dz}{\cos^2 z}$$

$$= \int \sec^2 z dz$$

$$= \tan z + c = \tan(xe^x) + c$$

FORM 4:

$\int(\sin^m x \cos^n x)\,dx$, where m, n belong to natural number.

Working Rule:

a. If one of them is odd, then substitute for the term of even power.
b. If both are odd, substitute either of them.
c. If both are even, use trigonometric identities only.
d. If m and n are rational numbers and $\left(\frac{m+n-2}{2}\right)$ is a negative integer, then substitute $\cot x = p$ or $\tan x = p$ whichsoever is found suitable.

Example 7.46 Find $\int \sin^5 x dx.$

Sol. [Here power of $\sin x$ is 5 which is an odd positive integer, therefore, put $z = \cos x$]

Let $z = \cos x$ then $dz = -\sin x\,dx$

$$\text{Now} \int \sin^5 x dx \int \sin^4 x \sin x dx$$

$$= \int (\sin^2 x)^2 \sin x dx$$

$$= \int (1-\cos^2 x)^2 \sin x dx$$

$$= \int (1-z^2)^2(-dz) \qquad [\because\ z = \cos x]$$

$$= -\int (1-2z^2+z^4)dz$$

$$= -\left[z - 2\frac{z^3}{3} + \frac{z^5}{5}\right] + c$$

$$= -z + \frac{2}{3}z^3 - \frac{z^5}{5} + c$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{\cos^5 x}{5} + c$$

Example 7.47 Find $\int \sin^3 x \cos^5 x dx.$

Sol. [Here, powers of both $\cos x$ and $\sin x$ are odd positive integers; therefore, put $z = \cos x$ or $z = \sin x$, but the power of $\cos x$ is greater, therefore, it is convenient to put $z = \cos x$]

Let $z = \cos x$, then $dz = -\sin x\,dx$

$$\text{Now} \int \sin^3 x \cos^5 x dx$$

$$= \int \sin^2 x \cos^5 x \sin x dx$$

$$= \int (1-\cos^2 x)\cos^5 x \sin x dx$$

$$= \int (1-z^2)z^5(-dz)$$

$$= -\int (z^5 - z^7)dz$$

$$= -\left(\frac{z^6}{6} - \frac{z^8}{8}\right) + c = -\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + c$$

Example 7.48 Find $\int \frac{dx}{\sin x \cos^3 x}.$

Sol. [Here, the power of $\sin x$ is -1 and that of $\cos x$ is -3. Since the sum of powers of $\sin x$ and $\cos x$ is -4 which is even and negative, therefore, put $z = \tan x$.]

Let $z = \tan x$, then $dz = \sec^2 x\,dx$

$$\text{Now } I = \int \frac{dx}{\sin x \cos^3 x}$$

$$= \int \frac{\sec^4 x\,dx}{\tan x}$$

$$= \int \frac{(1+\tan^2 x)\sec^2 x\,dx}{\tan x}$$

Let $\tan x = z$. Then, $\sec^2 x dx = dz$

$$\Rightarrow I = \int \frac{1+z^2}{z}dz = \int \left(\frac{1}{z} + z\right)dz = \log|z| + \frac{z^2}{2} + c$$

$$= \log|\tan x| + \frac{\tan^2 x}{2} + c$$

Example 7.49 Evaluate $\int \sin^2 x \cos^2 x dx.$

Sol. [Here, the power of neither $\sin x$ nor $\cos x$ is an odd positive integer, but the sum of their powers is an even positive integer. Hence, we will have to change $\sin^2 x \cos^2 x$ as sines or cosines of multiple angles.]

$$\text{Now} \int \sin^2 x \cos^2 x dx = \int \frac{1-\cos 2x}{2}\,\frac{1+\cos 2x}{2}dx$$

$$= \frac{1}{4}\int (1-\cos^2 2x)dx = \frac{1}{4}\int \sin^2 2x dx$$

$$= \frac{1}{4}\int \frac{1-\cos 4x}{2}dx = \frac{1}{8}\int (1-\cos 4x)dx$$

$$= \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right] + c$$

Concept Application Exercise 7.4

Evaluate the following:

1. $\int \frac{x^2 \tan^{-1} x^3}{1+x^6} dx$
2. $\int \frac{\sqrt{x}\, dx}{1+x}$
3. $\int \frac{\cot x}{\sqrt{\sin x}} dx$
4. $\int \frac{dx}{x+\sqrt{x}}$
5. $\int \frac{dx}{9+16\sin^2 x}$
6. $\int \frac{e^{2x}-2e^x}{e^{2x}+1} dx$
7. $\int \frac{ax^3+bx}{x^4+c^2} dx$
8. $\int \frac{dx}{x^{2/3}(1+x^{2/3})}$
9. $\int e^{3\log x}\left(x^4+1\right)^{-1} dx$
10. $\int \frac{\sec x\, dx}{\sqrt{\cos 2x}}$
11. $\int \sin^3 x \cos^2 x dx$

FORM 5:

$$\int \frac{dx}{\text{Quadratic}}$$

Standard Formulae

1. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$

Proof:
Let $x = a\tan\theta$, then $dx = a\sec^2\theta\, d\theta$

$$Now \int \frac{dx}{a^2+x^2} = \int \frac{a\sec^2\theta}{a^2+a^2\tan^2\theta} d\theta$$

$$= \int \frac{a\sec^2\theta}{a^2(1+\tan^2\theta)} d\theta$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$$

2. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{a+x}\right|+c$

Proof: $\int \frac{dx}{x^2-a^2}$

$$= \frac{1}{2a}\int\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx$$

$$= \frac{1}{2a}(\ln|x-a| - \ln|x+a|)+c$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+c$$

Example 7.50 Evaluate $\int \frac{1}{x^2-x+1}dx.$

Sol. $\int \frac{1}{x^2-x+1}dx$

$$= \int \frac{1}{(x-1/2)^2+3/4}dx$$

$$= \int \frac{1}{(x-1/2)^2+(\sqrt{3}/2)^2}dx$$

$$= \frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right)+C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$$

Example 7.51 Evaluate $\int \frac{1}{2x^2+x-1}dx.$

Sol. $\int \frac{1}{2x^2+x-1}dx$

$$= \frac{1}{2}\int \frac{1}{x^2+\frac{x}{2}-\frac{1}{2}}dx$$

$$= \frac{1}{2}\int \frac{1}{(x+1/4)^2-(3/4)^2}dx$$

$$= \frac{1}{2}\cdot\frac{1}{2(3/4)}\log\left|\frac{x+1/4-3/4}{x+1/4+3/4}\right|+C$$

$$= \frac{1}{3}\log\left|\frac{x-1/2}{x+1}\right|+C = \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right|+C$$

Example 7.52 Evaluate $\int \frac{\cos x}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}dx.$

Sol. $I = \int \frac{\cos x}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}dx$

$$= \int \frac{\cos x}{\sin^2 x - \sin^2\frac{\pi}{6}}dx$$

Let $\sin x = t$

$\Rightarrow \quad dt = \cos x\, dx$

$\Rightarrow \quad I = \int \frac{dt}{t^2-\frac{1}{4}}$

$$= \int \frac{dt}{t^2-\frac{1}{4}}$$

$$= \frac{1}{2\frac{1}{2}}\log\left|\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right|+C$$

$$= \log\left|\frac{2t-1}{2t+1}\right| + C$$

$$= \log\left|\frac{2\sin x-1}{2\sin x+1}\right| + C$$

FORM 6:

$$\int \frac{dx}{a\cos^2 x + b\sin^2 x}, \int \frac{dx}{a+b\sin^2 x},$$

$$\int \frac{1}{a+b\cos^2 x}dx, \int \frac{1}{(a\sin x + b\cos x)^2}dx,$$

$$\int \frac{1}{a+b\sin^2 x + \cos^2 x}dx$$

Working Rule:

To evaluate this type of integrals, divide both the numerator and denominator by $\cos^2 x$, replace $\sec^2 x$, if any, in the denominator by $(1 + \tan^2 x)$ and put $\tan x = t$. So that $\sec^2 x\,dx = dt$.

Example 7.53 Evaluate $\int \frac{\sin x}{\sin 3x}dx$

Sol. $I = \int \frac{\sin x}{\sin 3x}dx = \int \frac{\sin x}{3\sin x - 4\sin^3 x}dx$

$$= \int \frac{1}{3-4\sin^2 x}dx$$

$$= \int \frac{\sec^2 x}{3\sec^2 x - 4\tan^2 x}dx \quad \text{[Dividing } N^r \text{ and } D^r \text{ by } \cos^2 x]$$

Putting $\tan x = t$ and $\sec^2 x\,dx = dt$, we get

$$I = \int \frac{dt}{3(1+t^2) - 4t^2} = \int \frac{dt}{3-t^2} = \int \frac{1}{(\sqrt{3})^2 - t^2}dt$$

$$= \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + C = \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + C$$

Example 7.54 Evaluate $\int \frac{1}{3+\sin 2x}dx.$

Sol. $I = \int \frac{1}{3+\sin 2x}dx$

$$= \int \frac{1}{3(\sin^2 x + \cos^2 x) + 2\sin x\cos x}dx$$

$$\Rightarrow \quad I = \int \frac{\sec^2 x}{3\tan^2 x + 2\tan x + 3}dx$$

[Dividing N^r and D^r by $\cos^2 x$]

Putting $\tan x = t$ and $\sec^2 x\,dx = dt$, we get

$$\therefore \quad I = \int \frac{dt}{3t^2 + 2t + 3} = \frac{1}{3}\int \frac{dt}{t^2 + \frac{2}{3}t + 1}$$

$$= \frac{1}{3}\int \frac{dt}{\left(t+\frac{1}{3}\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2}$$

$$\therefore \quad I = \frac{1}{3}\,\frac{1}{\left(\frac{2\sqrt{2}}{3}\right)}\tan^{-1}\left(\frac{t+\frac{1}{3}}{\frac{2\sqrt{2}}{3}}\right) + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{3t+1}{2\sqrt{2}}\right) + C$$

FORM 7:

$$\int \frac{1}{a+b\sin x + c\cos x}dx$$

Working Rule:

Write $\sin x$ and $\cos x$ in terms of $\tan(x/2)$, and then substitute t for $\tan(x/2)$.

Example 7.55 Evaluate $\int \frac{1}{1+\sin x+\cos x}dx.$

Sol. Putting $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$ and $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$,

we have

$$I = \int \frac{1}{1+\sin x + \cos x}dx$$

$$= \int \frac{1}{1 + \frac{2\tan x/2}{1+\tan^2 x/2} + \frac{1-\tan^2 x/2}{1+\tan^2 x/2}}dx$$

$$= \int \frac{1+\tan^2 x/2}{1+\tan^2 x/2 + 2\tan x/2 + 1 - \tan^2 x/2}dx$$

$$= \int \frac{\sec^2 x/2}{2+2\tan x/2}dx$$

Putting $\tan\frac{x}{2} = t$ and $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

or, $\sec^2\frac{x}{2}\,dx = 2dt$, we get

$$I = \int \frac{2dt}{2+2t} = \int \frac{1}{t+1}dt = \log|t+1| + C$$

$$= \log\left|\tan\frac{x}{2} + 1\right| + C$$

FORM 8:

$$\int \frac{p\cos x + q\sin x + r}{a\cos x + b\sin x + c}dx$$

Working Rule:

In this integral, express numerator as λ (denominator) + μ (differentiation of denominator) + γ.

Find λ, μ and γ by comparing coefficients of $\sin x$, $\cos x$ and constant term and splitting the integral into the sum of three integrals.

$$\lambda\int dx+\mu\int\frac{\text{differentiation of (denominator)}}{\text{denominator}}dx+n\int\frac{dx}{a\sin x+b\cos x+c}$$

Example 7.56 Evaluate $\int\frac{3\sin x+2\cos x}{3\cos x+2\sin x}dx$.

Sol. We have, $I=\int\frac{3\sin x+2\cos x}{3\cos x+2\sin x}dx$

Let $3\sin x+2\cos x=\mu\frac{d}{dx}(3\cos x+2\sin x)+\lambda(3\cos x+2\sin x)$

$\Rightarrow$ $3\sin x+2\cos x=\mu(-3\sin x+2\cos x)+\lambda(3\cos x+2\sin x)$

Comparing the coefficients of $\sin x$ and $\cos x$ on both sides, we get

$-3\mu+2\lambda=3$ and $2\mu+3\lambda=2\Rightarrow\lambda=\frac{12}{13}$ and $\mu=-\frac{5}{13}$

$$\therefore\ I=\int\frac{\mu(-3\sin x+2\cos x)+\lambda(3\cos x+2\sin x)}{3\cos x+2\sin x}dx$$

$$=\lambda\int 1\,dx+\mu\int\frac{-3\sin x+2\cos x}{3\cos x+2\sin x}dx$$

$$=\lambda x+\mu\int\frac{dt}{t},\text{ where } t=3\cos x+2\sin x$$

$$=\lambda x+\mu\log|t|+C=\frac{12}{13}x+\frac{-5}{13}\log|3\cos x+2\sin x|+C$$

Bi-quadratic Form

Example 7.57 Evaluate $\int\frac{x^2+1}{x^4+1}dx$.

Sol. $$I=\int\frac{x^2+1}{x^4+1}dx=\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx=\int\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx$$

Let $x-\frac{1}{x}=t\Rightarrow d\left(x-\frac{1}{x}\right)=dt\Rightarrow\left(1+\frac{1}{x^2}\right)dx=dt$

$$\therefore\quad I=\int\frac{dt}{t^2+(\sqrt{2})^2}=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+C$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right)+C$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\,x}\right)+C$$

Example 7.58 Evaluate $\int\frac{x^2-1}{x^4+x^2+1}dx$.

Sol. $$I=\int\frac{x^2-1}{x^4+x^2+1}dx=\int\frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}dx$$

$$=\int\frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-1^2}dx$$

Let $x+\frac{1}{x}=u$, then $d\left(x+\frac{1}{x}\right)=du\Rightarrow\left(1-\frac{1}{x^2}\right)dx=du$

$$\Rightarrow\quad I=\int\frac{du}{u^2-1^2}=\frac{1}{2(1)}\log\left|\frac{u-1}{u+1}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right|+C=\frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right|+C$$

Example 7.59 Evaluate $\int\frac{x^2+4}{x^4+16}dx$.

Sol. $$I=\int\frac{x^2+4}{x^4+16}dx=\int\frac{1+\frac{4}{x^2}}{x^2+\frac{16}{x^2}}dx$$

$$=\int\frac{1+\frac{4}{x^2}}{x^2+\left(\frac{4}{x}\right)^2-8+8}dx$$

$$=\int\frac{1+\frac{4}{x^2}}{\left(x-\frac{4}{x}\right)^2+8}dx$$

Let $x-\frac{4}{x}=t$, then $d\left(x-\frac{4}{x}\right)=dt\Rightarrow\left(1+\frac{4}{x^2}\right)dx=dt$

$$\therefore\quad I=\int\frac{dt}{t^2+(2\sqrt{2})^2}=\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t}{2\sqrt{2}}\right)+C$$

$$=\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{4}{x}}{2\sqrt{2}}\right)+C$$

$$=\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-4}{2x\sqrt{2}}\right)+C$$

Example 7.60 Evaluate $\int\sqrt{\tan\theta}\,d\theta$.

Sol. Let $I=\int\sqrt{\tan\theta}\,d\theta$

Let $\tan\theta = x^2$. Then, $d(\tan\theta) = d(x^2) \Rightarrow \sec^2\theta\, d\theta = 2x\,dx$

or, $d\theta = \dfrac{2x.dx}{\sec^2\theta} = \dfrac{2x\,dx}{1+\tan^2\theta} = \dfrac{2x\,dx}{1+x^4}$

$$I = \int \sqrt{x^2} \times \frac{2x\,dx}{1+x^4} = \int \frac{2x^2}{x^4+1}dx$$

$$= \int \frac{2}{x^2+1/x^2}dx$$

$$I = \int \frac{1+1/x^2+1-1/x^2}{x^2+1/x^2}dx$$

$$= \int \frac{1+1/x^2}{x^2+1/x^2}dx + \int \frac{1-1/x^2}{x^2+1/x^2}dx$$

$$= \int \frac{1+1/x^2}{(x-1/x)^2+2}dx + \int \frac{1-1/x^2}{(x+1/x)^2-2}dx$$

Putting $x - \dfrac{1}{x} = u$ in 1st integral and $x + \dfrac{1}{x} = v$ in 2nd integral, we get

$$I = \int \frac{du}{u^2+(\sqrt{2})^2} + \int \frac{dv}{v^2-(\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{x+1/x-\sqrt{2}}{x+1/x+\sqrt{2}}\right| + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right| + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan\theta-1}{\sqrt{2\tan\theta}}\right)$$

$$+ \frac{1}{2\sqrt{2}}\log\left|\frac{\tan\theta-\sqrt{2\tan\theta}+1}{\tan\theta+\sqrt{2\tan\theta}+1}\right| + C$$

Example 7.61 Evaluate $\displaystyle\int \frac{x^2-1}{(x^2+1)\sqrt{1+x^4}}dx$.

Sol. $$I = \int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}}dx$$

$$= \int \frac{x^2(1-1/x^2)}{x^2(x+1/x)\sqrt{x^2+1/x^2}}dx$$

$$= \int \frac{(1-1/x^2)dx}{(x+1/x)\sqrt{(x+1/x)^2-2}}$$

Putting $x + 1/x = t$, we have $I = \displaystyle\int \frac{dt}{t\sqrt{t^2-2}}$

Again putting $t^2 - 2 = y^2$, $2t\,dt = 2y\,dy$,

$$I = \int \frac{y\,dy}{(y^2+2)y} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{y}{\sqrt{2}} = \frac{1}{2}\tan^{-1}\frac{\sqrt{x^2+1/x^2}}{\sqrt{2}} + c.$$

Concept Application Exercise 7.5

Evaluate the following:

1. $\displaystyle\int \frac{1}{2x^2+x-1}dx$
2. $\displaystyle\int \frac{x}{x^4+x^2+1}dx$
3. $\displaystyle\int \frac{(4x+1)\,dx}{x^2+3x+2}$
4. $\displaystyle\int \frac{x^3+x+1}{x^2-1}dx$
5. $\displaystyle\int \frac{x^2-1}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)}dx$
6. $\displaystyle\int \frac{1}{x^4+1}dx$
7. $\displaystyle\int \frac{1}{\sin^4 x+\cos^4 x}dx$

Some Standard Trigonometric Substitutions

Expression	Substitution
a^2+x^2	$x = a\tan\theta$ or $a\cot\theta$
a^2-x^2	$x = a\sin\theta$ or $a\cos\theta$
x^2-a^2	$x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
$\sqrt{\dfrac{a-x}{a+x}}$ or $\sqrt{\dfrac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\dfrac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(x-\beta)}$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$

Example 7.62 Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}dx$.

Sol. Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\Rightarrow \int \frac{1}{x^2\sqrt{1+x^2}}dx = \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sec\theta}$$

$$= \int \text{cosec}\,\theta\cot\theta\, d\theta$$

$$= -\text{cosec}\,\theta + c$$

$$= \frac{-\sqrt{x^2+1}}{x} + c$$

Example 7.63 Evaluate $\displaystyle\int \frac{dx}{(a^2+x^2)^{3/2}}$.

Sol. $I = \displaystyle\int \frac{dx}{(a^2+x^2)^{3/2}}$

Put $x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$

$$\therefore \quad I = \int \frac{a\sec^2\theta}{(a^2 + a^2\tan^2\theta)^{3/2}}\, d\theta$$

$$= \int \frac{a\sec^2\theta}{a^3(\sec^2\theta)^{3/2}}\, d\theta$$

$$\Rightarrow \quad I = \frac{1}{a^2}\int \frac{d\theta}{\sec\theta} = \frac{1}{a^2}\int \cos\theta\, d\theta = \frac{1}{a^2}\sin\theta + c$$

$$\Rightarrow \quad I = \frac{x}{a^2(x^2 + a^2)^{1/2}} + c$$

FORM 9:

$$\int \frac{dx}{\sqrt{\text{Quadratic}}}$$

Standard Formulae

1. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$

Proof:
Let $x = a\sin\theta$, then $dx = a\cos\theta\, d\theta$.

$$\text{Now } \int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a\cos\theta}{\sqrt{a^2 - a^2\sin^2\theta}}\, d\theta$$

$$= \int \frac{a\cos\theta}{\sqrt{a^2(1 - \sin^2\theta)}}\, d\theta$$

$$= \int \frac{a\cos\theta}{a\cos\theta}\, d\theta = \int d\theta = \theta + c = \sin^{-1}\frac{x}{a} + c$$

$$\left[\because \sin\theta = \frac{x}{a} \therefore \theta = \sin^{-1}\frac{x}{a}\right]$$

2. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log(x + \sqrt{x^2 + a^2}) + c$

Proof: *Here integrand involves an expression of the form $\sqrt{a^2 + x^2}$; therefore substitution $x = a\tan\theta$ may be tried.*

Let $x = a\tan\theta$, then $dx = a\sec^2\theta\, d\theta$.

$$\text{Now, } \int \frac{dx}{\sqrt{a^2 + x^2}}$$

$$= \int \frac{a\sec^2\theta}{\sqrt{a^2 + a^2\tan^2\theta}}\, d\theta$$

$$= \int \frac{a\sec^2\theta}{\sqrt{a^2(1 + \tan^2\theta)}}\, d\theta$$

$$= \int \frac{a\sec^2\theta}{a\sec\theta}\, d\theta = \int \sec\theta\, d\theta = \log|\sec\theta + \tan\theta| + c \quad (1)$$

$$\because x = a\tan\theta \Rightarrow \tan\theta = \frac{x}{a}$$

$$\Rightarrow \sec\theta = \sqrt{1 + \tan^2\theta} = \sqrt{1 + \frac{x^2}{a^2}} = \frac{\sqrt{a^2 + x^2}}{a}$$

Now from equation (1), $\int \frac{dx}{\sqrt{a^2 + x^2}} = \log\left|\frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a}\right| + c$

$$= \log\left|\frac{\sqrt{a^2 + x^2} + x}{a}\right| + c$$

$$= \log|\sqrt{a^2 + x^2} + x| - \log|a| + c$$

$$= \log|x + \sqrt{a^2 + x^2}| + k.$$

3. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|\sqrt{x^2 - a^2} + x\right| + c$

Proof similar to 2.

Example 7.64 Evaluate $\int \frac{1}{\sqrt{(x-1)(x-2)}}\, dx.$

Sol.

$$I = \int \frac{1}{\sqrt{x^2 - 3x + 2}}\, dx$$

$$= \int \frac{1}{\sqrt{x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2}}\, dx$$

$$= \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\, dx$$

$$= \log\left|\left(x - \frac{3}{2}\right) + \sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + C$$

$$= \log\left|\left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2}\right| + C$$

Example 7.65 Evaluate $\int \frac{\sec^2 x}{\sqrt{16 + \tan^2 x}}\, dx.$

Sol.

$$I = \int \frac{\sec^2 x}{\sqrt{16 + \tan^2 x}}\, dx = \int \frac{\sec^2 x}{\sqrt{4^2 + \tan^2 x}}\, dx.$$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow \quad I = \int \frac{dt}{\sqrt{16 + t^2}} = \int \frac{dt}{\sqrt{4^2 + t^2}}$$

$$= \log\left|t + \sqrt{4^2 + t^2}\right| + C$$

$$= \log\left|\tan x + \sqrt{16 + \tan^2 x}\right| + C$$

Example 7.66 Evaluate $\int \frac{e^x}{\sqrt{4-e^{2x}}}dx.$

Sol. $$I=\int \frac{e^x}{\sqrt{4-e^{2x}}}dx=\int \frac{e^x}{\sqrt{2^2-\left(e^x\right)^2}}dx$$

Let $e^x=t \Rightarrow e^x\,dx=dt$

$$\therefore\quad I=\int \frac{dt}{\sqrt{4-t^2}}=\int \frac{dt}{\sqrt{2^2-t^2}}$$

$$=\sin^{-1}\left(\frac{t}{2}\right)+C=\sin^{-1}\left(\frac{e^x}{2}\right)+C$$

Example 7.67 Evaluate $\int \frac{e^x}{e^{2x}+6e^x+5}dx.$

Sol. $$I=\int \frac{e^x}{e^{2x}+6e^x+5}dx=\int \frac{e^x}{\left(e^x\right)^2+6e^x+5}dx$$

Let $e^x=t \Rightarrow e^x\,dx=dt$

$$\therefore\quad I=\int \frac{dt}{t^2+6t+5}$$

$$=\int \frac{1}{(t+3)^2-2^2}dt$$

$$=\frac{1}{2\times 2}\log\left|\frac{t+3-2}{t+3+2}\right|+C=\frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right|+C$$

FORM 10:

(i) $\int \frac{(px+q)\,dx}{ax^2+bx+c}$ (ii) $\int \frac{(px+q)dx}{\sqrt{ax^2+bx+c}}$

Working Rule:

This linear factor $(px+q)$ is expressed in terms of the derivative of the quadratic factor ax^2+bx+c together with a constant as: $px+q=\lambda\frac{d}{dx}\{ax^2+bx+c\}+\mu$

$\Rightarrow px+q=\lambda(2ax+b)+\mu$

Here, we have to find λ and μ and replace $(px+q)$ by $\{\lambda(2ax+b)+\mu\}$ in (i) and (ii).

Example 7.68 Evaluate $\int \frac{4x+1}{x^2+3x+2}dx.$

Sol. $$I=\int \frac{4x+1}{x^2+3x+2}dx$$

$$=\int \frac{2(2x+3)-5}{x^2+3x+2}dx$$

$$=2\int \frac{2x+3}{x^2+3x+2}dx-5\int \frac{1}{x^2+3x+2}dx$$

$$=2\log|x^2+3x+2|-5\int \frac{1}{x^2+3x+(9/4)-(9/4)+2}dx$$

$$=2\log|x^2+3x+2|-5\int \frac{1}{(x+3/2)^2-(1/2)^2}dx$$

$$=2\log|x^2+3x+2|-5\,\frac{1}{2(1/2)}\log\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right|+C$$

$$=2\log|x^2+3x+2|-5\log\left|\frac{x+1}{x+2}\right|+C$$

Example 7.69 Evaluate $\int \sqrt{\frac{1+x}{x}}dx.$

Sol. $\int \sqrt{\frac{1+x}{x}}dx$

$$=\int \sqrt{\frac{1+x}{x}}\sqrt{\frac{1+x}{1+x}}dx=\int \frac{1+x}{\sqrt{x(1+x)}}dx$$

$$=\int \frac{1+x}{\sqrt{x^2+x}}dx$$

$$=\frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x}}dx+\frac{1}{2}\int \frac{1}{\sqrt{x^2+x}}dx$$

$$=\frac{1}{2}\int \frac{1}{\sqrt{t}}dt+\frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx, \text{ where } t=x^2+x$$

$$=\sqrt{t}+\frac{1}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|+C$$

$$=\sqrt{x^2+x}+\frac{1}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|+C$$

Concept Application Exercise 7.6

Evaluate the following:

1. $\int \frac{x^2}{\sqrt{1-x^6}}dx$
2. $\int \sqrt{\frac{x}{a^3-x^3}}dx$
3. $\int \frac{1}{\sqrt{1-e^{2x}}}dx$
4. $\int \frac{2x+3}{\sqrt{x^2+4x+1}}dx$
5. $\int \frac{x^{5/2}}{\sqrt{1+x^7}}dx$
6. $\int x^3 d(\tan^{-1}x)$

INTEGRATION BY PARTS

Theorem

If u and v are two functions of x, then

$$\int uv\,dx=u\left(\int v\,dx\right)-\int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

That is the integral of product of two functions = (first function) × (integral of second function) − integral of (differential of first function × integral of second).

Proof:
For any two functions $f(x)$ and $g(x)$, we have

$$\frac{d}{dx}\{f(x)g(x)\}=f(x)\frac{d}{dx}(g(x))+g(x)\frac{d}{dx}\{f(x)\}$$

$$\Rightarrow \int\left(f(x)\frac{d}{dx}\{g(x)\}+g(x)\frac{d}{dx}\{f(x)\}\right)dx=f(x)\,g(x)$$

$$\Rightarrow \int\left(f(x)\frac{d}{dx}\{g(x)\}\right)dx+\int\left(g(x)\frac{d}{dx}\{f(x)\}\right)dx=f(x)g(x)$$

$$\Rightarrow \int\left(f(x)\frac{d}{dx}\{g(x)\}\right)dx$$

$$=f(x)\,g(x)-\int\left(g(x)\frac{d}{dx}\{f(x)\}\right)dx$$

Let, $f(x)=u$ and $\frac{d}{dx}\{g(x)\}=v$. So that $g(x)=\int v dx$

$$\therefore \int uv\,dx=u\left(\int v\,dx\right)-\int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

Note:

- *While applying the above rule, care has to be taken in the selection of the first function (u) and the second function (v). Normally we use the following methods:*
  (i) *If in the product of the two functions, one of the functions is not directly integrable (e.g., $\log|x|$, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, ...), then we take it as the first function and the remaining function is taken as the second function, e.g., in the integration of $\int x\tan^{-1}x\,dx$, $\tan^{-1}x$ is taken as the first function and x as the second function.*
  (ii) *If there is no other function, then unity is taken as the second function, e.g., in the integration of $\int \tan^{-1}x\,dx$, $\tan^{-1}x$ is taken as the first function and 1 as the second function.*
  (iii) *If both the functions are directly integrable, then the first function is chosen in such a way that the derivative of the function thus obtained under the integral sign is easily integrable. Usually, we use the following preference order for the first function: Inverse, Logarithmic, Algebraic, Trigonometric, Exponent.*

In the above stated order, the function on the left is always chosen as the first function. This rule is called as ***ILATE.***

Example 7.70 Evaluate $\int x\sin 3x dx$.

