Chapter

5

Conic Section

Contents

Apollonius

The synthetic approach to the subject of geometry as given by Euclid and in Sulbasutras etc. was continued for some 1300 yrs. In the 200 B.C. Apollonius wrote a book called 'The conic' which was all about conic sections with many important discoveries that have remained unsurpassed for eighteen centuries.

Many important discoveries, both in mathematics and science, have been linked to the conic-sections.

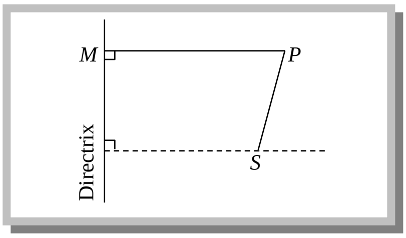
5.0 Conic Section : General

5.0.1. Introduction

The curves obtained by intersection of a plane and a double cone in different orientation are called conic section.

In other words "Graph of a quadratic equation (in two variables) is a "Conic section".

A conic section or conic is the locus of a point *P*, which moves in such a way that its distance from a fixed point *S* always bears a constant ratio to its distance from a fixed straight line, all being in the same plane.



$$\frac{SP}{PM} = \text{constant} = e \text{ (eccentricity)}$$

or $SP = e.\,PM$

5.0.2. Definitions of Various important Terms

(1) **Focus :** The fixed point is called the focus of the conic-section.

(2) **Directrix :** The fixed straight line is called the directrix of the conic section.

In general, every central conic has four foci, two of which are real and the other two are imaginary. Due to two real foci, every conic has two directrices corresponding to each real focus.

(3) **Eccentricity :** The constant ratio is called the eccentricity of the conic section and is denoted by *e*.

If $e = 1$, the conic is called **Parabola**.

If $e < 1$, the conic is called **Ellipse**.

If $e > 1$, the conic is called **Hyperbola**.

If $e = 0$, the conic is called **Circle**.

If $e = \infty$, the conic is called **Pair of the straight lines**.

(4) **Axis:** The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic section. A conic is always symmetric about its axis.

(5) **Vertex:** The points of intersection of the conic section and the axis are called vertices of conic section.

(6) **Centre:** The point which bisects every chord of the conic passing through it, is called the centre of conic.

(7) **Latus-rectum:** The latus-rectum of a conic is the chord passing through the focus and perpendicular to the axis.

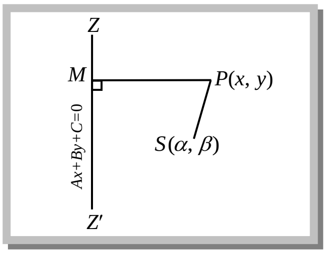
(8) **Double ordinate:** The double ordinate of a conic is a chord perpendicular to the axis.

(9) **Focal chord:** A chord passing through the focus of the conic is called a focal chord.

(10) **Focal distance:** The distance of any point on the conic from the focus is called the focal distance of the point.

5.0.3. General equation of a Conic section when its Focus, Directrix and Eccentricity are given

Let $S(\alpha, \beta)$ be the focus, $Ax + By + C = 0$ be the directrix and e be the eccentricity of a conic. Let $P(h, k)$ be any point on the conic. Let PM be the perpendicular from P, on the directrix. Then by definition



$$SP = ePM \Rightarrow SP^2 = e^2 PM^2$$

$$\Rightarrow (h-\alpha)^2 + (k-\beta)^2 = e^2 \left(\frac{Ah + Bk + C}{\sqrt{A^2 + B^2}} \right)^2$$

Thus the locus of (h, k) is $(x-\alpha)^2 + (y-\beta)^2 = e^2 \frac{(Ax + By + C)^2}{(A^2 + B^2)}$ this is the cartesian equation of the conic section which, when simplified, can be written in the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, which is general equation of second degree.

5.0.4. Recognisation of Conics

The equation of conics is represented by the general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad ......(i)$$

and discriminant of above equation is represented by Δ, where

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

Case I: When $\Delta = 0$

In this case equation (i) represents the degenerate conic whose nature is given in the following table.

S. No.	Condition	Nature of conic
1.	$\Delta = 0$ and $ab - h^2 = 0$`	A pair of coincident straight lines
2.	$\Delta = 0$ and $ab - h^2 < 0$	A pair of intersecting straight lines
3.	$\Delta = 0$ and $ab - h^2 > 0$	A point

Case II: When $\Delta \neq 0$

In this case equation (i) represents the non-degenerate conic whose nature is given in the following table.

S. No.	Condition	Nature of conic
1.	$\Delta \neq 0,\ h = 0,\ a = b$	A circle
2.	$\Delta \neq 0,\ ab - h^2 = 0$	A parabola
3.	$\Delta \neq 0,\ ab - h^2 > 0$	An ellipse
4.	$\Delta \neq 0,\ ab - h^2 < 0$	A hyperbola
5.	$\Delta \neq 0,\ ab - h^2 < 0$ and $a + b = 0$	A rectangular hyperbola

5.0.5. Method to find centre of a Conic

Let $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ be the given conic. Find $\frac{\partial S}{\partial x};\frac{\partial S}{\partial y}$

Solve $\frac{\partial S}{\partial x} = 0, \frac{\partial S}{\partial y} = 0$ for x, y we shall get the required centre (x, y)

$$(x,y) = \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$$

Example: 1 The equation $x^2 - 2xy + y^2 + 3x + 2 = 0$ represents **[UPSEAT 2001]**

(a) A parabola (b) An ellipse (c) A hyperbola (d) A circle

Solution: (a) Comparing the given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Here, $a = 1, b = 1, h = -1, g = \frac{3}{2}, f = 0, c = 2$

Now $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$\Rightarrow \Delta = (1)(1)(2) + 2\left(\frac{3}{2}\right)(0)(-1) - (1)(0)^2 - 1\left(\frac{3}{2}\right)^2 - 2(-1)^2 \Rightarrow \Delta = \frac{-9}{4}$ *i.e.*, $\Delta \neq 0$ and $h^2 - ab = 1 - 1 = 0$ *i.e.*, $h^2 = ab$

So given equation represents a parabola.

Example: 2 The centre of $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$ is **[BIT Ranchi1986]**

(a) (2, 3) (b) (2, –3) (c) (–2, 3) (d) (–2, –3)

Solution: (a) Centre of conic is $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

Here, $a = 14$, $h = -2$, $b = 11$, $g = -22$, $f = -29$, $c = 71$

Centre $\equiv \left(\frac{(-2)(-29) - (11)(-22)}{(14)(11) - (-2)^2}, \frac{(-22)(-2) - (14)(-29)}{(14)(11) - (-2)^2}\right)$

Centre $\equiv (2, 3)$.

