4

Limit

Contents :

4.1 Introduction

We have studied function in 11th Standard. We studied in the chapter that when we substitute a particular value of a variable in the function, we got the corresponding value of the function. For example, if we substitute $x = 2$ in the function $f(x) = 2x + 3$, we get $f(2) = 7$. And if we substitute $x = 1$ in the function $f(x) = \frac{3-x}{3x+2}$, we get $f(1) = \frac{2}{5}$. But this is not possible for all functions and all values of x. Let us consider a function $f(x) = \frac{x^2-9}{x-3}$ and if we substitute $x = 3$ in $f(x)$, we get $f(3) = \frac{0}{0}$ which is an indeterminate value. To find approximate value of $f(3)$ for this function, we need to know the concept of limit of a function. So, limits can be used to approximate the value of a function when the value of the function is indeterminate.

We consider the following illustration to clarify the above concept.

Assume that we are watching a football game through internet. Unfortunately, the connection is choppy and we missed what happened at 14:00 (14 minutes after the commencement of match.)

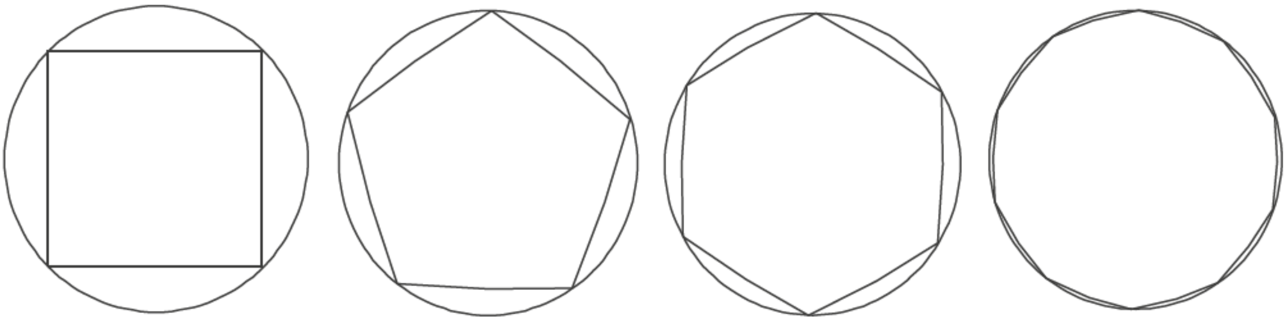
What would be the position of the ball at 14:00 ? We have seen the position of the ball at 13:58 (13 minutes and 58 seconds after the commencement of match), 13:59, 14:01, 14:02.

We will see the neighbouring instants of 14:00, (13:59 and 14:01) and estimate the position of the ball at 14:00. Our estimation is “At 14:00, the ball was somewhere between its position at 13:59 and 14:01.” With a slow-motion camera, we might even say “At 14:00, the ball was somewhere between its positions at 13:59.99 and 14:00.01”. It means that our estimation improves as we take closer and closer instants to 14:00. The approximate position of the ball thus obtained will be the limiting value of the position of the ball.

Thus, we can say that, “Limit is a method for finding confident approximate value.”

We consider one more illustration.

Suppose we want to find the area of a circle. We can estimate the area of circle from the area of polygon drawn inside the circle.



4 side | 5 side | 6 side | 10 side

We can see from the above figures that as the number of sides of polygon increases, area of the polygon approaches nearer the area of circle. The limiting value of the area of polygon is the best approximate value of the area of the circle.

Thus, limit can be used to approximate the unknown values by using its nearby values. Closer the neighbouring values, better is the approximation.

To understand the concept of limit, we shall understand the following basic terms.

4.2 Real Line and its Interval

Real line : The real line or real number line is a line where its points are the real numbers.

Interval : A set of real numbers between any two real numbers is an interval. We shall study different types of intervals.

Closed Interval : If $a \in R$, $b \in R$ and $a < b$ then the set of all real numbers between a and b including a and b is called a closed interval. It is denoted by $[a, b]$.

$$[a, b] = \{x \mid a \le x \le b, \ x \in R\}$$

Open Interval : If $a \in R$, $b \in R$ and $a < b$ then the set of all real numbers between a and b not including a and b is called an open interval. It is denoted by (a, b).

$$(a, b) = \{x \mid a < x < b, \ x \in R\}$$

Closed-Open Interval : If $a \in R$, $b \in R$ and $a < b$ then the set of all real numbers between a and b including a but not including b is called a closed open interval. It is denoted by $[a, b)$.

$[a, b) = \{x \mid a \le x < b, \ x \in R\}$

Open-Closed Interval : If $a \in R$, $b \in R$ and $a < b$ then set of all real numbers between a and b not including a but including b is called an open closed interval. It is denoted by $(a, b]$.

$(a, b] = \{x \mid a < x \le b, \ x \in R\}$

4.3 Modulus

If $x \in R$ then

$|x| = x$ if $x \ge 0$

$= -x$ if $x < 0$

Modulus of any real number is always non-negative.

e.g. $|3| = 3$, $|-4| = 4$, $|0| = 0$

Meaning of $|x-a| < \delta$ (*Delta*)

Using the definition of modulus

$|x-a| < \delta = (x-a) < \delta$ if $x \ge a$ **or** $x < a+\delta$ if $x \ge a$

$= (a-x) < \delta$ if $x < a$ **or** $x > a-\delta$ if $x < a$

$\therefore \ |x-a| < \delta \iff x \in (a-\delta, \ a+\delta)$

4.4 Neighbourhood

Any open interval containing a, $a \in R$ is called a **neighbourhood** of a.

δ neighbourhood of a :

If $a \in R$ and δ is non-negative real number then the open interval $(a-\delta, \ a+\delta)$ is called δ neighbourhood of a. It is denoted by $N(a, \ \delta)$.

Here, it can be understood that

$N(a, \ \delta) = \{x \mid a-\delta < x < a+\delta, \ x \in R\}$

$= \{x \mid |x-a| < \delta, \ x \in R\}$

δ neighbourhood of a can be expressed in the following different ways.

Neighbourhood form	Modulus form	Interval form
$N(a, \ \delta)$	$\lvert x-a \rvert < \delta$	$(a-\delta, \ a+\delta)$

Illustration 1 : Express $N(5, 0.2)$ in modulus and interval form.

