Exponents

Exponential Form of Numbers

Many a times, we come across very large numbers, especially in Science. For example, the speed of light in air is approximately 300000000 m/s. As we advance to higher classes, we will also be required to solve questions, in which we will have to perform calculations involving very large numbers. Now, the speed of light has digit 3 followed by eight 0s. If we mistakenly add or remove even one 0 from this, then our calculations will be wrong.

For this reason, large numbers are often written in exponential form, which is less confusing.

To understand the concept of exponential form, look at the following video.

We may also come across some cases where the base is a negative number. For example, let us consider $(-2)^5$. Is its value same as that of 2^5?

$(-2)^5 = (-2) \times (-2) \times (-2) \times (-2) \times (-2) = -32$

$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$

Thus, the values of $(-2)^5$ and 2^5 are not equal.

Now, let us look at the values of $(-2)^6$ and 2^6.

$(-2)^6 = (-2) \times (-2) \times (-2) \times (-2) \times (-2) \times (-2) = 64$

$2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$

In this case, the values of $(-2)^6$ and 2^6 are equal. Why is this so?

If the exponent of a negative base is odd, then the value of the exponential form is negative. However, if the exponent of a negative base is even, then the value of the exponential form is positive.

Let us now find the values of $(-1)^5$ and $(-1)^6$.

$(-1)^5 = (-1) \times (-1) \times (-1) \times (-1) \times (-1) = -1$

$(-1)^6 = (-1) \times (-1) \times (-1) \times (-1) \times (-1) \times (-1) = 1$

Thus, we can conclude that

$(-1)^{\text{odd number}} = -1$

$(-1)^{\text{even number}} = 1$

We already know how to prime factorize numbers. For example, the prime factorization of 1800 can be done as

1800 = 2 × 2 × 2 × 3 × 3 × 5 × 5

Now, let us try and express it using exponents.

We can break this expression into smaller groups as

$$1800 = \underbrace{2\times2\times2}_{\text{Product of three 2s}} \times \underbrace{3\times3}_{\text{Product of two 3s}} \times \underbrace{5\times5}_{\text{Product of two 5s}}$$

Now, $2 \times 2 \times 2 = 2^3$, $3 \times 3 = 3^2$, and $5 \times 5 = 5^2$

Thus, we can restate the prime factorization of 1800 as

1800 = 2 × 2 × 2 × 3 × 3 × 5 × 5

$= 2^3 \times 3^2 \times 5^2$

Now, if we have to write a negative number, say −1800, in exponential form then we proceed as follows:

−1800 = (−1) × (1800)

−1800 = (−1) × 2 × 2 × 2 × 3 × 3 × 5 × 5

$-1800 = (-1) \times 2^3 \times 3^2 \times 5^2$

Similarly, the prime factorizations of the numbers 1000 and −432 can be done as follows:

$1000 = 2^3 \times 5^3$

$-432 = (-1) \times 2^4 \times 3^3$

Let us look at some examples:

Number	Expanded form	Exponential form	Base and exponent
10000	10 × 10 × 10 × 10	10^4	base 10, exponent 4
$\frac{1}{243}$	$\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}$	$\left(\frac{1}{3}\right)^5$	base $\frac{1}{3}$, exponent 5
64	2 × 2 × 2 × 2 × 2 × 2	2^6	base 2, exponent 6
64	(–2) × (–2) × (–2) × (–2) × (–2) × (–2)	$(-2)^6$	base –2, exponent 6
–32	(–2) × (–2) × (–2) × (–2) × (–2)	$(-2)^5$	base –2, exponent 5

Let us solve some more examples to understand the concept better.