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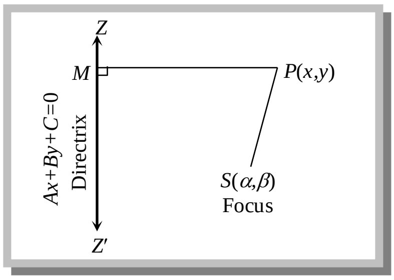
5.1 Parabola

5.1.1 Definition

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point (*i.e.*, focus) in the plane is always equal to its distance from a fixed straight line (*i.e.*, directrix) in the same plane.

General equation of a parabola : Let S be the focus, ZZ' be the directrix and let P be any point on the parabola. Then by definition,



$$SP = PM \qquad (\because\ e = 1)$$

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = \frac{Ax + By + C}{\sqrt{A^2 + B^2}}$$

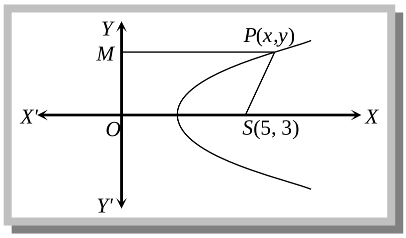
Or $\quad (A^2 + B^2)\{(x-\alpha)^2 + (y-\beta)^2\} = (Ax + By + C)^2$

Example: 1 The equation of parabola whose focus is (5, 3) and directrix is $3x - 4y + 1 = 0$, is **[MP PET 2002]**

(a) $(4x + 3y)^2 - 256x - 142y + 849 = 0$ (b) $(4x - 3y)^2 - 256x - 142y + 849 = 0$

(c) $(3x + 4y)^2 - 142x - 256y + 849 = 0$ (d) $(3x - 4y)^2 - 256x - 142y + 849 = 0$

Solution: (a) $PM^2 = PS^2 \Rightarrow (x-5)^2 + (y-3)^2 = \left(\frac{3x - 4y + 1}{\sqrt{9+16}}\right)^2$



$\Rightarrow 25(x^2 + 25 - 10x + y^2 + 9 - 6y)$

$= 9x^2 + 16y^2 + 1 - 12xy + 6x - 8y - 12xy$

$\Rightarrow 16x^2 + 9y^2 - 256x - 142y + 24xy + 849 = 0$

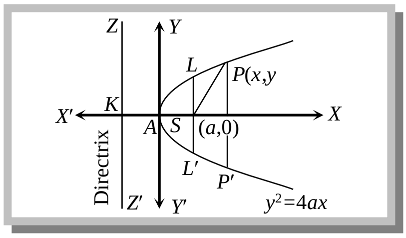
$\Rightarrow (4x + 3y)^2 - 256x - 142y + 849 = 0$

5.1.2 Standard equation of the Parabola

Let S be the focus ZZ' be the directrix of the parabola and (x, y) be any point on parabola.

Let $AS = AK = a (> 0)$ then coordinate of S is $(a, 0)$ and the equation of KZ is $x = -a$ or $x + a = 0$

Now $SP = PM \Rightarrow (SP)^2 = (PM)^2$

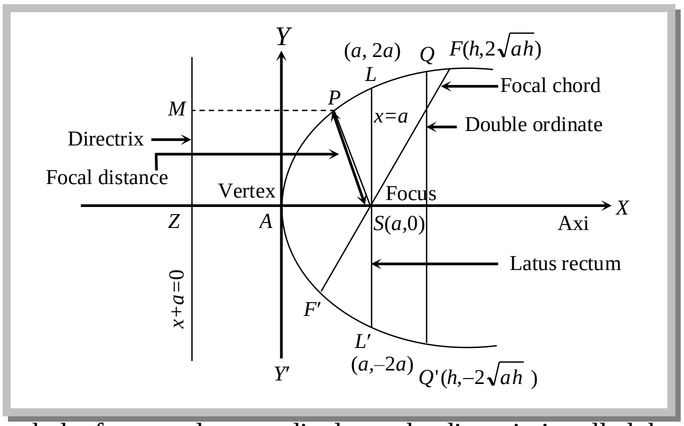


$\Rightarrow (x - a)^2 + (y - 0)^2 = (a + x)^2$

$\therefore \boxed{y^2 = 4ax}$

which is the equation of the parabola in its standard form.

Some terms related to parabola



For the parabola $y^2 = 4ax$,

(1) **Axis :** A straight line passes through the focus and perpendicular to the directrix is called the axis of parabola.

For the parabola $y^2 = 4ax$, x-axis is the axis. Here all powers of y are even in $y^2 = 4ax$. Hence parabola $y^2 = 4ax$ is symmetrical about x-axis.

(2) **Vertex :** The point of intersection of a parabola and its axis is called the vertex of the parabola. The vertex is the middle point of the focus and the point of intersection of axis and the directrix.

For the parabola $y^2 = 4ax$, $A(0,0)$ i.e., the origin is the vertex.

(3) **Double-ordinate :** The chord which is perpendicular to the axis of parabola or parallel to directrix is called double ordinate of the parabola.

Let QQ' be the double-ordinate. If abscissa of Q is h then ordinate of Q, $y^2 = 4ah$ or $y = 2\sqrt{ah}$ (for I^{st} Quadrant) and ordinate of Q' is $y = -2\sqrt{ah}$ (for IVth Quadrant). Hence coordinates of Q and Q' are $(h, 2\sqrt{ah})$ and $(h, -2\sqrt{ah})$ respectively.

(4) **Latus-rectum :** If the double-ordinate passes through the focus of the parabola, then it is called latus-rectum of the parabola.

Coordinates of the extremeties of the latus rectum are $L(a, 2a)$ and $L'(a, -2a)$ respectively.

Since $LS = L'S = 2a$ $\therefore$ Length of latus rectum $LL' = 2(LS) = 2(L'S) = 4a$.

(5) **Focal Chord :** A chord of a parabola which is passing through the focus is called a focal chord of the parabola. Here PP' and LL' are the focal chords.

(6) **Focal distance (Focal length) :** The focal distance of any point P on the parabola is its distance from the focus S *i.e.*, SP.

Here, Focal distance $SP = PM = x + a$

Note : ❑ If length of any double ordinate of parabola $y^2 = 4ax$ is $2l$, then coordinates of end points of this double ordinate are $\left(\frac{l^2}{4a}, l\right)$ and $\left(\frac{l^2}{4a}, -l\right)$.

Important Tips

☞ *The area of the triangle inscribed in the parabola $y^2 = 4ax$ is $\frac{1}{8a}(y_1 \sim y_2)(y_2 \sim y_3)(y_3 \sim y_1)$, where $y_1, y_2 y_3$ are the ordinate of the vertices*

☞ *The length of the side of an equilateral triangle inscribed in the parabola $y^2 = 4ax$ is $8a\sqrt{3}$ (one angular point is at the vertex).*

Example: 2 The point on the parabola $y^2 = 18x$, for which the ordinate is three times the abscissa, is **[MP PET 2003]**

(a) (6, 2) (b) (–2, –6) (c) (3, 18) (d) (2, 6)

Solution: (d) Given $y=3x$, then $(3x)^2=18x \Rightarrow 9x^2=18x \Rightarrow x=2$ and $y=6$.