Comparing $N(5, 0.2)$ with $N(a, \delta)$, we get $a = 5$ and $\delta = 0.2$.

Modulus form : $|x-a| < \delta$

Putting $a = 5$ and $\delta = 0.2$,

$N(5, 0.2) = |x-5| < 0.2$

Interval form : $(a - \delta,\ a + \delta)$

Putting $a = 5$ and $\delta = 0.2$,

$$N(5, 0.2) = (5-0.2,\ 5+0.2)$$
$$= (4.8,\ 5.2)$$

Illustration 2 : Express 0.001 neighbourhood of 3 in modulus and interval form.

Comparing 0.001 neighbourhood of 3 with δ neighbourhood of a, we get $a = 3$ and $\delta = 0.001$.

Modulus form : $|x-a| < \delta$

Putting $a = 3$ and $\delta = 0.001$,

0.001 neighbourhood of 3 = $|x-3| < 0.001$

Interval form : $(a - \delta,\ a + \delta)$

Putting $a = 3$ and $\delta = 0.001$,

0.001 neighbourhood of 3 $= (3-0.001,\ 3+0.001)$
$= (2.999,\ 3.001)$

Illustration 3 : Express $|x+1| < 0.5$ in neighbourhood and interval form.

Comparing $|x+1| < 0.5$ with $|x-a| < \delta$, we get $a = -1$ and $\delta = 0.5$.

Neighbourhood form : $N(a, \delta)$

Putting $a = -1$ and $\delta = 0.5$,

$|x+1| < 0.5 = N(-1, 0.5)$

Interval form : $(a - \delta,\ a + \delta)$

Putting $a = -1$ and $\delta = 0.5$,

$|x+1| < 0.5 = (-1 - 0.5,\ -1 + 0.5)$
$= (-1.5,\ -0.5)$

Illustration 4 : Express $(0.9, 1.1)$ in neighbourhood and modulus form.

Comparing $(0.9, 1.1)$ with $(a - \delta,\ a + \delta)$, we get $a - \delta = 0.9$ and $a + \delta = 1.1$.

Adding $a - \delta = 0.9$ and $a + \delta = 1.1$, we get $2a = 2$ $\therefore$ $a = 1$.

Putting $a = 1$ in $a + \delta = 1.1$, we get $\delta = 0.1$.

Neighbourhood form : $N(a, \delta)$

Putting $a = 1$ and $\delta = 0.1$,

$(0.9,\ 1.1) = N(1,\ 0.1)$

Modulus form : $|x-a| < \delta$

Putting $a = 1$ and $\delta = 0.1$,

$(0.9,\ 1.1) = |x-1| < 0.1$

Punctured δ neighbourhood of a :

If $a \in R$ and δ is a non-negative real number then the open interval $(a-\delta,\ a+\delta) - \{a\}$ is called punctured δ neighbourhood of *a*. It is denoted by $N^*(a,\ \delta)$.

Here, it can be understood that

$$N^*(a,\ \delta) = N(a,\ \delta) - \{a\}$$
$$= \{x \mid a-\delta < x < a+\delta,\ x \neq a,\ x \in R\}$$
$$= \{x \mid |x-a| < \delta,\ x \neq a,\ x \in R\}$$

e.g. $N^*(5,\ 2) = N(5,\ 2) - \{5\}$

$$= \{x \mid 3 < x < 7,\ x \neq 5,\ x \in R\}$$
$$= \{x \mid |x-5| < 2,\ x \neq 5,\ x \in R\}$$

EXERCISE 4.1

1. Express the following in modulus and interval form :

(1) 0.4 neighbourhood of 4 (2) 0.02 neighbourhood of 2

(3) 0.05 neighbourhood of 0 (4) 0.001 neighbourhood of −1

2. Express the following in interval and neighbourhood form :

(1) $|x-2| < 0.01$ (2) $|x+5| < 0.1$

(3) $|x| < \frac{1}{3}$ (4) $|x+3| < 0.15$

3. Express the following in modulus and neighbourhood form :

(1) $3.8 < x < 4.8$ (2) $1.95 < x < 2.05$

(3) $-0.4 < x < 1.4$ (4) $1.998 < x < 2.002$

4. Express $N(16,\ 0.5)$ in the interval and modulus form.

5. If $N(3,\ b) = (2.95,\ k)$ then find the values of b and k.

6. If $|x-10| < k_1 = (k_2,\ 10.01)$ then find the values of k_1 and k_2.

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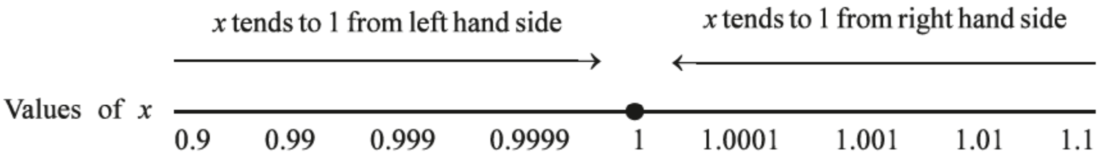
Meaning of $x \to a$:

If the value of variable x is brought very close to a number 'a' by increasing or decreasing its value then it can be said that x tends to a. It is denoted by $x \to a$.

It is necessary here to note that $x \to a$ means value of x approaches very close to a but it will not be equal to a.

e.g. Let us understand the meaning of $x \to 1$.

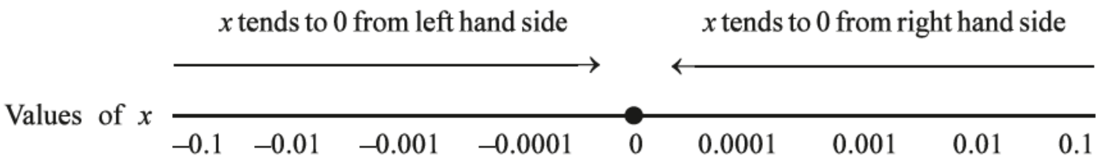


Meaning of $x \to 0$:

If by decreasing the positive value of a variable x or by increasing negative value of the variable x, the value of x is brought very close to '0' then it can be said that x tends to 0. It is denoted by $x \to 0$.

It is necessary here to note that $x \to 0$ means, the value of x approaches very close to 0 but it will not be equal to 0.

Let us understand the meaning of $x \to 0$.