Example 1:
Write the following in exponential form.

a. Minus nine to the power of six

b. One fourth to the power of five

c. Three square to the power of five

Solution:

a. Minus nine to the power of six = $(-9)^6$

b. One fourth to the power of five = $\left(\frac{1}{4}\right)^5$

c. Three square to the power of five = $(3^2)^5$

Example 2:
Write the base and the exponent for the following.

a. $\left(\frac{1}{3}\right)^2$
b. $(-2.5)^5$

Solution:

a. $\left(\frac{1}{3}\right)^2$

Here, base = $\frac{1}{3}$, exponent = 2

b. $(–2.5)^5$

Here, base = –2.5, exponent = 5

Example 3:
Expand the following expressions.

a. 5^4
b. $(3^2)^3$
c. $(−2)^2$

Solution:
a. $5^4 = 5 \times 5 \times 5 \times 5$

b. $(3^2)^3 = 3^2 \times 3^2 \times 3^2$

c. $(−2)^2 = (−2) \times (−2)$

Example 4:
Write the exponents for the base given.

a. −125 with (–5) as base
b. 16 with (–2) as base

Solution:
a.
$−125 = (−5) \times (−5) \times (−5) = (−5)^3$
Thus, The exponent is $(−5)^3$.

b.
$16 = (–2) \times (–2) \times (–2) \times (–2) = (–2)^4$
Thus, the exponent is $(–2)^4$.

Example 5:

If 400 can be prime factorized as $2^x \times 5^2$, then what is the value of *x*?

Solution:

Let us first perform the prime factorization of 400.

2	400
2	200
2	100
2	50
5	25
5	5
	1

∴ $400 = 2 \times 2 \times 2 \times 2 \times 5 \times 5 = 2^4 \times 5^2$

Now, comparing $2^4 \times 5^2$ with $2^x \times 5^2$, we observe that $2^4 = 2^x$

Thus, the value of *x* is 4.

Example 6:

What is the value of the expression $2^2 \times (-3)^3 \times (-5)^2$?

Solution:

$2^2 \times (-3)^3 \times (-5)^2 = (2 \times 2) \times [(-3) \times (-3) \times (-3)] \times [(-5) \times (-5)]$

$= 4 \times (-27) \times 25$

$= -2700$

Example 7:

How can the number 3960 be written as a product of the powers of its prime factors?

Solution:

$3960 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 11 = 2^3 \times 3^2 \times 5 \times 11$

Example 8:

Express 500 using base 2 and exponents.

Solution:

$$\begin{aligned} 500 &= 256 + 128 + 64 + 32 + 16 + 4 \\ &= 2^8 + 2^7 + 2^6 + 2^5 + 2^4 + 2^2 \\ &= 1 \cdot 2^8 + 1 \cdot 2^7 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 \end{aligned}$$

Comparing and Ordering Numbers with Exponents

We know how to compare positive and negative integers. For example, among the numbers 32, 64, 27, and 25, the smallest number is 25, while the greatest number is 64.

Similarly, among the numbers −128, 16, −81, and 125, the smallest number is −128, while the greatest number is 125.

Let us now try and extend this knowledge to compare numbers written in exponential forms.

What if we were asked to write 2^5, 2^6, 3^3, and 5^2 in ascending order? We first have to find the number that represents each exponential form, and then compare these numbers. Let us see how.

$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$

$2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$

$3^3 = 3 \times 3 \times 3 = 27$

$5^2 = 5 \times 5 = 25$

Now, we can easily compare the numbers 32, 64, 27, and 25, just as we did in the beginning.

We know that $25 < 27 < 32 < 64$

$\therefore 5^2 < 3^3 < 2^5 < 2^6$

Let us solve some more examples to understand the concept better.

Let us now find the greatest and the least numbers among $(-2)^7$, 2^4, $(-3)^4$, and 5^3.

$(-2)^7 = (-2) \times (-2) \times (-2) \times (-2) \times (-2) \times (-2) \times (-2) = -128$

$2^4 = 2 \times 2 \times 2 \times 2 = 16$

$(-3)^4 = (-3) \times (-3) \times (-3) \times (-3) = 81$

$5^3 = 5 \times 5 \times 5 = 125$

As seen here, the greatest number is 125, while the smallest number is −128.

Thus, the greatest and the least numbers are 5^3 and $(-2)^7$ respectively.

We can also compare numbers given in exponential forms without simplifying them. To understand this concept, look at this video.

Let us solve some more examples to understand the concept better.

Example 1:

Arrange 5^4, $(-4)^6$, and 6^3 in decreasing order.