Example: 3 The equation of the directrix of parabola $5y^2=4x$ is **[UPSEAT 1998]**

(a) $4x-1=0$ (b) $4x+1=0$ (c) $5x+1=0$ (d) $5x-1=0$

Solution: (c) The given parabola is $y^2=\frac{4}{5}x$. Here $a=\frac{1}{5}$. Directrix is $x=-a=\frac{-1}{5} \Rightarrow 5x+1=0$

Example: 4 The point on the parabola $y^2=8x$. Whose distance from the focus is 8, has *x*-coordinate as

(a) 0 (b) 2 (c) 4 (d) 6

Solution: (d) If $P(x_1,y_1)$ is a point on the parabola $y^2=4ax$ and S is its focus, then $SP=x_1+a$

Here $4a=8 \Rightarrow a=2;\ SP=8$

$\therefore\ 8=x_1+2 \Rightarrow x_1=6$

Example: 5 If the parabola $y^2=4ax$ passes through (–3, 2), then length of its latus rectum is **[Rajasthan PET 1986, 95]**

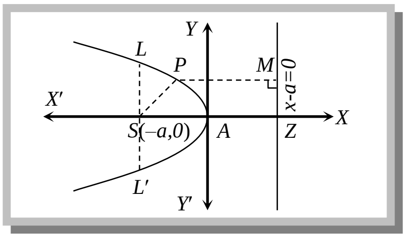
(a) 2/3 (b) 1/3 (c) 4/3 (d) 4

Solution: (c) The point $(-3,2)$ will satisfy the equation $y^2=4ax \Rightarrow 4=-12a \Rightarrow$ Latus rectum $=4|a|=4\times\left|-\frac{1}{3}\right|=\frac{4}{3}$

5.1.3 Some other standard forms of Parabola

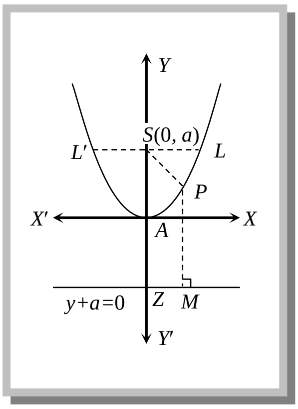
(1) Parabola opening to left

(*i.e.* $y^2=-4ax$); $(a>0)$



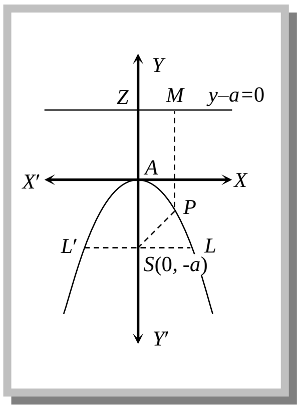
(2) Parabola opening upwards

(*i.e.* $x^2=4ay$); $(a>0)$



(3) Parabola opening down wards

(*i.e.* $x^2=-4ay$); $(a>0)$



Important terms	$y^2=4ax$	$y^2=-4ax$	$x^2=4ay$	$x^2=-4ay$
Coordinates of vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Coordinates of focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Equation of the directrix	$x=-a$	$x=a$	$y=-a$	$y=a$
Equation of the axis	$y=0$	$y=0$	$x=0$	$x=0$
Length of the latusrectum	$4a$	$4a$	$4a$	$4a$
Focal distance of a point $P(x,y)$	$x+a$	$a-x$	$y+a$	$a-y$

Example: 6 Focus and directrix of the parabola $x^2=-8ay$ are **[Rajasthan PET 2001]**

(a) $(0,-2a)$ and $y=2a$ (b) $(0,2a)$ and $y=-2a$ (c) $(2a,0)$ and $x=-2a$ (d) $(-2a,0)$ and $x=2a$

Solution: (a) Given equation is $x^2 = -8ay$

Comparing the given equation with $x^2 = -4AY$, $A = 2a$

Focus of parabola (0,–*A*) i.e. (0, –2*a*)

Directrix $y = A$, *i.e.* $y = 2a$

Example: 7 The equation of the parabola with its vertex at the origin, axis on the *y*-axis and passing through the point (6, –3) is **[MP PET 2001]**

(a) $y^2 = 12x + 6$ (b) $x^2 = 12y$ (c) $x^2 = -12y$ (d) $y^2 = -12x + 6$

Solution: (c) Since the axis of parabola is *y*-axis with its vertex at origin.

$\therefore$ Equation of parabola $x^2 = 4ay$. Since it passes through (6, –3) ; $\therefore 36 = -12a \Rightarrow a = -3$

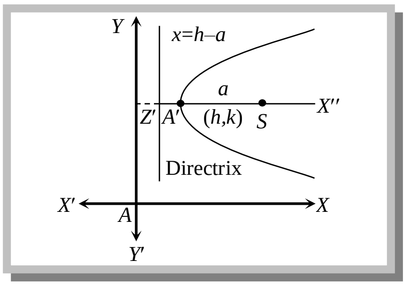
$\therefore$ Equation of parabola is $x^2 = -12y$.

5.1.4 Special form of Parabola $(y - k)^2 = 4a(x - h)$

The equation of a parabola with its vertex at (*h*, *k*) and axis as parallel to *x*-axis is $(y - k)^2 = 4a(x - h)$

If the vertex of the parabola is (p, q) and its axis is parallel to *y*-axis, then the equation of the parabola is $(x - p)^2 = 4b(y - q)$

When origin is shifted at $A'(h, k)$ without changing the direction of axes, its equation becomes $(y - k)^2 = 4a(x - h)$ or $(x - p)^2 = 4b(y - q)$



Equation of Parabola	Vertex	Axis	Focus	Directrix	Equation of L.R.	Length of L.R.
$(y - K)^2 = 4a(x - h)$	(h, k)	$y = k$	$(h + a, k)$	$x + a - h = 0$	$x = a + h$	$4a$
$(x - p)^2 = 4b(y - q)$	(p, q)	$x = p$	$(p, b + q)$	$y + b - q = 0$	$y = b + q$	$4b$

Important Tips

- ☞ $y^2 = 4a(x + a)$ *is the equation of the parabola whose focus is the origin and the axis is x-axis.*
- ☞ $y^2 = 4a(x - a)$ *is the equation of parabola whose axis is x-axis and y-axis is directrix.*
- ☞ $x^2 = 4a(y + a)$ *is the equation of parabola whose focus is the origin and the axis is y-axis.*
- ☞ $x^2 = 4a(y - a)$ *is the equation of parabola whose axis is y-axis and the directrix is x-axis.*
- ☞ *The equation to the parabola whose vertex and focus are on x-axis at a distance a and a' respectively from the origin is* $y^2 = 4(a' - a)(x - a)$.
- ☞ *The equation of parabola whose axis is parallel to x-axis is* $x = Ay^2 + By + C$ *and* $y = Ax^2 + Bx + C$ *is a parabola with its axis parallel to y-axis.*

Example: 8 Vertex of the parabola $x^2 + 4x + 2y - 7 = 0$ is **[MP PET 1990]**

(a) (–2,11 / 2) (b) (–2,2) (c) (–2,11) (d) (2,11)

Solution: (a) Equation of the parabola is $(x + 2)^2 = -2y + 7 + 4 \Rightarrow (x + 2)^2 = -2\left(y - \frac{11}{2}\right)$. Hence vertex is $\left(-2, \frac{11}{2}\right)$.