Sol. Here, both the functions, viz., x and $\sin 3x$ are easily integrable and the derivative of x is one, a less complicated function. Therefore, we take x as the first function and $\sin 3x$ as the second function.

$$\therefore \int \underset{\text{I}}{x}\,\underset{\text{II}}{\sin 3x}\,dx$$

$$=x\left\{\int \sin 3x\,dx\right\}-\int\left\{\frac{d}{dx}(x)\int \sin 3x\,dx\right\}dx$$

$$=-x\frac{\cos 3x}{3}-\int 1\left\{-\frac{\cos 3x}{3}\right\}dx$$

$$=-\frac{1}{3}x\cos 3x+\frac{1}{3}\int \cos 3x\,dx$$

$$=-\frac{1}{3}x\cos 3x+\frac{1}{9}\sin 3x+C$$

Example 7.71 Evaluate $\int x\log x dx$.

Sol. $\int \underset{\text{II}}{x}\,\underset{\text{I}}{\log x}\,dx$

$$=\log x\left\{\int x\,dx\right\}-\int\left\{\frac{d}{dx}(\log x)\int x\,dx\right\}dx$$

$$=(\log x)\frac{x^2}{2}-\int \frac{1}{x}\frac{x^2}{2}\,dx$$

$$=\frac{x^2}{2}\log x-\frac{1}{2}\int x\,dx$$

$$=\frac{x^2}{2}\log x-\frac{1}{2}\left(\frac{x^2}{2}\right)+C=\frac{x^2}{2}\log x-\frac{1}{4}x^2+C$$

Example 7.72 Evaluate $\int \sin^{-1}x dx$.

Sol. Let $\sin^{-1}x=t$. Then $x=\sin t\Rightarrow dx=\cos t\,dt$

$$\therefore \int \sin^{-1}x\,dx$$

$$=\int \underset{\text{I}}{t}\,\underset{\text{II}}{\cos t}\,dt$$

$$=t\sin t-\int 1(\sin t)\,dt$$

$$=t\sin t-\int \sin t\,dt$$

$$=t\sin t+\cos t+C=x\sin^{-1}x+\sqrt{1-\sin^2 t}+C$$

$$=x\sin^{-1}x+\sqrt{1-x^2}+C$$

Example 7.73 Evaluate $\int \frac{x-\sin x}{1-\cos x}dx$.

Sol. $\int \frac{x-\sin x}{1-\cos x}dx=\int \frac{x}{1-\cos x}-\int \frac{\sin x}{1-\cos x}dx$

$$=\int \frac{x}{2}\operatorname{cosec}^2\frac{x}{2}\,dx$$

$$-\int \frac{2\sin x/2\cos x/2}{2\sin^2 x/2}\,dx$$

$$= \frac{1}{2}\int \underset{\text{I}}{x}\, \underset{\text{II}}{\text{cosec}^2} \frac{x}{2} - \int \cot\frac{x}{2}\,dx$$

$$= \frac{1}{2}\left\{x\left(-2\cot\frac{x}{2}\right) - \int 1\left(-2\cot\frac{x}{2}\right)dx\right\}$$

$$- \int \cot\frac{x}{2}\,dx + C$$

$$= -x\cot\frac{x}{2} + \int \cot\frac{x}{2}\,dx - \int \cot\frac{x}{2}\,dx + C$$

$$= -x\cot\frac{x}{2} + C$$

Example 7.74 If $f(x)$ is a polynomial function of the nth degree, prove that $\int e^x f(x)dx = e^x[f(x) - f'(x) + f''(x) - f'''(x) + \cdots + (-1)^n f^{(n)}(x)]$ where $f^{(n)}(x)$ denotes $\frac{d^n f}{dx^n}$.

Sol. $I = \int e^x f(x)dx$

$= f(x)\,e^x - \int e^x f'(x)dx$

$= f(x)\,e^x - f'(x)\,e^x + \int e^x f''(x)dx$

$= f(x)\,e^x - f'(x)\,e^x + f''(x)\,e^x - \int e^x f'''(x)dx$

$= f(x)\,e^x - f'(x)\,e^x + f''(x)\,e^x - f'''(x)\,e^x + \int e^x f^{iv}(x)dx$

continuing this way we get $I = e^x[f(x) - f'(x) + f''(x) - f'''(x) \cdots + (-1)^n f^{(n)}(x)]$

Example 7.75 Evaluate $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$.

Sol. $I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$

Let $x = a\tan^2\theta \Rightarrow dx = 2a\tan\theta\sec^2\theta\, d\theta$

$\Rightarrow \int \sin^{-1}\sqrt{\sin^2\theta}\, 2a\sec^2\theta\tan\theta\, d\theta$

$$= 2a\int \underset{\text{I}}{\theta}\, \underset{\text{II}}{\sec^2\theta\tan\theta}\, d\theta$$

$$= 2a\left[\theta\frac{\tan^2\theta}{2} - \int \frac{\tan^2\theta}{2}d\theta\right]$$

$$= a\left[\theta\tan^2\theta - \int (\sec^2\theta - 1)d\theta\right]$$

$= a\left[\theta\tan^2\theta - \tan\theta + \theta\right] + c$, where,

$$\theta = \tan^{-1}\sqrt{\frac{x}{a}}$$

INTEGRATION BY CANCELLATION

Example 7.76 Evaluate $\int \left(\frac{\cos x}{x} - \log x^{\sin x}\right)dx$.

Sol. $\int \left(\frac{\cos x}{x} - \log x^{\sin x}\right)dx$

$= \int \frac{\cos x}{x}dx - \int \sin x\log x\,dx$

$= \cos x\log x - \int -\sin x\log x\,dx - \int \sin x\log x\,dx$

(integration of 1st integral by parts)

$= \cos x\log x + c$

Example 7.77 Find an anti-derivative of the function $f(x)\,g''(x) - f''(x)\,g(x)$.

Sol. The required anti-derivative is given by

$\int\{f(x)g''(x) - f''(x)g(x)\}dx$

$$= \int \underset{\text{I}}{f(x)}\, \underset{\text{II}}{g''(x)}\, dx - \int \underset{\text{II}}{f''(x)}\, \underset{\text{I}}{g(x)}\, dx$$

$$= \left\{f(x)\,g'(x) - \int f'(x)\,g'(x)\,dx\right\}$$

$$- \left\{g(x)\,f'(x) - \int f'(x)\,g'(x)\,dx\right\} + C$$

$= f(x)\,g'(x) - g(x)f'(x) + C$

Example 7.78 Evaluate $\int \left(3x^2\tan\frac{1}{x} - x\sec^2\frac{1}{x}\right)dx$.

Sol. $\int\left(3x^2\tan\frac{1}{x} - x\sec^2\frac{1}{x}\right)dx$

$= \int 3x^2\tan\frac{1}{x}dx - \int x\sec^2\frac{1}{x}dx$

$= \tan\frac{1}{x}x^3 - \int\left(\sec^2\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)x^3dx - \int x\sec^2\frac{1}{x}dx$

$= x^3\tan\frac{1}{x} + c$

Formula

$\int e^x(f(x) + f'(x))dx = e^x f(x) + C$

Example 7.79 Evaluate $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx$.

Sol. $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx$

$$= \int e^x\left(\frac{1}{x} + \left(\frac{1}{x}\right)'\right)dx$$

$$= \frac{1}{x}e^x + C$$

Example 7.80 Evaluate $\int e^x \frac{x}{(x+1)^2}dx$.

Sol. $\int e^x \frac{x}{(x+1)^2}dx$

$= \int e^x \frac{x+1-1}{(x+1)^2}dx$

$= \int e^x \left\{\frac{1}{x+1} + \left(\frac{1}{x+1}\right)'\right\}dx = \frac{1}{x+1}e^x + c$

Example 7.81 Evaluate $\int e^x \left(\frac{1-\sin x}{1-\cos x}\right)dx$.

Sol. $\int e^x \left(\frac{1-\sin x}{1-\cos x}\right)dx$

$= \int e^x \left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$

$= \int e^x \left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right)dx$

$= -\int e^x \left(\cot\frac{x}{2} + \left(\cot\frac{x}{2}\right)'\right)dx$

$= -e^x \cot\frac{x}{2} + c$

Example 7.82 Evaluate $\int\{\sin(\log x)+\cos(\log x)\}dx$.

Sol. $I = \int\{\sin(\log x)+\cos(\log x)\}\,dx$

Let $\log x = t$. Then $x = e^t \Rightarrow dx = e^t\,dt$

$\Rightarrow \quad I = \int e^t(\sin t + \cos t)\,dt$

$= e^t \sin t + c$

$= x\sin(\log x) + c$

Example 7.83 Evaluate $\int \frac{\log x}{(1+\log x)^2}dx$.

Sol. $I = \int \frac{\log x}{(1+\log x)^2}dx$

Let $\log x = t$. Then $x = e^t \Rightarrow dx = e^t\,dt$.

$\therefore \quad I = \int \frac{t\,e^t}{(t+1)^2}dt$

$= \int e^t\left(\frac{1}{(t+1)} - \frac{1}{(t+1)^2}\right)dt$

$= \frac{e^t}{t+1} + c = \frac{x}{(\log x+1)} + c$

Formula

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx) + c$$

Proof: $I = \int e^{ax}\sin bx\,dx$

$= e^{ax}\int \sin bx\,dx - \int ae^{ax}\frac{-\cos bx}{b}dx + c$

$= -e^{ax}\frac{\cos bx}{b} + \frac{a}{b}\int e^{ax}\cos bx\,dx + c$

$= -e^{ax}\frac{\cos bx}{b} + \frac{a}{b}\left[e^{ax}\int\cos bx\,dx - \int ae^{ax}\frac{\sin bx}{b}dx\right] + c$

$= -e^{ax}\frac{\cos bx}{b} + \frac{a}{b}\left[e^{ax}\frac{\sin bx}{b} - \frac{a}{b}I\right] + c$

$\Rightarrow \left(1+\frac{a^2}{b^2}\right)I = -e^{ax}\frac{\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx + c$

$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

Similarly, we can prove that

$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx) + c$

Example 7.84 Evaluate $\int e^{2x}\sin 3x\,dx$.

Sol. $\int e^{2x}\sin 3x\,dx$

$= \frac{e^{2x}}{2^2+3^2}(2\sin 3x - 3\cos 3x) + c$

$= \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + c$

Example 7.85 Evaluate $\int \sin(\log x)dx$.

Sol. Let $I = \int \sin(\log x)dx$

Let $\log x = t$. Then $x = e^t \Rightarrow dx = e^t dt$

$\therefore I = \int \sin t\, e^t\,dt = \frac{e^t}{2}(\sin t - \cos t) + c$

Hence, $\int \sin(\log x)dx = \frac{x}{2}[\sin(\log x) - \cos(\log x)] + c$

Example 7.86 Evaluate $\int e^{\sin^{-1}x}dx$.

Sol. $I = \int e^{\sin^{-1}x}dx$

let $\sin^{-1}x = t \Rightarrow x = \sin t \Rightarrow dx = \cos t\,dt$

$\Rightarrow \quad I = \int e^t \cos t\,dt$

$$= \frac{e^t}{2}(\sin t + \cos t) + c$$

$$= \frac{e^{\sin^{-1} x}}{2}\left(x + \sqrt{1-x^2}\right) + c$$

Concept Application Exercise 7.7

Evaluate the following:

1. $\int x \sin^2 x dx$
2. If $\int f(x)\, dx = g(x)$, then $\int f^{-1}(x)\, dx$
3. If $\int g(x)dx = g(x)$, then $\int g(x)\{f(x) + f'(x)\}dx$
4. $\int \cos\sqrt{x}\, dx$
5. $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$
6. $\int \tan^{-1} \sqrt{x}\, dx$
7. $\int \cos x \log\left(\tan \frac{x}{2}\right) dx$
8. $\int \left(\frac{\log x - 1}{1 + (\log x)^2}\right)^2 dx$
9. $\int \frac{e^x(2 - x^2)dx}{(1-x)\sqrt{1-x^2}}$
10. $\int e^x (1 + \tan x + \tan^2 x)\, dx$
11. $\int \sin^2(\log x) dx$
12. $\int [f(x)g''(x) - f''(x)g(x)]\, dx$
13. $\int \tan^{-1} \sqrt{\frac{1-x}{1+x}}\, dx$

INTEGRATION BY PARTIAL FRACTIONS

Some Definitions

Polynomial of Degree n

An expression of the type $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ where $a_0, a_1, a_2, \ldots, a_n$ are real numbers, $a_0 \neq 0$ and n, a positive integer, is called a polynomial of degree n.

Rational Function

A function of the form $\frac{P}{Q}$, where P and Q are polynomials is called *rational function* e.g. $\frac{x}{x^2+1}, \frac{x^3+3x}{x^4-x^3+x}$.

Proper and Improper Fractions

Any rational algebraic function is called a *proper fraction* if the degree of numerator is less than that of its denominator, otherwise it is called an *improper fraction.*

For example $\frac{x^2+x+2}{x^3+4x^2-7x+1}$ is a proper fraction.

Whereas $\frac{x^4-9x^2-10x+7}{x^2+4x+5}$

$= \left\{(x^2 - 4x + 2) + \left|\frac{2x-3}{x^2+x+5}\right.\right\}$ is an improper fraction.

To integrate the rational function on the L.H.S., it is enough to integrate the two fractions on the R.H.S, which easy. This is known as the method of partial fractions. Here, we assume that the denominator can be fractional into linear or quadratic factors.

Partial Fractions

Consider the rational function $\frac{x+7}{(2x-3)(3x+4)}$ $= \frac{1}{2x-3} - \frac{1}{3x+4}$. The two fractions on the R.H.S. are called the *partial fractions*.

Note:

While using the method of partial fractions, we must have the degree of polynomial in numerator P(x) always less than that of denominator Q(x). If it is not so, then we carry out the division of P(x) by Q(x) and reduce the degree of the numerator to less than that of the denominator.

i.e., $\frac{P(x)}{Q(x)} = P_1(x) + \frac{P_2(x)}{Q(x)}$, *where the degree of* $P_2(x)$ < *degree of* $Q(x)$, *then to integrate, we apply the method of partial fractions to* $\frac{P_2(x)}{Q(x)}$.

The partial fractions depend on the nature of the factors of Q (x). We have to deal with the following different type when the factors of Q (x) are

(i) *linear and non-repeated.*

(ii) *linear and repeated.*

(iii) *quadratic and non-repeated.*

Case I: *When denominator is expressible as the product of non-repeated linear factors:*

Let $Q(x) = (x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n)$.

Then, we assume that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \frac{A_3}{(x-a_3)} + \cdots + \frac{A_n}{(x-a_n)}$$

where $A_1, A_2, \ldots, A_n$ are constants and can be determined by equating the numerator on R.H.S. to the numerator on L.H.S. and then substituting $x = a_1, a_2, \ldots, a_n$.

Shortcut method: *Consider $x - a_1 = 0$, then $x = a_1$, put this value of x in all the expressions other than $x - a_1$ and so on, e.g.,* $\frac{x^2+1}{x(x-1)(x+1)} = \frac{0+1}{x(0-1)(0+1)}$

$$+\frac{1+1}{1(x-1)(1+1)}+\frac{1+1}{-1(-1-1)(x+1)}$$

Case II: *When the denominator g(x) is expressible as the product of the linear factors such that some of them are repeating.* **(Linear and Repeated).**

Let $Q(x) = (x-a)^k (x-a_1)(x-a_2) \ldots (x-a_r)$. Then we assume that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots$$

$$\frac{A_k}{(x-a)^k} + \frac{B_1}{(x-a_1)} + \frac{B_2}{(x-a_2)} + \cdots + \frac{B_r}{(x-a_r)}$$

Case III: *When some of the factors in denominator are quadratic but non-repeating. Corresponding to each quadratic factor $ax^2 + bx + c$, we assume the partial fraction of the type $\frac{Ax+B}{ax^2+bx+C}$, where A and B are constants to be determined by comparing the coefficients of similar power of x in numerator on both the sides.*

Example 7.87 Evaluate $\int \frac{2x-1}{(x-1)(x+2)(x-3)} dx$.

Sol. Since all the factors in the denominator are linear, we have

$$\int \frac{2x-1}{(x-1)(x+2)(x-3)} dx$$

$$= \int \left[\frac{1}{(x-1)(3)(-2)} + \frac{-5}{(-3)(x+2)(-5)} + \frac{5}{(2)(5)(x-3)}\right] dx$$

$$= -\frac{1}{6}\log|x-1| - \frac{1}{3}\log|x+2| + \frac{1}{2}\log|x-3| + C$$

Example 7.88 Evaluate $\int \frac{2x}{(x^2+1)(x^2+2)} dx$.

Sol. Let $I = \int \frac{2x}{\left(x^2+1\right)\left(x^2+2\right)} dx$

Putting $x^2 = t$ and $2x\,dx = dt$, we get

$$I = \int \frac{dt}{(t+1)(t+2)} = \int \left(\frac{1}{t+1} - \frac{1}{t+2}\right) dt$$

$$= \log|t+1| - \log|t+2| + C$$

$$= \log|x^2+1| - \log|x^2+2| + C$$

Example 7.89 Evaluate $\int \frac{1}{\sin x - \sin 2x} dx$.

Sol. $I = \int \frac{1}{\sin x - \sin 2x} dx$

$$= \int \frac{1}{(\sin x - 2\sin x \cos x)} dx$$

$$= \int \frac{1}{\sin x(1-2\cos x)} dx$$

$$= \int \frac{\sin x}{\sin^2 x(1-2\cos x)} dx$$

$$= \int \frac{\sin x}{\left(1-\cos^2 x\right)(1-2\cos x)} dx$$

Putting $\cos x = t$, and $-\sin x\,dx = dt$ or $\sin x\,dx = -dt$, we get

$$I = \int \frac{-dt}{\left(1-t^2\right)(1-2t)}$$

$$= \int \frac{1}{(t-1)(1+t)(1-2t)} dt$$

$$= \int \left(\frac{1}{(t-1)(2)(-1)} + \frac{1}{(-2)(1+t)(3)} + \frac{1}{(-1/2)(3/2)(1-2t)}\right) dt$$

$$= -\frac{1}{2}\log|1-t| - \frac{1}{6}\log|1+t| + \frac{2}{3}\log|1-2t| + C$$

$$= -\frac{1}{2}\log|1-\cos x| - \frac{1}{6}\log|1+\cos x| + \frac{2}{3}\log|1-2\cos x| + C$$

Example 7.90 Evaluate $\int \frac{1-\cos x}{\cos x(1+\cos x)} dx$.

Sol. Let $I = \int \frac{1-\cos x}{\cos x(1+\cos x)} dx$.

Let $\cos x = y$.

Then $\frac{1-\cos x}{\cos x(1+\cos x)} = \frac{1-y}{y(1+y)} = \frac{1}{y} - \frac{2}{1+y}$

$$= \frac{1}{\cos x} - \frac{2}{1+\cos x}$$

$$\therefore I = \int \frac{1-\cos x}{\cos x(1+\cos x)} dx = \int \frac{1}{\cos x} dx - \int \frac{2}{1+\cos x} dx$$

$$\Rightarrow \quad I = \int \sec x\,dx - \int \frac{2}{2\cos^2 x/2} dx$$

$$= \int \sec x - \int \sec^2 x/2\,dx$$

$$\Rightarrow \quad I = \log|\sec x + \tan x| - 2\tan x/2 + C$$

Example 7.91 Evaluate $\int \frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}dx$.

Sol. Improper fraction

$$I=\int \frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}\,dx$$ proper fraction

$$=\int\left[1+\frac{(x-1)(x-2)(x-3)-(x-4)(x-5)(x-6)}{(x-4)(x-5)(x-6)}\right]dx$$

(adding and subtracting 1)

$$=\int\left[1+\frac{3\times2\times1}{(x-4)(-1)(-2)}+\frac{4\times3\times2}{1(x-5)(-1)}+\frac{5\times4\times3}{(2)(1)(x-6)}\right]dx$$

$$=1+3\log|x-4|-24\log|x-5|+30\log|x-6|+C$$

Example 7.92 Evaluate $\int \frac{x^2+1}{(x-1)^2(x+3)}dx$.

Sol. $E=\int \frac{x^2+1}{(x-1)^2\,(x+3)}dx$

Let $\frac{x^2+1}{(x-1)^2\,(x+3)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+3}$ (1)

$\Rightarrow x^2+1=A\,(x-1)\,(x+3)+B\,(x+3)+C\,(x-1)^2$ (2)

Putting $x-1=0$, i.e., $x=1$ in equation (2), we get $2=4B\Rightarrow B=\frac{1}{2}$. Putting $x+3=0$, i.e., $x=-3$ in equation (2), we get $10=16C\Rightarrow C=\frac{5}{8}$

Equating the coefficient of x^2 on both the sides of the identity of equation (2), we get $1=A+C\Rightarrow A=1-C=1-\frac{5}{8}=\frac{3}{8}$

Substituting the values of A, B in equation(1), we get

$$\frac{x^2+1}{(x-1)^2\,(x+3)}=\frac{3}{8}\frac{1}{x-1}+\frac{1}{2}\frac{1}{(x-1)^2}+\frac{5}{8}(x+3)$$

$$\Rightarrow I=\frac{3}{8}\int\frac{1}{x-1}dx+\frac{1}{2}\int\frac{1}{(x-1)^2}dx+\frac{5}{8}\int\frac{1}{x+3}dx$$

$$\Rightarrow I=\frac{3}{8}\log|x-1|-\frac{1}{2(x-1)}+\frac{5}{8}\log|x+3|+C$$

Example 7.93 Evaluate $\int \frac{x}{(x-1)(x^2+4)}dx$.

Sol. $\int \frac{x}{(x-1)(x^2+4)}dx$

Let $\frac{x}{(x-1)\,(x^2+4)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+4}$ (1)

$\Rightarrow \quad x=A\,(x^2+4)+(Bx+C)\,(x-1)$ (2)

Putting $x=1$ in equation (2), we get $1=5A$

Putting $x=0$ in equation (2), we get $0=4A-C$

Putting $x=-1$ in equation (2), we get $-1=5A+2B-2C$

Solving these equations, we obtain $A=\frac{1}{5}, B=-\frac{1}{5}$ and $C=\frac{4}{5}$

Substituting the values of A, B and C in equation (1), we obtain

$$\frac{x}{(x-1)\,(x^2+4)}=\frac{1}{5(x-1)}+\frac{-\frac{1}{5}x+\frac{4}{5}}{x^2+4}$$

$$=\frac{1}{5(x-1)}-\frac{1}{5}\frac{(x-4)}{(x^2+4)}$$

$$\Rightarrow I=\frac{1}{5}\int\frac{1}{x-1}dx-\frac{1}{5}\int\frac{x-4}{x^2+4}dx$$

$$=\frac{1}{5}\int\frac{1}{x-1}dx-\frac{1}{10}\int\frac{2x}{x^2+4}dx+\frac{4}{5}\int\frac{1}{x^2+4}dx$$

$$=\frac{1}{5}\log|x-1|-\frac{1}{10}\log(x^2+4)+\frac{4}{5}\times\frac{1}{2}\tan^{-1}\frac{x}{2}+C$$

$$=\frac{1}{5}\log|x-1|-\frac{1}{10}\log(x^2+4)+\frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right)+C$$

Example 7.94 Evaluate $\int \frac{x^2}{(x^2+1)(x^2+4)}dx$.

Sol. $\int \frac{x^2}{(x^2+1)(x^2+4)}dx=\frac{1}{3}\int\left[\frac{4}{x^2+4}-\frac{1}{x^2+1}\right]dx$

$$=-\frac{1}{3}\int\frac{1}{x^2+1}dx+\frac{4}{3}\int\frac{1}{x^2+4}dx$$

$$=-\frac{1}{3}\tan^{-1}x+\frac{4}{3}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$$

$$=-\frac{1}{3}\tan^{-1}x+\frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right)+C$$

Example 7.95 Evaluate $\int \frac{\sin x}{\sin 4x}dx$.

Sol. $I=\int\frac{\sin x}{\sin 4x}\,dx=\int\frac{\sin x}{2\sin 2x\,\cos 2x}dx$

$$=\int\frac{\sin x}{4\sin x\cos x\,\cos 2x}dx$$

$$=\frac{1}{4}\int\frac{1}{\cos x\cos 2x}dx=\frac{1}{4}\int\frac{\cos x}{\cos^2 x\cos 2x}dx$$

$$=\frac{1}{4}\int\frac{\cos x}{(1-\sin^2 x)\,(1-2\sin^2 x)}dx$$

Putting $\sin x=t$ and $\cos x\,dx=dt$, we get

$$I=\frac{1}{4}\int\frac{dt}{(1-t^2)\,(1-2t^2)}$$

$$I=\frac{1}{4}\int\left[\frac{2}{1-2t^2}-\frac{1}{1-t^2}\right]dt$$

$$= -\frac{1}{4}\int\frac{1}{1-t^2}dt + \frac{2}{4}\int\frac{1}{1-\left(\sqrt{2}t\right)^2}dt$$

$$= -\frac{1}{4}\times\frac{1}{2}\log\left|\frac{1+t}{1-t}\right| + \frac{1}{2}\cdot\frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}\,t}{1-\sqrt{2}\,t}\right| + C$$

$$= -\frac{1}{8}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + C$$

Concept Application Exercise 7.8

Evaluate the following:

1. $\int\frac{1}{(x^2-4)\sqrt{x+1}}dx$
2. $\int\frac{x^2+1}{x\left(x^2-1\right)}dx$
3. $\int\frac{\sin x}{\sin 4x}dx$
4. $\int\frac{x^3}{(x-1)(x-2)}dx$
5. $\int\frac{dx}{\sin x(3+\cos^2 x)}$
6. $\int\frac{\cos 2x\sin 4x dx}{\cos^4 x(1+\cos^2 2x)}$

INTEGRATIONS OF IRRATIONAL FUNCTIONS

FORM 11:

$\int\sqrt{\text{Quadratic}}\,dx$

Standard Formulae

1. $\int\sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| + C$

Proof: $I = \int\sqrt{a^2+x^2}\,dx$

$$= \sqrt{a^2+x^2}\int 1dx - \int\left[\frac{d}{dx}\left(\sqrt{a^2+x^2}\right)\int 1dx\right]dx + C$$

$$= x\sqrt{a^2+x^2} - \int\frac{x}{\sqrt{a^2+x^2}}x dx + C$$

$$= x\sqrt{a^2+x^2} - \int\frac{a^2+x^2-a^2}{\sqrt{a^2+x^2}}dx + C$$

$$= x\sqrt{a^2+x^2} - \int\sqrt{a^2+x^2}\,dx + \int\frac{a^2}{\sqrt{a^2+x^2}}dx + C$$

$$\Rightarrow \quad 2I = x\sqrt{a^2+x^2} + a^2\ln\left|x+\sqrt{a^2+x^2}\right| + C$$

$$\Rightarrow \quad I = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| + C$$

2. $\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

3. $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

Example 7.96 Evaluate $\int\sqrt{x^2+2x+5}dx$.

Sol. $\int\sqrt{x^2+2x+5}\,dx = \int\sqrt{(x+1)^2+4}\,dx$

$$= \frac{1}{2}(x+1)\sqrt{(x+1)^2+2^2} + \frac{1}{2}\cdot(2)^2\log\left|(x+1)+\sqrt{(x+1)^2+2^2}\right| + C$$

$$= \frac{1}{2}(x+1)\sqrt{x^2+2x+5} + 2\log\left|(x+1)+\sqrt{x^2+2x+5}\right| + C$$

Example 7.97 Evaluate $\int\frac{\sqrt{1-\sqrt{x}}}{\sqrt{1+\sqrt{x}}}dx$.

Sol. $I = \int\frac{\sqrt{1-\sqrt{x}}}{\sqrt{1+\sqrt{x}}}dx$

Putting $x = t^2$ and $dx = 2t\,dt$, we get $I = 2\int\sqrt{\frac{1-t}{1+t}}\,t\,dt$

$$\Rightarrow I = 2\int\frac{t(1-t)}{\sqrt{1-t^2}}dt$$

$$I = \int\frac{2t}{\sqrt{1-t^2}}dt + 2\int\frac{-t^2}{\sqrt{1-t^2}}dt$$

$$I = \int\frac{2t}{\sqrt{1-t^2}}dt + 2\int\frac{\left(1-t^2\right)-1}{\sqrt{1-t^2}}dt$$

$$I = -\int\frac{-2t}{\sqrt{1-t^2}}dt + 2\int\sqrt{1-t^2}\,dt - 2\int\frac{1}{\sqrt{1-t^2}}dt$$

$$I = -2\sqrt{1-t^2} + 2\times\frac{1}{2}\left[t\sqrt{1-t^2} + \sin^{-1}t\right] - 2\sin^{-1}t + C$$

$$I = -2\sqrt{1-t^2} + t\sqrt{1-t^2} - \sin^{-1}t + C$$

$$I = \sqrt{1-x}\left(\sqrt{x}-2\right) - \sin^{-1}\sqrt{x} + C$$

FORM 12:

$\int\text{Linear}\sqrt{\text{Quadratic}}\,dx$

Working Rule:

Substitute for Linear = m (Quadratic)$'$ + n, where find m and n by comparing co-efficient of x and constant term.