4.5 Limit of a function

When the value of a variable x is brought closer and closer to a number 'a', the value of function $f(x)$ reaches closer and closer to a definite number 'l' then we can say that as x tends to a, $f(x)$ tends to l. i.e. as $x \to a$, $f(x) \to l$. Symbolically it can be written as $\lim_{x \to a} f(x) = l$. l is called the limiting value of the function.

Definition : The function $f(x)$ has a limit l as x tends to 'a' if for each given predetermined $\varepsilon > 0$, however small, we can find a positive number δ such that $\left|f(x) - l\right| < \varepsilon$ (*Epsilon*) for all x such that $|x - a| < \delta$.

Now, we shall understand how limit of a function is obtained.

Suppose, we want to find the value of the function $f(x) = \frac{x^2-1}{x-1}$ at $x = 1$.

If we put $x = 1$ in $f(x) = \frac{x^2-1}{x-1}$ we get $f(1) = \frac{0}{0}$ which is indeterminate. So, we cannot find the value of $f(1)$ but assuming value of x very close to 1, we can approximate the value of $f(1)$. Let us see the changes in $f(x)$ as x tends to 1.

x (towards 1 from LHS of 1)	$f(x)$	x (towards 1 from RHS of 1)	$f(x)$
0.9	1.9	1.1	2.1
0.99	1.99	1.01	2.01
0.999	1.999	1.001	2.001
0.9999	1.9999	1.0001	2.0001
.	.	.	.
.	.	.	.
.	.	.	.

We can assume any value of x close to 1. Generally, we start with a value at a distance 0.1 on both sides of $x = 1$. i.e. we start with $x = 0.9$ and 1.1 and bring values of x closer to 1 from both the sides.

It is clear from the table that when the value of x is brought nearer to 1 by increasing or decreasing its values, the value of $f(x)$ approaches to 2.

This can symbollically be expressed as $\lim\limits_{x \to 1} \frac{x^2-1}{x-1} = 2$.

Limit of a function is obtained by putting different values of x in $f(x)$ and tabulating them. So, this method of obtaining the limit of a function is called a **tabular method.**

Illustration 5 : Find $\lim\limits_{x \to 3} 2x+5$ by tabular method.

We have $f(x) = 2x+5$. We shall take the values of x very near to 3 and prepare a table in the following way :

x	$f(x)$	x	$f(x)$
2.9	10.8	3.1	11.2
2.99	10.98	3.01	11.02
2.999	10.998	3.001	11.002
2.9999	10.9998	3.0001	11.0002
.	.	.	.
.	.	.	.
.	.	.	.

It is clear from the table that when the value of x is brought nearer to 3 by increasing or decreasing its values, the value of $f(x)$ approaches to 11. That is, when $x \to 3$, $f(x) \to 11$.

$\therefore \lim\limits_{x \to 3} 2x+5 = 11$

Illustration 6 : Find $\lim\limits_{x \to -1} \frac{x^2-1}{x+1}$, $x \in R - \{-1\}$ by preparing table.

We have $f(x) = \frac{x^2-1}{x+1}$. We shall take the values of x very near to –1 and prepare a table in the following way :

x	$f(x)$	x	$f(x)$
–1.1	–2.1	–0.9	–1.9
–1.01	–2.01	–0.99	–1.99
–1.001	–2.001	–0.999	–1.999
–1.0001	–2.0001	–0.9999	–1.9999
. . .	. . .	. . .	. . .

It is clear from the table that when the value of x is brought nearer to –1 by increasing or decreasing its value, the value of $f(x)$ approaches to –2. That is, when $x \to -1$, $f(x) \to -2$.

$$\therefore \lim_{x \to -1} \frac{x^2-1}{x+1} = -2$$

Illustration 7 : Find $\lim\limits_{x \to 0} \frac{2x^2+3x}{x}$, $x \in R - \{0\}$ using tabular method.

We have $f(x) = \frac{2x^2+3x}{x}$. We shall take the values of x very near to 0 and prepare a table in the following way :

x	$f(x)$	x	$f(x)$
–0.1	2.8	0.1	3.2
–0.01	2.98	0.01	3.02
–0.001	2.998	0.001	3.002
–0.0001	2.9998	0.0001	3.0002
. . .	. . .	. . .	. . .

It is clear from the table that when the value of x is brought nearer to 0 by increasing or decreasing its value, the value of $f(x)$ approaches to 3. That is, when $x \to 0$, $f(x) \to 3$.

$$\therefore \lim_{x \to 0} \frac{2x^2+3x}{x} = 3$$

Illustration 8 : Find $\lim\limits_{x \to 1} \frac{1}{x-1}$, $x \in R - \{1\}$ by tabular method.

We have $f(x) = \frac{1}{x-1}$. We shall take the values of x very near to 1 and prepare a table in the following way :

x	$f(x)$	x	$f(x)$
0.9	–10	1.1	10
0.99	–100	1.01	100
0.999	–1000	1.001	1000
0.9999	–10000	1.0001	10000
. . .	. . .	. . .	. . .

It is clear from the table that when the value of x is brought nearer to 1 by increasing or decreasing its value, the value of $f(x)$ does not approach to a particular value. That is, when $x \to 1$, $f(x)$ does not tend to a particular value. Thus, limit of the function does not exist.

$\therefore$ $\lim\limits_{x \to 1} \frac{1}{x-1}$ does not exist.

Illustration 9 : Find $\lim\limits_{x \to 2} \frac{3x^2 - 4x - 4}{x^2 - 4}$, $x \in R - \{2\}$ by tabular method.

We have $f(x) = \frac{3x^2 - 4x - 4}{x^2 - 4}$. We can obtain the value of limit as calculated in previous illustrations. But for simplification we shall obtain the value of limit of $f(x)$ after eliminating the common factor $(x - 2)$ from numerator and denominator.

$$\lim_{x \to 2} \frac{3x^2 - 4x - 4}{x^2 - 4} = \lim_{x \to 2} \frac{(x-2)(3x+2)}{(x-2)(x+2)}$$

$$= \lim_{x \to 2} \frac{3x + 2}{x + 2} \qquad (\because \; x - 2 \neq 0)$$

We shall take the values of x very near to 2 and prepare a table in the following way :

x	$f(x)$	x	$f(x)$
1.9	1.9744	2.1	2.02439
1.99	1.9975	2.01	2.002494
1.999	1.9997	2.001	2.0002499
1.9999	1.9999	2.0001	2.000025
. . .	. . .	. . .	. . .