Example: 9 The focus of the parabola $4y^2 - 6x - 4y = 5$ is **[Rajasthan PET 1997]**

(a) $(-8/5, 2)$ (b) $(-5/8, 1/2)$ (c) $(1/2, 5/8)$ (d) $(6/8, -1/2)$

Solution: (b) Given equation of parabola when written in standard form, we get

$4\left(y - \frac{1}{2}\right)^2 = 6(x+1) \Rightarrow \left(y - \frac{1}{2}\right)^2 = \frac{3}{2}(x+1) \Rightarrow Y^2 = \frac{3}{2}X$ where, $Y = y - \frac{1}{2}, X = x + 1$

$\therefore y = Y + \frac{1}{2}, x = X - 1$ ....(i)

Focus $\Rightarrow X = a, \quad Y = 0$; $\therefore 4a = \frac{3}{2} \Rightarrow a = \frac{3}{8} \Rightarrow x = \frac{3}{8} - 1 = -\frac{5}{8}$; $y = 0 + \frac{1}{2} = \frac{1}{2} \Rightarrow$ Focus $= \left(-\frac{5}{8}, \frac{1}{2}\right)$

Example: 10 The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is **[IIT Screening 2001]**

(a) $x = -1$ (b) $x = 1$ (c) $x = \frac{-3}{2}$ (d) $x = \frac{3}{2}$

Solution: (d) Here, $y^2 + 4y + 4 + 4x - 2 = 0$ or $(y+2)^2 = -4\left(x - \frac{1}{2}\right)$

Let $y + 2 = Y$, $\frac{1}{2} - x = X$. Then the parabola is $Y^2 = 4X$. $\therefore$ The directrix is $X + 1 = 0$ or $\frac{1}{2} - x + 1 = 0$, $\therefore x = \frac{3}{2}$

Example: 11 The line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$. Then one of the values of k is **[IIT Screening 2000]**

(a) $\frac{1}{8}$ (b) 8 (c) 4 (d) $\frac{1}{4}$

Solution: (c) The parabola is $y^2 = 4\frac{k}{4}\left(x - \frac{8}{k}\right)$. Putting $y = Y$, $x - \frac{8}{k} = X$. The equation is $Y^2 = 4.\frac{k}{4}.X$

$\therefore$ The directrix is $X + \frac{k}{4} = 0$ *i.e.*, $x - \frac{8}{k} + \frac{k}{4} = 0$. But $x - 1 = 0$ is the directrix. So $\frac{8}{k} - \frac{k}{4} = 1 \Rightarrow k = -8, 4$.

Example: 12 Equation of the parabola with its vertex at (1, 1)and focus (3, 1)is **[Karnataka CET 2001, 2002]**

(a) $(x-1)^2 = 8(y-1)$ (b) $(y-1)^2 = 8(x-3)$ (c) $(y-1)^2 = 8(x-1)$ (d) $(x-3)^2 = 8(y-1)$

Solution: (c) Given vertex of parabola $(h,k) \equiv (1,1)$ and its focus $(a+h,k) \equiv (3,1)$ or $a + h = 3$ or $a = 2$. We know that as the y-coordinates of vertex and focus are same, therefore axis of parabola is parallel to x-axis. Thus equation of the parabola is $(y-k)^2 = 4a(x-h)$ or $(y-1)^2 = 4 \times 2(x-1)$ or $(y-1)^2 = 8(x-1)$.

5.1.5 Parametric equations of a Parabola

The simplest and the best form of representing the coordinates of a point on the parabola $y^2 = 4ax$ is $(at^2 2at)$ because these coordinates satisfy the equation $y^2 = 4ax$ for all values of t. The equations $x = at^2, y = 2at$ taken together are called the parametric equations of the parabola $y^2 = 4ax$, t being the parameter.

The following table gives the parametric coordinates of a point on four standard forms of the parabola and their parametric equation.

Parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Parametric Coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Parametric Equations	$x = at^2$ $y = 2at$	$x = -at^2$ $y = 2at$	$x = 2at$ $y = at^2$	$x = 2at$, $y = -at^2$

Note : ❑ The parametric equation of parabola $(y-k)^2 = 4a(x-h)$ are $x = h + at^2$ and $y = k + 2at$

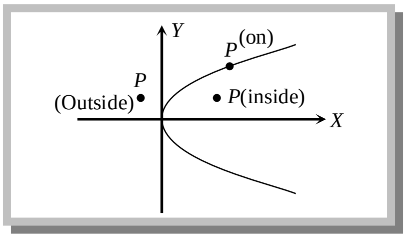
Example: 13 $x - 2 = t^2, y = 2t$ are the parametric equations of the parabola

(a) $y^2 = 4x$ (b) $y^2 = -4x$ (c) $x^2 = -4y$ (d) $y^2 = 4(x-2)$

Solution: (d) Here $\frac{y}{2} = t$ and $x - 2 = t^2 \Rightarrow (x-2) = \left(\frac{y}{2}\right)^2 \Rightarrow y^2 = 4(x-2)$

5.1.6 Position of a point and a Line with respect to a Parabola

(1) **Position of a point with respect to a parabola**: The point $P(x_1, y_1)$ lies outside on or inside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 >, =, \text{or} < 0$



(2) **Intersection of a line and a parabola:** Let the parabola be $y^2 = 4ax$ .....(i)

And the given line be $y = mx + c$ .....(ii)

Eliminating y from (i) and (ii) then $(mx + c)^2 = 4ax$ or $m^2x^2 + 2x(mc - 2a) + c^2 = 0$ .....(iii)

This equation being quadratic in *x*, gives two values of *x*. It shows that every straight line will cut the parabola in two points, may be real, coincident or imaginary, according as discriminate of (iii) >, = or < 0

$\therefore$ The line $y = mx + c$ does not intersect, touches or intersect a parabola $y^2 = 4ax$, according as $c >, =, < \frac{a}{m}$

Condition of tangency : The line $y = mx + c$ touches the parabola, if $c = \frac{a}{m}$

Example: 14 The equation of a parabola is $y^2 = 4x$. $P(1,3)$ and $Q(1,1)$ are two points in the *xy*-plane. Then, for the parabola

(a) *P* and *Q* are exterior points
(b) P is an interior point while *Q* is an exterior point
(c) *P* and *Q* are interior points
(d) *P* is an exterior point while *Q* is an interior point

Solution: (d) Here, $S \equiv y^2 - 4x = 0$; $S(1,3) = 3^2 - 4.1 > 0 \Rightarrow P(1,3)$ is an exterior point.