Example 7.98 Evaluate $\int(x-5)\sqrt{x^2+x}dx$.

Sol. Let $(x-5) = \lambda\frac{d}{dx}\left(x^2+x\right) + \mu$. Then,

$x-5 = \lambda(2x+1)+\mu$

Comparing coefficient of like power of x, we get $1 = 2\lambda$ and

$\lambda + \mu = -5 \Rightarrow \lambda = \frac{1}{2}$ and $\mu = -\frac{11}{2}$

$$\therefore \int (x-5)\sqrt{x^2+x}\, dx$$

$$= \int \left[\frac{1}{2}(2x+1) - \frac{11}{2}\right]\sqrt{x^2+x}\, dx$$

$$= \frac{1}{2}\int (2x+1)\sqrt{x^2+x}\, dx - \frac{11}{2}\int \sqrt{x^2+x}\, dx$$

$$= \frac{1}{2}\int \sqrt{t}\, dt - \frac{11}{2}\int \sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\, dx,$$

where $t = x^2 + x$

$$= \frac{1}{2}\frac{t^{3/2}}{3/2} - \frac{11}{2}\left[\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right.$$

$$\left. + \frac{1}{2}\left(\frac{1}{2}\right)^2 \log\left|\left(x+\frac{1}{2}\right) + \sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right|\right] + C$$

FORM 13:

$$\int \frac{1}{(\text{Linear})_1\sqrt{(\text{Linear})_2}}dx,\ \int \frac{(\text{Linear})_1}{\sqrt{(\text{Linear})_2}}dx,\ \int \frac{\sqrt{(\text{Linear})_2}}{(\text{Linear})_1}dx$$

Working Rule:

Substitute t^2 for $(\text{Linear})_2$

Example 7.99 Evaluate $\int \frac{1}{(x-3)\sqrt{x+1}}dx.$

Sol. Let $I = \int \frac{1}{(x-3)\sqrt{x+1}}dx$

Let $x + 1 = t^2$ and $dx = 2t\, dt$

$$\therefore I = \int \frac{1}{(t^2-1-3)}\frac{2t}{\sqrt{t^2}}dt$$

$$\Rightarrow I = 2\int \frac{dt}{t^2 - 2^2} = 2 \times \frac{1}{2(2)}\log\left|\frac{t-2}{t+2}\right| + C$$

$$\Rightarrow I = \frac{1}{2}\log\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right| + C$$

FORM 14:

$\int \frac{1}{\text{Linear}\sqrt{\text{Quadratic}}}dx$, Substitute for $\frac{1}{t} = \text{Linear}$

Example 7.100 Evaluate $\int \frac{1}{(x+1)\sqrt{x^2-1}}dx.$

Sol. Let $I = \int \frac{1}{(x+1)\sqrt{x^2-1}}dx$

Putting $x + 1 = \frac{1}{t}$ and $dx = -\frac{1}{t^2}dt$, we get

$$\therefore \quad I = \int \frac{1}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2 - 1}}\left(-\frac{1}{t^2}\right)dt$$

$$= -\int \frac{dt}{\sqrt{1-2t}} = -\int (1-2t)^{-1/2}\, dt$$

$$= -\frac{(1-2t)^{1/2}}{(-2)\left(\frac{1}{2}\right)} + C = \sqrt{1-2t} + C$$

$$= \sqrt{1 - \frac{2}{x+1}} + C = \sqrt{\frac{x-1}{x+1}} + C$$

FORM 15:

$$\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$$

Working Rule: Substitute for $x = \frac{1}{t}$, then the integrand reduces to $\int \frac{t\,dt}{(pt^2+q)\sqrt{rt^2+s}}$, and then substitute u^2 for $rt^2 + s$.

Example 7.101 Evaluate $\int \frac{1}{\left(1-x^2\right)\sqrt{1+x^2}}dx.$

Sol. Putting $x = \frac{1}{t}$ and $dx = -\frac{1}{t^2}dt$, we get

$$I = \int \frac{\left(-\frac{1}{t^2}\right)dt}{\left(1-\frac{1}{t^2}\right)\sqrt{1+\frac{1}{t^2}}} = -\int \frac{t\, dt}{\left(t^2-1\right)\sqrt{t^2+1}}$$

Let $t^2 + 1 = u^2$, we get $\quad 2tdt = 2udu$

$$\Rightarrow I = -\int \frac{du}{u^2 - (\sqrt{2})^2}$$

$$= -\frac{1}{2\sqrt{2}}\log\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right| + C$$

$$= -\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{t^2+1}-\sqrt{2}}{\sqrt{t^2+1}+\sqrt{2}}\right| + C$$

$$= -\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{\frac{1}{x^2}+1}-\sqrt{2}}{\sqrt{\frac{1}{x^2}+1}+\sqrt{2}}\right| + C$$

$$= -\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{1+x^2}-\sqrt{2}\,x}{\sqrt{1+x^2}+\sqrt{2}\,x}\right| + C$$

Concept Application Exercise 7.9

Evaluate the following:

1. $\int \frac{1}{(x+1)\sqrt{x^2-1}}dx$
2. $\int \frac{x^2-1}{(x^2+1)\sqrt{1+x^4}}dx$
3. $\int \sec^3 x\, dx$
4. $\int \frac{x+1}{(x-1)\sqrt{x+2}}dx$
5. $\int \frac{x}{(x^2+4)\sqrt{x^2+1}}dx$
6. $\int \frac{1}{(x+1)\sqrt{x^2+x+1}}dx$
7. $\int \frac{x^3+1}{\sqrt{x^2+x}}dx$

MISCELLANEOUS SOLVED PROBLEMS

1. Evaluate $\int \frac{dx}{x^2\sqrt{1+x^2}}$.

Sol. $I = \int \frac{dx}{x^2\sqrt{1+x^2}} = \int \frac{dx}{x^3\sqrt{1+\frac{1}{x^2}}}$

let $t = \sqrt{1+\frac{1}{x^2}} \Rightarrow \frac{dt}{dx} = \frac{1\left(-\frac{2}{x^3}\right)}{2\sqrt{1+\frac{1}{x^2}}}$

$\Rightarrow \frac{dx}{x^3} = -t\,dt$

$\Rightarrow I = -\int \frac{t\,dt}{t} = -t + C = -\sqrt{1+\frac{1}{x^2}} + C$

$= -\frac{1}{x}\sqrt{1+x^2} + C$

2. Evaluate $\int x^{-11}(1+x^4)^{-1/2}dx$.

Sol. $I = \int \frac{dx}{x^{11}(1+x^4)^{1/2}} = \int \frac{dx}{x^{11}.\,x^2(1+1/x^4)^{1/2}}$

Let $1+\frac{1}{x^4} = t^2$ and $\frac{-4}{x^5}dx = 2t\,dt$

$\Rightarrow I = \int \frac{dx}{x^{13}(1+1/x^4)^{1/2}}$

$= -\frac{1}{4}\int \frac{2t\,dt}{x^8\, t}$

$= -\frac{1}{2}\int (t^2-1)^2\,dt$

$= -\frac{1}{2}\int (t^4 - 2t^2 + 1)\,dt$

$= -\frac{1}{2}\left[\frac{t^5}{5} - \frac{2t^3}{3} + t\right] + C$, where $t = \sqrt{1+\frac{1}{x^4}}$

3. Evaluate $\int \frac{(x-x^3)^{1/3}}{x^4}dx$.

Sol. $I = \int \frac{(x-x^3)^{1/3}}{x^4}dx = \int \frac{\left(\frac{1}{x^2}-1\right)^{1/3}}{x^3}dx$

Putting $\frac{1}{x^2} = t$, $\frac{1}{x^3}dx = -\frac{dt}{2}$, we get

$I = -\frac{1}{2}\int t^{1/3}dt = -\frac{3}{8}t^{4/3} + C$

$= -\frac{3}{8}\left(\frac{1}{x^2}-1\right)^{4/3} + C$

4. Evaluate $\int \frac{1}{[(x-1)^3(x+2)^5]^{1/4}}dx$.

Sol. $I = \int \frac{1}{\left(\frac{x-1}{x+2}\right)^{3/4}(x+2)^2}dx$

Let $\frac{x-1}{x+2} = t \Rightarrow \frac{3dx}{(x+2)^2} = dt$

$\Rightarrow I = \frac{1}{3}\int \frac{1}{t^{3/4}}dt$

$= \frac{1}{3}\left(\frac{t^{1/4}}{1/4}\right) + C$

$= \frac{4}{3}t^{1/4} + C$

$= \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$

5. Evaluate $\int \frac{\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx$.

Sol. Put $\log(x+\sqrt{1+x^2}) = t \Rightarrow \frac{1}{\sqrt{1+x^2}}dx = dt$, then

$\int \frac{\log(x+\sqrt{1\ x^2})}{\sqrt{1+x^2}}dx$

$= \int t\,dt = \frac{1}{2}\left[\log(x+\sqrt{1+x^2})\right]^2 + c$

6. Evaluate $\int \frac{\sqrt{\tan x}}{\sin x \cos x}dx$.

Sol. $I = \int \frac{\sqrt{\tan x}}{\sin x\cos x}dx = \int \frac{\sqrt{\tan x}}{\tan x}\sec^2 x\,dx$

$$= \int \frac{1}{\sqrt{t}}\, dt \text{, where } t = \tan x$$

$$I = 2t^{1/2} + C = 2\sqrt{\tan x} + C$$

7. $I = \int \frac{dx}{\sqrt[3]{\sin^{11} x \cos x}}$.

Sol. Here both the exponents $\left(-\frac{11}{3} \text{ and } -\frac{1}{3}\right)$ are negative numbers and their sum $\left[-\frac{11}{3} - \frac{1}{3}\right]$ is -4 which is an even number, therefore we put $\tan x = t$; $\frac{dx}{\cos^2 x} = dt$

$$I = \int \frac{dx}{\sin^{11/3} x \cos^{1/3} x}$$

$$= \int \frac{dx}{\tan^{11/3} x \cos^4 x}$$

$$= \int \frac{(1+\tan^2 x)\sec^2 x\, dx}{\tan^{11/3} x}$$

$$= \int \frac{(1+t^2)\,dt}{t^{11/3}}$$

$$= -\frac{3}{8} t^{-\frac{8}{3}} - \frac{3}{2} t^{-\frac{2}{3}} + C \text{ (where } t = \tan x)$$

8. $\int \frac{\sin x}{2+\sin 2x}\, dx$.

Sol. $\int \frac{\sin x}{2+\sin 2x}\, dx$

$$= \frac{1}{2}\int \frac{\sin x + \cos x - (\cos x - \sin x)}{2+\sin 2x}\, dx$$

$$= \frac{1}{2}\int \frac{\sin x + \cos x}{2+\sin 2x}\, dx - \frac{1}{2}\int \frac{\cos x - \sin x}{2+\sin 2x}\, dx$$

$$= \frac{1}{2}\int \frac{\sin x + \cos x}{3-(\sin x - \cos x)^2}\, dx - \frac{1}{2}\int \frac{\cos x - \sin x}{1+(\sin x + \cos x)^2}\, dx$$

$$= \frac{1}{2}\int \frac{dt}{3-t^2} - \frac{1}{2}\int \frac{du}{1+u^2} \left(\begin{array}{l}\text{where } t = \sin x - \cos x \\ \text{and } u = \sin x + \cos x\end{array}\right)$$

$$= \frac{1}{2}\frac{1}{2\sqrt{3}} \log\left|\frac{\sqrt{3}-t}{\sqrt{3}+t}\right| - \frac{1}{2}\tan^{-1} u + c$$

$$= \frac{1}{4\sqrt{3}} \log\left|\frac{\sqrt{3}-(\sin x - \cos x)}{\sqrt{3}+(\sin x - \cos x)}\right|$$

$$- \frac{1}{2}\tan^{-1}(\sin x + \cos x) + c$$

Concept Application Exercise 7.10

Evaluate the following:

1. $\int \frac{dx}{x^2(1+x^5)^{4/5}}$

2. $\int \frac{1+x^4}{(1-x^4)^{3/2}}\, dx$

3. $\int \frac{1}{x^2(x^4+1)^{3/4}}\, dx$

4. $\int \frac{(x^4-x)^{1/4}}{x^5}\, dx$

5. $\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}\, dx$

6. $\int x^x \ln(ex)\, dx$

7. $\int \frac{dx}{(x-p)\sqrt{(x-p)(x-q)}}$

Li — put $x-p = 1/t$ or $I = \int \frac{dx}{\left(\frac{x-p}{x-q}\right)^{3/2} \cdot (x-q)^2}$ put $\left(\frac{x-p}{x-q}\right) = t$

8. $\int \frac{\left[\sqrt{1+x^2}+x\right]^n}{\sqrt{1+x^2}}\, dx$

9. $\int \frac{dx}{\cos^3 x \sqrt{\sin 2x}}$

10. $\int \sec^5 x \operatorname{cosec}^3 x\, dx$

EXERCISES

Subjective Type

Solutions on page 7.38

1. Evaluate $\int \sqrt{\left(\frac{1+x^2}{x^2-x^4}\right)}\, dx$.

2. Evaluate $\int \frac{(\cos 2x)^{1/2}}{\sin x}\, dx$.

3. Evaluate $\int \frac{x^2-1}{(x^2+1)\sqrt{1+x^4}}\, dx$.

4. If $I_n = \int \cos^n x\, dx$, prove that

$$I_n = \frac{1}{n}\left(\cos^{n-1} x \sin x\right) + \left(\frac{n-1}{x}\right) I_{n-2}.$$

5. Evaluate $\int \frac{(1-x\sin x)\,dx}{x\left(1-x^3e^{3\cos x}\right)}$.

6. Evaluate

$$\int \frac{e^{\tan^{-1}x}}{(1+x^2)}\left[\left(\sec^{-1}\sqrt{1+x^2}\right)^2+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]dx\ (x>0).$$

7. Evaluate $\int \frac{x^2-1}{x\sqrt{\left(x^2+\alpha x+1\right)\left(x^2+\beta x+1\right)}}\,dx.$

8. Evaluate $\int \frac{2x}{(1-x^2)\sqrt{x^4-1}}\,dx.$

9. Evaluate $\int \frac{dx}{x^3\sqrt{x^2-1}}$.

10. Evaluate $\int \sqrt{\frac{3-x}{3+x}}.\sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right)dx.$

11. Evaluate $\int \sqrt{\sec x-1}\,dx.$

12. Evaluate $\int \sqrt{1+\operatorname{cosec} x}\,dx\ (0<x<\pi/2).$

13. Evaluate $\int \frac{\cos^4 x}{\sin^3 x\{\sin^5 x+\cos^5 x\}^{3/5}}\,dx.$

Objective Type

Solutions on page 7.41

Each question has four choices a, b, c, and d, out of which *only* *one* is correct.

1. $\int \frac{\sin 2x}{\sin 5x\sin 3x}dx$ is equal to
   - **a.** $\log\sin 3x-\log\sin 5x+c$
   - **b.** $\frac{1}{3}\log\sin 3x+\frac{1}{5}\log\sin 5x+c$
   - **c.** $\frac{1}{3}\log\sin 3x-\frac{1}{5}\log\sin 5x+c$
   - **d.** $3\log\sin 3x-5\log\sin 5x+c$

2. $\int \sqrt{1+\sin x}\,dx$ is equal to
   - **a.** $-2\sqrt{1-\sin x}+C$
   - **b.** $\sin(x/2)+\cos(x/2)+C$
   - **c.** $\cos(x/2)-\sin(x/2)+C$
   - **d.** $2\sqrt{1-\sin x}+C$

3. $\int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}dx$ is equal to
   - **a.** $\frac{1}{2}\sin 2x+C$
   - **b.** $-\frac{1}{2}\sin 2x+C$
   - **c.** $-\frac{1}{2}\sin x+C$
   - **d.** $-\sin^2 x+C$

4. If $\int \frac{\cos 4x+1}{\cot x-\tan x}dx=A\cos 4x+B$, then
   - **a.** $A=-1/2$
   - **b.** $A=-1/8$
   - **c.** $A=-1/4$
   - **d.** None of these

5. The primitive of the function $x\,|\cos x|$ when $\frac{\pi}{2}<x<\pi$ is given by
   - **a.** $\cos x+x\sin x+C$
   - **b.** $-\cos x-x\sin x+C$
   - **c.** $x\sin x-\cos x+C$
   - **d.** None of these $+\,C$

6. $\int \frac{dx}{x\left(x^n+1\right)}$ is equal to
   - **a.** $\frac{1}{n}\log\left(\frac{x^n}{x^n+1}\right)+c$
   - **b.** $\frac{1}{n}\log\left(\frac{x^n+1}{x^n}\right)+c$
   - **c.** $\log\left(\frac{x^n}{x^n+1}\right)+c$
   - **d.** None of these

7. $\int 4\sin x\cos\frac{x}{2}\cos\frac{3x}{2}dx$ is equal to
   - **a.** $\cos x+\frac{1}{2}\cos 2x-\frac{1}{3}\cos 3x+C$
   - **b.** $\cos x-\frac{1}{2}\cos 2x-\frac{1}{3}\cos 3x+C$
   - **c.** $\cos x+\frac{1}{2}\cos 2x+\frac{1}{3}\cos 3x+C$
   - **d.** $\cos x-\frac{1}{2}\cos 2x+\frac{1}{3}\cos 3x+C$

8. Let $x=f''(t)\cos t+f'(t)\sin t$ and $y=-f''(t)\sin t+f'(t)\cos t$. Then $\int\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{1/2}dt$ equals
   - **a.** $f'(t)+f''(t)+c$
   - **b.** $f''(t)+f'''(t)+c$
   - **c.** $f(t)+f''(t)+c$
   - **d.** $f'(t)-f''(t)+c$

9. $\int \frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}dx,\ \alpha\neq n\pi,\ n\in Z$ is equal to
   - **a.** $-2\operatorname{cosec}\alpha(\cos\alpha-\tan x\sin\alpha)^{1/2}+C$
   - **b.** $-2(\cos\alpha+\cot x\sin\alpha)^{1/2}+C$
   - **c.** $-2\operatorname{cosec}\alpha(\cos\alpha+\cot x\sin\alpha)^{1/2}+C$
   - **d.** $-2\operatorname{cosec}\alpha(\sin\alpha+\cot x\cos\alpha)^{1/2}+C$

10. $\int \frac{px^{p+2q-1}-qx^{q-1}}{x^{2p+2q}+2x^{p+q}+1}dx$ is equal to

a. $-\dfrac{x^p}{x^{p+q}+1}+C$ b. $\dfrac{x^q}{x^{p+q}+1}+C$

c. $-\dfrac{x^q}{x^{p+q}+1}+C$ d. $\dfrac{x^p}{x^{p+q}+1}+C$

11. If $I_n=\int(\ln x)^n\,dx$, then I_n+nI_{n-1}

a. $\dfrac{(\ln x)^n}{x}+C$ b. $x(\ln x)^{n-1}+C$

c. $x(\ln x)^n+C$ d. None of these

12. $\int\dfrac{x+2}{(x^2+3x+3)\sqrt{x+1}}dx$ is equal to

a. $\dfrac{1}{\sqrt3}\tan^{-1}\left(\dfrac{x}{\sqrt{3(x+1)}}\right)$ b. $\dfrac{2}{\sqrt3}\tan^{-1}\left(\dfrac{x}{\sqrt{3(x+1)}}\right)$

c. $\dfrac{2}{\sqrt3}\tan^{-1}\left(\dfrac{x}{\sqrt{x+1}}\right)$ d. None of these

13. $\int\dfrac{\sin 2x}{\sin^4 x+\cos^4 x}dx$ is equal to

a. $\cot^{-1}(\tan^2 x)+c$ b. $\tan^{-1}(\tan^2 x)+c$

c. $\cot^{-1}(\cot^2 x)+c$ d. $\tan^{-1}(\cot^2 x)+c$

LO **14.** $\int\dfrac{\sec x\,dx}{\sqrt{\sin(2x+A)+\sin A}}$ is equal to

a. $\dfrac{\sec A}{\sqrt2}\sqrt{\tan x\cos A-\sin A}+c$

b. $\sqrt2\sec A\sqrt{\tan x\cos A-\sin A}+c$

c. $\sqrt2\sec A\sqrt{\tan x\cos A+\sin A}+c$

d. None of these

15. If $\int\sqrt{1+\sin x}\,f(x)\,dx=\dfrac23(1+\sin x)^{3/2}+c$, then $f(x)$ equals

a. $\cos x$ b. $\sin x$

c. $\tan x$ d. 1

16. Let $\int e^x\{f(x)-f'(x)\}dx=\phi(x)$. Then $\int e^x f(x)dx$ is

a. $\phi(x)=e^x f(x)$ b. $\phi(x)-e^x f(x)$

c. $\dfrac12\{\phi(x)+e^x f(x)\}$ d. $\dfrac12\{\phi(x)+e^x f'(x)\}$

17. Let $f(x)=\int\dfrac{x^2dx}{(1+x^2)(1+\sqrt{1+x^2})}$ and $f(0)=0$, then the value of $f(1)$ be

a. $\log(1+\sqrt2)$ b. $\log(1+\sqrt2)-\dfrac{\pi}{4}$

c. $\log(1+\sqrt2)+\dfrac{\pi}{2}$ d. None of these

18. If $y=\int\dfrac{dx}{(1+x^2)^{\frac32}}$ and $y=0$ when $x=0$, find the value of y when $x=1$ is

a. $\dfrac{1}{\sqrt2}$ b. $\sqrt2$

c. $2\sqrt2$ d. None of these

19. $\int\sqrt{x}\,(1+x^{1/3})^4\,dx$ is equal to

a. $2\left\{x^{2/3}+\dfrac{4}{11}x^{11/6}+\dfrac{6}{13}x^{13/6}+\dfrac{4}{15}x^{5/2}+\dfrac{1}{17}x^{17/6}\right\}+c$

b. $6\left\{x^{2/3}-\dfrac{4}{11}x^{11/6}+\dfrac{6}{13}x^{13/6}-\dfrac{4}{15}x^{5/2}+\dfrac{1}{17}x^{17/6}\right\}+c$

c. $6\left\{x^{2/3}+\dfrac{4}{11}x^{11/6}+\dfrac{6}{13}x^{13/6}+\dfrac{4}{15}x^{5/2}+\dfrac{1}{17}x^{17/6}\right\}+c$

d. None of these

20. If $\int x^5(1+x^3)^{2/3}dx=A(1+x^3)^{8/3}+B(1+x^3)^{5/3}+c$, then

a. $A=\dfrac14, B=\dfrac15$ b. $A=\dfrac18, B=-\dfrac15$

c. $A=-\dfrac18, B=\dfrac15$ d. None of these

21. The value of the integral $\int\dfrac{(1-\cos\theta)^{2/7}}{(1+\cos\theta)^{9/7}}d\theta$ is

a. $\dfrac{7}{11}\left(\tan\dfrac{\theta}{2}\right)^{\frac{11}{7}}+C$ b. $\dfrac{7}{11}\left(\cos\dfrac{\theta}{2}\right)^{\frac{11}{7}}+C$

c. $\dfrac{7}{11}\left(\sin\dfrac{\theta}{2}\right)^{\frac{11}{7}}+C$ d. None of these

22. If $\int\dfrac{1-x^7}{x(1+x^7)}dx=a\ln|x|+b\ln|x^7+1|+c$, then

a. $a=1, b=\dfrac27$ b. $a=-1, b=\dfrac27$

c. $a=1, b=-\dfrac27$ d. $a=-1, b=-\dfrac27$

23. $\int\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)dx$ is equal to

a. $x\tan^{-1}x - \ln|\sec(\tan^{-1}x)| + c$
b. $x\tan^{-1}x + \ln|\sec(\tan^{-1}x)| + c$
c. $x\tan^{-1}x - \ln|\cos(\tan^{-1}x)| + c$
d. None of these

24. $\int \frac{\ln(\tan x)}{\sin x \cos x} dx$ is equal to

a. $\frac{1}{2}\ln(\tan x) + c$
b. $\frac{1}{2}\ln(\tan^2 x) + c$
c. $\frac{1}{2}(\ln(\tan x))^2 + c$
d. None of these

25. $\int \frac{2\sin x}{(3+\sin 2x)} dx$ is equal to

a. $\frac{1}{2}\ln\left|\frac{2+\sin x-\cos x}{2-\sin x+\cos x}\right| - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right) + c$

b. $\frac{1}{2}\ln\left|\frac{2+\sin x-\cos x}{2-\sin x+\cos x}\right| - \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right) + c$

c. $\frac{1}{4}\ln\left|\frac{2+\sin x-\cos x}{2-\sin x+\cos x}\right| - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right) + c$

d. None of these

26. $\int \frac{x^9\, dx}{(4x^2+1)^6}$ is equal to

a. $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5} + c$
b. $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5} + c$
c. $\frac{1}{10}\left(1+4x^2\right)^{-5} + c$
d. $\frac{1}{10}\left(4+\frac{1}{x^2}\right)^{-5} + c$

27. $\int e^{\tan^{-1}x}(1+x+x^2)\; d(\cot^{-1}x)$ is equal to

a. $-e^{\tan^{-1}x} + c$
b. $e^{\tan^{-1}x} + c$
c. $-x\,e^{\tan^{-1}x} + c$
d. $x\,e^{\tan^{-1}x} + c$

28. If $\int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}} = a\sqrt{\cot x} + b\sqrt{\tan^3 x} + c$, then

a. $a = -1, b = 1/3$
b. $a = -3, b = 2/3$
c. $a = -2, b = 4/3$
d. None of these

29. $\int \frac{\cos 4x - 1}{\cot x - \tan x} dx$ is equal to

a. $\frac{1}{2}\ln|\sec 2x| - \frac{1}{4}\cos^2 2x + c$
b. $\frac{1}{2}\ln|\sec 2x| + \frac{1}{4}\cos^2 x + c$
c. $\frac{1}{2}\ln|\cos 2x| - \frac{1}{4}\cos^2 2x + c$
d. $\frac{1}{2}\ln|\cos 2x| + \frac{1}{4}\cos^2 x + c$

30. If $\int \frac{1}{x\sqrt{1-x^3}} dx = a\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right| + b$, then a is equal to

a. 1/3
b. 2/3
c. –1/3
d. –2/3

31. If $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}} = -[f(x)]^{1/n} + c$, then $f(x)$ is

a. $(1+x^n)$
b. $1+x^{-n}$
c. x^n+x^{-n}
d. None of these

32. $\int \frac{\sqrt{x-1}}{x\sqrt{x+1}} dx$ is equal to

a. $\ln|x-\sqrt{x^2-1}| - \tan^{-1}x + c$
b. $\ln|x+\sqrt{x^2-1}| - \tan^{-1}x + c$
c. $\ln|x-\sqrt{x^2-1}| - \sec^{-1}x + c$
d. $\ln|x+\sqrt{x^2-1}| - \sec^{-1}x + c$

33. If $I = \int \frac{dx}{\left(2ax+x^2\right)^{3/2}}$, then I is equal to

LO

a. $-\frac{x+a}{\sqrt{2ax+x^2}} + c$
b. $-\frac{1}{a}\frac{x+a}{\sqrt{2ax+x^2}} + c$
c. $-\frac{1}{a^2}\frac{x+a}{\sqrt{2ax+x^2}} + c$
d. $-\frac{1}{a^3}\frac{x+a}{\sqrt{2ax+x^3}} + c$