It is clear from the table that when the value of x is brought very near to 2 by increasing or decreasing its value, the value of $f(x)$ approaches to 2. That is, when $x \to 2$, $f(x) \to 2$.

$\therefore$ $\lim\limits_{x \to 2} \frac{3x^2 - 4x - 4}{x^2 - 4} = 2$

EXERCISE 4.2

1. **Find the values of the following using tabular method :**

(1) $\lim_{x \to 1} 2x+1$

(2) $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3}$

(3) $\lim_{x \to 2} \frac{2x^2 + 3x - 14}{x - 2}$

(4) $\lim_{x \to -3} \frac{2x^2 + 9x + 9}{x + 3}$

(5) $\lim_{x \to 2} x$

2. Using tabular method, show that $\lim_{x \to 3} \frac{2}{x - 3}$ does not exist.

3. If $y = \frac{x^2 + x - 6}{x - 2}$, show that as $x \to 2$ then $y \to 5$ using tabular method.

4. If $y = 5 - 2x$, show that as $x \to -1$ then $y \to 7$ using tabular method.

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4.6 Working rules of limit

The following rules will be accepted without proof :

If $f(x)$ and $g(x)$ are two real valued functions of a real variable x and $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then

(1) $\lim_{x \to a} [f(x) \pm g(x)] = l \pm m$

The limit of the sum (or difference) of two functions is equal to the sum (or difference) of their limits.

(2) $\lim_{x \to a} [f(x) \times g(x)] = l \times m$

The limit of the product of two functions is equal to the product of their limits.

(3) $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{l}{m}$, $\quad m \neq 0$

The limit of the division of two functions is equal to the division of their limits, provided the limit of the function in denominator is not zero.

(4) $\lim_{x \to a} k\, f(x) = kl$, k is the constant.

The limit of the product of a function with a constant is equal to the product of the limit of the function with the same constant.

4.7 Standard forms of limit

(1) **Limit of a polynominl**

Let $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ then using the working rules of limit

$$\lim_{x \to b} f(x) = a_0 + a_1 b + a_2 b^2 + ... + a_n b^n$$

(2) $\lim_{x \to a} \left[\frac{x^n - a^n}{x - a}\right] = n a^{n-1}, \quad n \in Q$

We will see some illustrations based on the standard forms and working rules of limit.

Illustration 10 : Find the value of $\lim_{x \to 0} \frac{x^2 + 5x + 6}{x^2 + 2x + 3}$.

$$\lim_{x \to 0} \frac{x^2 + 5x + 6}{x^2 + 2x + 3} = \frac{(0)^2 + 5(0) + 6}{(0)^2 + 2(0) + 3}$$

$$= \frac{6}{3}$$

$$= 2$$

Illustration 11 : Find the value of $\lim_{x \to 2} \frac{2x + 3}{x - 1}$.

$$\lim_{x \to 2} \frac{2x + 3}{x - 1} = \frac{2(2) + 3}{2 - 1}$$

$$= \frac{7}{1}$$

$$= 7$$

Illustration 12 : Find the value of $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 5x + 6}$.

If we put $x = 3$ in the function $f(x)$, we get the value of the function as $\frac{0}{0}$, which is indeterminate. Hence, we shall factorize numerator and denominator. Since $x \to 3$, $(x - 3)$ will be a common factor of numerator and denominator.

Note : If we put $x = a$ in the given function and we get $\frac{0}{0}$ then $(x - a)$ will be the common factor of numerator and denominator.

Numerator $= x^2 - 2x - 3$

$= x^2 - 3x + x - 3$

$= x(x - 3) + 1(x - 3)$

$= (x - 3)\ (x + 1)$

Denominator $= x^2 - 5x + 6$

$= x^2 - 3x - 2x + 6$

$= x(x - 3) - 2(x - 3)$

$= (x - 3)\ (x - 2)$

Now, $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 5x + 6} = \lim_{x \to 3} \frac{(x - 3)\ (x + 1)}{(x - 3)(x - 2)}$

$$= \lim_{x \to 3} \frac{(x + 1)}{(x - 2)} \qquad (\because\ x - 3 \neq 0)$$

$$= \frac{3+1}{3-2}$$

$$= \frac{4}{1}$$

$$= 4$$

Illustration 13 : Find the value of $\lim_{x \to 1} \frac{2x^2 + x - 3}{x^2 - 1}$.

If we put $x = 1$ in the function $f(x)$, we get the value of the function as $\frac{0}{0}$, which is indeterminate.

Numerator $= 2x^2 + x - 3$

$= 2x^2 + 3x - 2x - 3$

$= x(2x + 3) - 1(2x + 3)$

$= (2x + 3)\ (x - 1)$

Denominator $= x^2 - 1$

$= (x + 1)\ (x - 1)$

Now, $\lim_{x \to 1} \frac{2x^2 + x - 3}{x^2 - 1} = \lim_{x \to 1} \frac{(2x + 3)(x - 1)}{(x + 1)(x - 1)}$

$= \lim_{x \to 1} \frac{2x + 3}{x + 1} \qquad (\because \ x - 1 \neq 0)$

$= \frac{2(1) + 3}{1 + 1}$

$= \frac{5}{2}$

Illustration 14 : Find the value of $\lim_{x \to -3} \frac{2x^2 + 7x + 3}{3x^2 + 8x - 3}$.

If we put $x = -3$ in the function $f(x)$, we get the value of the function as $\frac{0}{0}$, which is indeterminate.

Numerator $= 2x^2 + 7x + 3$

$= 2x^2 + 6x + x + 3$

$= 2x(x + 3) + 1(x + 3)$

$= (x + 3)\ (2x + 1)$

Denominator $= 3x^2 + 8x - 3$

$= 3x^2 + 9x - x - 3$

$= 3x(x + 3) - 1(x + 3)$

$= (x + 3)\ (3x - 1)$

Now, $\lim_{x \to -3} \frac{2x^2 + 7x + 3}{3x^2 + 8x - 3} = \lim_{x \to -3} \frac{(x+3)(2x+1)}{(x+3)(3x-1)}$

$= \lim_{x \to -3} \frac{2x+1}{3x-1} \qquad (\because \ x + 3 \neq 0)$

$= \frac{2(-3)+1}{3(-3)-1}$

$= \frac{-6+1}{-9-1}$

$= \frac{-5}{-10}$

$= \frac{1}{2}$

Illustration 15 : Find the value of $\lim_{x \to -\frac{1}{2}} \frac{2x^2 - x - 1}{4x^2 + 8x + 3}$.