$S(1,1) = 1^2 - 4.1 < 0 \Rightarrow Q(1,1)$ is an interior point.

Example: 15 The ends of a line segment are $P(1,3)$ and $Q(1,1)$. *R* is a point on the line segment *PQ* such that $PQ : QR = 1 : \lambda$. If *R* is an interior point of the parabola $y^2 = 4x$, then

(a) $\lambda \in (0, 1)$
(b) $\lambda \in \left(-\frac{3}{5}, 1\right)$
(c) $\lambda \in \left(\frac{1}{2}, \frac{3}{5}\right)$
(d) None of these

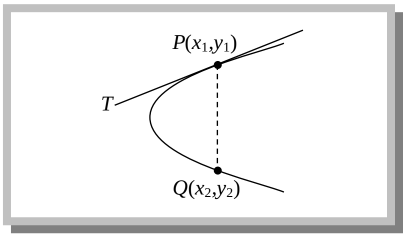
Solution: (a) $R = \left(1, \frac{1+3\lambda}{1+\lambda}\right)$ It is an interior point of $y^2 - 4x = 0$ *iff* $\left(\frac{1+3\lambda}{1+\lambda}\right)^2 - 4 < 0$

$\Rightarrow \left(\frac{1+3\lambda}{1+\lambda} - 2\right)\left(\frac{1+3\lambda}{1+\lambda} + 2\right) < 0 \Rightarrow \left(\frac{\lambda - 1}{1+\lambda}\right)\left(\frac{5\lambda+3}{1+\lambda}\right) < 0 \Rightarrow (\lambda - 1)\left(\lambda + \frac{3}{5}\right) < 0$

Therefore, $-\frac{3}{5} < \lambda < 1$. But $\lambda > 0$ $\therefore 0 < \lambda < 1 \Rightarrow \lambda \in (0, 1)$.

5.1.7 Equation of Tangent in Different forms

(1) **Point Form:** The equation of the tangent to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is $yy_1 = 2a(x + x_1)$



Equation of tangent of all other standard parabolas at (x_1, y_1)	
Equation of parabolas	**Tangent at (x_1, y_1)**
$y^2 = -4ax$	$yy_1 = -2a(x + x_1)$

$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

Note : ❑ The equation of tangent at (x_1, y_1) to a curve can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$ and xy by $\frac{xy_1 + x_1y}{2}$ provided the equation of curve is a polynomial of second degree in x and y.

(2) **Parametric form :** The equation of the tangent to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is $ty = x + at^2$

Equations of tangent of all other standard parabolas at '*t*'		
Equations of parabolas	**Parametric co-ordinates '*t*'**	**Tangent at '*t*'**
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$

(3) **Slope Form:** The equation of a tangent of slope m to the parabola $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ is $y = mx + \frac{a}{m}$

Equation of parabolas	**Point of contact in terms of slope (*m*)**	**Equation of tangent in terms of slope (*m*)**	**Condition of Tangency**
$y^2 = 4ax$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$	$c = \frac{a}{m}$
$y^2 = -4ax$	$\left(-\frac{a}{m^2}, -\frac{2a}{m}\right)$	$y = mx - \frac{a}{m}$	$c = -\frac{a}{m}$
$x^2 = 4ay$	$(2am, am^2)$	$y = mx - am^2$	$c = -am^2$
$x^2 = -4ay$	$(-2am, -am^2)$	$y = mx + am^2$	$c = am^2$

Important Tips

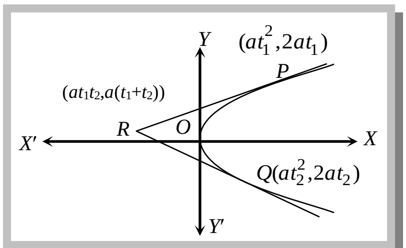
☞ *If the straight line* $lx + my + n = 0$ *touches the parabola* $y^2 = 4ax$ *then* $ln = am^2$.

☞ *If the line* $x\cos\alpha + y\sin\alpha = p$ *touches the parabola* $y^2 = 4ax$*, then* $P\cos\alpha + a\sin^2\alpha = 0$ *and point of contact is* $(a\tan^2\alpha, -2a\tan\alpha)$

☞ *If the line* $\frac{x}{l} + \frac{y}{m} = 1$ *touches the parabola* $y^2 = 4a(x + b)$*, then* $m^2(l + b) + al^2 = 0$

5.1.8 Point of intersection of Tangents at any two points on the Parabola

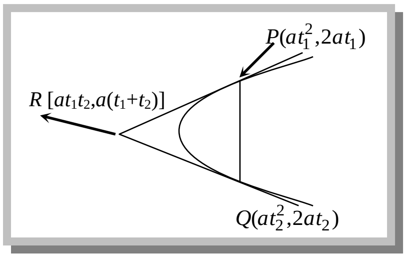
The point of intersection of tangents at two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $(at_1t_2, a(t_1 + t_2))$.



The locus of the point of intersection of tangents to the parabola $y^2 = 4ax$ which meet at an angle α is $(x+a)^2 \tan^2 \alpha = y^2 - 4ax$.

Director circle: The locus of the point of intersection of perpendicular tangents to a conic is known as its director circle. The director circle of a parabola is its directrix.

Note : ❑ Clearly, *x*-coordinates of the point of intersection of tangents at *P* and *Q* on the parabola is the G.M of the *x*-coordinate of *P* and *Q* and y-coordinate is the A.M. of *y*-coordinate of *P* and *Q*.

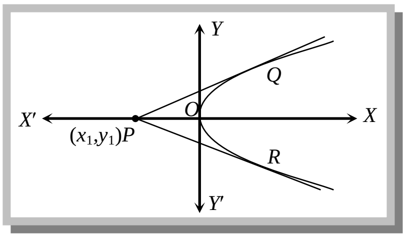


❑ The equation of the common tangents to the parabola $y^2 = 4ax$ and $x^2 = 4by$ is $a^{\frac{1}{3}}x + b^{\frac{1}{3}}y + a^{\frac{2}{3}}b^{\frac{2}{3}} = 0$

❑ The tangents to the parabola $y^2 = 4ax$ at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ intersect at *R*. Then the area of triangle *PQR* is $\frac{1}{2}a^2(t_1 - t_2)^3$

5.1.9 Equation of Pair of Tangents from a point to a Parabola

If $y_1^2 - 4ax_1 > 0$, then any point $P(x_1, y_1)$ lies out side the parabola and a pair of tangents *PQ*, *PR* can be drawn to it from *P*



The combined equation of the pair of the tangents drawn from a point to a parabola is $SS' = T^2$ where $S = y^2 - 4ax; S' = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$

Note : ❑ The two tangents can be drawn from a point to a parabola. The two tangent are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the parabola.