34. If $f'(x) = \frac{1}{-x+\sqrt{x^2+1}}$ and $f(0) = -\frac{1+\sqrt{2}}{2}$, then $f(1)$, is equal to

LO

a. $-\log(\sqrt{2}+1)$
b. 1
c. $1+\sqrt{2}$
d. None of these

35. $\int e^x\left(\frac{2\tan x}{1+\tan x}+\cot^2\left(x+\frac{\pi}{4}\right)\right)dx$ is equal to

a. $e^x\tan\left(\frac{\pi}{4}-x\right)+c$ b. $e^x\tan\left(x-\frac{\pi}{4}\right)+c$

c. $e^x\tan\left(\frac{3\pi}{4}-x\right)+c$ c. None of these

36. The value of the integral $\int(x^2+x)(x^{-8}+2x^{-9})^{1/10}\,dx$ is

a. $\frac{5}{11}(x^2+2x)^{11/10}+c$ b. $\frac{5}{6}(x+1)^{11/10}+c$

c. $\frac{6}{7}(x+1)^{11/10}+c$ d. None of these

37. If $\int\frac{dx}{(x+2)(x^2+1)}=a\ln(1+x^2)+b\tan^{-1}x+\frac{1}{5}\ln|x+2|+C$, then

a. $a=-\frac{1}{10}, b=-\frac{2}{5}$ b. $a=\frac{1}{10}, b=-\frac{2}{5}$

c. $a=-\frac{1}{10}, b=\frac{2}{5}$ d. $a=\frac{1}{10}, b=\frac{2}{5}$

38. If $\int\frac{3\sin x+2\cos x}{3\cos x+2\sin x}dx=ax+b\ln|2\sin x+3\cos x|+C$, then

a. $a=-\frac{12}{13}, b=\frac{15}{39}$ b. $a=-\frac{7}{13}, b=\frac{6}{13}$

c. $a=\frac{12}{13}, b=-\frac{15}{39}$ d. $a=-\frac{7}{13}, b=-\frac{6}{13}$

39. $\int\frac{3e^x-5e^{-x}}{4e^x+5e^{-x}}dx=ax+b\ln(4e^x+5e^{-x})+C$, then

a. $a=-\frac{1}{8}, b=\frac{7}{8}$ b. $a=\frac{1}{8}\ b=\frac{7}{8}$

c. $a=-\frac{1}{8}, b=-\frac{7}{8}$ d. $a=\frac{1}{8}, b=-\frac{7}{8}$

40. $\int\sqrt{\frac{\cos x-\cos^3 x}{1-\cos^3 x}}\,dx$ is equal to

a. $\frac{2}{3}\sin^{-1}(\cos^{3/2}x)+C$ b. $\frac{3}{2}\sin^{-1}(\cos^{3/2}x)+C$

c. $\frac{2}{3}\cos^{-1}(\cos^{3/2}x)+C$ d. None of these

41. If $l^r(x)$ means log log log … x, the log being repeated r times, then $\int[x\,l(x)l^2(x)l^3(x)\ldots l^r(x)]^{-1}dx$ is equal to

a. $l^{r+1}(x)+C$ b. $\frac{l^{r+1}(x)}{r+1}+C$

c. $l^r(x)+C$ d. None of these

42. If $I=\int\left(\sqrt{\cot x}-\sqrt{\tan x}\right)dx$, then I equals

a. $\sqrt{2}\log\left(\sqrt{\tan x}-\sqrt{\cot x}\right)+C$

b. $\sqrt{2}\log|\sin x+\cos x+\sqrt{\sin 2x}|+C$

b. $\sqrt{2}\log|\sin x-\cos x+\sqrt{2}\sin x\cos x|+C$

d. $\sqrt{2}\log|\sin(x+\pi/4)+\sqrt{2}\sin x\cos x|+C$

43. If $I=\int\frac{dx}{\left(a^2-b^2x^2\right)^{3/2}}$, then I equals

a. $\frac{x}{\sqrt{a^2-b^2x^2}}+C$ b. $\frac{x}{a^2\sqrt{a^2-b^2x^2}}+C$

c. $\frac{ax}{\sqrt{a^2-b^2x^2}}+C$ d. None of these

44. $\int e^{x^4}(x+x^3+2x^5)e^{x^2}dx$ is equal to

a. $\frac{1}{2}xe^{x^2}e^{x^4}+c$ b. $\frac{1}{2}x^2e^{x^4}+c$

c. $\frac{1}{2}e^{x^2}e^{x^4}+c$ d. $\frac{1}{2}x^2e^{x^2}e^{x^4}+c$

45. $\int x\left(\frac{\ln a^{a^{x/2}}}{3a^{5x/2}b^{3x}}+\frac{\ln b^{b^x}}{2a^{2x}b^{4x}}\right)dx$ (where $a, b\in R^+$) is equal to

a. $\frac{1}{6\ln a^2b^3}a^{2x}b^{3x}\ln\frac{a^{2x}b^{3x}}{e}+k$

b. $\frac{1}{6\ln a^2b^3}\frac{1}{a^{2x}b^{3x}}\ln\frac{1}{ea^{2x}b^{3x}}+k$

c. $\frac{1}{6\ln a^2b^3}\frac{1}{a^{2x}b^{3x}}\ln(a^{2x}b^{3x})+k$

d. $-\frac{1}{6\ln a^2b^3}\frac{1}{a^{2x}b^{3x}}\ln(a^{2x}b^{3x})+k$

46. If $\int x\frac{\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx=a\sqrt{1+x^2}\ln(x+\sqrt{1+x^2})+bx+c$, then

a. $a=1, b=-1$ b. $a=1, b=1$

c. $a=-1, b=1$ d. $a=-1, b=-1$

47. $\int \frac{\operatorname{cosec}^2 x - 2005}{\cos^{2005} x} dx$ is equal to

a. $\frac{\cot x}{(\cos x)^{2005}} + c$ b. $\frac{\tan x}{(\cos x)^{2005}} + c$

c. $\frac{-(\tan x)}{(\cos x)^{2005}} + c$ d. None of these

48. If $x f(x) = 3f^2(x) + 2$, then $\int \frac{2x^2 - 12xf(x) + f(x)}{(6f(x) - x)(x^2 - f(x))^2} dx$ equals

a. $\frac{1}{x^2 - f(x)} + c$ b. $\frac{1}{x^2 + f(x)} + c$

c. $\frac{1}{x - f(x)} + c$ d. $\frac{1}{x + f(x)} + c$

49. If $\int f(x) \sin x \cos x \, dx = \frac{1}{2(b^2 - a^2)} \ln f(x) + c$, then $f(x)$ is equal to

a. $\frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}$ b. $\frac{1}{a^2 \sin^2 x - b^2 \cos^2 x}$

c. $\frac{1}{a^2 \cos^2 x + b^2 \sin^2 x}$ d. $\frac{1}{a^2 \cos^2 x - b^2 \sin^2 x}$

50. The value of integral $\int e^x \left(\frac{1}{\sqrt{1+x^2}} + \frac{1-2x^2}{\sqrt{(1+x^2)^5}} \right) dx$ is equal to

a. $e^x \left(\frac{1}{\sqrt{1+x^2}} + \frac{x}{\sqrt{(1+x^2)^3}} \right) + c$

b. $e^x \left(\frac{1}{\sqrt{1+x^2}} - \frac{x}{\sqrt{(1+x^2)^3}} \right) + c$

c. $e^x \left(\frac{1}{\sqrt{1+x^2}} + \frac{x}{\sqrt{(1+x^2)^5}} \right) + c$

d. None of these

51. $\int \frac{dx}{(1 + \sqrt{x})\sqrt{(x - x^2)}}$ is equal to

a. $\frac{1+\sqrt{x}}{(1-x)^2} + c$ b. $\frac{1+\sqrt{x}}{(1+x)^2} + c$

c. $\frac{1-\sqrt{x}}{(1-x)^2} + c$ d. $\frac{2(\sqrt{x}-1)}{\sqrt{(1-x)}} + c$

52. The value of $\int \frac{(ax^2 - b)\, dx}{x\sqrt{c^2 x^2 - (ax^2 + b)^2}}$ is equal to

a. $\frac{1}{c} \sin^{-1} \left(ax + \frac{b}{x} \right) + k$ b. $c \sin^{-1} \left(a + \frac{b}{x} \right) + c$

c. $\sin^{-1} \left(\frac{ax + \frac{b}{x}}{c} \right) + k$ d. None of these

53. If $\int \frac{dx}{\cos^3 x \sqrt{\sin 2x}} = a(\tan^2 x + b)\sqrt{\tan x} + c$, then

a. $a = \frac{\sqrt{2}}{5}, b = \frac{1}{\sqrt{5}}$ b. $a = \frac{\sqrt{2}}{5}, b = 5$

c. $a = \frac{\sqrt{2}}{5}, b = -\frac{1}{\sqrt{5}}$ d. $a = \frac{\sqrt{2}}{5}, b = \sqrt{5}$

54. If $\int x \log(1 + 1/x) dx = f(x) \log(x+1) + g(x) x^2 + Ax + C$, then

a. $f(x) = \frac{1}{2}x^2$ b. $g(x) = \log x$

c. $A = 1$ d. None of these

55. If $I = \int \frac{dx}{x^3 \sqrt{x^2 - 1}}$, then I equals

a. $\frac{1}{2} \left(\frac{\sqrt{x^2 - 1}}{x^3} + \tan^{-1} \sqrt{x^2 - 1} \right) + C$

b. $\frac{1}{2} \left(\frac{\sqrt{x^2 - 1}}{x^2} + x \tan^{-1} \sqrt{x^2 - 1} \right) + C$

c. $\frac{1}{2} \left(\frac{\sqrt{x^2 - 1}}{x} + \tan^{-1} \sqrt{x^2 - 1} \right) + C$

d. $\frac{1}{2} \left(\frac{\sqrt{x^2 - 1}}{x^2} + \tan^{-1} \sqrt{x^2 - 1} \right) + C$

56. If $I_{m,n} = \int \cos^m x \sin nx \, dx$, then $7 I_{4,3} - 4 I_{3,2}$ is equal to

a. constant b. $-\cos^2 x + C$

c. $-\cos^4 x \cos 3x + C$ d. $\cos 7x - \cos 4x + C$

57. If $\int \frac{dx}{x^2 (x^n + 1)^{(n-1)/n}} = -[f(x)]^{1/n} + C$, then $f(x)$ is

a. $(1 + x^n)$ b. $1 + x^{-n}$

c. $x^n + x^{-n}$ d. None of these

58. $4\int \frac{\sqrt{a^6+x^8}}{x}\,dx$ is equal to

a. $\sqrt{a^6+x^8}+\frac{a^3}{2}\ln\left|\frac{\sqrt{a^6+x^8}+a^3}{\sqrt{a^6+x^8}-a^3}\right|+c$

b. $a^6\ln\left|\frac{\sqrt{a^6+x^8}-a^3}{\sqrt{a^6+x^8}+a^3}\right|+c$

c. $\sqrt{a^6+x^8}+\frac{a^3}{2}\ln\left|\frac{\sqrt{a^6+x^8}-a^3}{\sqrt{a^6+x^8}+a^3}\right|+c$

d. $a^6\ln\left|\frac{\sqrt{a^6+x^8}+a^3}{\sqrt{a^6+x^8}-a^3}\right|+c$

59. If $I=\int e^{-x}\log(e^x+1)\,dx$, then I equals

a. $x+(e^{-x}+1)\log(e^x+1)+C$
b. $x+(e^x+1)\log(e^x+1)+C$
c. $x-(e^{-x}+1)\log(e^x+1)+C$
d. None of these

60. If $\int xe^x\cos x\,dx = ae^x(b(1-x)\sin x+cx\cos x)+d$, then

a. $a=1, b=1, c=-1$ b. $a=\frac{1}{2}, b=-1, c=1$

c. $a=1, b=-1, c=1$ d. $a=\frac{1}{2}, b=1, c=-1$

61. If $I=\int\sqrt{\frac{5-x}{2+x}}\,dx$, then I equals

a. $\sqrt{x+2}\,\sqrt{5-x}+3\sin^{-1}\sqrt{\frac{x+2}{3}}+C$

b. $\sqrt{x+2}\,\sqrt{5-x}+7\sin^{-1}\sqrt{\frac{x+2}{7}}+C$

c. $\sqrt{x+2}\,\sqrt{5-x}+5\sin^{-1}\sqrt{\frac{x+2}{5}}+C$

d. None of these

62. $\int e^{\tan x}(\sec x-\sin x)dx$, is equal to

a. $e^{\tan x}\cos x+C$ b. $e^{\tan x}\sin x+C$

c. $-e^{\tan x}\cos x+C$ d. $e^{\tan x}\sec x+C$

63. $\int\frac{x^3dx}{\sqrt{1+x^2}}$ is equal to

a. $\frac{1}{3}\sqrt{1+x^2}(2+x^2)+C$ b. $\frac{1}{3}\sqrt{1+x^2}(x^2-1)+C$

c. $\frac{1}{3}(1+x^2)^{3/2}+C$ d. $\frac{1}{3}\sqrt{1+x^2}(x^2-2)+C$

64. If $I=\int\frac{dx}{\sec x+\operatorname{cosec} x}$, then I equals

a. $\frac{1}{2}\left(\cos x+\sin x-\frac{1}{\sqrt{2}}\log(\operatorname{cosec} x-\cos x)\right)+C$

b. $\frac{1}{2}\left(\sin x-\cos x-\frac{1}{\sqrt{2}}\log|\operatorname{cosec} x+\cot x|\right)+C$

c. $\frac{1}{\sqrt{2}}\left(\sin x+\cos x+\frac{1}{2}\log|\operatorname{cosec} x-\cos x|\right)+C$

d. $\frac{1}{2}[\sin x-\cos x]-\frac{1}{\sqrt{2}}\log\left|\operatorname{cosec}(x+\pi/4)-\cot(x+\pi/4)\right|+C$

65. If $I=\int\frac{\sin 2x}{(3+4\cos x)^3}\,dx$, then I equals

a. $\frac{3\cos x+8}{(3+4\cos x)^2}+C$ b. $\frac{3+8\cos x}{16(3+4\cos x)^2}+C$

c. $\frac{3+\cos x}{(3+4\cos x)^2}+C$ d. $\frac{3-8\cos x}{16(3+4\cos x)^2}+C$

66. $\int\frac{\ln\left(\frac{x-1}{x+1}\right)}{x^2-1}\,dx$ is equal to

a. $\frac{1}{2}\left(\ln\left(\frac{x-1}{x+1}\right)\right)^2+C$ b. $\frac{1}{2}\left(\ln\left(\frac{x+1}{x-1}\right)\right)^2+C$

c. $\frac{1}{4}\left(\ln\left(\frac{x-1}{x+1}\right)\right)^2+C$ d. $\frac{1}{4}\left(\ln\left(\frac{x+1}{x-1}\right)\right)$

67. $\int\sqrt{e^x-1}\,dx$ is equal to

a. $2\left[\sqrt{e^x-1}-\tan^{-1}\sqrt{e^x-1}\right]+c$

b. $\sqrt{e^x-1}-\tan^{-1}\sqrt{e^x-1}+c$

c. $\sqrt{e^x-1}+\tan^{-1}\sqrt{e^x-1}+c$

d. $2\left[\sqrt{e^x-1}+\tan^{-1}\sqrt{e^x-1}\right]+c$

68. $\int x \sin x \sec^3 x\, dx$ is equal to

a. $\frac{1}{2}\left[\sec^2 x - \tan x\right] + c$ b. $\frac{1}{2}\left[x\sec^2 x - \tan x\right] + c$

c. $\frac{1}{2}[x \sec^2 x + \tan x] + c$ d. $\frac{1}{2}[\sec^2 x + \tan x] + c$

69. $\int e^x \frac{(x^2+1)}{(x+1)^2} dx$ is equal to

a. $\left(\frac{x-1}{x+1}\right)e^x + c$ b. $e^x\left(\frac{x+1}{x-1}\right) + c$

c. $e^x(x+1)(x-1) + c$ d. None of these

70. $\int \left(\frac{x+2}{x+4}\right)^2 e^x\, dx$ is equal to

a. $e^x\left(\frac{x}{x+4}\right) + c$ b. $e^x\left(\frac{x+2}{x+4}\right) + c$

c. $e^x\left(\frac{x-2}{x+4}\right) + c$ d. $\left(\frac{2xe^2}{x+4}\right) + c$

71. $\int \frac{3+2\cos x}{(2+3\cos x)^2} dx$ is equal to

a. $\left(\frac{\sin x}{3\cos x+2}\right) + c$ b. $\left(\frac{2\cos x}{3\sin x+2}\right) + c$

c. $\left(\frac{2\cos x}{3\cos x+2}\right) + c$ d. $\left(\frac{2\sin x}{3\sin x+2}\right) + c$

Multiple Correct Answers Type

Solutions on page 7.50

Each question has four choices a, b, c, and d, out of which *one or more* answers are correct.

1. $\int \frac{x^2 + \cos^2 x}{x^2+1} \operatorname{cosec}^2 x\, dx$ is equal to

a. $\cot x - \cot^{-1} x + c$ b. $c - \cot x + \cot^{-1} x$

c. $-\tan^{-1} x - \frac{\operatorname{cosec} x}{\sec x} + c$ d. $-e^{\log \tan^{-1} x} - \cot x + c$

2. If $\int \sin x\, d(\sec x) = f(x) - g(x) + c$, then

a. $f(x) = \sec x$ b. $f(x) = \tan x$

c. $g(x) = 2x$ d. $g(x) = x$

3. $\int \sqrt{1 + \operatorname{cosec} x}\, dx$ equals

a. $2\sin^{-1}\sqrt{\sin x} + c$ b. $\sqrt{2}\cos^{-1}\sqrt{\cos x} + c$

c. $c - 2\sin^{-1}(1 - 2\sin x)$ d. $\cos^{-1}(1 - 2\sin x) + c$

4. If $I = \int \sec^2 x \operatorname{cosec}^4 x\, dx = A\cot^3 x + B\tan x + C\cot x + D$, then

a. $A = -\frac{1}{3}$ b. $B = 2$

c. $C = -2$ d. None of these

5. A curve $g(x) = \int x^{27}(1+x+x^2)^6 (6x^2+5x+4)dx$ is passing through origin, then

a. $g(1) = \frac{3^7}{7}$ b. $g(1) = \frac{2^7}{7}$

c. $g(-1) = \frac{1}{7}$ d. $g(-1) = \frac{3^7}{14}$

6. If $\int \sqrt{\operatorname{cosec} x + 1}\, dx = k\, f o g(x) + c$, where k is a real constant, then

a. $k = -2, f(x) = \cot^{-1} x, g(x) = \sqrt{\operatorname{cosec} x - 1}$

b. $k = -2, f(x) = \tan^{-1} x, g(x) = \sqrt{\operatorname{cosec} x - 1}$

c. $k = 2, f(x) = \tan^{-1} x, g(x) = \frac{\cot x}{\sqrt{\operatorname{cosec} x - 1}}$

d. $k = 2, f(x) = \cot^{-1} x, g(x) = \frac{\cot x}{\sqrt{\operatorname{cosec} x + 1}}$

7. If $I = \int \frac{\sin x + \sin^3 x}{\cos 2x} dx = P\cos x + Q\log|f(x)| + R$, then

a. $P = 1/2, Q = -\frac{3}{4\sqrt{2}}$

b. $P = 1/4, Q = -\frac{1}{\sqrt{2}}$

c. $f(x) = \frac{\sqrt{2}\cos x + 1}{\sqrt{2}\cos x - 1}$

d. $f(x) = \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}$

8. If $\int \frac{e^{x-1}}{(x^2-5x+4)} 2x\, dx = AF(x-1) + BF(x-4) + C$ and $F(x) = \int \frac{e^x}{x} dx$, then

a. $A = -2/3$ b. $B = (4/3)e^3$

c. $A = 2/3$ d. $B = (8/3)e^3$

9. If $\int x^2 e^{-2x} dx = e^{-2x}(ax^2 + bx + c) + d$, then

a. $a = 1$ b. $b = 2$

c. $c = \frac{1}{2}$ d. $d \in R$

10. If $\int \frac{x^4+1}{x^6+1}\,dx = \tan^{-1} f(x) - \frac{2}{3}\tan^{-1} g(x) + C$, then

a. both $f(x)$ and $g(x)$ are odd functions

b. $f(x)$ is monotonic function

c. $f(x) = g(x)$ has no real roots

d. $\int \frac{f(x)}{g(x)}\,dx = -\frac{1}{x} + \frac{3}{x^3} + c$

11. If $\int \frac{x^2-x+1}{(x^2+1)^{\frac{3}{2}}} e^x dx = e^x f(x) + c$, then

a. $f(x)$ is an even function

b. $f(x)$ is a bounded function

c. The range of $f(x)$ is (0, 1]

d. $f(x)$ has two points of extrema

12. If $\int \frac{\cos 4x + 1}{\cot x - \tan x}\,dx = A f(x) + B$, then

a. $A = -\frac{1}{8}$

b. $B = \frac{1}{2}$

c. $f(x)$ has fundamental period $\frac{\pi}{2}$

d. $f(x)$ is an odd function

13. If $\int \sin^{-1} x \cos^{-1} x\,dx = f^{-1}(x)$

$\left[Ax - x f^{-1}(x) - 2\sqrt{1-x^2}\right] + 2x + C$, then

a. $f(x) = \sin x$

b. $f(x) = \cos x$

c. $A = \frac{\pi}{4}$

d. $A = \frac{\pi}{2}$

14. If $f(x) = \int \frac{x^8+4}{x^4-2x^2+2}\,dx$ and $f(0) = 0$, then

a. $f(x)$ is an odd function

b. $f(x)$ has range R

c. $f(x)$ has at least one real root

d. $f(x)$ is a monotonic function

15. $\int \frac{dx}{x^2+ax+1} = f(g(x)) + c$, then

a. $f(x)$ is inverse trigonometric function for $|a| > 2$

b. $f(x)$ is logarithmic function for $|a| < 2$

c. $g(x)$ is quadratic function for $|a| > 2$

d. $g(x)$ is rational function for $|a| < 2$

Reasoning Type

Solutions on page 7.52

Each question has four choices a, b, c, and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. if both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1

b. if both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1

c. if STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.

d. if STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** $\int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + c.$

Statement 2: $\int e^x (f(x) + f'(x))\,dx = e^x f(x) + c.$

2. **Statement 1:** For $-1 < a < 4$, $\int \frac{dx}{x^2+2(a-1)x+a+5}$ $= \lambda \log|g(x)| + c$, where λ and c are constants.

Statement 2: For $-1 < a < 4$, $\frac{1}{x^2+2(a-1)x+a+5}$ is a continuous function.

3 **Statement 1:** $\int \frac{\sin x\,dx}{x}$ $(x > 0)$ cannot be evaluated.

Statement 2: Only differentiable functions can be integrated.

4. **Statement 1:** $\int \frac{dx}{x^3\sqrt{1+x^4}} = -\frac{1}{2}\sqrt{1+\frac{1}{x^4}} + C.$

Statements 2: For integration by parts we have to follow ILATE rule.

5. **Statement 1:** If the primitive of $f(x) = \pi \sin \pi x + 2x - 4$ has the value 3 for $x = 1$, then there are exactly two values of x for which primitive of $f(x)$ vanishes.

Statement 2: $\cos \pi x$ has period 2.

6. **Statement 1:** $\int \frac{\{f(x)\,\phi'(x) - f'(x)\,\phi(x)\}}{f(x)\,\phi(x)} \{\log \phi(x) - \log f(x)\}\,dx = \frac{1}{2}\left\{\log \frac{\phi(x)}{f(x)}\right\}^2 + c.$

Statement 2: $\int (h(x))^n h'(x)dx = \frac{(h(x))^{n+1}}{n+1} + c.$

Linked Comprehension Type

Solutions on page 7.52

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c, and d, out of which *only one* is correct.

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For Problems 1–3

$y = f(x)$ is a polynomial function passing through point (0, 1) and which increases in the intervals (1, 2) and $(3, \infty)$ and decreases in the intervals $(-\infty, 1)$ and (2, 3).

1. If $f(1) = -8$, then the value of $f(2)$ is
   a. $1-3$ b. -6
   c. -20 d. -7

2. If $f(1) = -8$, then the range of $f(x)$ is
   a. $[3, \infty)$ b. $[-8, \infty)$
   c. $[-7, \infty)$ d. $(-\infty, 6]$

3. If $f(x) = 0$ has four real roots, then the range of values of leading co-efficient of polynomial is
   a. [4/9, 1/2] b. [4/9, 1]
   c. [1/3, 1/2] d. None of these

For Problems 4–6

If A is square matrix and e^A is defined as $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \frac{1}{2}\begin{bmatrix} f(x) & g(x) \\ g(x) & f(x) \end{bmatrix}$, where $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$ and $0 < x < 1$, I is an identity matrix.

4. $\int \frac{g(x)}{f(x)} dx$ is equal to
   a. $\log(e^x + e^{-x}) + c$ b. $\log|e^x - e^{-x}| + c$
   c. $\log|e^{2x} - 1| + c$ d. None of these

5. $\int (g(x) + 1)\sin x dx$ is equal to
   a. $\frac{e^x}{2}(\sin x - \cos x)$
   b. $\frac{e^{2x}}{5}(2\sin x - \cos x)$
   c. $\frac{e^x}{5}(\sin 2x - \cos 2x)$
   d. None of these

6. $\int \frac{f(x)}{\sqrt{g(x)}} dx$ is equal to
   a. $\frac{1}{2\sqrt{e^x - 1}} - \operatorname{cosec}^{-1}(e^x) + c$
   b. $\frac{2}{\sqrt{e^x - e^{-x}}} - \sec^{-1}(e^x) + c$
   c. $\frac{1}{2\sqrt{e^{2x} - 1}} + \sec^{-1}(e^x) + c$
   d. None of these

For Problems 7–9

Euler's substitution

Integrals of the form $\int R(x, \sqrt{ax^2 + bx + c})\, dx$ are calculated with the aid of one of the three Euler substitutions

1. $\sqrt{ax^2 + bx + c} = t \pm x\sqrt{a}$ if $a > 0$;
2. $\sqrt{ax^2 + bx + c} = tx \pm \sqrt{c}$ if $c > 0$;
3. $\sqrt{ax^2 + bx + c} = (x - a)t$ if $ax^2 + bx + c = a(x - a)(x - b)$ i.e., if α is a real root of $ax^2 + bx + c = 0$.

7. Which of the following functions does not appear in the primitive of $\frac{1}{1 + \sqrt{x^2 + 2x + 2}}$ if t is a function of x?
   a. $\log_e|t+1|$ b. $\log_e|t+2|$
   c. $\frac{1}{t+2}$ d. None of these

8. Which of the following functions does not appear in the primitive of $\frac{dx}{x + \sqrt{x^2 - x + 1}}$ if t is a function of x?
   a. $\log_e|t|$ b. $\log_e|t-2|$
   c. $\log_e|t-1|$ d. $\log|t+1|$

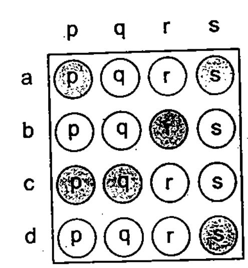
9. $\int \frac{x dx}{\left(\sqrt{7x - 10 - x^2}\right)^3}$ can be evaluated by substituting for x as
   a. $x = \frac{5 + 2t^2}{t^2 + 1}$ b. $x = \frac{5 - t^2}{t^2 + 2}$
   c. $x = \frac{2t^2 - 5}{3t^2 - 1}$ d. None of these

Matrix-Match Type

Solutions on page 7.54

Each question contains statements given in two columns which have to be matched.

Statements (a, b, c, d) in column 1 have to be matched with statements (p, q, r, s) in column 2. If the correct match are a-p, s, b-r, c-p, q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:



1.

Column 1	Column 2
a. If $\int \frac{2^x}{\sqrt{1-4^x}}dx = k\sin^{-1}(f(x))+C$, then k is greater than	**p.** 0
b. If $\int \frac{(\sqrt{x})^5}{(\sqrt{x})^7+x^6}dx = a\ln\frac{x^k}{x^k+1}+c$, then ak is less than	**q.** 1
c. $\int \frac{x^4+1}{x\left(x^2+1\right)^2}dx = k\ln\|x\| + \frac{m}{1+x^2}+n$, where n is the constant of integration, then mk is greater than	**r.** 3
d. $\int \frac{dx}{5+4\cos x} = k\tan^{-1}\left(m\tan\frac{x}{2}\right)+C$, then k/m is greater than	**s.** 4

2.

Column 1	Column 2
a. $\int \frac{e^{2x}-1}{e^{2x}+1}dx$ is equal to	**p.** $x-\log\left[1+\sqrt{1-e^{2x}}\right]+c$
b. $\int \frac{1}{(e^x+e^{-x})^2}dx$ is equal to	**q.** $\log(e^x+1)-x-e^{-x}+c$
c. $\int \frac{e^{-x}}{1+e^x}dx$ is equal to	**r.** $\log(e^{2x}+1)-x+c$
d. $\int \frac{1}{\sqrt{1-e^{2x}}}dx$ is equal to	**s.** $-\frac{1}{2\left(e^{2x}+1\right)}+c$

3.