If we put $x = -\frac{1}{2}$ in the function $f(x)$, we get the value of the function as $\frac{0}{0}$, which is indeterminate.

Numerator $= 2x^2 - x - 1$

$= 2x^2 - 2x + x - 1$

$= 2x(x-1) + 1(x-1)$

$= (x-1)(2x+1)$

Denominator $= 4x^2 + 8x + 3$

$= 4x^2 + 6x + 2x + 3$

$= 2x(2x+3) + 1(2x+3)$

$= (2x+3)(2x+1)$

Now, $\lim_{x \to -\frac{1}{2}} \frac{2x^2 - x - 1}{4x^2 + 8x + 3} = \lim_{x \to -\frac{1}{2}} \frac{(x-1)(2x+1)}{(2x+3)(2x+1)}$

$= \lim_{x \to -\frac{1}{2}} \frac{x-1}{2x+3} \qquad (\because \ 2x + 1 \neq 0)$

$= \frac{-\frac{1}{2} - 1}{2\left(-\frac{1}{2}\right) + 3}$

$= \frac{-\frac{3}{2}}{-1+3}$

$= \frac{-\frac{3}{2}}{2}$

$= -\frac{3}{4}$

Illustration 16 : Find the value of $\lim\limits_{x \to 2} \left[\frac{1}{x-2} - \frac{2}{x^2-2x} \right]$.

$$\lim_{x \to 2} \left[\frac{1}{x-2} - \frac{2}{x^2-2x} \right] = \lim_{x \to 2} \left[\frac{1}{x-2} - \frac{2}{x(x-2)} \right]$$

$$= \lim_{x \to 2} \left[\frac{x-2}{x(x-2)} \right]$$

$$= \lim_{x \to 2} \frac{1}{x} \qquad (\because \ x-2 \neq 0)$$

$$= \frac{1}{2}$$

Illustration 17 : Find the value of $\lim\limits_{x \to 0} \frac{1}{x} \left[\frac{2x+3}{3x-5} + \frac{3}{5} \right]$.

$$\lim_{x \to 0} \frac{1}{x} \left[\frac{2x+3}{3x-5} + \frac{3}{5} \right] = \lim_{x \to 0} \frac{1}{x} \left[\frac{5(2x+3) + 3(3x-5)}{5(3x-5)} \right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\frac{10x+15+9x-15}{5(3x-5)} \right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\frac{19x}{5(3x-5)} \right]$$

$$= \lim_{x \to 0} \frac{19}{5(3x-5)} \qquad (\because \ x \neq 0)$$

$$= \frac{19}{5[3(0)-5]}$$

$$= \frac{19}{5(-5)}$$

$$= -\frac{19}{25}$$

Illustration 18 : If $f(x) = x^2 + x$ then find the value of $\lim\limits_{x \to 2} \frac{f(x) - f(2)}{x^2 - 4}$.

Here, $f(x) = x^2 + x$

$$\therefore \ f(2) = (2)^2 + 2$$

$$= 4 + 2$$

$$= 6$$

Now, $\lim\limits_{x\to 2} \dfrac{f(x) - f(2)}{x^2 - 4} = \lim\limits_{x\to 2} \dfrac{\left(x^2 + x\right) - 6}{x^2 - 4}$

Numerator $= x^2 + x - 6$

$= x^2 + 3x - 2x - 6$

$= x(x + 3) - 2(x + 3)$

$= (x + 3)(x - 2)$

Denominator $= x^2 - 4$

$= (x + 2)(x - 2)$

So, $\lim\limits_{x\to 2} \dfrac{x^2 + x - 6}{x^2 - 4} = \lim\limits_{x\to 2} \dfrac{(x + 3)(x - 2)}{(x + 2)(x - 2)}$

$= \lim\limits_{x\to 2} \dfrac{x + 3}{x + 2} \quad (\because \; x - 2 \neq 0)$

$= \dfrac{2 + 3}{2 + 2}$

$= \dfrac{5}{4}$

Illustration 19 : If $f(x) = x^3$ then find the value of $\lim\limits_{h\to 0} \dfrac{f(3 + h) - f(3 - h)}{2h}$.

Here, $f(x) = x^3$

$\therefore \; f(3 + h) = (3 + h)^3$

$= 27 + 27h + 9h^2 + h^3$

and

$f(3 - h) = (3 - h)^3$

$= 27 - 27h + 9h^2 - h^3$

Now, $\lim\limits_{h\to 0} \dfrac{f(3 + h) - f(3 - h)}{2h} = \lim\limits_{h\to 0} \dfrac{\left(27 + 27h + 9h^2 + h^3\right) - \left(27 - 27h + 9h^2 - h^3\right)}{2h}$

$= \lim\limits_{h\to 0} \dfrac{27 + 27h + 9h^2 + h^3 - 27 + 27h - 9h^2 + h^3}{2h}$

$$= \lim_{h \to 0} \frac{54h + 2h^3}{2h}$$

$$= \lim_{h \to 0} \frac{h\left(54 + 2h^2\right)}{2h}$$

$$= \lim_{h \to 0} \frac{54 + 2h^2}{2} \qquad \left(\because \quad h \neq 0\right)$$

$$= \frac{54 + 2(0)^2}{2}$$

$$= \frac{54}{2}$$

$$= 27$$

Illustration 20 : Find the value of $\lim\limits_{x \to 0} \dfrac{\sqrt{3+x} - \sqrt{3}}{x}$.

$$\lim_{x \to 0} \frac{\sqrt{3+x} - \sqrt{3}}{x}$$

(multiplying numerator and denominator by $\sqrt{3+x} + \sqrt{3}$)

$$= \lim_{x \to 0} \frac{\sqrt{3+x} - \sqrt{3}}{x} \times \frac{\sqrt{3+x} + \sqrt{3}}{\sqrt{3+x} + \sqrt{3}}$$

$$= \lim_{x \to 0} \frac{\left(\sqrt{3+x}\right)^2 - \left(\sqrt{3}\right)^2}{x\left(\sqrt{3+x} + \sqrt{3}\right)}$$

$$= \lim_{x \to 0} \frac{3 + x - 3}{x\left(\sqrt{3+x} + \sqrt{3}\right)}$$

$$= \lim_{x \to 0} \frac{x}{x\left(\sqrt{3+x} + \sqrt{3}\right)}$$

$$= \lim_{x \to 0} \frac{1}{\left(\sqrt{3+x} + \sqrt{3}\right)} \qquad \left(\because \quad x \neq 0\right)$$

$$= \frac{1}{\sqrt{3+0} + \sqrt{3}}$$

$$= \frac{1}{\sqrt{3} + \sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}$$

