Fill Ups of Inverse Trigonometric Functions

Fill in the Blanks

Q. 1. Let a, b, c be positive real numbers. Let

$$\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}.$$

Then tan θ = _________

Ans. 0

Solution. Let a + b + c = u, then

$$\theta = \tan^{-1}\sqrt{\frac{au}{bc}} + \tan^{-1}\sqrt{\frac{bu}{ca}} + \tan^{-1}\sqrt{\frac{cu}{ab}}$$

$$\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ when } xy > 1$$

$$\sqrt{\frac{au}{bc}} \times \sqrt{\frac{bu}{ca}} = \frac{u}{c} = \frac{a+b+c}{c} > 1; a, b, c$$

being +ve real no's.

$$\therefore \quad \text{We get } \theta = \pi + \tan^{-1}\left[\frac{\sqrt{\frac{au}{bc}} + \sqrt{\frac{bu}{ca}}}{1 - \sqrt{\frac{au}{bc}}\sqrt{\frac{bu}{ca}}}\right] + \tan^{-1}\sqrt{\frac{cu}{ab}}$$

$$\theta = \pi + \tan^{-1}\left[\frac{\frac{a+b}{\sqrt{abc}}\sqrt{u}}{1-\frac{u}{c}}\right] + \tan^{-1}\sqrt{\frac{cu}{ab}}$$

$$\theta = \pi + \tan^{-1}\left[\frac{(u-c)\sqrt{u}}{\sqrt{abc}} \times \frac{c}{-(u-c)}\right] + \tan^{-1}\sqrt{\frac{cu}{ab}}$$

$$\theta = \pi - \tan^{-1}\sqrt{\frac{uc}{ab}} + \tan^{-1}\sqrt{\frac{cu}{ab}}$$

[Using $\tan^{-1}(-x) = -\tan^{-1} x$] $= \pi$

$\therefore \tan\theta = \tan\pi = 0$

Q. 2. The numerical value of tan $\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right\}$ is **equal to _________**

Ans. -7/17

Solution.

$$\tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right) = \tan\left[\tan^{-1}\left(\frac{2/5}{1-(1/5)^2}\right) - \tan^{-1}(1)\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{5}{12}\right) - \tan^{-1}(1)\right] = \tan\left[\tan^{-1}\left(\frac{5/12-1}{1+5/12}\right)\right]$$

$$= \tan(\tan^{-1}(-7/17)) = -7/17$$

Q. 3. The greater of the two angles A = 2 tan $^{-1}$ (2 √2 - 1) and B = 3 sin $^{-1}$ (1/3) + sin $^{-1}$ (3/5) is __________ .

Ans. A

Solution. We have

$$A = 2\tan^{-1}(2\sqrt{2}-1) = 2\tan^{-1}(2\times 1.414 - 1)$$

$$= 2\tan^{-1}(1.828) > 2\tan^{-1}\sqrt{3} = 2\pi/3$$

$\Rightarrow A > 2\pi/3$ ... (1)

Also $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5)$

$$= \sin^{-1}\left[3\times\frac{1}{3} - 4\times\frac{1}{27}\right] + \sin^{-1}(3/5)$$

$$= \sin^{-1}\frac{23}{27} + \sin^{-1}(0.6) = \sin^{-1}(0.852) + \sin^{-1}(0.6) < \sin^{-1}(\sqrt{3}/2) + \sin^{-1}(\sqrt{3}/2) = 2.\pi/3$$

$\Rightarrow B < 2\pi/3$ ....(2)

From (1) and (2) we conclude A > B.

Subjective Questions of Inverse Trigonometric

Subjective Questions

Q. 1. Find the value of : $\cos(^2\cos^{-1}x + \sin^{-1}x)$ at $x = 1/5$, where $0 \leq \cos^{-1} x \leq \pi$ and $-\pi/2 \leq \sin^{-1} x \leq \pi/2$.

Ans. $\frac{-2\sqrt{6}}{5}$

Solution. We have $\cos (2 \cos^{-1} x + \sin^{-1} x)$

$= \cos (\cos^{-1} x + \cos^{-1} x + \sin^{-1}x)$

$= \cos (\cos^{-1} x + \pi/2)$ {Using $\cos^{-1} x + \sin^{-1} x = \pi/2$}

$= -\sin (\cos^{-1} x)$

$$= -\sqrt{1-\cos^2(\cos^{-1}x)} = -\sqrt{1-[\cos(\cos^{-1}x)]^2}$$

$$= -\sqrt{1-x^2} = -\sqrt{1-1/25} \qquad [\text{for } x = 1/5]$$

$$= -\frac{\sqrt{24}}{5} = \frac{-2\sqrt{6}}{5}$$

Q. 2. Find all the solution of $4 \cos^2 x \sin x - 2 \sin^2 x = 3 \sin x$

Ans. $x = n\pi,\ n\pi + (-1)^n \frac{\pi}{10},\ n\pi + (-1)^n \left(\frac{-3\pi}{10}\right)$ where $n \in N$