Column 1	Column 2 (which of the following functions appear in integration of function in Column 1)
a. $\int \frac{x^2-x+1}{x^3-4x^2+4x}dx$	**p.** $\log\|x\|$
b. $\int \frac{x^2-1}{x\left(x-2\right)^3}dx$	**q.** $\log\|x-2\|$
c. $\int \frac{x^3+1}{x\left(x-2\right)^2}dx$	**r.** $\frac{1}{(x-2)}$
d. $\int \frac{x^5+1}{x\left(x-2\right)^3}dx$	**s.** x

Integer Type

Solutions on page 7.55

1. Let $f(x) = \int x^{\sin x}(1 + x\cos x\cdot\ln x + \sin x)\,dx$ and $f\left(\frac{\pi}{2}\right)=\frac{\pi^2}{4}$, then the value of $|\cos(f(\pi))|$ is

2. Let $g(x) = \int \frac{1+2\cos x}{(\cos x+2)^2}dx$ and $g(0)=0$, then the value of $8g(\pi/2)$ is

3. Let $k(x) = \int \frac{(x^2+1)\,dx}{\sqrt[3]{x^3+3x+6}}$ and $k(-1)=\frac{1}{\sqrt[3]{2}}$, then the value of $k(-2)$ is

4. If $\int x^2 \cdot e^{-2x}\, dx = e^{-2x}(ax^2 + bx + c) + d$, then the value of $|a/bc|$ is

5. If $f(x) = \int \frac{3x^2+1}{(x^2-1)^3}dx$ and $f(0) = 0$, then the value of $|2/f(2)|$ is

6. If $f(x) = \sqrt{x}$, $g(x) = e^x - 1$, and $\int fog(x)dx = A fog(x) + B \tan^{-1}(fog(x)) + C$, then $A + B$ is equal to

7. If $\int \frac{2\cos x - \sin x + \lambda}{\cos x + \sin x - 2} dx = A \ln|\cos x + \sin x - 2| + Bx + C$. Then the value of $A + B + |\lambda|$ is

8. If $\int \left[\left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x\right] \ln x\, dx = A\left(\frac{x}{e}\right)^x + B\left(\frac{e}{x}\right)^x + C$, then the value of $A + B$ is

Archives

Solutions on page 7.56

Subjective

1. Evaluate $\int \frac{\sin x}{\sin x - \cos x} dx$. **(IIT-JEE, 1978)**

2. Evaluate $\int \frac{x^2}{(a+bx)^2} dx$. **(IT-JEE, 1979)**

3. Evaluate the following integrals:

a. $\int \sqrt{1 + \sin\left(\frac{x}{2}\right)}dx$ b. $\int \frac{x^2}{\sqrt{1-x}} dx$.

(IIT-JEE, 1980)

4. Evaluate $\int (e^{\log x} + \sin x) \cos x\, dx$. **(IIT-JEE, 1981)**

5. Evaluate $\int \frac{(x-1)\, e^x}{(x+1)^3} dx$. **(IIT-JEE, 1983)**

6. Evaluate $\int \frac{dx}{x^2(x^4+1)^{3/4}}$. **(IIT-JEE, 1984)**

7. Evaluate $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$. **(IIT-JEE, 1985)**

8. Evaluate $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}$. **(IIT-JEE, 1986)**

9. Evaluate $\int \frac{\sqrt{\cos 2x}}{\sin x} dx$. **(IIT-JEE, 1987)**

10. Evaluate $\int (\sqrt{\tan x} + \sqrt{\cot x})\, dx$. **(IIT-JEE, 1989)**

11. Evaluate $\int \left(\frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} + \frac{\ln\left(1 + \sqrt[6]{x}\right)}{\sqrt[3]{x} + \sqrt{x}}\right) dx$.

(IIT-JEE, 1992]

12. Evaluate $\int \cos 2\theta \ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) d\theta$. **(IIT-JEE, 1994)**

13. Evaluate $\int \frac{x+1}{x(1+xe^x)^2} dx$. **(IIT-JEE, 1996)**

14. Evaluate $\int \frac{1}{x}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$. **(IIT-JEE, 1997)**

15. Evaluate $\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)} dx$. **(IIT-JEE, 1999)**

16. Evaluate $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right) dx$. **(IIT-JEE, 2001)**

17. Evaluate for $m \in N$,

$\int (x^{3m} + x^{2n} + x^m)(2x^{2m} + 3x^m + 6)^{1/m}\, dx, x > 0$.

(IIT-JEE, 2002)

Objective

Fill in the blanks

1. $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = Ax + B\log(9e^{2x} - 4) + C$, then

A = ______, B = ______, C = ______.

Multiple choice questions with one correct answer

1. The value of the integral $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx$ is

a. $\sin x - 6\tan^{-1}(\sin x) + C$

b. $\sin x - 2(\sin x)^{-1} + C$

c. $\sin x - 2(\sin x)^{-1} - 6\tan^{-1}(\sin x) + C$

d. $\sin x - 2(\sin x)^{-1} + 5\tan^{-1}(\sin x) + C$

2. $\int \frac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}} dx$ is equal to

a. $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^3} + C$

b. $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x} + C$

c. $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^2} + C$

d. $\frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + C$

ANSWERS AND SOLUTIONS

Subjective Type

1. $\int \sqrt{\left(\frac{1+x^2}{x^2-x^4}\right)}dx = \int \frac{1+x^2}{x\sqrt{(1-x^4)}}dx$

$= \int \frac{dx}{x\sqrt{(1-x^4)}} + \int \frac{xdx}{\sqrt{(1-x^4)}}$

$= \int \frac{x^3dx}{x^4\sqrt{(1-x^4)}} + \int \frac{xdx}{\sqrt{(1-x^4)}}$

$= -\frac{1}{2}\int \frac{udu}{(1-u^2)u} + \frac{1}{2}\int \frac{dv}{\sqrt{(1-v^2)}}$

(Putting $1-x^4 = u^2, -4x^3\,dx = 2u\,du$ in the first integral and $x^2 = v, 2x\,dx = dv$ in the second integral)

$= \frac{1}{2}\int \frac{du}{u^2-1} + \frac{1}{2}\sin^{-1} v$

$= \frac{1}{2}.\frac{1}{2\cdot 1}\log\left|\frac{u-1}{u+1}\right| + \frac{1}{2}\sin^{-1} v + c$

$= \frac{1}{4}\log\left|\frac{\sqrt{(1-x^4)}-1}{\sqrt{(1-x^4)}+1}\right| + \frac{1}{2}\sin^{-1}(x^2) + c$

2. $I = \int \frac{(2\cos^2 x-1)^{1/2}dx}{\sin x}$

$= \int \frac{(2\cos^2 x-1)\sin x dx}{\sin^2 x\sqrt{(2\cos^2 x-1)}}$

$= -\int \frac{(2t^2-1)dt}{(1-t^2)\sqrt{(2t^2-1)}}$

(Putting $\cos x = t, -\sin x\,dx = dt$)

$= -\int \frac{-2(1-t^2)+1}{(1-t^2)\sqrt{(2t^2-1)}}dt$

$= 2\int \frac{dt}{\sqrt{(2t^2-1)}} - \int \frac{dt}{(1-t^2)\sqrt{(2t^2-1)}}$

$= I_1 + I_2$ (say) (1)

Now $I_1 = \frac{2}{\sqrt{2}}\log\left|\sqrt{2}t + \sqrt{2t^2-1}\right| + C_1$

And putting $t = 1/z, dt = (-1/z^2)dz$, we get

$I_2 = \int \frac{zdz}{(z^2-1)\sqrt{(2-z^2)}} = \int \frac{dv}{v^2-1}$

Putting $2-z^2 = v^2, -z\,dz = v\,dv$

$\Rightarrow \quad I_2 = \frac{1}{2\times 1}\log\left(\frac{v-1}{v+1}\right) + C_2$

$= \frac{1}{2}\log\left|\frac{\sqrt{(2-z^2)}-1}{\sqrt{(2-z^2)}+1}\right| + C_2$

$= \frac{1}{2}\log\left|\frac{\sqrt{(2t^2-1)}-t}{\sqrt{(2t^2-1)}+t}\right| + C_2$

$= \frac{1}{2}\log\left|\frac{\sqrt{(\cos 2x)}-\cos x}{\sqrt{(\cos 2x)}+\cos x}\right| + C_2$

Hence, from equation (1), we get

$I = \sqrt{2}\log|\sqrt{2}\cos x + \sqrt{(\cos 2x)}|$

$+\frac{1}{2}\log\left|\frac{\sqrt{(\cos 2x)}-\cos x}{\sqrt{(\cos 2x)}+\cos x}\right| + C$

3. $I = \int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}}dx$

$= \int \frac{x^2(1-1/x^2)}{x^2(x+1/x)\sqrt{x^2+1/x^2}}dx$

$= \int \frac{(1-1/x^2)dx}{(x+1/x)\sqrt{(x+1/x)^2-2}}$

Putting $x + 1/x = t$, we have $I = \int \frac{dt}{t\sqrt{t^2-2}}$

Again putting $t^2 - 2 = y^2, 2t\,dt = 2y\,dy$, we get

$I = \int \frac{ydy}{(y^2+2)y} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{y}{\sqrt{2}}$

$= \frac{1}{2}\tan^{-1}\frac{\sqrt{x^2+1/x^2}}{\sqrt{2}} + c$

4. $I_n = \int \cos^n x dx$

$= \cos^{n-1} x \int \cos x dx + (n-1)\int (\sin^2 x)\cos^{n-2} x dx$

$= (\cos^{n-1} x \sin x) + (n-1)\int \cos^{n-2} x(1-\cos^2 x)dx$

$= (\cos^{n-1} x \sin x) + (n-1)\int [\cos^{n-2} x - \cos^n x]dx$

$\Rightarrow I_n + (n-1)I_n = (\cos^{n-1} x \sin x) + (n-1)(I_{n-2})$

$\Rightarrow I_n = \frac{1}{n}(\cos^{n-1} x\sin x) + \left(\frac{n-1}{n}\right)I_{n-2}$

5. Here, $I = \int \frac{(1-x\sin x)dx}{x\left(1-\left(xe^{\cos x}\right)^3\right)}$,

put $xe^{\cos x} = t$ so that $(xe^{\cos x}(-\sin x) + e^{\cos x})\,dx = dt$

$\therefore \quad I = \int \frac{dt}{t(1-t^3)}$

$= \int \frac{dt}{t^4\left(\frac{1}{t^3}-1\right)}$

Let $\frac{1}{t^3}-1 = y \Rightarrow dy = \frac{-3}{t^4}dt$

$\Rightarrow \quad I = -\frac{1}{3}\int \frac{dy}{y} = -\frac{1}{3}\log|y| + C$

$= -\frac{1}{3}\log\left|\frac{1}{t^3}-1\right| + C$

$\Rightarrow \quad I = \int \frac{dt}{t} + \frac{1}{3}\int \frac{dt}{1-t} + \int \frac{\left(-\frac{2}{3}t - \frac{1}{3}\right)}{1+t+t^2}dt$

$= \log|t| - \frac{1}{3}\log|1-t| - \frac{1}{3}\log|1+t+t^2|$

where, $t = xe^{\cos x}$

6. Note that $\sec^{-1}\sqrt{1+x^2} = \tan^{-1}x$;

$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x$ (for $x > 0$)

$\Rightarrow \quad I = \int \frac{e^{\tan^{-1}x}}{1+x^2}\left[(\tan^{-1}x)^2 + 2\tan^{-1}x\right]dx$

[put $\tan^{-1}x = t$]

$= \int e^t(t^2 + 2t)\,dt$

$= e^t t^2 = e^{\tan^{-1}x}\left(\tan^{-1}x\right)^2 + C$

7. $I = \int \frac{x^2\left(1-\frac{1}{x^2}\right)dx}{x^2\left[\sqrt{\left(x+\frac{1}{x}+\alpha\right)}\sqrt{x+\frac{1}{x}+\beta}\right]}$

Put $x + \frac{1}{x} = z \quad \therefore \left(1-\frac{1}{x^2}\right)dx = dz$

$\Rightarrow \quad I = \int \frac{dz}{\sqrt{(z+\alpha)(z+\beta)}}$

$= \int \frac{dz}{\sqrt{z^2 + (\alpha+\beta)z + \alpha\beta}}$

$= \int \frac{dz}{\sqrt{\left(z+\frac{\alpha+\beta}{2}\right)^2 - \left(\frac{\alpha-\beta}{2}\right)^2}}$

$= \log\left|z + \frac{\alpha+\beta}{2} - \sqrt{(z+\alpha)(z+\beta)}\right| + c$

$= \log\left|x + \frac{1}{x} + \frac{\alpha+\beta}{2} - \sqrt{(z+\alpha)(z+\beta)}\right| + c$

$= \log\left|x + \frac{1}{x} + \frac{\alpha+\beta}{2} - \sqrt{\left(x+\frac{1}{x}+\alpha\right)\left(x+\frac{1}{x}+\beta\right)}\right| + c$

$= \log\left[\frac{2x^2 + (\alpha+\beta)x + 2}{2x} - \frac{\sqrt{(x^2+\alpha x+1)(x^2+\beta x+1)}}{x}\right] + c$

$= \log\frac{1}{2}\left(\frac{\sqrt{x^2+\alpha x+1} - \sqrt{x^2+\beta x+1}}{\sqrt{x}}\right)^2 + c$

$= 2\log\left(\frac{\sqrt{x^2+\alpha x+1} - \sqrt{x^2+\beta x+1}}{\sqrt{x}}\right)^2 + c$

8. $\int \frac{2x}{(1-x^2)\sqrt{x^4-1}}dx$

$= \int \frac{-2x}{(x^2-1)^{3/2}\sqrt{x^2+1}}dx$

Put $\sqrt{\frac{x^2+1}{x^2-1}} = z$

$\therefore \frac{1}{2}\left(\frac{x^2-1}{x^2+1}\right)^{1/2} \frac{(x^2-1)2x - (x^2+1)2x}{(x^2-1)^2}dx = dz$

$\Rightarrow \frac{\sqrt{x^2-1}}{\sqrt{x^2+1}} \frac{-2x}{(x^2-1)^2}dx = dz$

$\Rightarrow \frac{-2x}{(x^2-1)^{3/2}\sqrt{x^2+1}}dx = dz$

$\therefore$ given integral $= \int dz = z + c = \sqrt{\frac{x^2+1}{x^2-1}} + c$

9. Write $I = \int \frac{x\,dx}{x^4\sqrt{x^2-1}}$ and put $x^2 - 1 = t^2$, so that $2x\,dx = 2t\,dt$ and

$I = \int \frac{t}{(t^2+1)^2 t}dt = \int \frac{dt}{(t^2+1)^2}$

But $\tan^{-1}t = \int \frac{dt}{t^2+1} = \int 1.\frac{1}{t^2+1}dt$

$$= \frac{t}{t^2+1} + \int t \frac{2t}{\left(t^2+1\right)^2} dt$$

$$= \frac{t}{t^2+1} + 2\int \frac{t^2+1-1}{\left(t^2+1\right)^2} dt$$

$$= \frac{t}{t^2+1} + 2\tan^{-1} t - 2I$$

$$\therefore \quad I = \frac{1}{2}\frac{t}{t^2+1} + \frac{1}{2}\tan^{-1} t + C$$

$$= \frac{1}{2}\left(\frac{\sqrt{x^2-1}}{x^2} + \tan^{-1}\sqrt{x^2-1}\right) + C$$

10. Here, $I = \int \sqrt{\frac{3-x}{3+x}} \sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right) dx,$

put $x = 3\cos 2\theta \Rightarrow dx = -6\sin 2\theta\, d\theta$

$$= \int \sqrt{\frac{3-3\cos 2\theta}{3+3\cos 2\theta}} \sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-3\cos 2\theta}\right) \times (-6\sin 2\theta)\, d\theta$$

$$= \int \frac{\sin\theta}{\cos\theta} \sin^{-1}(\sin\theta)\, (-6\sin 2\theta)\, d\theta$$

$$= -6\int \theta\, (2\sin^2\theta)\, d\theta$$

$$= -6\int \theta\,(1-\cos 2\theta)\, d\theta$$

$$= -6\left\{\frac{\theta^2}{2} - \int \theta \cos 2\theta\, d\theta\right\}$$

$$= -6\left\{\frac{\theta^2}{2} - \left(\theta\,\frac{\sin 2\theta}{2} - \int 1\left(\frac{\sin 2\theta}{2}\right) d\theta\right)\right\}$$

$$= -3\theta^2 + 6\left\{\frac{\theta \sin 2\theta}{2} + \frac{\cos 2\theta}{4}\right\} + c$$

$$= \frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}\cos^{-1}\left(\frac{x}{2}\right) + 2x\right\} + c.$$

11. $I = \int \sqrt{\sec x - 1}\, dx = \int \sqrt{\frac{1-\cos x}{\cos x}} dx$

$$= \int \sqrt{\frac{(1-\cos x)}{\cos x} \times \frac{(1+\cos x)}{(1+\cos x)}}\, dx$$

$$= \int \sqrt{\frac{1-\cos^2 x}{\cos x + \cos^2 x}}\, dx$$

$$= \int \frac{\sin x}{\sqrt{\cos^2 x + \cos x}}\, dx$$

Let $\cos x = t$. Then $d(\cos x) = dt \Rightarrow -\sin x\, dx = dt$

$$\therefore \quad I = \int \frac{-dt}{\sqrt{t^2+t}}$$

$$= -\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$= -\log\left|\left(t+\frac{1}{2}\right) + \sqrt{\left(t+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + C$$

$$= -\log\left|\left(t+\frac{1}{2}\right) + \sqrt{t^2+t}\right| + C$$

$$= -\log\left|\left(\cos x + \frac{1}{2}\right) + \sqrt{\cos^2 x + \cos x}\right| + C$$

12. $I = \int \sqrt{1+\operatorname{cosec} x}\, dx$

$$= \int \frac{\sqrt{1+\sin x}}{\sqrt{\sin x}}\, dx$$

$$= \int \frac{\sin\frac{x}{2} + \cos\frac{x}{2}}{\sqrt{2\sin\frac{x}{2}\cos\frac{x}{2}}} \qquad (\because 0 < x < \pi/2)$$

$$= \int \frac{\sin\frac{x}{2} + \cos\frac{x}{2}}{\sqrt{1-\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2}}\, dx$$

Put $\sin\frac{x}{2} - \cos\frac{x}{2} = t \Rightarrow \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) dx = 2dt$

$$\therefore \quad I = \int \frac{2dt}{\sqrt{1-t^2}} = 2\sin^{-1} t + c$$

$$= 2\sin^{-1}\left(\sin x\,\frac{x}{2} - \cos\frac{x}{2}\right) + c$$

13. $I = \int \frac{\cos^4 x}{\sin^3 x(\sin^5 x + \cos^5 x)^{\frac{3}{5}}}\, dx$

$$= \int \frac{\cos^4 x}{\sin^6 x(1+\cot^5 x)^{\frac{3}{5}}}\, dx$$

$$= \int \frac{\sec^2 x\, dx}{\tan^6 x\left(1+\frac{1}{\tan^5 x}\right)^{\frac{3}{5}}}$$

Let $\tan x = p$, then $\sec^2 x\,dx = dp$

$$\Rightarrow \quad I = \int \frac{dp}{p^6\left(1+\frac{1}{p^5}\right)^{3/5}}$$

Let $\left(1+\frac{1}{p^5}\right) = k \Rightarrow -5\frac{1}{p^6}dp = dk$

$$\Rightarrow \quad I = -\frac{1}{5}\int (k)^{-3/5}dk$$

$$= -\frac{1}{5}(k^{2/5})\left(\frac{5}{2}\right)+c$$

$$\Rightarrow \quad I = -\frac{1}{2}\left[\frac{p^5+1}{p^5}\right]^{2/5}$$

$$= -\frac{1}{2}\left[\frac{\tan^5 x+1}{\tan^5 x}\right]^{2/5} = -\frac{1}{2}(1+\cot^5 x)^{2/5}+c$$

Objective Type

1. c. $\int \frac{\sin 2x}{\sin 5x \sin 3x}dx$

$$= \int \frac{\sin(5x-3x)}{\sin 5x \sin 3x}$$

$$= \int \frac{\sin 5x \cos 3x - \cos 5x \sin 3x}{\sin 5x \sin 3x}dx$$

$$= \frac{1}{3}\log \sin 3x - \frac{1}{5}\log \sin 5x + C$$

2. a. $I = \int \frac{\sqrt{1+\sin x}\,\sqrt{1-\sin x}}{\sqrt{1-\sin x}}dx$

$$= \int \frac{\cos x}{\sqrt{1-\sin x}}dx = -2\sqrt{1-\sin x}+C$$

3. b. $\int \frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x}dx$

$$= \int \frac{(\sin^2 x-\cos^2 x)(\sin^4 x+\cos^4 x)}{1-2\sin^2 x\cos^2 x}$$

$$= \int -\cos 2x\,dx = -\frac{1}{2}\sin 2x + C$$

4. b. $\int \frac{\cos 4x+1}{\cot x - \tan x}dx$

$$= \int \frac{2\cos^2 2x}{\cos^2 x-\sin^2 x}\sin x\cos x\,dx$$

$$= \int \cos 2x \sin 2x\,dx$$

$$= \frac{1}{4}\int \sin 4x\,dx = -\frac{1}{8}\cos 4x + C$$

5. b. $f(x) = x|\cos x|$, $\frac{\pi}{2} < x < \pi$ $= -x\cos x$, because $\cos x$ is negative in $\left(\frac{\pi}{2}, \pi\right)$.

$\therefore$ the required primitive function $= \int -x\cos x\,dx$

Now, use integration by parts

6. a. $I = \int \frac{dx}{x(x^n+1)} = \int \frac{x^{n-1}}{x^n(x^n+1)}dx$

Putting $x^n = t$ so that $n x^{n-1}dx = dt$

$\Rightarrow x^{n-1}dx = \frac{1}{n}dt$

$$\therefore \quad I = \int \frac{\frac{1}{n}dt}{t(t+1)} = \frac{1}{n}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$$

$$= \frac{1}{n}(\log t - \log(t+1)) + C$$

$$= \frac{1}{n}\log\left(\frac{x^n}{x^n+1}\right)+C$$

7. b. $I = \int 4\sin x \cos\frac{x}{2}\cos\frac{3x}{2}dx$

$$= \int 2\sin x(\cos 2x + \cos x)dx$$

$$= \int (\sin 3x - \sin x + \sin 2x)dx$$

$$= \cos x - \frac{1}{3}\cos 3x - \frac{1}{2}\cos 2x + C$$

8. c. $\frac{dx}{dt} = f'''(t)\cos t - f''(t)\sin t + f''(t)\sin t + f'(t)\cos t$

$$= [f'''(t)+f'(t)]\cos t$$

$$\frac{dy}{dt} = -f'''(t)\sin t - f''(t)\cos t + f''(t)\cos t - f'(t)\sin t$$

$$= -[f'''(t)+f'(t)]\sin t$$

$$\Rightarrow \quad \left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{1/2}$$

$$= [(f'''(t)+f'(t))^2(\cos^2 t+\sin^2 t)]^{1/2}$$

$$= f'''(t)+f'(t)$$

$$\Rightarrow \quad \int\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{1/2}dt = f''(t)+f(t)+C$$

9. c. $\sin^3 x \sin(x+\alpha)$

$= \sin^3 x(\sin x\cos\alpha + \cos x\sin\alpha)$

$= \sin^4 x(\cos\alpha + \cot x\sin\alpha)$

$$I = \int \frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}dx$$

$$= \int \frac{1}{\sin^2 x \sqrt{\cos\alpha + \cot x \sin\alpha}} dx$$

$$= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}} dx$$

Putting $\cos\alpha + \cot x \sin\alpha = t$ and $-\operatorname{cosec}^2 x \sin\alpha dx = dt$, we have

$$I = \int -\frac{1}{\sin\alpha \sqrt{t}} dt = -\frac{1}{\sin\alpha}\int t^{-1/2} dt$$

$$= \frac{1}{\sin\alpha}\left(\frac{t^{1/2}}{1/2}\right) + C$$

$$\Rightarrow \quad I = -2\operatorname{cosec}\alpha \sqrt{t} + C$$

$$= -2\operatorname{cosec}\alpha(\cos\alpha + \cot x \sin\alpha)^{1/2} + C$$

10. c. $\int \frac{px^{p+2q-1} - qx^{q-1}}{(x^{p+q}+1)^2} dx$

$$= \int \frac{px^{p-1} - qx^{-q-1}}{(x^p + x^{-q})^2} dx$$

(Dividing N^r and D^r by x^{2q})

$$= \int \frac{dt}{t^2} = -\frac{1}{t} + C = -\frac{1}{x^p + x^{-q}} + C = -\frac{x^q}{x^{p+q}+1} + C$$

11. c. $I_n = x(\ln x)^n - \int \frac{x(n)(\ln x)^{n-1}}{x} dx$

$$= x(\ln x)^n - n I_{(n-1)}$$

$$\Rightarrow I_n + n I_{n-1} = x(\ln x)^n$$

12. b. Let $I = \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}} dx$

Putting $x + 1 = t^2$, $dx = 2t\,dt$, we get

$$I = 2\int \frac{t^2+1}{t^4+t^2+1} dt$$

$$= 2\int \frac{1+(1/t)^2}{\left(t-\frac{1}{t}\right)^2 + 3} dt$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{3}}\right) + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right) + C$$

13. b. $I = \int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$

$$= \int \frac{2\sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$= \int \frac{2\tan x \sec^2 x}{1+\tan^4 x} dx$$

Let $\tan^2 x = t \Rightarrow 2\tan x \sec^2 x\, dx = dt$,

$$\Rightarrow \quad I = \int \frac{dt}{1+t^2} = \tan^{-1} t + C = \tan^{-1}(\tan^2 x) + C$$

14. c. $I = \int \frac{\sec x\, dx}{\sqrt{2\sin(x+A)\cos x}}$

$$= \int \frac{\sec^2 x\, dx}{\sqrt{\frac{2\sin(x+A)}{\cos x}}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{\sec^2 x\, dx}{\sqrt{\tan x \cos A + \sin A}}$$

$$= \frac{\sec A}{\sqrt{2}} \int \frac{2p\,dp}{p}$$

($\tan x \cos A + \sin A = p^2$, then $\cos A \sec^2 x\, dx = 2p\,dp$)

$$I = \sqrt{2}\sec A \int dp = \sqrt{2}\sec A\sqrt{\tan x \cos A + \sin A} + C$$

15. a. Differentiating both sides, we get

$$\sqrt{1+\sin x}\; f(x) = \frac{2}{3}\frac{3}{2}(1+\sin x)^{1/2}\cos x$$

$$\Rightarrow f(x) = \cos x.$$

16. c. Here, $\int e^x\{f(x) - f'(x)\}dx = \phi(x)$

and $\int e^x\{f(x) + f'(x)\}dx = e^x f(x)$

On adding, we get $2\int e^x f(x)dx = \phi(x) + e^x f(x)$

17. b. $f(x) = \int \frac{x^2 dx}{(1+x^2)(1+\sqrt{1+x^2})}$

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta = (1+x^2)\, d\theta$

$$\Rightarrow \quad f(x) = \int \frac{x^2 dx}{(1+x^2)(1+\sqrt{1+x^2})}$$

$$= \int \frac{\tan^2\theta \sec^2\theta\, d\theta}{\sec^2\theta(1+\sec\theta)}$$

$$= \int \frac{\tan^2\theta\, d\theta}{1+\sec\theta}$$

$$= \int \frac{\sin^2\theta\, d\theta}{\cos\theta(1+\cos\theta)}$$

$$= \int \frac{1-\cos^2\theta\, d\theta}{\cos\theta(1+\cos\theta)}$$

$$= \int \frac{(1-\cos\theta)\, d\theta}{\cos\theta}$$

$$= \int \sec\theta d\theta - \int d\theta$$

$$= \log(x + \sqrt{1+x^2}) - \tan^{-1}x + C$$

Given $f(0) = 0$

$\Rightarrow \quad 0 = \log 1 - 0 + C$

$\Rightarrow \quad C = 0$

$\Rightarrow \quad f(1) = \log(1 + \sqrt{1+1}) - \tan^{-1}(1)$

$$= \log(1+\sqrt{2}) - \frac{\pi}{4}$$

18. a. Let $x = \tan\theta$, then $dx = \sec^2\theta\, d\theta$

Now $y = \int \frac{dx}{(1+x^2)^{\frac{3}{2}}} = \int \frac{\sec^2\theta}{(1+\tan^2\theta)^{\frac{3}{2}}} d\theta$

$$= \int \frac{\sec^2\theta}{(\sec^2\theta)^{\frac{3}{2}}} d\theta$$

$$= \int \frac{\sec^2\theta}{\sec^3\theta} d\theta = \int \frac{d\theta}{\sec\theta} = \int \cos\theta d\theta$$

Hence, $y = \sin\theta + c = \frac{x}{\sqrt{1+x^2}} + c$ (1)

$$\left[\because \tan\theta = x = \frac{x}{1} \therefore \sin\theta = \frac{x}{\sqrt{1^2+x^2}}\right]$$

Given when $x = 0, y = 0 \Rightarrow$ from equation (1), $0 = 0 + c$

$\Rightarrow c = 0$

$\Rightarrow$ from equation (1), $y = \frac{x}{\sqrt{1+x^2}}$

$\Rightarrow$ when $x = 1, y = \frac{1}{\sqrt{2}}$

19. c. Let $x = t^6 \Rightarrow dx = 6t^5\, dt$

$\Rightarrow I = \int t^3 (1+t^2)^4 6t^5\, dt$

$\Rightarrow I = 6\int t^8(1 + 4t^2 + 6t^4 + 4t^6 + t^8)\, dt$

$$= 6\int (t^8 + 4t^{10} + 6t^{12} + 4t^{14} + t^{16})\, dt$$

$$= 6\left\{\frac{t^9}{9} + \frac{4t^{11}}{11} + \frac{6t^{13}}{13} + \frac{4t^{15}}{15} + \frac{t^{17}}{17}\right\} + C$$

$$= 6\left\{x^{2/3} + \frac{4}{11}x^{11/6} + \frac{6}{13}x^{13/6} + \frac{4}{15}x^{5/2} + \frac{1}{17}x^{17/6}\right\} + C$$