Illustration 21 : Find the value of $\lim\limits_{x \to 2} \dfrac{\sqrt{x+7} - 3}{x - 2}$.

$$\lim_{x \to 2} \frac{\sqrt{x+7} - 3}{x - 2}$$

(multiplying numerator and denominator by $\sqrt{x+7} + 3$)

$$= \lim_{x \to 2} \frac{\sqrt{x+7} - 3}{x - 2} \times \frac{\sqrt{x+7} + 3}{\sqrt{x+7} + 3}$$

$$= \lim_{x \to 2} \frac{\left(\sqrt{x+7}\right)^2 - (3)^2}{(x-2)\left(\sqrt{x+7} + 3\right)}$$

$$= \lim_{x \to 2} \frac{x + 7 - 9}{(x-2)\left(\sqrt{x+7} + 3\right)}$$

$$= \lim_{x \to 2} \frac{x - 2}{(x-2)\left(\sqrt{x+7} + 3\right)}$$

$$= \lim_{x \to 2} \frac{1}{\sqrt{x+7} + 3} \qquad (\because \ x - 2 \neq 0)$$

$$= \frac{1}{\sqrt{2+7} + 3}$$

$$= \frac{1}{\sqrt{9} + 3}$$

$$= \frac{1}{3 + 3}$$

$$= \frac{1}{6}$$

Illustration 22 : Find the value of $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ where $f(x) = \sqrt{x}$, $x > 0$.

$$f(x) = \sqrt{x}$$

$$\therefore \ f(x+h) = \sqrt{x+h}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

(multiplying numerator and denominator by $\sqrt{x+h} + \sqrt{x}$)

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \qquad (\because \ h \neq 0)$$

$$= \frac{1}{\sqrt{x+0}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Illustration 23 : Find the value of $\lim_{x\to 2} \frac{x^3-8}{\sqrt{x}-\sqrt{2}}$.

$$\lim_{x\to 2} \frac{x^3-8}{\sqrt{x}-\sqrt{2}}$$

(multiplying numerator and denominator by $\sqrt{x}+\sqrt{2}$)

$$= \lim_{x\to 2} \frac{x^3-8}{\sqrt{x}-\sqrt{2}} \times \frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}$$

$$= \lim_{x\to 2} \frac{(x-2)\left(x^2+2x+4\right)\left(\sqrt{x}+\sqrt{2}\right)}{\left(\sqrt{x}\right)^2-\left(\sqrt{2}\right)^2}$$

$$= \lim_{x\to 2} \frac{(x-2)\left(x^2+2x+4\right)\left(\sqrt{x}+\sqrt{2}\right)}{(x-2)}$$

$$= \lim_{x\to 2} \left(x^2+2x+4\right)\left(\sqrt{x}+\sqrt{2}\right) \qquad (\because \ x-2 \neq 0)$$

$$= \left[(2)^2+2(2)+4\right]\left[\sqrt{2}+\sqrt{2}\right]$$

$$= (4+4+4)\left(2\sqrt{2}\right)$$

$$= 12\left(2\sqrt{2}\right)$$

$$= 24\sqrt{2}$$

Illustration 24 : Find the value of $\lim\limits_{x\to 1} \dfrac{\sqrt{x+3}-2}{\sqrt{x+8}-3}$.

$$\lim_{x\to 1} \frac{\sqrt{x+3}-2}{\sqrt{x+8}-3}$$

(multiplying numerator and denominator by $\sqrt{x+3}+2$ and $\sqrt{x+8}+3$)

$$= \lim_{x\to 1} \frac{\sqrt{x+3}-2}{\sqrt{x+8}-3} \times \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} \times \frac{\sqrt{x+8}+3}{\sqrt{x+8}+3}$$

$$= \lim_{x\to 1} \frac{\left(\sqrt{x+3}\right)^2 - (2)^2}{\left(\sqrt{x+8}\right)^2 - (3)^2} \times \frac{\sqrt{x+8}+3}{\sqrt{x+3}+2}$$

$$= \lim_{x\to 1} \frac{(x+3-4) \times \left(\sqrt{x+8}+3\right)}{(x+8-9) \times \left(\sqrt{x+3}+2\right)}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{x+8}+3\right)}{(x-1)\left(\sqrt{x+3}+2\right)}$$

$$= \lim_{x\to 1} \frac{\sqrt{x+8}+3}{\sqrt{x+3}+2} \qquad (\because \; x-1 \neq 0)$$

$$= \frac{\sqrt{1+8}+3}{\sqrt{1+3}+2}$$

$$= \frac{\sqrt{9}+3}{\sqrt{4}+2}$$

$$= \frac{3+3}{2+2}$$

$$= \frac{6}{4}$$

$$= \frac{3}{2}$$

Illustration 25 : Find the value of $\lim\limits_{x\to 2} \dfrac{x^5-32}{x-2}$.

$$\lim_{x\to 2} \frac{x^5-32}{x-2} = \lim_{x\to 2} \frac{x^5-2^5}{x-2}$$

$$= 5(2)^{5-1} \qquad \left[\because \; \lim_{x\to a} \frac{x^n-a^n}{x-a} = n a^{n-1}\right]$$

$$= 5(2)^4$$

$$= 5(16)$$

$$= 80$$

Illustration 26 : Find the value of $\lim\limits_{x \to 3} \dfrac{x^5 - 243}{x^3 - 27}$.

$$\lim_{x \to 3} \frac{x^5 - 243}{x^3 - 27} = \lim_{x \to 3} \frac{x^5 - 3^5}{x^3 - 3^3}$$