Solution. Given eq. is,

$4 \cos^2 x \sin x - 2 \sin^2 x = 3 \sin x$

$\Rightarrow 4 \cos^2 x \sin x - 2 \sin^2 x - 3 \sin x = 0$

$\Rightarrow 4 (1 - \sin^2 x) \sin x - 2 \sin^2 x - 3 \sin x = 0$

$\Rightarrow \sin x [4 \sin^2 x + 2 \sin x - 1] = 0$

$\Rightarrow$ either $\sin x = 0$ or $4 \sin^2 x + 2 \sin x - 1 = 0$

If sin x = 0 ⇒ x = nπ

⇒ If 4 $\sin^2$ x + 2 sin x – 1 = 0 ⇒ $\sin x = \frac{-1 \pm \sqrt{5}}{4}$

$$\text{If } \sin x = \frac{-1 \pm \sqrt{5}}{4} = \sin 18° = \sin \frac{\pi}{10}$$

$$\text{then } x = nx + (-1)^4 \frac{\pi}{10}$$

$$\text{If} \sin x = -\left(\frac{\sqrt{5}+1}{4}\right) = \sin\ (-54°) = \sin \left(\frac{-3\pi}{10}\right)$$

$$\text{then } x = n\pi + (-1)^n \left(\frac{-3\pi}{10}\right)$$

$$\text{Hence, } x = n\pi,\ n\pi + (-1)^n \frac{\pi}{10} \text{ or } n\pi + (-1)^n \left(\frac{-3\pi}{10}\right)$$

Where n is some integer

Q. 3. Prove that cos $\tan^{-1}$ sin $\cot^{-1}$ x $x = \sqrt{\frac{x^2+1}{x^2+2}}$

Solution. To prove that cos $\tan^{-1}$ sin $\cot^{-1}$ $x = \sqrt{\frac{x^2+1}{x^2+2.}}$

L.H.S. = cos [$\tan^{-1}$ (sin ($\cot^{-1}$x))]

$$= \cos\ [\tan^{-1} (\sin (\sin^{-1} \frac{1}{\sqrt{1+x^2}}))] \text{ if } x > 0$$

$$\text{and } \cos\ [\tan^{-1} (\sin (\pi - \sin^{-1} \frac{1}{\sqrt{1+x^2}}))] \text{ if } x < 0$$

In each case,

$$= \cos\left[\tan^{-1} \frac{1}{\sqrt{1+x^2}}\right] = \cos\ [\cos^{-1} \sqrt{\frac{1+x^2}{2+x^2}}]$$

$$= \sqrt{\frac{x^2+1}{x^2+2}} = R.H.S. \qquad \textit{Hence Proved.}$$

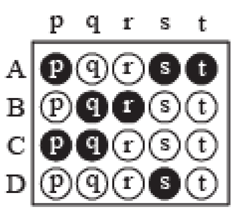
Match the following Question of Inverse Trigonometric Functions

DIRECTIONS (Q. 1 & 2): Each question contains statements given in two columns, which have to be matched.

The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II.

The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example:

If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.



Q. 1. Match the following

Column I	**Column II**
(A) $\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{1}{2i^2}\right) = t$, then tan t =	(p) 1
(B) Sides a, b, c of a triangle ABC are in AP and $\cos\theta_1 = \frac{a}{b+c}, \cos\theta_2 = \frac{b}{a+c}, \cos\theta_3 = \frac{c}{a+b}$, then $\tan^2\left(\frac{\theta_1}{2}\right) + \tan^2\left(\frac{\theta_3}{2}\right) =$	**(q)** $\frac{\sqrt{5}}{3}$
(C) A line is perpendicular to x + 2y + 2z = 0 and passes through (0, 1, 0). The perpendicular distance of this line from the origin is	**(r) 2/3**

Ans. (A)-(p), (B)-(r), (C)-(q)

Solution.