20. b. Here, $\int x^5(1+x^3)^{2/3}\, dx$

Let $1 + x^3 = t^2$ and $3x^2\, dx = 2t\, dt$

$\therefore \int x^5 (1+x^3)^{2/3}\, dx$

$$= \int x^3 (1+x^3)^{2/3} x^2\, dx$$

$$= \int (t^2-1)(t^2)^{2/3} x^2\, dx$$

$$= \frac{2}{3}\int (t^2-1)\, t^{7/3}\, dt$$

$$= \frac{2}{3}\int (t^{13/3} - t^{7/3})\, dt$$

$$= \frac{2}{3}\left\{\frac{3}{16}t^{16/3} - \frac{3}{10}t^{10/3}\right\} + C$$

$$= \frac{1}{8}(1+x^3)^{8/3} - \frac{1}{5}(1+x^3)^{5/3} + C$$

21. a. Let $I = \int \frac{(1-\cos\theta)^{2/7}}{(1+\cos\theta)^{9/7}} d\theta$

$$I = \int \frac{(2\sin^2\theta/2)^{2/7}}{(2\cos^2\theta/2)^{9/2}} d\theta = \frac{1}{2}\int \frac{(\sin\theta/2)^{4/7}}{(\cos\theta/2)^{18/7}} d\theta$$

Put $\frac{\theta}{2} = t \therefore \frac{d\theta}{2} = dt$

$\Rightarrow \quad I = \int \frac{(\sin t)^{4/7}}{(\cos t)^{18/7}} dt$ (Here $m + n = -2$)

$$= \int (\tan t)^{4/7} \sec^2 t dt$$

Put $\tan t = u \therefore \sec^2 t\, dt = du$

$\Rightarrow \quad I = \int u^{4/7} du = \frac{u^{11/7}}{11/7} + c = \frac{7}{11}(\tan t)^{11/7} + C$

$$= \frac{7}{11}\left(\tan\frac{\theta}{2}\right)^{11/7} + C$$

22. c. $I = \int \frac{1-x^7}{x(1+x^7)} dx = a\ln|x| + b\ln|1+x^7| + C$

Diff. both sides, we get $\frac{1-x^7}{x(1+x^7)} = \frac{a}{x} + b\frac{7x^6}{1+x^7}$

$\Rightarrow 1 - x^7 = a(1+x^7) + 7bx^7$

$\Rightarrow \quad a = 1, a + 7b = -1$

$\Rightarrow \quad b = -2/7$

23. d. $I = \int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$, let $x = \tan\theta$

$\Rightarrow \quad dx = \sec^2\theta\, d\theta$

$\Rightarrow \quad I = \int \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \sec^2\theta d\theta$

$$= 2\int \theta \sec^2\theta\, d\theta$$

$$= 2(\theta\tan\theta - \ln|\sec\theta|) + C$$

$$= 2(x\tan^{-1}x - \ln|\sec(\tan^{-1}x)|) + C$$

24. c. $I = \int \frac{\ln(\tan x)}{\sin x \cos x} dx$, let $t = \ln(\tan x)$

$$\Rightarrow \quad \frac{dt}{dx} = \frac{\sec^2 x}{\tan x}$$

$$\Rightarrow \quad dt = \frac{dx}{\sin x \cos x}$$

$$\Rightarrow \quad I = \int t\,dt = \frac{1}{2}t^2 + C = \frac{1}{2}(\ln(\tan x))^2 + C$$

25. c. $I = \int \frac{2\sin x}{(3+\sin 2x)} dx$

$$= \int \frac{\sin x + \cos x + \sin x - \cos x}{(3+\sin 2x)} dx$$

$$= \underbrace{\int \frac{\sin x + \cos x}{3+\sin 2x} dx}_{I_1} - \underbrace{\int \frac{-\sin x + \cos x}{(3+\sin 2x)} dx}_{I_2}$$

Putting $t_1 = \sin x - \cos x$ in I_1 and $t_2 = \sin x + \cos x$ in I_2, we get

$$I = \int \frac{dt_1}{[3+(1-t_1^2)]} - \int \frac{dt_2}{[3+(t_2^2-1)]}$$

$$= \int \frac{dt_1}{4-t_1^2} - \int \frac{dt_2}{2+t_2^2}$$

$$= \frac{1}{4}\ln\left|\frac{2+t_1}{2-t_1}\right| - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t_2}{\sqrt{2}}\right) + C$$

$$= \frac{1}{4}\ln\left|\frac{2+\sin x - \cos x}{2-\sin x + \cos x}\right| - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right) + C$$

26. d. $I = \int \frac{x^9\,dx}{(4x^2+4)^6}$

$$= \int \frac{dx}{x^3\left(4+\frac{1}{x^2}\right)^6}$$

$$= -\frac{1}{2}\int \frac{d\left(4+\frac{1}{x^2}\right)}{\left(4+\frac{1}{x^2}\right)^6}$$

$$= -\frac{1}{2}\,\frac{\left(4+\frac{1}{x^2}\right)^{-5}}{-5} + C = \frac{1}{10}\left(4+\frac{1}{x^2}\right)^{-5} + C$$

27. c. $I = \int e^{\tan^{-1}x}(1+x+x^2)\left(-\left(\frac{1}{1+x^2}\right)dx\right)$

$$= -\int e^{\tan^{-1}x}\left(1+\frac{x}{1+x^2}\right)dx$$

$$= -\int e^{\tan^{-1}x}dx - \int x\,\frac{e^{\tan^{-1}x}}{1+x^2}dx$$

$$= -\int e^{\tan^{-1}x}dx - x\,e^{\tan^{-1}x} + \int e^{\tan^{-1}x}dx + C$$

$$= -x\,e^{\tan^{-1}x} + C$$

28. d. $I = \int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}}$

$$= \int \frac{dx}{\sqrt{\frac{\sin^3 x}{\cos^3 x}\cos^8 x}}$$

$$= \int \frac{\sec^4 x}{\sqrt{\tan^3 x}} dx$$

$$= \int \frac{(1+\tan^2 x)\sec^2 x}{\sqrt{\tan^3 x}} dx$$

Let $t = \tan x \Rightarrow dt = \sec^2 x\,dx$

$$\Rightarrow \quad I = \int \frac{1+t^2}{t^{3/2}} dt$$

$$= \int (t^{-3/2} + t^{1/2})\,dt$$

$$= -2t^{-\frac{1}{2}} + \frac{2}{3}t^{3/2} + C$$

$$= -2\sqrt{\cot x} + \frac{2}{3}\sqrt{\tan^3 x} + C$$

$$\Rightarrow \quad a = -2, b = \frac{2}{3}$$

29. c. $I = \int \frac{\cos 4x - 1}{\cot x - \tan x} dx$

$$= \int \frac{-2\sin^2 2x(\sin x \cos x)}{(\cos^2 x - \sin^2 x)} dx$$

$$= -\int \frac{\sin^2 2x \sin 2x}{\cos 2x} x$$

$$= \int \frac{(\cos^2 2x - 1)\sin 2x}{\cos 2x} dx$$

Let $t = \cos 2x \Rightarrow dt = -2\sin 2x\,dx$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{(1-t^2)}{t} dt = \frac{1}{2}\ln|t| - \frac{t^2}{4} + C$$

$$= \frac{1}{2}\ln|\cos 2x| - \frac{1}{4}\cos^2 2x + c$$

30. a. Putting $1-x^3 = y^2, -3x^2dx = 2y\,dy$, we get

$$\int \frac{1}{x\sqrt{1-x^3}} dx$$

$$= -\frac{2}{3}\int \frac{1}{1-y^2} dy$$

$$= \frac{1}{3}\log\left|\frac{y-1}{y+1}\right| + C$$

31

32.

33. (

34. d.

$$= \frac{1}{3}\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right| + C \Rightarrow a = \frac{1}{3}$$

31. b. We have $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}}$

$$= \int \frac{dx}{x^2\, x^{n-1}\left(1+\frac{1}{x^n}\right)^{(n-1)/n}}$$

$$= \int \frac{dx}{x^{n+1}(1+x^{-n})^{(n-1)/n}}$$

Put $1 + x^{-n} = t$

$\therefore\ -nx^{-n-1}\,dx = dt \Rightarrow \frac{dx}{x^{n+1}} = -\frac{dt}{n}$

$$\Rightarrow \int \frac{dx}{x^2(x^n+1)^{(n-1)/n}} = -\frac{1}{n}\int \frac{dt}{t^{(n-1)/n}}$$

$$= -\frac{1}{n}\int t^{1/n-1}dt = -\frac{1}{n}\,\frac{t^{1/n-1+1}}{1/n-1+1} + C$$

$$= -t^{1/n} + C = -(1+x^{-n})^{1/n} + C$$

32. d. $I = \int \frac{\sqrt{x-1}}{x\sqrt{x+1}}dx$

$$= \int \frac{x-1}{x\sqrt{x^2-1}}dx$$

$$= \int \frac{dx}{\sqrt{x^2-1}} - \int \frac{dx}{x\sqrt{x^2-1}}$$

$$= \ln|x+\sqrt{x^2+1}| - \sec^{-1}x + c$$

33. c. Write $2ax + x^2 = (x+a)^2 - a^2$, and put $x + a = a\sec\theta$, so that $dx = a\sec\theta\tan\theta\, d\theta$

$$\therefore \quad I = \int \frac{a\sec\theta\tan\theta}{a^3\tan^3\theta}d\theta$$

$$= \frac{1}{a^2}\int \frac{\cos\theta}{\sin^2\theta}d\theta$$

$$= -\frac{1}{a^2\sin\theta} + C$$

$$= -\frac{1}{a^2}\frac{\sec\theta}{\tan\theta} + C = -\frac{1}{a^2}\frac{x+a}{\sqrt{2ax+x^2}} + C$$

34. d. By rationalizing the integrand, the given integral can be written as

$$f(x) = \int\left(x + \sqrt{x^2+1}\right)dx$$

$$= \frac{x^2}{2} + \frac{x}{2}\sqrt{x^2+1} + \frac{1}{2}\log\left|x + \sqrt{x^2+1}\right| + C$$

Putting $x = 0$, we have $f(0) = C$ so $C = -1/2 - 1/\sqrt{2}$

and $f(1) = \frac{1}{2} + \frac{1}{2}\sqrt{2} + \frac{1}{2}\log|1+\sqrt{2}| + \left(-\frac{1}{2} - \frac{1}{\sqrt{2}}\right)$

$$= \frac{1}{2}\log(1+\sqrt{2}) = -\log(\sqrt{2}-1)$$

35. b. $\int e^x\left(\frac{2\tan x}{1+\tan x} + \tan^2\left(x - \frac{\pi}{4}\right)\right)dx$

$$= \int e^x\left(\tan\left(x-\frac{\pi}{4}\right) + \sec^2\left(x-\frac{\pi}{4}\right)\right)dx$$

$$= e^x\tan\left(x-\frac{\pi}{4}\right) + C$$

36. a. Given that $I = \int (x^2+x)(x^{-8}+2x^{-9})^{1/10}dx$

or $\quad I = \int (x+1)(x^2+2x)^{1/10}dx$

Now put $x^2 + 2x = t \Rightarrow (x+1)dx = \frac{dt}{2}$

$$\Rightarrow \quad I = \int t^{1/10}\frac{dt}{2} = \frac{1}{2}\times\frac{10}{11}t^{11/10} = \frac{5}{11}t^{11/10} + C$$

$$= \frac{5}{11}(x^2+2x)^{11/10} + C$$

37. c. $\int \frac{dx}{(x+2)(x^2+1)} = a\ln(1+x^2) + b\tan^{-1}x + \frac{1}{5}\ln|x+2| + C$

Differentiating both sides, we get

$$\frac{1}{(x+2)(x^2+1)} = \frac{2ax}{(1+x^2)} + \frac{b}{(1+x^2)} + \frac{1}{5(x+2)}$$

$$\Rightarrow \quad \frac{1}{(x+2)(x^2+1)} = \frac{(x+2)(5b+10ax)+1+x^2}{5(1+x^2)(x+2)}$$

$$\Rightarrow \quad 5 = (1+x^2) + 5(b+2ax)(x+2)$$

Comparing the like powers of x on both sides, we get

$1 + 10a = 0, b + 4a = 0, 10b + 1 = 5$

$$\Rightarrow \quad a = -\frac{1}{10}, b = \frac{2}{5}$$

38. c. Differentiating both sides, we get

$$\frac{3\sin x + 2\cos x}{3\cos x + 2\sin x} = a + \frac{b(2\cos x - 3\sin x)}{(2\sin x + 3\cos x)}$$

$$= \frac{\sin x\,(2a-3b) + \cos x\,(3a+2b)}{(3\cos x + 2\sin x)}$$

Comparing like terms on both sides, we get

$$3 = 2a - 3b, 2 = 3a + 2b \Rightarrow a = \frac{12}{13}, b = -\frac{15}{39}$$

39. a. $\int \frac{3e^x - 5e^{-x}}{4e^x + 5e^{-x}} = ax + b\ln(4e^x + 5e^{-x}) + C$

Differentiating both sides, we get

$$\frac{3e^x - 5e^{-x}}{4e^x + 5e^{-x}} = a + b\frac{(4e^x - 5e^{-x})}{4e^x + 5e^{-x}}$$

$$\Rightarrow \quad 3e^x - 5e^{-x} = a(4e^x + 5e^{-x}) + b(4e^x - 5e^{-x})$$

Comparing the coefficient of like terms on both sides, we get

$3 = 4(a+b), -5 = 5a - 5b \Rightarrow a = -\frac{1}{8},\ b = \frac{7}{8}$

40. c. $\int\sqrt{\frac{\cos x - \cos^3 x}{1-\cos^3 x}}dx = \int\sqrt{\frac{\cos x}{1-\cos^3 x}}\sin x dx$

$= \int\sqrt{\frac{t}{1-t^3}}dt = -\int\frac{\sqrt{t}}{\sqrt{1-(t^{3/2})^2}}dt$, where $t = \cos x$

$= -\frac{2}{3}\int\frac{\frac{3}{2}\sqrt{t}}{\sqrt{1-(t^{3/2})^2}}dt = \frac{2}{3}\cos^{-1}(t^{3/2}) + C$

41. a. Putting,

$l^{r+1}(x) = t$ and $\frac{1}{xl(x)l^2(x)\ldots l^r(x)}dx = dt$, we get

$\int\frac{1}{xl^2(x)l^3(x)\ldots l^r(x)} = \int 1\,dt = t + C = l^{r+1}(x) + C$

42. b. $I = \int\frac{\cos x - \sin x}{\sqrt{\cos x \sin x}}dx$

Put $\sin x + \cos x = t$, so that $2\sin x\cos x = t^2 - 1$

$\therefore \quad I = \sqrt{2}\int\frac{dt}{\sqrt{t^2-1}} = \sqrt{2}\log\left|t + \sqrt{t^2-1}\right| + C$

$= \sqrt{2}\log|\sin x + \cos x + \sqrt{\sin 2x}| + C$

43. b. Write $I = \int\frac{dx}{x^3\left(a^2/x^2 - b^2\right)^{3/2}}$

and put $a^2/x^2 = t + b^2$, so that $(-2a^2/x^3)\,dx = dt$

$\therefore \quad I = \int\frac{\left(-1/2a^2\right)dt}{t^{3/2}}$

$= -\frac{1}{2a^2}\int t^{-3/2}\,dt = \frac{1}{a^2\sqrt{t}} + C$

$= \frac{1}{a^2\left(a^2/x^2 - b^2\right)^{1/2}} + C$

$= \frac{x}{a^2\left(a^2 - b^2x^2\right)^{1/2}} + C$

44. d. Putting $x^2 = t$,

$I = \frac{1}{2}\int e^{t^2}(1 + t + 2t^2)e^t dt$

$= \frac{1}{2}\int e^t[te^{t^2} + (e^{t^2} + 2t^2e^{t^2})]dt$

$= \frac{1}{2}\int e^t[f(t) + f'(t)]dt = \frac{1}{2}e^t(te^{t^2}) + C$

where $t = x^2$

45. b. $I = \int x\left(\frac{\ln a^{a^{x/2}}}{3a^{5x/2}b^{3x}} + \frac{\ln b^{b^x}}{2a^{2x}b^{4x}}\right)dx = \int\frac{\ln a^{2x}b^{3x}}{6a^{2x}b^{3x}}dx$

let $a^{2x}b^{3x} = t$, then $t\ln a^2b^3\,dx = dt$

$\Rightarrow \quad I = \int\frac{1}{6\ln a^2b^3}\frac{\ln t}{t^2}dt$

$= \frac{1}{6\ln a^2b^3}\left(\frac{-\ln t}{t} - \int\frac{-1}{t^2}dt\right)$

$= -\frac{1}{6\ln a^2b^3}\left(\frac{\ln et}{t}\right) + k$

$= -\frac{1}{6\ln a^2b^3}\left(\frac{\ln a^{2x}b^{3x}e}{a^{2x}b^{3x}}\right) + k$

46. a. $I = \int x\frac{\ln(x + \sqrt{x^2+1})}{\sqrt{x^2+1}}dx$, let $t = \sqrt{x^2+1}$

$\Rightarrow \quad \frac{dt}{dx} = \frac{x}{\sqrt{x^2+1}}$

$\Rightarrow \quad I = \int\ln(t + \sqrt{t^2-1})dt$

$= \ln(t + \sqrt{t^2-1})\,t - \int\frac{1 + \frac{t}{\sqrt{t^2-1}}}{t + \sqrt{t^2-1}}t\,dt$

$= t\ \ln(t + \sqrt{t^2-1}) - \frac{1}{2}\int\frac{2t}{\sqrt{t^2-1}}dt$

$= t\ln(t + \sqrt{t^2-1}) - \sqrt{t^2-1} + C$

$= \sqrt{1+x^2}\ \ln(x + \sqrt{1+x^2}) - x + C$

$\Rightarrow \quad a = 1, b = -1$

47. d. $\int\frac{\text{cosec}^2 x - 2005}{\cos^{2005} x}dx$

$= \int(\cos x)^{-2005}\text{cosec}^2 x dx - 2005\int\frac{dx}{\cos^{2005} x}$

$= (\cos x)^{-2005}(-\cot x)$

$- \int(-2005)(\cos x)^{-2006}(-\sin x)(-\cot x)dx - 2005\int\frac{dx}{\cos^{2005} x}$

$= -\frac{\cot x}{(\cos x)^{2005}} + C$

48. a. $f'(x) = \frac{f(x)}{6f(x) - x}$

Now $\quad I = \int\frac{2x(x - 6f(x)) + f(x)}{(6f(x) - x)(x^2 - f(x))^2}dx$

$\Rightarrow \quad I = -\int\frac{2x - f'(x)}{(x^2 - f(x))^2}dx = \frac{1}{x^2 - f(x)} + C$

49. a. Differentiating, we get

$\frac{f'(x)}{f(x)^2} = 2(b^2 - a^2)\sin x\cos x$

Integrating both sides w.r.t. x

$$\Rightarrow \quad -\frac{1}{f(x)} = -b^2\cos^2 x - a^2\sin^2 x$$

$$\Rightarrow \quad f(x) = \frac{1}{a^2\sin^2 x + b^2\cos^2 x}$$

50. a. $\int e^x\left(\frac{1}{\sqrt{1+x^2}} - \frac{x}{\sqrt{(1+x^2)^3}} + \frac{x}{\sqrt{(1+x^2)^3}} + \frac{1-2x^2}{\sqrt{(1+x^2)^5}}\right)$

$$= e^x\frac{1}{\sqrt{1+x^2}} + e^x\frac{x}{\sqrt{(1+x^2)^3}} = e^x\left(\frac{1}{\sqrt{1+x^2}} + \frac{x}{\sqrt{(1+x^2)^3}}\right) + C$$

Using $\int e^x(f(x)+f'(x))dx$, we get

$= e^x f(x) + c$

51. d. Let $I = \int \frac{dx}{(1+\sqrt{x})\sqrt{(x-x^2)}}$

If $\sqrt{x} = \sin p$, then $\frac{1}{2\sqrt{x}}\,dx = \cos p\,dp$

$$I = \int \frac{2\sin p\cos p\,dp}{(1+\sin p)\sin p\cos p}$$

$$= 2\int \frac{dp}{(1+\sin p)}$$

$$= 2\int \frac{(1-\sin p)\,dp}{\cos^2 p}$$

$$= 2\left\{\int \sec^2 p\,dp - \int(\tan p\sec p)\,dp\right\}$$

$$= 2(\tan p - \sec p) + C$$

$$= 2\left(\sqrt{\frac{x}{(1-x)}} - \frac{1}{\sqrt{(1-x)}}\right) + C = \frac{2(\sqrt{x}-1)}{\sqrt{(1-x)}} + C$$

52. c. Let $I = \int \frac{(ax^2-b)\,dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$

$$= \int \frac{\left(a-\frac{b}{x^2}\right)dx}{\sqrt{c^2-\left(ax+\frac{b}{x}\right)^2}}, \quad \left\{\begin{array}{l}\text{put } ax+\frac{b}{x}=t\\ \therefore \left(a-\frac{b}{x^2}\right)dx = dt\end{array}\right.$$

$$= \int \frac{dt}{\sqrt{c^2-t^2}} = \sin^{-1}\left(\frac{t}{c}\right) + k = \sin^{-1}\left(\frac{ax+\frac{b}{x}}{c}\right) + C$$

53. b. $I = \int \frac{dx}{\cos^3 x\sqrt{\sin 2x}}$

$$= \int \frac{dx}{\cos^3 x\sqrt{\frac{2\sin x\cos x}{\cos^2 x}\cos^2 x}}$$

$$= \int \frac{\sec^4 dx}{\sqrt{2\tan x}} = \frac{1}{\sqrt{2}}\int \frac{\sec^2 x(1+\tan^2 x)}{\sqrt{\tan x}}dx$$

let $t = \sqrt{\tan x}$

$$\Rightarrow \quad dt = \frac{\sec^2 x dx}{2\sqrt{\tan x}}$$

$$\Rightarrow \quad I = \frac{2}{\sqrt{2}}\int(1+t^4)dt$$

$$= \sqrt{2}\left(t + \frac{t^5}{5}\right) + C$$

$$= \frac{\sqrt{2}}{5}t(t^4+5) + C = \frac{\sqrt{2}}{5}\sqrt{\tan x}\,(\tan^2 x + 5) + C$$

$$\Rightarrow \quad a = \frac{\sqrt{2}}{5}, b = 5$$

54. d. $\int x\log\left(1+\frac{1}{x}\right)dx$

$$= \int x\log(x+1)dx - \int x\log x dx$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\int\frac{x^2}{x+1}dx - \frac{x^2}{2}\log x + \frac{1}{2}\int\frac{x^2}{x}dx$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\int\left(x-1+\frac{1}{x+1}\right)dx - \frac{x^2}{2}\log x + \frac{1}{4}x^2$$

$$= \frac{x^2}{2}\log(x+1) - \frac{x^2}{2}\log x - \frac{1}{2}\left(\frac{x^2}{2} - x\right) - \frac{1}{2}\log(x+1) + \frac{1}{4}x^2 + C$$

$$= \frac{x^2}{2}\log(x+1) - \frac{x^2}{2}\log x - \frac{1}{2}\log(x+1) + \frac{1}{2}x + C$$

Hence, $f(x) = \frac{x^2}{2} - \frac{1}{2}$, $g(x) = -\frac{1}{2}\log x$ and $A = \frac{1}{2}$

55. d. $I = \int \frac{x\,dx}{x^4\sqrt{x^2-1}}$

Let $x^2 - 1 = t^2 \Rightarrow 2x\,dx = 2t\,dt$

$$\Rightarrow \quad I = \int \frac{t}{(t^2+1)^2 t}dt = \int \frac{dt}{(t^2+1)^2}$$

But $\tan^{-1} t = \int \frac{dt}{t^2+1} = \int 1\cdot\frac{1}{t^2+1}dt$

$$= \frac{t}{t^2+1} + \int t\frac{2t}{(t^2+1)^2}dt$$

$$= \frac{t}{t^2+1} + 2\int \frac{t^2+1-1}{(t^2+1)^2} dt$$

$$= \frac{t}{t^2+1} + 2\tan^{-1} t - 2I$$

$$\therefore \quad I = \frac{1}{2}\frac{t}{t^2+1} + \frac{1}{2}\tan^{-1} t + C$$

$$= \frac{1}{2}\left(\frac{\sqrt{x^2-1}}{x^2} + \tan^{-1}\sqrt{x^2-1}\right) + C$$

56. c. $I_{4,3} = \int \cos^4 x \sin 3x dx$

Integrating by parts, we have

$$I_{4,3} = -\frac{\cos 3x \cos^4 x}{3} - \frac{4}{3}\int \cos^3 x \sin x \cos 3x dx$$

But $\sin x \cos 3x = -\sin 2x + \sin 3x \cos x$, so

$$I_{4,3} = -\frac{\cos x \cos^4 x}{3} + \frac{4}{3}\int \cos^3 x \sin 2x dx$$

$$- \frac{4}{3}\int \cos^4 x \sin 3x dx + C$$

$$= -\frac{\cos 3x \cos^4 x}{3} + \frac{4}{3} I_{3,2} - \frac{4}{3} I_{4,3} + C$$

Therefore, $\frac{7}{3} I_{4,3} - \frac{4}{3} I_{3,2} = -\frac{\cos 3x \cos^3 x}{3} + C$

or $7I_{4,3} - 4I_{3,2} = -\cos 3x \cos^4 x + C.$

57. b. We have $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}}$

$$= \int \frac{dx}{x^2 x^{n-1}\left(1+\frac{1}{x^n}\right)^{(n-1)/n}}$$

$$= \int \frac{dx}{x^{n+1}(1+x^{-n})^{(n-1)/n}}$$

Put $1 + x^{-n} = t \;\therefore\; -nx^{-n-1} dx = dt \Rightarrow \frac{dx}{x^{n+1}} = -\frac{dt}{n}$

$$\Rightarrow \quad \int \frac{dx}{x^2(x^n+1)^{(n-1)/n}} = -\frac{1}{n}\int \frac{dt}{t^{(n-1)/n}}$$

$$= -\frac{1}{n}\int t^{-1+\frac{1}{n}} dt = \frac{-1}{n}\cdot\frac{t^{1/n}}{1/n} + C$$

$$= -t^{1/n} + C$$

58. c. Putting $a^6 + x^8 = t^2$, we get

$$\Rightarrow \quad I = \int \frac{t^2}{t^2-a^6} dt = t + \frac{a^3}{2}\ln\left|\frac{t-a^3}{t+a^3}\right| + C$$

59. c. $I = -e^{-x}\log\left(e^x+1\right) + \int \frac{e^{-x} e^x}{e^x+1} dx$

$$= -e^{-x}\log(e^x+1) + \int \frac{e^{-x}}{e^{-x}+1} dx$$

$$= -e^{-x}\log(e^x+1) - \log(e^{-x}+1) + C$$

$$= -e^{-x}\log(e^x+1) - \log(1+e^x) + x + C$$

$$= -(e^{-x}+1)\log(e^x+1) + x + C$$

60. b. $I = \int xe^x \cos x dx$

$$= xe^x \sin x - \int (xe^x + e^x)\sin x dx$$

$$= xe^x \sin x - xe^x(-\cos x) - \int (xe^x + e^x)\cos x \, dx$$

$$- \int e^x \sin x dx$$

$$= xe^x \sin x + x e^x \cos x - \int xe^x \cos x dx$$

$$- \int e^x(\cos x + \sin x) dx$$

$\Rightarrow \quad 2I = xe^x(\sin x + \cos x) - e^x \sin x + d$

$\Rightarrow \quad 2I = e^x((x-1)\sin x + x\cos x) + d$

$\Rightarrow \quad I = \frac{1}{2}e^x((x-1)\sin x + x\cos x) + d$

$\Rightarrow \quad a = \frac{1}{2}, b = -1, c = 1$

61. b. Put $2 + x = t^2$, so that $dx = 2t\,dt$ and

$$I = \int \frac{\sqrt{7-t^2}}{t}(2t)\,dt \;\; 2\int \sqrt{7-t^2}\;\, dt$$

$$= t\sqrt{7-t^2} + 7\sin^{-1}\left(\frac{t}{\sqrt{7}}\right) + C$$

$$= \sqrt{x+2}\,\sqrt{5-x} + 7\sin^{-1}\left(\frac{\sqrt{x+2}}{\sqrt{7}}\right) + C$$

62. c. $I = \int e^{\tan x}(\sin x - \sec x)dx$

$$= \int \sin x\, e^{\tan x} dx - \int \sec x\, e^{\tan x} dx$$

$$= -e^{\tan x}\cos x + \int \cos x\, e^{\tan x}\sec^2 x dx - \int \sec x\, e^{\tan x} dx$$

$$= -\cos x\; e^{\tan x} + C$$

63. d. $I = \int \frac{x^3 dx}{\sqrt{1+x^2}} = \int \frac{x \times x^2 dx}{\sqrt{1+x^2}}$, let $t = \sqrt{1+x^2}$

$$\Rightarrow \quad \frac{dt}{dx} = \frac{x}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad I = \int (t^2-1)dt$$

$$= \frac{t^3}{3} - t + C = \frac{t}{3}(t^2-3) + C$$

$$= \frac{1}{3}\sqrt{1+x^2}\,(x^2-2)+C$$

64. d. $I = \int \frac{\sin x \cos x}{\sin x+\cos x}\,dx$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x)^2 - 1}{\sin x + \cos x}\,dx$$

$$= \frac{1}{2}\int \left[\sin x + \cos x - \frac{1}{\sqrt{2}\sin(x+\pi/4)}\right]dx$$

$$= \frac{1}{2}[\sin x - \cos x] - \frac{1}{2\sqrt{2}}\log\left|\operatorname{cosec}(x+\pi/4) - \cot(x+\pi/4)\right| + C$$