Important Tips

- *Tangents at the extremities of any focal chord of a parabola meet at right angles on the directrix.*
- *Area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.*
- *If the tangents at the points P and Q on a parabola meet in T, then ST is the geometric mean between SP and SQ, i.e.* $ST^2 = SP.SQ$
- *Tangent at one extremity of the focal chord of a parabola is parallel to the normal at the other extremity.*
- *The angle of intersection of two parabolas* $y^2 = 4ax$ *and* $x^2 = 4by$ *is given by* $\tan^{-1}\frac{3a^{1/3}b^{1/3}}{2(a^{2/3}+b^{2/3})}$

Example: 16 The straight line $y = 2x + \lambda$ does not meet the parabola $y^2 = 2x$, if **[MP PET 1993; MNR 1977]**

(a) $\lambda < \frac{1}{4}$ (b) $\lambda > \frac{1}{4}$ (c) $\lambda = 4$ (d) $\lambda = 1$

Solution: (b) $y = 2x + \lambda$ does not meet the parabola $y^2 = 2x$, If $\lambda > \frac{a}{m} = \frac{1}{2.2} = \frac{1}{4} \Rightarrow \lambda > \frac{1}{4}$

Example: 17 If the parabola $y^2 = 4ax$ passes through the point $(1, -2)$, then the tangent at this point is **[MP PET 1998]**

(a) $x + y - 1 = 0$ (b) $x - y - 1 = 0$ (c) $x + y + 1 = 0$ (d) $x - y + 1 = 0$

Solution: (c) $\because$ Parabola passes through the point (1, –2), then $4 = 4a \Rightarrow a = 1$. From $yy_1 = 2a(x + x_1) \Rightarrow -2y = 2(x + 1)$

$\therefore$ Required tangent is $x + y + 1 = 0$

Example: 18 The equation of the tangent to the parabola $y^2 = 16x$, which is perpendicular to the line $y = 3x + 7$ is **[MP PET 1998]**

(a) $y - 3x + 4 = 0$ (b) $3y - x + 36 = 0$ (c) $3y + x - 36 = 0$ (d) $3y + x + 36 = 0$

Solution: (a) A line perpendicular to the given line is $3y + x = \lambda \Rightarrow y = -\frac{1}{3}x + \frac{\lambda}{3}$

Here $m = -\frac{1}{3}$, $c = \frac{\lambda}{3}$. If we compare $y^2 = 16x$ with $y^2 = 4ax$, then $a = 4$

Condition for tangency is $c = \frac{a}{m} \Rightarrow \frac{\lambda}{3} = \frac{4}{(-1/3)} \Rightarrow \lambda = -36$. $\therefore$ Required equation is $x + 3y + 36 = 0$.

Example: 19 If the tangent to the parabola $y^2 = ax$ makes an angle of 45^o with x-axis, then the point of contact is

[Rajasthan PET 1985, 90, 2003]

(a) $\left(\frac{a}{2}, \frac{a}{2}\right)$ (b) $\left(\frac{a}{4}, \frac{a}{4}\right)$ (c) $\left(\frac{a}{2}, \frac{a}{4}\right)$ (d) $\left(\frac{a}{4}, \frac{a}{2}\right)$

Solution: (d) Parabola is $y^2 = ax$ *i.e.* $y^2 = 4\left(\frac{a}{4}\right)x$ .....(i)

Let point of contact is (x_1, y_1). $\therefore$ Equation of tangent is $y - y_1 = \frac{2(a/4)}{y_1}(x - x_1) \Rightarrow y = \frac{a}{2y_1}(x) - \frac{ax_1}{2y_1} + y_1$

Here, $m = \frac{a}{2y_1} = \tan 45^o \Rightarrow \frac{a}{2y_1} = 1 \Rightarrow y_1 = \frac{a}{2}$. From (i), $x_1 = \frac{a}{4}$, So point is $\left(\frac{a}{4}, \frac{a}{2}\right)$.

Example: 20 The line $x - y + 2 = 0$ touches the parabola $y^2 = 8x$ at the point **[Roorkee 1998]**

(a) (2, –4) (b) $(1, 2\sqrt{2})$ (c) $(4, -4\sqrt{2})$ (d) (2, 4)

Solution: (d) The line $x - y + 2 = 0$ i.e. $x = y - 2$ meets parabola $y^2 = 8x$, if

$\Rightarrow y^2 = 8(y-2) = 8y - 16 \Rightarrow y^2 - 8y + 16 = 0 \Rightarrow (y-4)^2 = 0 \Rightarrow y = 4,4$

$\because$ Roots are equal, $\therefore$ Given line touches the given parabola.

$\therefore$ $x = 4 - 2 = 2$, Thus the required point is (2, 4).

Example: 21 The equation of the tangent to the parabola at point $(a/t^2, 2a/t)$ is **[Rajasthan PET 1996]**

(a) $ty = xt^2 + a$ (b) $ty = x + at^2$ (c) $y = tx + at^2$ (d) $y = tx + (a/t^2)$

Solution: (a) Equation of the tangent to the parabola, $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

$\Rightarrow y.\frac{2a}{t} = 2a\left(x + \frac{a}{t^2}\right) \Rightarrow \frac{y}{t} = \left(x + \frac{a}{t^2}\right) \Rightarrow \frac{y}{t} = \frac{t^2x + a}{t^2} \Rightarrow ty = t^2x + a$

Example: 22 Two tangents are drawn from the point (–2, –1) to the parabola $y^2 = 4x$. If α is the angle between these tangents, then $\tan\alpha =$

(a) 3 (b) 1/3 (c) 2 (d) 1/2

Solution: (a) Equation of pair of tangent from $(-2,-1)$ to the parabola is given by $SS_1 = T^2$ *i.e.* $(y^2 - 4x)(1+8) = [y(-1) - 2(x-2)]^2$

$\Rightarrow 9y^2 - 36x = [-y - 2x + 4]^2 \Rightarrow 9y^2 - 36x = y^2 + 4x^2 + 16 + 4xy - 16x - 8y \Rightarrow 4x^2 - 8y^2 + 4xy + 20x - 8y + 16 = 0$

$\therefore \tan\alpha = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\frac{2\sqrt{4 - 4(-8)}}{4-8}\right| = \left|\frac{12}{-4}\right| = 3$

Example: 23 If $\left(\frac{a}{b}\right)^{1/3} + \left(\frac{b}{a}\right)^{1/3} = \frac{\sqrt{3}}{2}$, then the angle of intersection of the parabola $y^2 = 4ax$ and $x^2 = 4by$ at a point other than the origin is

(a) $\pi/4$ (b) $\pi/3$ (c) $\pi/2$ (d) None of these

Solution: (b) Given parabolas are $y^2 = 4ax$ ....(i) and $x^2 = 4by$ .....(ii)