(multiplying numerator and denominator by $(x-3)$)

$$= \lim_{x \to 3} \frac{x^5 - 3^5}{x - 3} \times \frac{x - 3}{x^3 - 3^3}$$

$$= \lim_{x \to 3} \left[\frac{x^5 - 3^5}{x - 3} \div \frac{x^3 - 3^3}{x - 3} \right]$$

$$= \frac{5(3)^{5-1}}{3(3)^{3-1}} \quad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1} \right]$$

$$= \frac{5(3)^4}{3(3)^2}$$

$$= \frac{5 \times 81}{3 \times 9}$$

$$= 15$$

Illustration 27 : Find the value of $\lim\limits_{x \to -2} \dfrac{x^7 + 128}{x + 2}$.

$$\lim_{x \to -2} \frac{x^7 + 128}{x + 2} = \lim_{x \to -2} \frac{x^7 - (-2)^7}{x - (-2)}$$

$$= 7(-2)^{7-1} \quad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1} \right]$$

$$= 7(-2)^6$$

$$= 7(64)$$

$$= 448$$

Illustration 28 : Find the value of $\lim\limits_{h \to 0} \dfrac{(x + h)^5 - x^5}{h}$.

$$\lim_{h \to 0} \frac{(x + h)^5 - x^5}{h}$$

(Taking $x + h = t$, when $h \to 0$, $t \to x$)

$$= \lim_{t \to x} \frac{t^5 - x^5}{t - x} \quad (\because x + h = t)$$

$$= 5(x)^{5-1} \quad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1} \right]$$

$$= 5x^4$$

Illustration 29 : Find the value of $\lim\limits_{x \to 0} \dfrac{\sqrt[n]{x+1} - 1}{x}$.

$$\lim_{x \to 0} \frac{\sqrt[n]{x+1} - 1}{x} = \lim_{x \to 0} \frac{(x+1)^{\frac{1}{n}} - 1^{\frac{1}{n}}}{x}$$

(Taking $x + 1 = t$, when $x \to 0$, $t \to 1$)

$$= \lim_{t \to 1} \frac{t^{\frac{1}{n}} - 1^{\frac{1}{n}}}{t - 1} \quad (\because x + 1 = t \quad \therefore x = t - 1)$$

$$= \frac{1}{n}(1)^{\frac{1}{n} - 1} \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}\right]$$

$$= \frac{1}{n} \times 1$$

$$= \frac{1}{n}$$

Summary

- **Neighbourhood :** Let $a \in R$. Any open interval containing 'a' is called a neighbourhood of 'a'.
- δ **neighbourhood of a :** If $a \in R$ and δ is a non-negative real number then the open interval $(a - \delta, a + \delta)$ is called δ neighbourhood of a .
- **Meaning of $x \to a$:** If the value of a variable x is brought very close to a number 'a' by increasing or decreasing then it can be said that x tends to a. It is symbolically denoted by $x \to a$.
- **Meaning of $x \to 0$:** If by decreasing the positive values of a variable x or by increasing negative values of a variable x is brought very close to '0' then it can be said that x tends to 0. It is symbolically denoted by $x \to 0$.
- **Limit of a function**

  The function $f(x)$ has limit l as x tends to 'a' if for each given predetermined $\varepsilon > 0$, however small, we can find a positive number δ such that $|f(x) - l| < \varepsilon$ for all x such that $|x - a| < \delta$.

List of Formulae :

- $\lim_{x \to a} [f(x) \pm g(x)] = l \pm m$
- $\lim_{x \to a} [f(x) \times g(x)] = l \times m$
- $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{l}{m}, \quad m \neq 0$
- $\lim_{x \to a} k\, f(x) = kl$, k is constant.
- If $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ then

  $\lim_{x \to b} f(x) = a_0 + a_1b + a_2b^2 + ... + a_nb^n$
- $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}, \quad n \in Q$

EXERCISE 4

Section A

Choose the correct option for the following multiple choice questions :

1. What is the modulus form of 0.3 neighbourhood of 3 ?

(a) $|x - 0.3| < 3$ (b) $|x - 3| < 0.3$ (c) $|x + 3| < 0.3$ (d) $|x - 3| > 0.3$

2. What is the interval form of 0.02 neighbourhood of –2 ?

(a) $(1.98, 2.02)$ (b) $(-1.98, 2.02)$ (c) $(-2.02, -1.98)$ (d) $(-2.02, 1.98)$

3. What is the interval form of $|x - 5| < 0.25$?

(a) $(4.75, 5.25)$ (b) $(-4.75, +5.25)$ (c) $(-5.25, -4.75)$ (d) $(-5.25, 4.75)$

4. What is the interval form of $|2x + 1| < \frac{1}{5}$?

(a) $\left(-\frac{6}{5}, -\frac{4}{5}\right)$ (b) $\left(-\frac{6}{10}, -\frac{4}{10}\right)$ (c) $\left(\frac{4}{10}, \frac{6}{10}\right)$ (d) $\left(-\frac{6}{10}, \frac{4}{10}\right)$

5. What is the modulus form of $N(5, 0.02)$?

(a) $|x + 5| < 0.02$ (b) $|x - 0.02| < 5$ (c) $|x - 5| > 0.02$ (d) $|x - 5| < 0.02$

6. If modulus form of $N(a, 0.07)$ is $|x - 10| < k$ then what will be the value of k?

(a) a (b) 0.7 (c) 0.07 (d) 9.93

7. What is the value of $\lim_{x \to 3} 3x - 1$?

(a) 9 (b) 10 (c) $\frac{4}{3}$ (d) 8

8. What is the value of $\lim_{x \to 4} \sqrt{4x+9}$?

(a) 5 (b) 25 (c) $\frac{7}{4}$ (d) 7

9. What is the value of $\lim_{x \to -2} 10$?

(a) 10 (b) –2 (c) 8 (d) Indeterminate

10. What is the value of $\lim_{x \to 3} \frac{x^4 - 81}{x - 3}$?

(a) 192 (b) 324 (c) 36 (d) 108

11. What is the value of $\lim_{x \to -1} \frac{x^5 + 1}{x + 1}$?

(a) –5 (b) 5 (c) 4 (d) –4

12. If $y = 10 - 3x$ and $x \to -3$ then y tends to which value ?

(a) 1 (b) 9 (c) 19 (d) 7

Section B

Answer the following questions in one sentence :

1. Express 0.09 neighbourhood of 0 in interval form.
2. Express 0.001 neighbourhood of –5 in modulus form.
3. Express $|x - 10| < \frac{1}{10}$ in neighbourhood form.
4. Express $|2x| < \frac{1}{2}$ in interval form.
5. Express $N(50,\ 0.8)$ in modulus form.
6. If $N(a,\ 0.2) = |x - 7| < b$ then find the value of a.
7. If $|x + 4| < 0.04 = (k,\ -3.96)$ then find the value of k.
8. Find the value of $\lim_{x \to 5} (3x + 5)$.
9. Find the value of $\lim_{x \to -3} \sqrt[3]{2 - 2x}$.
10. Find the value of $\lim_{x \to 0} \left(\frac{3x^2 - 4x + 10}{2x + 5}\right)$.
11. Find the value of $\lim_{x \to 2} \frac{x^5 - 32}{x - 2}$.
12. Find the value of $\lim_{x \to -a} \frac{x^m + a^m}{x + a}$ where m is an odd number.