(A) $t = \sum_{i=1}^{\infty} \tan^{-1}\left(\frac{1}{2i^2}\right) = \sum_{i=1}^{\infty} \tan^{-1}\left[\frac{(2i+1)-(2i-1)}{1+4i^2-1}\right]$

$= \sum_{i=1}^{\infty} [\tan^{-1}(2i+1) - \tan^{-1}(2i-1)]$

t = $\tan^{-1} 3 - \tan^{-1} 1 + \tan^{-1}5 - \tan^{-1}3 + \ldots\ldots + \tan^{-1}(2n + 1) - \tan^{-1} (2n - 1) + \ldots\ldots \infty$

$\Rightarrow \quad t = \lim_{n\to\infty} [\tan^{-1}(2n+1) - \tan^{-1} 1]$

$= \lim_{n\to\infty} \tan^{-1}\left[\frac{2n}{1+(2n+1)}\right] = \lim_{n\to\infty} \tan^{-1}\left[\frac{1}{1+1/n}\right]$

$\Rightarrow t = \tan^{-1}(1) = \frac{\pi}{4} \Rightarrow \tan t = 1, \quad (A) \to (p)$

(B) $\therefore$ a, b, c are in AP $\Rightarrow$ 2b = a + c

$\cos\theta_1 = \frac{a}{b+c}$

$\Rightarrow \quad \frac{1-\tan^2\theta_1/2}{1+\tan^2\theta_1/2} = \frac{a}{b+c} \Rightarrow \tan^2\frac{\theta_1}{2} = \frac{b+c-a}{b+c+a}$

Similarly, $\tan^2\frac{\theta_3}{2} = \frac{a+b-c}{a+b+c}$

$\Rightarrow \tan^2\frac{\theta_1}{2} + \tan^2\frac{\theta_3}{2} = \frac{2b}{a+b+c} = \frac{2b}{3b} = \frac{2}{3}, \ (B) \to (r)$

(C) Equation of line through (0, 1, 0) and perpendicular to

$x + 2y + 2z = 0$ is $\frac{x}{1} = \frac{y-1}{2} = \frac{z}{2} = \lambda$

For some value of λ, the foot of perpendicular from origin to line is (λ, 2λ + 1, 2λ)

Dr 's of this ^ from origin are λ , 2λ + 1, 2λ

$\therefore 1.\lambda + 2(2\lambda+1) + 2.2\,\lambda = 0 \Rightarrow \lambda = -\frac{2}{9}$

∴ Foot of perpendicular is $\left(-\frac{2}{9}, \frac{5}{9}, -\frac{4}{9}\right)$

∴ Required distance

$$= \sqrt{\frac{4}{81} + \frac{25}{81} + \frac{16}{81}} = \sqrt{\frac{45}{81}} = \frac{\sqrt{5}}{3} \text{ (C)} \rightarrow \text{(q)}$$

Q. 2. Let (x, y) be such that $\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}$.

Match the statements in Column I with statements in Column II and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS.

Column I	Column II
(A) If a = 1 and b = 0, then (x, y)	**(p) lies on the circle $x2 + y2 = 1$**
(B) If a = 1 and b = 1, they (x, y)	**(q) lies on $(x2 – 1)(y2 – 1) = 0$**
(C) If a = 1 and b = 2, then (x, y)	**(r) lies on y = x**
(D) If a = 2 and b = 2, then (x, y)	**(s) lies on $(4x2 – 1) (y2 – 1) = 0$**

Ans. (A) → p, (B) → q, (C) → p , (D) → s

Solution.

$$\sin^{-1}(ax) + \cos^{-1} y + \cos^{-1}(bxy) = \frac{\pi}{2}$$

$$\Rightarrow \quad \cos^{-1} y + \cos^{-1}(bxy) = \frac{\pi}{2} - \sin^{-1}(ax) = \cos^{-1}(ax)$$

Let $\cos^{-1} y = \alpha$, $\cos^{-1}(bxy) = \beta$, $\cos^{-1}(ax) = \gamma$

⇒ y = cos α, bxy = cos β, αx = cos γ

∴ We get α + β = γ and cos β = bxy

⇒ cos (γ – α) = bxy

⇒ cos y cos α + sin γ sin α = bxy

$\Rightarrow$ axy + sin γ sin α = bxy $\Rightarrow$ (a – b) xy = – sin α sin γ

$\Rightarrow (a - b)^2 x^2y^2 = -\sin^2 \alpha \sin^2 \gamma$

$= (1 - \cos^2 \alpha)\ (1 - \cos^2 \gamma)$

$\Rightarrow (a - b)^2 x^2y^2 = (1 - a^2x^2)\ (1 - y^2)$ ....(1)