These meet at the points (0, 0), $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$

Tangent to (i) at $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$ is $y.4a^{2/3}b^{1/3} = 2a(x + 4a^{2/3}b^{1/3})$

Slope of the tangent $(m_1) = \frac{2a}{4a^{2/3}b^{1/3}} = \frac{a^{1/3}}{2b^{1/3}}$

Tangent to (ii) at $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$ is $x.4a^{1/3}b^{2/3} = 2b(y + 4a^{2/3}b^{1/3})$

Slope of the tangent $(m_2) = \frac{2a^{1/3}}{b^{1/3}}$

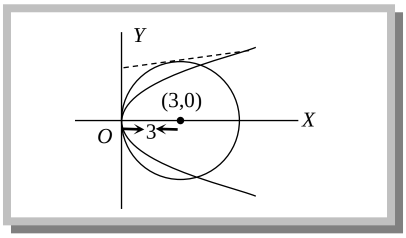
If θ is the angle between the two tangents, then $\Rightarrow$ $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{a^{1/3}}{2b^{1/3}} - \frac{2a^{1/3}}{b^{1/3}}}{1 + \frac{a^{1/3}}{2b^{1/3}} \cdot \frac{2a^{1/3}}{b^{1/3}}}\right|$

$$= \frac{3}{2} \cdot \frac{1}{\left(\frac{a}{b}\right)^{1/3} + \left(\frac{b}{a}\right)^{1/3}} = \frac{3}{2} \cdot \frac{1}{\frac{\sqrt{3}}{2}} = \sqrt{3}\ ;\ \therefore \theta = 60^o = \frac{\pi}{3}$$

Example: 24 The equation of the common tangent touching the circle $(x-3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis , is **[IIT Screening 2001]**

(a) $\sqrt{3}y = 3x + 1$ (b) $\sqrt{3}y = -(x+3)$ (c) $\sqrt{3}y = x + 3$ (d) $\sqrt{3}y = -(3x+1)$

Solution: (c) Any tangent to $y^2 = 4x$ is $y = mx + \frac{1}{m}$. It touches the circle if $3 = \left|\frac{3m + \frac{1}{m}}{\sqrt{1+m^2}}\right|$



or $9(1+m^2) = \left(3m + \frac{1}{m}\right)^2$ or $\frac{1}{m^2} = 3$, $\therefore m = \pm\frac{1}{\sqrt{3}}$

For the common tangent to be above the x-axis, $m = \frac{1}{\sqrt{3}}$

$\therefore$ Common tangent is $y = \frac{1}{\sqrt{3}}x + \sqrt{3} \Rightarrow \sqrt{3}y = x + 3$

Example: 25 If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ then

(a) $d^2 + (3b - 2c)^2 = 0$ (b) $d^2 + (3b + 2c)^2 = 0$ (c) $d^2 + (2b - 3c)^2 = 0$ (d) $d^2 + (2b + 3c)^2 = 0$

Solution: (d) Given parabolas are $y^2 = 4ax$ .....(i) and $x^2 = 4ay$ .....(ii)

from (i) and (ii) $\left(\frac{x^2}{4a}\right)^2 = 4ax \Rightarrow x^4 - 64a^3x = 0 \Rightarrow x = 0\ ,\ 4a \quad \therefore y = 0,\ 4a$

So points of intersection are $(0,0)$ and $(4a, 4a)$

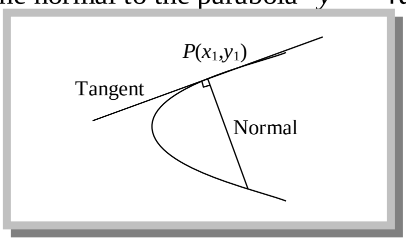
Given, the line $2bx + 3cy + 4d = 0$ passes through (0,0) and $(4a, 4a)$

$\therefore d = 0 \Rightarrow d^2 = 0$ and $(2b + 3c)^2 = 0$ $(\because a \neq 0)$

Therefore $d^2 + (2b + 3c)^2 = 0$

5.1.10 Equations of Normal in Different forms

(1) **Point form :** The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$



Equation of normals of all other standard parabolas at (x_1, y_1)	
Equation of parabolas	**Normal at (x_1, y_1)**
$y^2 = -4ax$	$y - y_1 = \frac{y_1}{2a}(x - x_1)$
$x^2 = 4ay$	$y - y_1 = -\frac{2a}{x_1}(x - x_1)$
$x^2 = -4ay$	$y - y_1 = \frac{2a}{x_1}(x - x_1)$

(2) **Parametric form**: The equation of the normal to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is $y + tx = 2at + at^3$

Equations of normal of all other standard parabola at *'t'*		
Equations of parabolas	**Parametric co-ordinates**	**Normals at *'t'***
$y^2 = -4ax$	$(-at^2, 2at)$	$y - tx = 2at + at^3$
$x^2 = 4ay$	$(2at, at^2)$	$x + ty = 2at + at^3$
$x^2 = -4ay$	$(2at, -at^2)$	$x - ty = 2at + at^3$

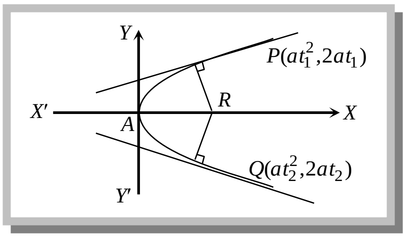
(3) **Slope form:** The equation of normal of slope m to the parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$ at the point $(am^2, -2am)$.

Equations of normal, point of contact, and condition of normality in terms of slope *(m)*			
Equations of parabola	**Point of contact in terms of slope (m)**	**Equations of normal in terms of slope (m)**	**Condition of normality**
$y^2 = 4ax$	$(am^2, -2am)$	$y = mx - 2am - am^3$	$c = -2am - am^3$
$y^2 = -4ax$	$(-am^2, 2am)$	$y = mx + 2am + am^3$	$c = 2am + am^3$
$x^2 = 4ay$	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$y = mx + 2a + \frac{a}{m^2}$	$c = 2a + \frac{a}{m^2}$
$x^2 = -4ay$	$\left(\frac{2a}{m}, -\frac{a}{m^2}\right)$	$y = mx - 2a - \frac{a}{m^2}$	$c = -2a - \frac{a}{m^2}$

Note : ❑ The line $lx + my + n = 0$ is a normal to the parabola $y^2 = 4ax$ if $al(l^2 + 2m^2) + m^2 n = 0$

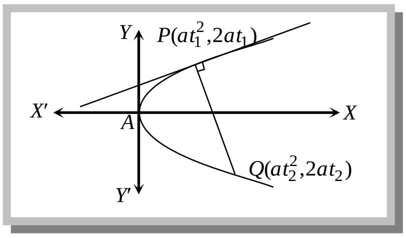
5.1.11 Point of intersection of normals at any two points on the Parabola