65. b. $I = \int \frac{\sin 2x}{(3+4\cos x)^3}\,dx$

and put $3 + 4\cos x = t$, so that $-4\sin x\,dx = dt$

$$I = \frac{-1}{8}\int \frac{(t-3)}{t^3}\,dt = \frac{1}{8}\left(\frac{1}{t} - \frac{3}{2}\frac{1}{t^2}\right) + C$$

$$= \frac{2t-3}{16t^2} = \frac{8\cos x + 3}{16(3+4\cos x)^2} + C$$

66. c. $I = \int \frac{\ln\left(\frac{x-1}{x+1}\right)}{x^2-1}\,dx$, let $t = \ln\left(\frac{x-1}{x+1}\right)$

$$\Rightarrow \frac{dt}{dx} = \frac{x+1}{x-1}\left\{\frac{x+1-(x-1)}{(x+1)^2}\right\} = \frac{2}{(x^2-1)}$$

$$\Rightarrow \frac{dx}{x^2-1} = \frac{dt}{2}$$

$$\Rightarrow I = \frac{1}{2}\int t\,dt = \frac{1}{4}t^2 + C = \frac{1}{4}\left(\ln\left(\frac{x-1}{x+1}\right)\right)^2 + C$$

67. a. $I = \int \sqrt{e^x - 1}\,dx$

Let $e^x - 1 = t^2 \Rightarrow e^x\,dx = 2t\,dt \Rightarrow dx = \frac{2t}{t^2+1}\,dt$

$$\Rightarrow I = \int t\,\frac{2t}{t^2+1}\,dt = \int \frac{2t^2}{t^2+1}\,dt$$

$$= \int \frac{2(t^2+1)-2}{t^2+1}\,dt = \int 2dt - \int \frac{2dt}{t^2+1}$$

$$= 2t - 2\tan^{-1} t + C$$

$$= 2\sqrt{e^x-1} - 2\tan^{-1}\sqrt{e^x-1} + C$$

68. b. $\int x \sin x \sec^3 x\,dx$

$$= \int x\sin x\,\frac{1}{\cos^3 x}\,dx$$

$$= \int x\tan x\sec^2 x\,dx$$

$$= x\int \sec x(\sec x\tan x)dx - \int[\sec x(\sec x\tan x)dx]dx + C$$

$$= x\frac{\sec^2 x}{2} - \int \frac{\sec^2 x}{2}\,dx + C$$

$$= x\frac{\sec^2 x}{2} - \frac{\tan x}{2} + C$$

69. a. $\int \frac{e^x(x^2+1)}{(x+1)^2}\,dx$

$$= \int \frac{e^x(x^2-1+2)}{(x+1)^2}\,dx$$

$$= \int e^x\left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right]dx$$

$= \int e^x[f(x) + f'(x)]\,dx$, where $f(x) = \frac{x-1}{x+1}$ and

$$f'(x) = \frac{2}{(x+1)^2}$$

$$= e^x\left(\frac{x-1}{x+1}\right) + C$$

70. a. $I = \int \left(\frac{x+2}{x+4}\right)^2 e^x\,dx = \int e^x\left[\frac{x^2+4x+4}{(x+4)^2}\right]dx$

$$\Rightarrow I = \int e^x\left[\frac{x(x+4)}{(x+4)^2} + \frac{4}{(x+4)^2}\right]dx$$

$$= \int e^x\left[\frac{x}{x+4} + \frac{4}{(x+4)^2}\right]dx$$

$$= e^x\left(\frac{x}{x+4}\right) + C$$

71. a. Let $I = \int \frac{3+2\cos x}{(2+3\cos x)^2}\,dx$. Multiplying N^r and D^r by $\operatorname{cosec}^2 x$, we get

$$\Rightarrow I = \int \frac{(3\operatorname{cosec}^2 x + 2\cot x\operatorname{cosec} x)}{(2\operatorname{cosec} x + 3\cot x)^2}\,dx$$

$$= -\int \frac{-3\operatorname{cosec}^2 x - 2\cot x\operatorname{cosec} x}{(2\operatorname{cosec} x + 3\cot x)^2}\,dx$$

$$= \frac{1}{2\operatorname{cosec} x + 3\cot x} + C = \left(\frac{\sin x}{2+3\cos x}\right) + C$$

Multiple Correct Answers Type

1. b, c, d.

$$I=\int \frac{x^2+\cos^2 x}{x^2+1}\operatorname{cosec}^2 x dx$$

$$=\int \frac{x^2+1+\cos^2 x-1}{x^2+1}\operatorname{cosec}^2 x dx$$

$$=\int\left(1-\frac{\sin^2 x}{x^2+1}\right)\operatorname{cosec}^2 x dx$$

$$=\int\left(\operatorname{cosec}^2 x-\frac{1}{x^2+1}\right)dx$$

$$=-\cot x-\tan^{-1}x+C$$

$$=-\cot x+\cot^{-1}x-\frac{\pi}{2}+C$$

$$=-\cot x+\cot^{-1}x+C$$

2. b, d.

$$\int \sin x d(\sec x)$$

$$=\int \sin x\frac{d(\sec x)}{dx}dx=\int \sin x\sec x\tan x dx$$

$$=\int \tan^2 x dx=\int\left(\sec^2 x-1\right)dx=\tan x-x+C$$

$$\Rightarrow f(x)=\tan x,\ g(x)=x$$

3. a, d.

$$I=\int\frac{\sqrt{(1+\sin x)(1-\sin x)}}{\sqrt{\sin x(1-\sin x)}}dx$$

$$=\int\frac{\cos x}{\sqrt{\sin x(1-\sin x)}}dx$$

$$=\int\frac{\cos x}{\sqrt{\frac{1}{4}-\left(\frac{1}{2}-\sin x\right)^2}}dx$$

$$=\int\frac{-dt}{\sqrt{\left(\frac{1}{2}\right)^2-t^2}}\qquad\left(\text{Putting }\frac{1}{2}-\sin x=t\right)$$

$$=-\sin^{-1}\left(\frac{t}{1/2}\right)+C=-\sin^{-1}(1-2\sin x)+C$$

$$=\cos^{-1}(1-2\sin x)+C-\frac{\pi}{2}$$

$$=\cos^{-1}(1-2\sin x)+C$$

$$=\cos^{-1}\left(1-2\left(\sqrt{\sin x}\right)^2\right)+C$$

$$=\cos^{-1}\left(1-2\sin^2 t\right)+C$$

$$\left(\text{Putting }\sqrt{\sin x}=\sin t\right)$$

$$=\cos^{-1}(\cos 2t)+C$$

$$=2t+C$$

$$\left(\because \sqrt{\sin x}>0\Rightarrow \sin t>0\Rightarrow t\in\left(0\ \frac{\pi}{2}\right)\right)$$

$$=2\sin^{-1}\left(\sqrt{\sin x}\right)+C$$

4. a, c. $I=\int \sec^2 x\operatorname{cosec}^4 x dx$

$$=\int\frac{\left(\sin^2 x+\cos^2 x\right)^2}{\cos^2 x\sin^4 x}dx$$

$$=\int\frac{\sin^4 x+\cos^4 x+2\sin^2 x\cos^2 x}{\cos^2 x\sin^4 x}dx$$

$$=\int\left(\sec^2 x+2\operatorname{cosec}^2 x+\frac{\cos^2 x}{\sin^4 x}\right)dx$$

$$=\tan x-2\cot x+\int\cot^2 x\operatorname{cosec}^2 x dx$$

$$=\tan x-2\cot x-\frac{\cot^3 x}{3}+D$$

5. a, c. $g(x)=\int x^{27}(1+x+x^2)^6(6x^2+5x+4)\,dx$

$$=\int(x^4+x^5+x^6)^6(6x^5+5x^4+4x^3)\,dx$$

let $x^6+x^5+x^4=t\Rightarrow(6x^5+5x^4+4x^3)dx=dt$

$$\therefore\quad g(x)=\int t^6dt=\frac{t^7}{7}+C=\frac{1}{7}(x^4+x^5+x^6)^7+C$$

$$g(0)=0\quad\Rightarrow x=0\quad\Rightarrow g(1)=\frac{3^7}{7}\ \text{also } g(-1)=\frac{1}{7}$$

6. b, d. $I=\int\sqrt{\operatorname{cosec} x+1}\,dx=\int\frac{\cot x}{\sqrt{\operatorname{cosec} x-1}}dx$

put $\operatorname{cosec} x-1=t^2\Rightarrow-\operatorname{cosec} x\cot x\,dx=2tdt$

$$\Rightarrow\qquad I=-\int\frac{-\cot x\operatorname{cosec} x}{\operatorname{cosec} x\sqrt{\operatorname{cosec} x-1}}dx=-\int\frac{2dt}{1+t^2}$$

$$=-2\tan^{-1}t+c=-2\tan^{-1}\sqrt{\operatorname{cosec} x-1}+C$$

$$=-2\left[\frac{\pi}{2}-\cot^{-1}\sqrt{\operatorname{cosec} x-1}\right]+C$$

$$=2\cot^{-1}\sqrt{\operatorname{cosec} x-1}+C$$

$$=2\cot^{-1}\frac{\cot x}{\sqrt{\operatorname{cosec} x+1}}+C$$

7. a, c. Let $\cos x=t$, $\Rightarrow\cos x=t\Rightarrow\cos 2x=2t^2-1$ and $dt=-\sin x\,dx$. Thus

$$I=\int\frac{t^2-2}{2t^2-1}dt=\frac{1}{2}\int\frac{2t^2-4}{2t^2-1}dt$$

$$= \frac{1}{2}\int dt - \frac{3}{2}\int \frac{dt}{2t^2 - 1}$$

$$= \frac{1}{2}t - \frac{3}{2\sqrt{2}} \times \frac{1}{2}\log\left|\frac{\sqrt{2}\,t - 1}{\sqrt{2}\,t + 1}\right| + C$$

$$= \frac{1}{2}\cos x - \frac{3}{4\sqrt{2}}\log\left|\frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}\right| + C$$

So, $P = 1/2, Q = -\frac{3}{4\sqrt{2}}, f(x) = \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}$

or $P = 1/2, Q = \frac{3}{4\sqrt{2}}, f(x) = \frac{\sqrt{2}\cos x + 1}{\sqrt{2}\cos x - 1}$

8. a, d.

$$\frac{2x}{(x-1)(x-4)} = \frac{C}{x-1} + \frac{D}{x-4}$$

$$2x = C(x-4) + D(x-1)$$

$\therefore \quad C = -2/3, D = 8/3$

$$\therefore \quad \int \frac{e^{x-1}}{(x-1)(x-4)} 2x\, dx = \int e^{x-1}\left(\frac{-2/3}{x-1} + \frac{8/3}{x-4}\right)dx$$

$$= -\frac{2}{3}F(x-1) + \frac{8}{3}e^3 F(x-4) + C$$

$\therefore \quad A = -2/3, B = 8/3\, e^3$

9. a, c, d.

$$\int x^2 e^{-2x} dx = e^{-2x}(ax^2 + bx + c) + d$$

Differentiating both sides, we get

$$x^2 e^{-2x} = e^{-2x}(2ax + b) + (ax^2 + bx + c)(-2e^{-2x})$$

$$= e^{-2x}(-2ax^2 + 2(a-b)x + b - 2c)$$

$\Rightarrow \quad a = 1, 2(a-b) = 0, b - 2c = 0$

$\Rightarrow \quad b = 1, c = \frac{1}{2}$

10. a, c, d.

Let $I = \int \frac{(x^4+1)}{(x^6+1)}dt$

$$= \int \frac{(x^2+1)^2 - 2x^2}{(x^2+1)(x^4 - x^2 + 1)}dx$$

$$= \int \frac{(x^2+1)dx}{(x^4 - x^2 + 1)} - 2\int \frac{x^2 dx}{(x^6+1)}$$

$$= \int \frac{\left(1 + \frac{1}{x^2}\right)dx}{\left(x^2 - 1 + \frac{1}{x^2}\right)} - 2\int \frac{x^2 dx}{(x^3)^2 + 1}$$

In the first integral, put $x - \frac{1}{x} = t$

$$\therefore \left(1 + \frac{1}{x^2}\right)dx = dt$$

and in the second integral put $x^3 = u$

$$\therefore \quad x^2 dx = \frac{du}{3}$$

then $I = \int \frac{dt}{1+t^2} - \frac{2}{3}\int \frac{du}{1+u^2}$

$$= \tan^{-1} t - \frac{2}{3}\tan^{-1} u + C$$

$$= \tan^{-1}\left(x - \frac{1}{x}\right) - \frac{2}{3}\tan^{-1}(x^3) + C$$

Here, $f(x) = x - \frac{1}{x}$ and $g(x) = x^3$

Both the functions are one-one.

Also $f'(x) = 1 + \frac{1}{x^2} \neq 0$. Hence, $f(x)$ is monotonic.

Also $\int \frac{f(x)}{g(x)}dx = \int \frac{x - \frac{1}{x}}{x^3}dx = \int\left(\frac{1}{x^2} - \frac{1}{x^4}\right)dx$

$$= -\frac{1}{x} + \frac{3}{x^3} + C$$

11. a, b, c.

$$I = \int \frac{x^2 - x + 1}{(x^2+1)^{3/2}} e^x dx$$

$$= \int e^x\left[\frac{x^2+1}{(x^2+1)^{3/2}} - \frac{x}{(x^2+1)^{3/2}}\right]dx$$

$$= \int e^x\left[\frac{1}{\sqrt{x^2+1}} + \left\{\frac{-x}{(x^2+1)^{3/2}}\right\}\right]dx$$

$= \int e^x[f(x) + f'(x)]dx$, where $f(x) = \frac{1}{\sqrt{x^2+1}}$

$$= e^x f(x) + c = \frac{e^x}{\sqrt{x^2+1}} + c$$

The graph of $f(x)$ is given in Fig. 7.1.

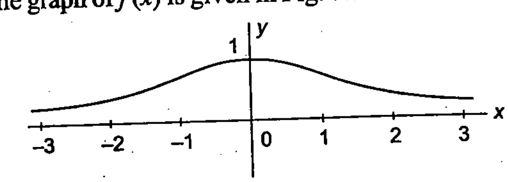


Fig. 7.1

From the graph, $f(x)$ is even, bounded function and has the range (0, 1].

12. a, c.

$$\int \frac{\cos^2 2x \sin 2x\, dx}{\cos 2x} = \frac{1}{2}\int \sin 4x\, dx = -\frac{1}{8}\cos 4x + B$$

13. a, d.

$$\int \sin^{-1} x \cos^{-1} x\,dx = \int\left[\frac{\pi}{2}\sin^{-1} x - \left(\sin^{-1} x\right)^2\right]dx$$

$$= \frac{\pi}{2}\left(x\sin^{-1}x+\sqrt{1-x^2}\right)-\left(x\left(\sin^{-1}x\right)^2+\sin^{-1}x\sqrt{1-x^2}-x\right)+C$$

(integrating by parts)

$$= \sin^{-1} x\left[\frac{\pi}{2}x - x\sin^{-1}x - 2\sqrt{1-x^2}\right]+\frac{\pi}{2}\sqrt{1-x^2}+2x+C$$

$\therefore f^{-1}(x) = \sin^{-1} x, f(x) = \sin x$

14. a, b, c, d.

$$\int\frac{\left(x^8+4+4x^4\right)-4x^4}{x^4-2x^2+2}dx$$

$$= \int\frac{\left(x^4+2\right)^2-\left(2x^2\right)^2}{\left(x^4-2x^2+2\right)}dx$$

$$= \int\frac{\left(x^4+2-2x^2\right)\left(x^4+2+2x^2\right)}{\left(x^4-2x^2+2\right)}dx$$

$$= \frac{x^5}{5}+\frac{2x^3}{3}+2x+C$$

15. a, b, d.

$$\int\frac{dx}{x^2+ax+1} = \int\frac{dx}{\left(x+\frac{a}{2}\right)^2+\left(1-\frac{a^2}{4}\right)}$$

Reasoning Type

1. a. $\int e^x \sin x dx$

$$= \frac{1}{2}\int e^x(\sin x+\cos x+\sin x-\cos x)dx$$

$$= \frac{1}{2}\left(\int e^x(\sin x+\cos x)dx-\int e^x(\cos x-\sin x)dx\right)$$

$$= \frac{1}{2}\left(e^x\sin x-e^x\cos x\right)+c$$

$$= \frac{1}{2}e^x\left(\sin x-\cos x\right)+c$$

2. d. For $x^2+2(a-1)x+a+5=0$

if $D<0 \Rightarrow 4(a-1)^2-4(a+5)<0$

$\Rightarrow a^2-3a-4<0$ or $(a-4)(a+1)<0$ or $-1<a<4$

Thus for these value of a, $x^2+2(a-1)x+a+5$ cannot be factorized, hence

$$\int\frac{dx}{x^2+2(a-1)x+a+5} = \lambda\tan^{-1}|g(x)|+c$$

Hence, statement 1 is false and statement 2 is true.

3. b. $\int\frac{\sin x dx}{x}$ cannot be evaluated as there does not exist any method for evaluating this (integration by parts also does not work); however, $\frac{\sin x}{x}$ $(x > 0)$ is a differentiable function. Hence, both the statements are true but statement 2 is not a correct explanation of statement 1.

4. b. $I = \int\frac{dx}{x^3\sqrt{1+x^4}} = \int\frac{dx}{x^5\sqrt{\frac{1}{x^4}+1}}$

Let $\frac{1}{x^4}+1 = t \Rightarrow dt = \frac{-4}{x^5}dx$

$$\Rightarrow I = -\frac{1}{4}\int\frac{dt}{\sqrt{t}} = -\frac{1}{2}\sqrt{t} = -\frac{1}{2}\sqrt{1+\frac{1}{x^4}}+C$$

Thus, both the statements are true but statement 2 is not a correct explanation of statement 1.

5. b. $f(x) = \pi\sin\pi x+2x-4$

$\Rightarrow g(x) = \int(\pi\sin\pi x+2x-4)dx = -\cos\pi x+x^2-4x+c$

Also $f(1) = 3 \Rightarrow 1+1-4+c = 3 \Rightarrow c = 0$

$\Rightarrow g(x) = -\cos\pi x+x^2-4x$

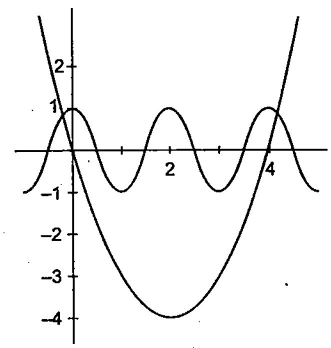


Fig. 7.2

Hence, both the statements are true but statement 2 is not a correct explanation of statement 1.

6. a. $I = \int\frac{\{f(x)\,\phi'(x)-f'(x)\,\phi(x)\}}{f(x)\,\phi(x)}\{\log\phi(x)-\log f(x)\}\,dx$

$$= \int\log\frac{\phi(x)}{f(x)}d\left\{\log\frac{\phi(x)}{f(x)}\right\} = \frac{1}{2}\left\{\log\frac{\phi(x)}{f(x)}\right\}^2+c$$

Linked Comprehension Type

For Problems 1–3

1. d., 2. b., 3. a.

Sol. From the given data, we can conclude that $\frac{dy}{dx}=0$, at $x = 1, 2, 3$.

Hence, $f'(x) = a(x-1)(x-2)(x-3), a>0$

$\Rightarrow\ f(x) = \int a(x^3 - 6x^2 + 11x - 6)dx$

$= a\int (x^3 - 6x^2 + 11x - 6)dx$

$= a\left(\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x\right) + C$

Also $f(0) = 1 \Rightarrow c = 1$

$\Rightarrow\ f(x) = a\left(\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x\right) + 1$ (1)

$f(1) = a\left(-\frac{9}{4}\right) + 1, f(2) = -2a + 1,$

$f(3) = a\left(-\frac{9}{4}\right) + 1$ (2)

$\Rightarrow$ The graph is symmetrical about line $x = 2$ and the range is $[f(1), \infty)$ or $[f(3), \infty)$.

1. d. $f(1) = -8 \Rightarrow a = 4$ (from (2))

$\Rightarrow\ f(2) = -7$

2. b. $f(3) = -8$. Hence the range is $[-8, \infty)$

3. a. If $f(2) = 0$, then $a = 1/2$

If $f(1) = 0$, then $a = 4/9$

$\Rightarrow$ For four roots of $f(x) = 0, a \in [4/9, 1/2]$

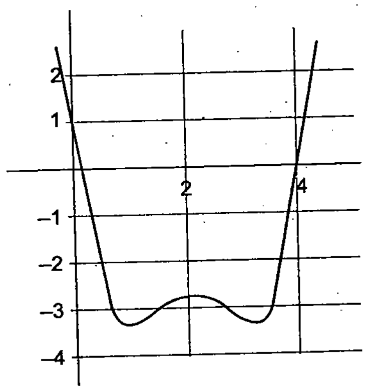


Fig. 7.3

For Problems 4–6

4. a., 5. b., 6. c.

Sol. $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 2x^2 & 2x^2 \\ 2x^2 & 2x^2 \end{bmatrix}, A^3 = \begin{bmatrix} 2^2x^3 & 2^2x^3 \\ 2^2x^3 & 2^2x^3 \end{bmatrix}$

and so on

Then $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots +$

$$= \begin{bmatrix} 1 + x + \frac{2x^2}{2!} + \frac{2^2x^3}{3!} + \cdots & x + \frac{2x^2}{2!} + \frac{2^2x^3}{3!} + \cdots \\ x + \frac{2x^2}{2!} + \frac{2^2x^3}{3!} + \cdots & 1 + x + \frac{2x^2}{2!} + \frac{2^2x^3}{3!} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\left(1 + 2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \cdots\right) + \frac{1}{2} & \frac{1}{2}\left(1 + 2x + \frac{2^2x^2}{2!} + \cdots\right) - \frac{1}{2} \\ \frac{1}{2}\left(1 + 2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \cdots\right) - \frac{1}{2} & \frac{1}{2}\left(1 + 2x + \frac{2^2x^2}{2!} + \cdots\right) + \frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^{2x} + 1 & e^{2x} - 1 \\ e^{2x} - 1 & e^{2x} + 1 \end{bmatrix}$$

$\Rightarrow\ f(x) = e^{2x} + 1$ and $g(x) = e^{2x} - 1$

4. a. $\int \frac{e^{2x} - 1}{e^{2x} + 1}dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \log|e^x - e^{-x}| + C$

5. b. $\int (g(x) + 1)\sin x dx$

$= \int e^{2x}\sin x dx$

$= \frac{e^{2x}}{5}(2\sin x - \cos x).$

6. c. $\int \frac{e^{2x} + 1}{\sqrt{e^{2x} - 1}}dx$

$= \int \frac{e^{2x}}{\sqrt{e^{2x} - 1}}dx + \int \frac{1}{\sqrt{e^{2x} - 1}}dx$

$= \int \frac{e^{2x}}{\sqrt{e^{2x} - 1}}dx + \int \frac{e^x}{e^x\sqrt{e^{2x} - 1}}dx$

$= \frac{1}{2\sqrt{e^{2x} - 1}} + \sec^{-1}(e^x) + C.$

For Problems 7–9

7. d., 8. b., 9. a

Sol.

7. d. Here $a = 1 > 0$; therefore we make the substitution $\sqrt{x^2 + 2x + 2} = t - x$. Squaring both sides of this equality and reducing the similar terms, we get

$2x + 2tx = t^2 - 2 \Rightarrow x = \frac{t^2 - 2}{2(1 + t)} \Rightarrow dx = \frac{t^2 + 2t + 2}{2(1 + t)^2}dt;$

$1 + \sqrt{x^2 + 2x + 2} = 1 + t - \frac{t^2 - 2}{2(1 + t)} = \frac{t^2 + 4t + 4}{2(1 + t)}.$

Substituting into the integral, we get

$I = \int \frac{2(1 + t)(t^2 + 2t + 2)}{(t^2 + 4t + 4)\,2(1 + t)^2}dt = \int \frac{(t^2 + 2t + 2)\,dt}{(1 + t)(t + 2)^2}$

Now let us expand the obtained proper rational fraction into partial fractions:

$\frac{t^2 + 2t + 2}{(t + 1)(t + 2)^3} = \frac{A}{t + 1} + \frac{B}{t + 2} + \frac{D}{(t + 2)^2}.$

8. b. $I=\int\frac{dx}{x+\sqrt{x^2-x+1}}$

Since here $c = 1 > 0$, we can apply the second Euler substitution:

$$\sqrt{x^2-x+1}=tx-1$$

$$\Rightarrow (2t-1)x=(t^2-1)x^2;\ x=\frac{2t-1}{t^2-1}$$

Substituting into I, we get an integral of a rational fraction:

$$\int\frac{dx}{x+\sqrt{x^2-x+1}}=\int\frac{-2t^2+2t-2}{t(t-1)(t+1)^2}\,dt,$$

Now $\frac{-2t^2+2t-2}{t(t-1)(t+1)}=\frac{A}{t}+\frac{B}{t-1}+\frac{D}{(t+1)^2}+\frac{E}{t+1}$

9. a. In this case, $a<0$ and $c<0$; therefore neither the first nor the second Euler substitution is applicable. But the quadratic $7x-10-x^2$ has real roots $\alpha=2, \beta=5$; therefore we use the third Euler substitution:

$$\sqrt{7x-10-x^2}=\sqrt{(x-2)(5-x)}=(x-2)t$$

$$\Rightarrow 5-x=(x-2)t^2$$

$$\Rightarrow \quad x=\frac{5+2t^2}{t^2+1}$$

Matrix-Match Type

1. a.→p, q, b.→r, s, c.→p., d.→p, q.

a. Let $I=\int\frac{2^x}{\sqrt{1-4^x}}dx=\frac{1}{\log 2}\int\frac{1}{\sqrt{1-t^2}}dt$

Putting $2^x=t$, $2^x\log 2\,dx=dt$

$$I=\frac{1}{\log 2}\sin^{-1}\left(\frac{t}{1}\right)+C=\frac{1}{\log 2}\sin^{-1}(2^x)+C$$

$$\therefore \quad K=\frac{1}{\log 2}$$

b. $\int\frac{dx}{(\sqrt{x})^2+(\sqrt{x})^7}=\int\frac{dx}{(\sqrt{x})^7\left(1+\frac{1}{(\sqrt{x})^5}\right)}$

Put $\frac{1}{(\sqrt{x})^5}=y$, $\frac{dy}{dx}=-\frac{5}{2(\sqrt{x})^7}$

$$\therefore I=\int\frac{-2dy}{5(1+y)}=-\frac{2}{5}\ln|1+y|+C=\frac{2}{5}\ln\left(\frac{1}{1+\frac{1}{(\sqrt{x})^5}}\right)$$

$$\Rightarrow a=\frac{2}{5}, k=\frac{5}{2}$$

c. Add and subtract $2x^2$ in the numerator, then $k=1$ and $m=1$.

d. $I=\int\frac{dx}{5+4\cos x}$

$$=\int\frac{dx}{5\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)+4\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$=\int\frac{dx}{9\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\int\frac{\sec^2\frac{x}{2}}{9+\tan^2\frac{x}{2}}dx$$

Let $t=\tan\frac{x}{2}\Rightarrow 2\,dt=\sec^2\frac{x}{2}dx$

$$\Rightarrow \quad I=\int\frac{2dt}{9+t^2}=\frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right)+C$$

$$=\frac{2}{3}\tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right)}{3}\right)+C$$

$$\Rightarrow \quad k=\frac{2}{3}, m=\frac{1}{3}$$

2. a.→r, b.→s, c→q, d→p.

a. $\int\frac{e^{2x}-1}{e^{2x}+1}dx$

$$=\int\frac{e^x-e^{-x}}{e^x+e^{-x}}dx$$

$$=\int\frac{\left(e^x+e^{-x}\right)'}{e^x+e^{-x}}dx$$

$$=\log(e^x+e^{-x})$$

$$=\log(e^{2x}+1)-x+C$$

b. $I=\int\frac{1}{\left(e^x+e^{-x}\right)^2}dx=\int\frac{e^{2x}}{\left(e^{2x}+1\right)^2}dx$

Put $e^{2x}+1=t\Rightarrow 2e^{2x}dx=dt$, we get

$$\Rightarrow I=\frac{1}{2}\int\frac{1}{t^2}dt=-\frac{1}{2}\frac{1}{t}+C=-\frac{1}{2\left(e^{2x}+1\right)}+C.$$

c. $I=\int\frac{e^{-x}}{1+e^x}dx=\int\frac{e^{-x}e^{-x}}{e^{-x}+1}dx$

Put $e^{-x}+1=t\Rightarrow -e^{-x}dx=dt$

$$\Rightarrow \quad I=-\int\frac{(t-1)}{t}dt=\int\left(\frac{1}{t}-1\right)dt$$

$$=\log t-t+C$$

$= \log(e^{-x}+1)-(e^{-x}+1)+C$

$= \log(e^{x}+1)-x-e^{-x}-1+C$

$= \log(e^{x}+1)-x-e^{-x}+C$

d. $I=\int\frac{1}{\sqrt{1-e^{2x}}}dx=\int\frac{e^{-x}}{\sqrt{e^{-2x}-1}}dx$

Put $e^{-x}=t \Rightarrow -e^{-x}\,dx=dt$,

$\Rightarrow \quad I=-\int\frac{1}{\sqrt{t^2-1}}dt$

$=-\log\left[t+\sqrt{t^2-1}\right]+C$

$=-\log\left[e^{-x}+\sqrt{e^{-2x}-1}\right]+C$

$=-\log\left[\frac{1}{e^x}+\frac{\sqrt{1-e^{2x}}}{e^x}\right]+C$

$=-\log\left[1+\sqrt{1-e^{2x}}\right]+\log e^x+C$

$=x-\log\left[1+\sqrt{1-e^{2x}}\right]+C$

3. a. → p, q, r, b. → p, q, r, c. → p, q, r, s.
d. → p, q, r, s

a. $\int\frac{x^2-x+1}{x^3-4x^2+4x}dx=\int\left[\frac{A}{x}+\frac{B}{x-2}+\frac{C}{(x-2)^2}\right]dx.$

b. $\int\frac{x^2-1}{x(x-2)^3}dx=\int\left[\frac{A}{x}+\frac{B}{x-2}+\frac{C}{(x-2)^2}+\frac{D}{(x-2)^3}\right]dx.$

c. $\int\frac{x^3+1}{x(x-2)^2}dx=\int\left[\left(\frac{x^3+1}{x(x-2)^2}-1\right)+1\right]dx$

$=\int\left[\left(\frac{x^3+1-x(x-2)^2}{x(x-2)^2}\right)+1\right]dx$

$=\int\left[\left(\frac{A}{x}+\frac{B}{x-2}+\frac{C}{(x-2)^2}\right)+1\right]dx$

d. $\int\frac{x^5+1}{x(x-2)^3}dx=\int\left[x+k+\frac{g(x)}{x(x-2)^3}\right]dx,$

where k is constant $\neq 0$ and $g(x)$ is a polynomial of degree less than 4.