(A) For a = 1, b = 0, equation (1) reduces to

$x^2y^2 = (1 - x^2)\ (1 - y^2) \Rightarrow x^2 + y^2 = 1$

(B) For a = 1, b = 1 equation (1) becomes

$(1 - x^2)\ (1 - y^2) = 0 \Rightarrow (x^2 - 1)\ (y^2 - 1) = 0$

(C) For a = 1, b = 2 equation (1) reduces to

$x^2y^2 = (1 - x^2)\ (1 - y^2) \Rightarrow x^2 + y^2 = 1$

(D) For a = 2, b = 2 equation (1) reduces to

$0 = (1 - 4x^2)\ (1 - y^2) \Rightarrow (4x^2 - 1)\ (y^2 - 1) = 0$

DIRECTIONS (Q. 3) : Following question has matching lists. The codes for the lists have choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

Q. 3. Match List I with List II and select the correct answer using the code given below the lists:

List

I

List II

P. $\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1} y) + y\sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)}\right)^2 + y^4\right)^{1/2}$ takes value

1. $\frac{1}{2}\sqrt{\frac{5}{3}}$

Q4. If cosx + cos y + cos z = 0 = sinx + sin y + sin z then possible value of

cos $\frac{x-y}{2}$ is **2. √2**

R. If $\cos\left(\frac{\pi}{4}-x\right)$ **cos 2x + sinx sin 2 secx = cosx sin2x secx**

+ **3. 1/2**

$\cos\left(\frac{\pi}{4}+x\right)$ **cos 2x then possible value of secx is**

S. If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right)=\sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right), x\neq 0,$

then possible value of x is

Codes:

	P	Q	R	S
(a)	4	3	1	2
(b)	4	3	2	1
(c)	3	4	2	1
(d)	3	4	1	2

Ans. (b)

Solution.

$$(P)\quad \left[\frac{1}{y^2}\left(\frac{\cos(\tan^{-1}y)+y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y)+\tan(\sin^{-1}y)}\right)^2+y^4\right]^{\frac{1}{2}}$$

$$=\left[\frac{1}{y^2}\left(\frac{\cos\left(\cos^{-1}\frac{1}{\sqrt{1+y^2}}\right)+y\sin\left(\sin^{-1}\frac{y}{\sqrt{1+y^2}}\right)}{\cot\left(\cot^{-1}\frac{\sqrt{1-y^2}}{y}\right)+\tan\left(\tan^{-1}\frac{y}{\sqrt{1-y^2}}\right)}\right)^2+y^4\right]^{\frac{1}{2}}$$

$$=\left[\frac{1}{y^2}\left(\frac{\sqrt{1+y^2}}{\frac{1}{y(\sqrt{1-y^2})}}\right)^2+y^4\right]^{\frac{1}{2}}$$

$$=\left(1-y^4+y^4\right)^{\frac{1}{2}}=1 \qquad \therefore \quad (P)\rightarrow(4)$$

(Q) We have cos x + cos y = – cos z

sin x + sin y = – sin z

Squaring and adding we get

$(\cos x + \cos y)^2 + (\sin x + \sin y)^2 = \cos^2 z + \sin^2 z$

$\Rightarrow 2 + 2\cos(x - y) = 1$

$$\Rightarrow \quad 4\cos^2\frac{x-y}{2} = 1 \quad \text{or} \quad \cos\frac{x-y}{2} = \frac{+1}{2}$$

$$\therefore \quad Q \to (3)$$

(R) We have

$$\cos\left(\frac{\pi}{4} - x\right)\cos 2x + \sin x \ \sin 2x \ \sec x$$

$$= \cos x \ \sin 2x \ \sec x + \cos\left(\frac{\pi}{4} + x\right)\cos 2x$$

$$\Rightarrow \cos 2x\left[\cos\left(\frac{\pi}{4} - x\right) - \cos\left(\frac{\pi}{4} + x\right)\right]$$

= sin2x secx (cosx – sinx)

$$\Rightarrow 2\sin\frac{\pi}{4}\sin x\cos 2x = 2\sin x(\cos x - \sin x)$$

$$\Rightarrow 2\sin x\left[\frac{1}{\sqrt{2}}(\cos^2 x - \sin^2 x) - (\cos x - \sin x)\right] = 0$$

$\therefore$ (R) $\rightarrow$ (2)

$$\text{(S)} \quad \cos\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin\left(\tan^{-1} x\sqrt{6}\right)$$

$$\Rightarrow \frac{x}{\sqrt{1-x^2}} = \frac{x\sqrt{6}}{\sqrt{1+6x^2}} \Rightarrow x = \pm\frac{5}{2\sqrt{3}}$$

$$\therefore \ (S) \to (1)$$

Hence $(P) \to (4), (Q) \to (3), (R) \to (2), (S) \to (1)$