Solution:

$5^4 = 5 \times 5 \times 5 \times 5 = 625$

$(-4)^6 = (-4) \times (-4) \times (-4) \times (-4) \times (-4) \times (-4) = 4096$

$6^3 = 6 \times 6 \times 6 = 216$

Now, $4096 > 625 > 216$

$\therefore (-4)^6 > 5^4 > 6^3$

Thus, the given numbers can be arranged in descending order as $(-4)^6$, 5^4, 6^3.

Example 2:

Arrange $(-5)^4$, $(-6)^3$, 6^3, and 5^5 in increasing order.

Solution:

$(-5)^4 = (-5) \times (-5) \times (-5) \times (-5) = 625$

$(-6)^3 = (-6) \times (-6) \times (-6) = -216$

$6^3 = 6 \times 6 \times 6 = 216$

$5^5 = 5 \times 5 \times 5 \times 5 \times 5 = 3125$

Now, $-216 < 216 < 625 < 3125$

$\therefore (-6)^3 < 6^3 < (-5)^4 < 5^5$

Thus, the given numbers can be arranged in ascending order as $(-6)^3$, 6^3, $(-5)^4$, 5^5.

Example 3:

Find the greater number out of 1.5 × 10^6 and 3.9 × 10^5.

Solution:

1.5 × 10^6 = 1.5 × 1000000 = 1500000

3.9 × 10^5 = 3.9 × 100000 = 390000

Since 1500000 > 390000,

1.5 × 10^6 > 3.9 × 10^5

Laws of Exponents of Integers

We know that for a non-zero integer *a*, $a^m \times a^n = a^{m+n}$, where *m* and *n* are natural numbers. Does this law also hold for negative exponents? Let us verify this by taking *a* = 10, *m* = −3, and *n* = −2.

$$\begin{aligned}
10^{-2} \times 10^{-3} &= \frac{1}{10^2} \times \frac{1}{10^3} && \left(a^{-m} = \frac{1}{a^m}\right) \\
&= \frac{1}{10^2 \times 10^3} \\
&= \frac{1}{10^{2+3}} && \left(a^m \times a^n = a^{m+n}\right) \\
&= \frac{1}{10^5} \\
&= 10^{-5} && \left(a^{-m} = \frac{1}{a^m}\right) \\
&= 10^{(-2)+(-3)} && \left\{(-2)+(-3) = -5\right\}
\end{aligned}$$

As seen in the above example, for a non-zero integer *a*, the relation $a^m \times a^n = a^{m+n}$ holds true for negative exponents as well.

In fact, all the relations for positive exponents hold true for negative exponents as well. For non-zero integers *a* and *b*, and integers *m* and *n.* The laws of exponents (Note that these laws are valid for both positive and negative exponents) can be summarized as:

(i)	$a^m \times a^n = a^{m+n}$
(ii)	$a^m \div a^n = a^{m-n}$
(iii)	$a^{-m} = \frac{1}{a^m}$
(iv)	$(a^m)^n = a^{mn}$
(v)	$a^m \times b^m = (ab)^m$
(vi)	$\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$
(vii)	$a^0 = 1$

Let us now discuss some examples where we need to use these laws of exponents.

Example 1:

Find the value of $\left(\frac{1}{3}\right)^{-2}$ **.**

Solution:

$$\left(\frac{1}{3}\right)^{-2} = \frac{1^{-2}}{3^{-2}} \qquad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

$$= \frac{\left(\frac{1}{1^2}\right)}{\left(\frac{1}{3^2}\right)} \qquad \left(a^{-m} = \frac{1}{a^m}\right)$$

$$= 1 \times \frac{3^2}{1}$$

$$= 9$$

Example 2:

Simplify the expression $(7)^2 \div (7)^{11}$ using positive exponents.

Solution:

$(7)^2 \div (7)^{11} = 7^{2-11}$ ($a^m \div a^n = a^{m-n}$)

$= 7^{-9}$

$= \frac{1}{7^9}$ $\left(a^{-m} = \frac{1}{a^m}\right)$

$= \frac{1^9}{7^9}$ $\left(1^9 = 1\right)$

$= \left(\frac{1}{7}\right)^9$ $\left(\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m\right)$

Thus, the simplified form of the expression $(7)^2 \div (7)^{11}$ is $\left(\frac{1}{7}\right)^9$.