Integer Type

1. (1) $f(x)=\int x^{\sin x}(1+x\cos x\cdot\ln x+\sin x)dx$

if $F(x)=x^{\sin x}=e^{\sin x\ln x}$

$\therefore f(x)=\int(F(x)+xF'(x))=xF(x)+C$

$f(x)=x\cdot x^{\sin x}+C$

$f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}\cdot\frac{\pi}{2}+C \Rightarrow C=0$

$\therefore f(x)=x(x)^{\sin x}; f(\pi)=\pi(\pi)^0=\pi$

2. (4) $g(x)=\int\frac{\cos x(\cos x+2)+\sin^2 x}{(\cos x+2)^2}dx$

$=\int\underbrace{\cos x}_{\text{II}}\cdot\underbrace{\frac{1}{(\cos x+2)}}_{\text{I}}dx+\int\frac{\sin^2 x}{\cos x+2}dx$

$=\frac{1}{\cos x+2}\cdot\sin x-\int\frac{\sin^2 x}{(\cos x+2)^2}dx+\int\frac{\sin^2 x}{(\cos x+2)^2}dx$

$\therefore g(x)=\frac{\sin x}{\cos x+2}+C$

$g(0)=0 \Rightarrow C=0$

$\therefore g(x)=\frac{\sin x}{\cos x+2} \Rightarrow g\left(\frac{\pi}{2}\right)=\frac{1}{2}$

3. (2) $k(x)=\int\frac{(x^2+1)dx}{(x^3+3x+6)^{1/3}}$

put $x^3+3x+6=t^3 \Rightarrow 3(x^2+1)\,dx=3t^2dt$

$k(x)=\int\frac{t^2\,dt}{t}=\frac{t^2}{2}+C$

$k(x)=\frac{1}{2}(x^3+3x+6)^{2/3}+C$

$k(-1)=\frac{1}{2}(2)^{2/3}+C \Rightarrow C=0$

$\therefore k(x)=\frac{1}{2}(x^3+3x+6)^{2/3}; f(-2)=\frac{1}{2}(-8)^{2/3}$

$=\frac{1}{2}[(-2)^3]^{2/3}=2$

4. (4) $\int x^2\cdot e^{-2x}dx=e^{-2x}(ax^2+bx+c)+d$

Differentiating both sides, we get

$x^2\cdot e^{-2x}=e^{-2x}(2ax+b)+(ax^2+bx+c)(-2e^{-2x})$

$=e^{-2x}(-2ax^2+2(a-b)x+b-2c)$

$\Rightarrow a=-\frac{1}{2}, 2(a-b)=0, b-2c=0$

$\Rightarrow a=-\frac{1}{2}, b=-\frac{1}{2}, c=-\frac{1}{4}$

5. (9) $f(x)=\int\frac{3x^2+1}{(x^2-1)^3}dx$

$=\int\frac{-(x^2-1)}{(x^2-1)^3}dx+\int\frac{4x^2}{(x^2-1)^3}dx$

$$=\int\left[\frac{-1}{(x^2-1)^2}+x\cdot\frac{4x}{(x^2-1)^3}\right]dx$$

$$=-\int\frac{dx}{(x^2-1)^2}+x\int\frac{4x\,dx}{(x^2-1)^3}-\int\left((x)'\int\frac{4x}{(x^2-1)^3}dx\right)dx$$

$$=x\left(\frac{-1}{(x^2-1)^2}\right)+C$$

$$=-\frac{x}{(x^2-1)^2}+C$$

$f(0)=0 \Rightarrow C=0$

$$\Rightarrow f(x)=-\frac{x}{(x^2-1)^2}$$

Now $f(2)=-\frac{2}{9}$

6. (0) $fog(x)=\sqrt{e^x-1}$

$\therefore\ I=\int\sqrt{e^x-1}\,dx$

$$=\int\frac{2t^2}{t^2+1}dt \ \{\text{where } \sqrt{e^x-1}=t\}$$

$=2t-2\tan^{-1}t+C$

$=2\sqrt{e^x-1}-2\tan^{-1}\left(\sqrt{e^x-1}\right)+C$

$=2fog(x)-2\tan^{-1}(fog(x))+C$

$\therefore\ A+B=2+(-2)=0$

7. (3) $\frac{d}{dx}(A\ln|\cos x+\sin x-2|+Bx+C)$

$$=A\frac{\cos x-\sin x}{\cos x+\sin x-2}+B$$

$$=\frac{A\cos x-A\sin x+B\cos x+B\sin x-2B}{\cos x+\sin x-2}$$

$\therefore\ 2=A+B, -1=-A+B, \lambda=-2B$

$\therefore\ A=3/2, B=1/2, \lambda=-1$

$\Rightarrow\ A+B+|\lambda|=3$

8. (0) $\int\left[\left(\frac{x}{e}\right)^x+\left(\frac{e}{x}\right)^x\right]\ln x\,dx$

put $\left(\frac{x}{e}\right)^x=t$

or $x\ln\left(\frac{x}{e}\right)=\ln t$

$$\therefore\ \left(x\cdot\frac{1}{x/e}\cdot\frac{1}{e}+\ln\left(\frac{x}{e}\right)\right)dx=\frac{1}{t}dt$$

$\therefore\ (1+\ln x-\ln e)\,dx=\frac{1}{t}\,dt$

$\therefore\ (\ln e+\ln x-\ln e)\,dx=\frac{1}{t}\,dt$

$\therefore\ (\ln x)dx=\frac{1}{t}\,dt$

or $I=\int\left(t+\frac{1}{t}\right)\frac{1}{t}dt=\int 1\cdot dt+\int\frac{1}{t^2}dt$

$=t-\frac{1}{t}+C$

or $I=\left(\frac{x}{e}\right)^x-\left(\frac{e}{x}\right)^x+C$

Archives

Subjective

1. $I=\int\frac{\sin x}{\sin x-\cos x}dx$

$$=\frac{1}{2}\int\frac{2\sin x}{\sin x-\cos x}dx$$

$$=\frac{1}{2}\int\frac{\sin x+\cos x+\sin x-\cos x}{\sin x-\cos x}$$

$$=\int\frac{\cos x+\sin x}{\sin x-\cos x}dx+\frac{1}{2}\int dx$$

$$=\frac{1}{2}\log|\sin x-\cos x|+\frac{x}{2}+C$$

2. $I=\int\frac{x^2\,dx}{(a+bx)^2}$

Let $a+bx=t \Rightarrow x=\left(\frac{t-a}{b}\right)$

$\Rightarrow\quad dx=\frac{dt}{b}$

$$\Rightarrow\quad I=\frac{1}{b^3}\int\frac{t^2-2at+a^2}{t^2}dt$$

$$=\frac{1}{b^3}\int\left(1-\frac{2a}{t}+\frac{a^2}{t^2}\right)dt$$

$$=\frac{1}{b^3}\left[t-2a\log|t|-\frac{a^2}{t}\right]+C$$

$$=\frac{1}{b^3}\left[a+bx-2a\log|a+bx|-\frac{a^2}{a+bx}\right]+C$$

3. a. $\int\sqrt{1+\sin\left(\frac{x}{2}\right)}dx$

$$=\int\sqrt{\sin^2\frac{x}{4}+\cos^2\frac{x}{4}+2\sin\frac{x}{4}\cos\frac{x}{4}}\,dx$$

$$=\pm\int\left(\sin\frac{x}{4}+\cos\frac{x}{4}\right)dx$$

$$= \pm\left[\frac{-\cos x/4}{1/4}+\frac{\sin x/4}{1/4}\right]+C$$

$$= \pm\, 4\left[\sin\frac{x}{4}-\cos\frac{x}{4}\right]+C$$

b. $I=\int\frac{x^2}{\sqrt{1-x}}\,dx$

Let $1-x=t^2 \Rightarrow dx=-2t\,dt$

$$\Rightarrow \quad I=\int\frac{(1-t^2)^2}{t}(-2t)\,dt$$

$$=-2\int(t^4-2t^2+1)\,dt$$

$$=-2\left[\frac{t^5}{5}-\frac{2t^3}{3}+t\right]+C$$

$$=-2\left[\frac{(1-x)^{5/2}}{5}-\frac{2(1-x)^{3/2}}{3}+\sqrt{1-x}\right]+C$$

4. $\int(e^{\log x}+\sin x)\cos x\,dx$

$$=\int(x+\sin x)\cos x\,dx \qquad [\text{Using } e^{\log x}=x]$$

$$=\int x\cos x+\frac{1}{2}\int\sin 2x\,dx$$

$$=x\sin x-\int\sin x\,dx+\frac{1}{2}\left(\frac{-\cos 2x}{2}\right)+C$$

$$=x\sin x+\cos x-\frac{1}{4}\cos 2x+C$$

5. $I=\int\frac{(x-1)e^x}{(x+1)^3}\,dx$

$$=\int\frac{(x+1-2)e^x}{(x+1)^3}\,dx$$

$$=\int\left[\frac{1}{(x+1)^2}-\frac{2}{(x+1)^3}\right]e^x\,dx$$

$$=\int\left[\frac{1}{(x+1)^2}+\left(\frac{1}{(x+1)^2}\right)'\right]e^x\,dx$$

$$=\frac{e^x}{(x+1)^2}+C$$

6. Let $\int\frac{dx}{x^3.x^2\left(1+\frac{1}{x^4}\right)^{3/4}}$

Put $1+\frac{1}{x^4}=t \Rightarrow \frac{-4}{x^5}dx=dt \Rightarrow \frac{dx}{x^5}=-\frac{dt}{4}$

$$\therefore \quad I=\int\frac{-dt}{4t^{3/4}}=\frac{-1}{4}\frac{t^{-3/4+1}}{\frac{-3}{4}+1}+C$$

$$=-t^{1/4}+C=-\left(1+\frac{1}{x^4}\right)^{1/4}+C$$

7. $I=\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

Put $x=\cos^2\theta \Rightarrow dx=-2\cos\theta\sin\theta\,d\theta$

$$\therefore \quad I=-\int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,2\sin\theta\cos\theta\,d\theta$$

$$=-\int\frac{\sin\theta/2}{\cos\theta/2}\,2\cdot2\sin\theta/2\cos\theta/2\cos\theta\,d\theta$$

$$=-2\int(1-\cos\theta)\cos\theta\,d\theta$$

$$=-2\int(\cos\theta-\cos^2\theta)\,d\theta$$

$$=-2\int\left(\cos\theta-\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$=-2\left[\sin\theta-\frac{1}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)\right]+C$$

$$=-2\sqrt{1-x}+\frac{2}{2}\left[\cos^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}\right]+C$$

[using $\sin\theta=\sqrt{1-x}$]

$$=-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}+C$$

8. Let $I=\int\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}\,dx$

We know that

$\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\pi/2$ (1)

Also $\cos^{-1}\sqrt{x}=\pi/2-\sin^{-1}\sqrt{x}$ (2)

Using equations (1) and (2), we get

$$I=\int\frac{\sin^{-1}\sqrt{x}-\left(\pi/2-\sin^{-1}\sqrt{x}\right)}{\pi/2}\,dx$$

$$=\frac{2}{\pi}\int(2\sin^{-1}\sqrt{x}-\pi/2)\,dx$$

$$=\frac{4}{\pi}\int\sin^{-1}\sqrt{x}\,dx-\int 1\,dx$$

Let $x=\sin^2\theta \Rightarrow dx=2\sin\theta\cos\theta\,d\theta$

$$= \frac{4}{\pi} \int \sin^{-1}(\sin\theta)\, 2\sin\theta\cos\theta\, d\theta - x + C$$

$$= \frac{4}{\pi} \int \theta \sin 2\theta\, d\theta - x + C$$

$$= \frac{4}{\pi}\left[\frac{-\theta\cos 2\theta}{2} + \int 1 \times \frac{\cos 2\theta}{2}\, d\theta\right] - x + C$$

[Integrating by parts]

$$= \frac{4}{\pi}\left[\frac{-\theta\cos 2\theta}{2} + \frac{\sin 2\theta}{4}\right] - x + C$$

$$= \frac{4}{4\times\pi}[-2\sin^{-1}\sqrt{x}\,(1-2x) + 2\sqrt{x}\,\sqrt{1-x}\,] - x + C$$

$$= \frac{2}{\pi}[\sqrt{x-x^2} - (1-2x)\sin^{-1}\sqrt{x}\,] - x + C$$

9. $I = \int \frac{\sqrt{\cos 2x}}{\sin x}\, dx$

$$= \int \frac{\sqrt{\cos^2 x - \sin^2 x}}{\sin x}\, dx$$

$$= \int \sqrt{\cot^2 x - 1}\, dx$$

Put $\cot^2 x - 1 = y^2$

$\Rightarrow \cot^2 x = 1 + y^2$

$\Rightarrow -2\cot x \operatorname{cosec}^2 x\, dx = 2y\, dy$

$$\Rightarrow \quad dx = \frac{-y\, dy}{\sqrt{1+y^2}\,(2+y^2)}$$

$$\Rightarrow \quad I = -\int \frac{y \times y dy}{\sqrt{1+y^2}\,(2+y^2)}$$

$$= -\int \frac{1}{\sqrt{y^2+1}}\, dy + 2\int \frac{dy}{(y^2+2)\sqrt{y^2+1}}$$

$$= -\log|y + \sqrt{y^2+1}| + 2I_1 \qquad (1)$$

where $I_1 = \int \frac{dy}{(y^2+2)\sqrt{y^2+1}}$

Put $\quad y = \frac{1}{t} \quad \Rightarrow dy = -\frac{dt}{t^2}$

$$\Rightarrow \quad I_1 = \int \frac{-\frac{dt}{t^2}}{\left(\frac{1}{t^2}+2\right)\sqrt{\frac{1}{t^2}+1}}$$

$$= -\int \frac{t\, dt}{(1+2t^2)\sqrt{t^2+1}}$$

Now let $t^2 + 1 = z^2$

$\Rightarrow \quad t\, dt = z\, dz$

$$\Rightarrow \quad I_1 = -\int \frac{z dz}{(1+2(z^2-1))\, z}$$

$$= -\int \frac{dz}{2z^2 - 1}$$

$$= -\frac{1}{2}\int \frac{dz}{z^2 - \frac{1}{2}}$$

$$= -\frac{1}{2\sqrt{2}} \log\left|\frac{z - \frac{1}{\sqrt{2}}}{z + \frac{1}{\sqrt{2}}}\right|$$

$$= -\frac{1}{2\sqrt{2}} \log\left|\frac{\sqrt{\frac{1}{y^2}+1} - \frac{1}{\sqrt{2}}}{\sqrt{\frac{1}{y^2}+1} + \frac{1}{\sqrt{2}}}\right| + C$$

$$= -\frac{1}{2\sqrt{2}} \log\left|\frac{\sqrt{2y^2+2} - y}{\sqrt{2y^2+2} + y}\right| + C$$

$$\Rightarrow I = -\log|y + \sqrt{y^2+1}| - \frac{1}{\sqrt{2}} \log\left|\frac{\sqrt{2y^2+2} - y}{\sqrt{2y^2+2} + y}\right| + C,$$

where $\cot^2 x = 1 + y^2$

10. $\int (\sqrt{\tan x} + \sqrt{\cot x})\, dx$

$$= \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\, dx$$

$$= \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$$

$$= \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}}\, dx$$

$$= \sqrt{2}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\sin^{-1} t + C$$

$$= \sqrt{2}\sin^{-1}(\sin x - \cos x) + C$$

11. $\int \left(\frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} + \frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}}\right) dx$

$$\text{Let } I = \underbrace{\int \frac{1}{\sqrt[3]{x} + \sqrt[4]{x}}\, dx}_{I_1} + \underbrace{\int \frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}}\, dx}_{I_2} \qquad (1)$$

$$I_1 = \int \frac{1}{\sqrt[3]{x} + \sqrt[4]{x}}\ dx$$

Let $x = y^{12}$ so that $dx = 12\, y^{11}\, dy$

$$\therefore I_1 = \int \frac{12\, y^{11}}{y^4 + y^3}\ dy = 12 \int \frac{y^8}{1+y}\ dy$$

$$= 12 \int \left(y^7 - y^6 + y^5 - y^4 + y^3 - y^2 + y - 1 + \frac{1}{y+1} \right) dy$$

$$= 12 \left[\frac{y^8}{8} - \frac{y^7}{7} + \frac{y^6}{6} - \frac{y^5}{5} + \frac{y^4}{4} - \frac{y^3}{3} + \frac{y^2}{2} - y + \log|y+1| \right] + C$$

$$= \frac{2}{3} x^{2/3} - \frac{12}{7} x^{7/12} + 2x^{1/2} - \frac{12}{5} x^{5/12} + 3x^{1/3}$$
$$- 4x^{1/4} + 6x^{1/6} - 12x^{1/12} + 12 \log|x^{1/12} + 1| + C_1 \quad (2)$$

$$I_2 = \int \frac{\ln(1 + x^{1/6})}{x^{1/3} + x^{1/2}} dx$$

Let $x = z^6$ so that $dx = 6z^5\, dz$

$$\Rightarrow \quad I_2 = \int \frac{\ln(1+z)}{z^2 + z^3}\ 6z^5\, dz$$

$$= \int \frac{6z^3 \ln(z+1)}{z+1}\ dz$$

Put $z + 1 = t \Rightarrow dz = dt$

$$\therefore I_2 = \int \frac{6\,(t-1)^3 \ln t}{t}\ dt$$

$$= 6 \int \left(t^2 - 3t + 3 - \frac{1}{t} \right) \ln t\, dt$$

$$= 6 \left[\int (t^2 - 3t + 3) \ln t\ dt - \int \frac{1}{t} \ln t\ dt \right]$$

$$= 6 \left[\left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t \right) \ln t - \int \left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t \right) \frac{1}{t} dt - \frac{(\ln t)^2}{2} \right] + C$$

$$= 6 \left[\left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t \right) \ln t - \left(\frac{t^3}{9} - \frac{3t^2}{4} + 3t \right) - \frac{(\ln t)^2}{2} \right] + C$$

(3)

Thus, we get the value of I on substituting the values of I_1 and I_2 from equations (2) and (3) in equation (1).

12. Let $\int \cos 2\theta \ln \left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} \right) d\theta$

$$= \ln \left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} \right) \int \cos 2\theta\ d\theta$$

$$- \int \frac{(\sin 2\theta)(\cos\theta - \sin\theta)}{2(\sin\theta + \cos\theta)} \frac{2}{(\cos\theta - \sin\theta)^2}$$

$$= \frac{\sin 2\theta}{2} \ln \left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} \right) - I_1$$

$$I_1 = \int \frac{(\sin 2\theta)}{(\sin\theta + \cos\theta)(-\sin\theta + \cos\theta)} d\theta$$

$$= \int \frac{\sin 2\theta}{\cos 2\theta}\ d\theta = \frac{1}{2} \ln|\sec 2\theta|$$

$$\Rightarrow I = \sin 2\theta \ln \sqrt{\frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}} - \frac{1}{2} \ln|\sec 2\theta| + C$$

13. $I = \int \frac{(x+1)}{x\,(1 + xe^x)^2}\ dx$

$$= \int \frac{e^x\,(x+1)}{x\, e^x\,(1 + xe^x)^2}\ dx$$

Put $1 + xe^x = t \Rightarrow (xe^x + e^x)\, dx = dt$

$$= \int \frac{dt}{(t-1)\, t^2}$$

$$= \int \left(\frac{1}{1-t} + \frac{1}{t} + \frac{1}{t^2} \right) dt$$

$$= -\log|1-t| + \log|t| - \frac{1}{t} + C$$

$$= -\log \left| \frac{t}{1-t} \right| - \frac{1}{t} + C$$

$$= -\log \left| \frac{1 + xe^x}{-xe^x} \right| - \frac{1}{1 + xe^x} + C$$

$$= -\log \left(\frac{1 + xe^x}{xe^x} \right) - \frac{1}{1 + xe^x} + C$$

14. Put $x = \cos^2\theta \Rightarrow dx = -2\cos\theta \sin\theta\, d\theta$

$$= \int \left(\frac{1 - \cos\theta}{1 + \cos\theta} \right)^{1/2} \left(\frac{-2\cos\theta \sin\theta\ d\theta}{\cos^2\theta} \right) dx$$

$$= -\int \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} . \frac{4 \sin\frac{\theta}{2} \cos\frac{\theta}{2}}{\cos\theta}\ d\theta$$

$$= -\int \frac{4 \sin^2 \frac{\theta}{2}}{\cos\theta}\ d\theta$$

$$= -4\int \frac{1-\cos\theta}{\cos\theta}\, d\theta$$

$$= -4\int (\sec\theta - 1)\, d\theta$$

$$= -4\,[\log|\sec\theta + \tan\theta| - \theta] + C$$

$$= -4\left[\log\left|\frac{1}{\sqrt{x}} + \frac{\sqrt{1-x}}{\sqrt{x}}\right| - \cos^{-1}\sqrt{x}\right] + C$$

$$= -4\left[\log\left|\frac{1+\sqrt{1-x}}{\sqrt{x}}\right| - \cos^{-1}\sqrt{x}\right] + C$$

15. $\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}\, dx$

$$= \int \frac{x\,(x^2+1) + 2\,(x+1)}{(x^2+1)^2\,(x+1)}\, dx$$

$$= \int \frac{x}{(x^2+1)\,(x+1)}\, dx + 2\int \frac{dx}{(x^2+1)^2}$$

$$= \int\left(\frac{x+1}{2\,(x^2+1)} - \frac{1}{2\,(x+1)}\right) dx + \int \frac{2}{(x^2+1)^2}\, dx$$

$$= \frac{1}{4}\log|x^2+1| - \frac{1}{2}\log|x+1| + \frac{1}{2}\tan^{-1}x + 2I$$

where $I = \int \frac{dx}{(x^2+1)^2}$. Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\Rightarrow \quad I = \int \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^2}$$

$$= \int \cos^2\theta\, d\theta$$

$$= \int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{1}{2}\left(\theta + \frac{\tan\theta}{1+\tan^2\theta}\right) + C$$

$$= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\,\frac{x}{1+x^2} + C$$

$\therefore$ the given integral

$$= \frac{1}{4}\log|x^2+1| - \frac{1}{2}\log|x+1| + \frac{1}{2}\tan^{-1}x + \tan^{-1}x + \frac{x}{1+x^2} + C$$

$$= \frac{1}{4}\log\left|\frac{x^2+1}{(x+1)^2}\right| + \frac{3}{2}\tan^{-1}x + \frac{x}{1+x^2} + C$$

16. $I = \int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right) dx$

$$= \int \sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2+3^2}}\right) dx$$

[put $2x+2 = 3\tan\theta \Rightarrow 2\,dx = 3\sec^2\theta\, d\theta$]

$$= \int \sin^{-1}\left(\frac{3\tan\theta}{3\sec\theta}\right)\frac{3}{2}\sec^2\theta\, d\theta$$

$$= \frac{3}{2}\int \theta\sec^2\theta\, d\theta$$

$$= \frac{3}{2}\left\{\theta\tan\theta - \int \tan\theta\, d\theta\right\}$$

$$= \frac{3}{2}\{\theta\tan\theta - \log|\sec\theta|\} + C$$

$$\Rightarrow I = \frac{3}{2}\left\{\frac{2x+2}{3}\tan^{-1}\left(\frac{2x+2}{3}\right) - \log\left(\sqrt{1+\left(\frac{2x+2}{3}\right)^2}\right)\right\} + C$$

$$= \frac{3}{2}\left\{\frac{2}{3}(x+1)\tan^{-1}\left(\frac{2}{3}(x+1)\right) - \log\sqrt{4x^2+8x+13}\right\} + C$$

17. $I = \int \left(x^{3m}+x^{2m}+x^m\right)\left(2x^{2m}+3x^m+6\right)^{1/m} dx$

$$= \int\left(\frac{x^{3m}+x^{2m}+x^m}{x}\right)(x)\left(2x^{2m}+3x^m+6\right)^{1/m} dx\, dx$$

$$= \int \left(x^{3m-1}+x^{2m-1}+x^{m-1}\right)\left(2x^{3m}+3x^{2m}+6x^m\right)^{1/m} dx$$

Let $2x^{3m}+3x^{2m}+6x^m = t \Rightarrow dt = 6m(x^{3m-1}+x^{2m-1}+x^{m-1})dx$

$$\Rightarrow \quad I = \int t^{1/m}\frac{dt}{6m} = \frac{1}{6m}\,\frac{t^{\frac{1}{m}+1}}{\frac{1}{m}+1} + C$$

$$= \frac{1}{6(m+1)}\left(2x^{3m}+3x^{2m}+6x^m\right)^{(m+1)/m} + C$$

Objective

Fill in the blanks

1. We have $\int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}\, dx = Ax + B\ln(9e^{2x}-4) + C$

Differentiating both sides w.r.t. x, we get

$$\Rightarrow \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}} = A + \frac{18\,B\,e^x}{9e^x-4e^{-x}}$$

$$\Rightarrow = \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} = \frac{(9A + 18B)e^x - 4Ae^{-x}}{9e^x - 4e^{-x}}$$

$$\Rightarrow 9A + 18B = 4; -4A = 6$$

$$\Rightarrow A = -3/2, B = \left(4 + \frac{27}{2}\right)\frac{1}{18} = \frac{35}{36}$$

C can have any real value.

Multiple choice questions with one correct answer

1. c. Let

$$I = \int \frac{\cos^3 + \cos^5 x}{\sin^2 x + \sin^4 x} dx$$

$$= \int \frac{(\cos^2 x + \cos^4 x) \cos x}{\sin^2 x (1 + \sin^2 x)} dx$$

$$= \int \frac{[1 - \sin^2 x + (1 - \sin^2 x)^2] \cos x}{\sin^2 x (1 + \sin^2 x)} dx$$

$$= \int \frac{(2 - 3\sin^2 x + \sin^4 x) \cos x}{\sin^2 x (1 + \sin^2 x)} dx$$

Put $\sin x = t \Rightarrow \cos x \, dx = dt$

$$\Rightarrow \quad I = \int \frac{2 - 3t^2 + t^4}{t^4 + t^2} dt$$

$$= \int \left(1 + \frac{2}{t^2} - \frac{6}{t^2 + 1}\right) dt$$

$$= t - \frac{2}{t} - 6\tan^{-1}(t) + C$$

$$= \sin x - 2(\sin x)^{-1} - 6\tan^{-1}(\sin x) + C$$

2. d. $I = \int \frac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}} dx$

$$= \frac{1}{4}\int \frac{\frac{4}{x^3} - \frac{4}{x^5}}{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}} dx$$

$$\Rightarrow \text{Put } 2 - \frac{2}{x^2} + \frac{1}{x^4} = t \Rightarrow \left(\frac{4}{x^3} - \frac{4}{x^5}\right)dx = dt$$

$$\Rightarrow I = \frac{1}{4}\int \frac{dt}{\sqrt{t}} = \frac{2\sqrt{t}}{4} + C$$

$$= \frac{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}}{2} + C$$

$$= \frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + C$$