13. If $\lim_{x\to -1} 4x + k = 6$ then find the value of k.

14. If $\lim_{x\to 3} \frac{2}{3x+k} = \frac{1}{7}$ then find the value of k.

Section C

Answer the following questions :

1. Define an open interval.
2. Define the δ neighbourhood of a.
3. Define the punctured δ neighbourhood of a.
4. Express the interval form $(-0.5, 0.5)$ in modulus form.
5. Express the interval form $(-8.75, -7.25)$ in neighbourhood form.
6. If $N(k_1, 0.5) = (19.5, k_2)$ then find the value of k_1 and k_2.
7. Express $|3x+1| < 2$ in neighbourhood and interval form.
8. If $|x - A_1| < 0.09 = (A_2, 4.09)$ then find the value of A_1 and A_2.
9. Explain the meaning of $x \to a$.
10. Explain the meaning of $x \to 0$.
11. Define limit of a function.
12. State multiplication working rule of limit.
13. State division working rule of limit.
14. State the standard form of limit of a polynomial.

Section D

Find the values of the following :

1. $\lim_{x\to 1} \frac{3x^2 - 4x + 1}{x - 1}$

2. $\lim_{x\to 3} \frac{x - 3}{2x^2 - 3x - 9}$

3. $\lim_{x\to -1} \frac{3x^2 - 2x - 5}{x + 1}$

4. $\lim_{x\to 1} \frac{x^2 + 2x - 3}{x^2 - 1}$

5. $\lim_{x\to \frac{1}{2}} \frac{2x^2 + 5x - 3}{4x^2 - 1}$

6. $\lim_{x\to -3} \frac{2x^2 + 9x + 9}{2x^2 + 7x + 3}$

7. $\lim_{x\to -\frac{1}{2}} \frac{2x^2 + 3x + 1}{2x^2 - x - 1}$

8. $\lim_{x\to -2} \frac{9x^2 + 5x - 26}{5x^2 + 17x + 14}$

9. $\lim_{x\to 0} \frac{1}{x}\left[\frac{5x+14}{3x+7} - 2\right]$

10. $\lim_{x\to 2} \left[\frac{2}{x-2} - \frac{4}{x^2 - 2x}\right]$

11. $\lim_{x\to 0} 1 + \frac{2}{3+\frac{4}{x}}$

12. $\lim_{x\to -p} \frac{x^4 - p^4}{x^3 + p^3}$

13. $\lim_{x\to 3} \frac{x^6 - 729}{x^4 - 81}$

14. $\lim_{x\to -2} \frac{x^{10} - 1024}{x^5 + 32}$

15. $\lim_{x\to -1} \frac{x^{2017} + 1}{x^{2018} - 1}$

16. $\lim_{x\to 1} \frac{x^{\frac{7}{2}} - 1}{x^{\frac{3}{2}} - 1}$

17. $\lim_{x\to 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$

Section E

I. Answer the following :

1. If $y = 5x + 7$ then using tabular method, prove that when $x \to 2$, $y \to 17$.

2. If $y = \frac{3x^2 + 16x + 16}{x + 4}$ then using tabular method, prove that when $x \to -4$, $y \to -8$.

3. Using tabular method, prove that $\lim_{x\to -1} \frac{3}{x+1}$ does not exist.

II. Find the values of the following using tabular method :

1. $\lim_{x\to 5} \frac{x^2 - 3x - 10}{x - 5}$

2. $\lim_{x\to 1} \frac{2x^2 + 3x - 5}{x - 1}$

3. $\lim_{x\to -1} \frac{4x^2 + 5x + 1}{x + 1}$

4. $\lim_{x\to 0} 3x - 1$

III. Find the values of the following :

1. $\lim_{h\to 0} \frac{(x+h)^7 - x^7}{h}$

2. $\lim_{x\to 0} \frac{\sqrt[10]{1+x} - 1}{x}$

3. $\lim_{x\to 0} \frac{(1+x)^n - 1}{x}$

4. $\lim_{x\to \frac{1}{2}} \frac{f(x) - f\left(\frac{1}{2}\right)}{2x - 1}$ where $f(x) = x^2 + x - 1$

5. $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ where $f(x) = x^3$

6. $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ where $f(x) = x^7$

7. $\lim_{x\to 2} \frac{f(x) - f(2)}{x - 2}$ where $f(x) = \sqrt{x+7}$

8. $\lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$ where $f(x) = 2x^2 + 3$

9. $\lim_{x\to 0} \frac{f(2+x) - f(2-x)}{2x}$ where $f(x) = x^2$

10. $\lim_{x\to 2} \frac{f(x) - f(2)}{x - 2}$ where $f(x) = x^2 + x$

Srinivasa Ramanujan
(1887 - 1920)

Srinivasa Ramanujan was one of the greatest mathematical geniuses of India. He made substantial contributions to the analytical theory of numbers and worked on elliptic functions, continued fractions and infinite series. Ramanujan independently discovered results of Gauss, Kummer and others on hyper geometric series. Ramanujan initially developed his own mathematical research in isolation; it was quickly recognized by Indian mathematicians. When his skills became obvious and known to the wider mathematical community, centered in Europe at the time, he began a famous partnership with the English mathematician G. H. Hardy, who realized that Ramanujan had rediscovered previously known theorems in addition to producing new ones. On 18th February 1918, Ramanujan was elected as fellow of the Cambridge Philosophical Society. On the 125th anniversary of his birth, India declared the birthday of Ramanujan, December 22nd as 'National Mathematics Day and also declared that the year 2012 would be celebrated as the National Year of Mathematics.

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