Example 3:

Find the value of the expression $\left(\frac{1}{6}\right)^{-2} + \left(\frac{1}{7}\right)^{-1} + \left(\frac{1}{11}\right)^{-1}$.

Solution:

$\left(\frac{1}{6}\right)^{-2} + \left(\frac{1}{7}\right)^{-1} + \left(\frac{1}{11}\right)^{-1}$

$= \frac{1^{-2}}{6^{-2}} + \frac{1^{-1}}{7^{-1}} + \frac{1^{-1}}{11^{-1}}$ $\left(\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}\right)$

$= \frac{6^2}{1^2} + \frac{7^1}{1^1} + \frac{11^1}{1^1}$ $\left(a^{-m} = \frac{1}{a^m}\right)$

$= 36 + 7 + 11$

$= 54$

Hence, the value of the expression $\left(\frac{1}{6}\right)^{-2} + \left(\frac{1}{7}\right)^{-1} + \left(\frac{1}{11}\right)^{-1}$ is 54.

Example 4:

Find the value of x for the equation $(-3)^{x} \div (-3)^{-2} = (-3)^5$.

Solution:

$(-3)^x \div (-3)^{-2} = (-3)^5$

$(-3)^{x-(-2)} = (-3)^5$ $\quad (a^m \div a^n = a^{m-n})$

$(-3)^{x+2} = (-3)^5$

Equating the powers of −3 on both sides:

$x + 2 = 5$

$x = 3$

Thus, the value of x for the equation $(-3)^x \div (-3)^{-2} = (-3)^5$ is 3.

Example 5:

Evaluate the following expressions.

(i) $\left[\dfrac{a^0 \div 1}{2}\right] \times \dfrac{4^2 \times 5}{2^3}$

(ii) $\dfrac{(x^2)^{-3} \times 125 \times 2^8}{2^{11} \times 5^3 \times x^{-7}}$

Solution:

(i)

$$\left[\frac{a^0 \div 1}{2}\right] \times \frac{4^2 \times 5}{2^3}$$

$$=\left(\frac{1\div 1}{2}\right)\times\frac{\left(2^2\right)^2\times 5}{2^3}\qquad\left[a^0=1 \text{ and } 4=2^2\right]$$

$$=\frac{1}{2}\times\frac{2^4\times 5}{2^3}\qquad\left[\left(a^m\right)^n=a^{mn}\right]$$

$$=\frac{2^4\times 5}{2^1\times 2^3}$$

$$=\frac{2^4\times 5}{2^{1+3}}$$

$$=\frac{2^4\times 5}{2^4}$$

$$=5$$

Thus, the value of the expression $\left[\frac{a^0\div 1}{2}\right]\times\frac{4^2\times 5}{2^3}$ is 5.

(ii)

$$\frac{\left(x^2\right)^{-3}\times 125\times 2^8}{2^{11}\times 5^3\times x^{-7}}$$

$$=\frac{(x)^{-6}\times(5)^3\times(2)^8}{(2)^{11}\times(5)^3\times(x)^{-7}}\qquad\left[\left(a^m\right)^n=a^{mn}\right]$$

$$=\frac{x^{-6}}{x^{-7}}\times\frac{(5)^3}{(5)^3}\times\frac{(2)^8}{2^{11}}$$

$$=x^{(-6)-(-7)}\times(5)^{3-3}\times 2^{8-11}\qquad\left[\frac{a^m}{a^n}=a^{m-n}\right]$$

$$=x^{-6+7}\times 5^0\times 2^{-3}$$

$$=x^1\times 1\times 2^{-3}\qquad\left[5^0=1\right]$$

$$=x\times\frac{1}{2^3}\qquad\left[a^{-m}=\frac{1}{a^m}\right]$$

$$=\frac{x}{8}$$

Thus, the value of the expression $\dfrac{\left(x^{2}\right)^{-3} \times 125 \times 2^{8}}{2^{11} \times 5^{3} \times x^{-7}}$ is $\dfrac{x}{8}$.