If R is the point of intersection then point of intersection of normals at any two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $R[2a + a(t_1^2 + t_2^2 + t_1 t_2), -at_1 t_2 (t_1 + t_2)]$



5.1.12 Relation between 't_1' and 't_2' if Normal at 't_1' meets the Parabola again at 't_2'

If the normal at the point $P(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again at $Q(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$



Important Tips

- *If the normals at points $(at_1^2, 2at)$ and $(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ meet on the parabola then $t_1t_2 = 2$*
- *If the normal at a point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ subtends a right angle at the vertex of the parabola then $t^2 = 2$.*
- *If the normal to a parabola $y^2 = 4ax$, makes an angle ϕ with the axis, then it will cut the curve again at an angle $\tan^{-1}\left(\frac{1}{2}\tan\phi\right)$.*
- *The normal chord to a parabola $y^2 = 4ax$ at the point whose ordinate is equal to abscissa subtends a right angle at the focus.*
- *If the normal at two points P and Q of a parabola $y^2 = 4ax$ intersect at a third point R on the curve. Then the product of the ordinate of P and Q is $8a^2$.*

Example: 26 If $x + y = k$ is a normal to the parabola $y^2 = 12x$, then k is **[IIT Screening 2000]**

(a) 3 (b) 9 (c) –9 (d) –3

Solution: (b) Any normal to the parabola $y^2 = 12x$ is $y + tx = 6t + 3t^3$. It is identical with $x + y = k$ if $\frac{t}{1} = \frac{1}{1} = \frac{6t + 3t^3}{k}$

$\therefore t = 1$ and $1 = \frac{6 + 3}{k} \Rightarrow k = 9$

Example: 27 The equation of normal at the point $\left(\frac{a}{4}, a\right)$ to the parabola $y^2 = 4ax$, is **[Rajasthan PET 1984]**

(a) $4x + 8y + 9a = 0$ (b) $4x + 8y - 9a = 0$ (c) $4x + y - a = 0$ (d) $4x - y + a = 0$

Solution: (b) From $y - y_1 = -\frac{-y_1}{2a}(x - x_1)$

$\Rightarrow y - a = \frac{-a}{2a}\left(x - \frac{a}{4}\right) \Rightarrow 2y + x = 2a + \frac{a}{4} = \frac{9a}{4} \Rightarrow 2y + x - \frac{9a}{4} = 0 \Rightarrow 4x + 8y - 9a = 0$

Example: 28 The point on the parabola $y^2 = 8x$ at which the normal is parallel to the line $x - 2y + 5 = 0$ is

(a) $(-1/2, 2)$ (b) $(1/2, -2)$ (c) $(2, -1/2)$ (d) $(-2, 1/2)$

Solution: (b) Let point be (h, k). Normal is $y - k = \frac{-k}{4}(x - h)$ or $-kx - 4y + kh + 4k = 0$

Gradient $= \frac{-K}{4} = \frac{1}{2} \Rightarrow k = -2$. Substituting (h, k) and $k = -2$ in $y^2 = 8x$, we get $h = \frac{1}{2}$. Hence point is $\left(\frac{1}{2}, -2\right)$

Trick: Here only point $\left(\frac{1}{2}, -2\right)$ will satisfy the parabola $y^2 = 8x$.

Example: 29 The equations of the normal at the ends of the latus rectum of the parabola $y^2 = 4ax$ are given by

(a) $x^2 - y^2 - 6ax + 9a^2 = 0$ (b) $x^2 - y^2 - 6ax - 6ay + 9a^2 = 0$

(c) $x^2 - y^2 - 6ay + 9a^2 = 0$ (d) None of these

Solution: (a) The coordinates of the ends of the latus rectum of the parabola $y^2 = 4ax$ are $(a, 2a)$ and $(a, -2a)$ respectively.

The equation of the normal at $(a, 2a)$ to $y^2 = 4ax$ is $y - 2a = \frac{-2a}{2a}(x - a)$ $\left\{\text{using } y - y_1 = \frac{-y_1}{2a}(x - x_1)\right\}$

Or $x + y - 3a = 0$ .....(i)

Similarly the equation of the normal at $(a, -2a)$ is $x - y - 3a = 0$ .....(ii)

The combined equation of (i) and (ii) is $x^2 - y^2 - 6ax + 9a^2 = 0$.

Example: 30 The locus of the point of intersection of two normals to the parabola $x^2 = 8y$, which are at right angles to each other, is **[Roorkee 1997]**

(a) $x^2 = 2(y - 6)$ (b) $x^2 = 2(y + 6)$ (c) $x^2 = -2(y - 6)$ (d) None of these

Solution: (a) Given parabola is $x^2 = 8y$ .....(i)

Let $(4t_1, 2t_1^2)$ and $Q(4t_2, 2t_2^2)$ be two points on the parabola (i)

Normal at P, Q are $y - 2t_1^2 = -\frac{1}{t_1}(x - 4t_1)$ ......(ii) and $y - 2t_2^2 = -\frac{1}{t_2}(x - 4t_2)$ ......(iii)

(ii)–(iii) gives $2(t_2^2 - t_1^2) = x\left(\frac{1}{t_2} - \frac{1}{t_1}\right) = x\frac{t_1 - t_2}{t_1 t_2}$, $\therefore x = -2t_1 t_2(t_2 + t_1)$ .....(iv)

From (ii), $y = 2t_1^2 - \frac{1}{t_1}(-2t_1 t_2(t_2 + t_1) - 4t_1) = 2t_1^2 + 2t_2(t_1 + t_2) + 4 \Rightarrow y = 2t_1^2 + 2t_1 t_2 + 2t_2^2 + 4$ .....(v)

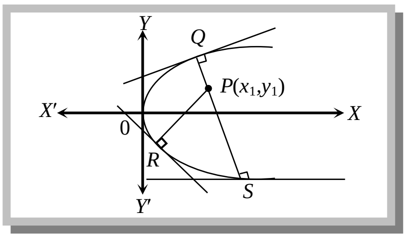
Since normals (ii) and (iii) are at right angles, $\therefore$ $\left(-\frac{1}{t_1}\right)\left(-\frac{1}{t_2}\right) = -1 \Rightarrow t_1 t_2 = -1$

$\therefore$ From (iv), $x = 2(t_1 + t_2)$ and from (v) $y = 2t_1^2 - 2 + 2t_2^2 + 4 \Rightarrow y = 2[t_1^2 + t_2^2 + 1] = 2[(t_1 + t_2)^2 - 2t_1 t_2 + 1]$

$\Rightarrow$ $y = 2[(t_1 + t_2)^2 + 2 + 1] = 2[(t_1 + t_2)^2 + 3 \Rightarrow y = 2\left[\frac{x^2}{4} + 3\right] = \frac{x^2}{2} + 6 \Rightarrow x^2 = 2(y - 6)$, which is the required locus.

5.1.13 Co-normal Points

The points on the curve at which the normals pass through a common point are called co-normal points.



Q, R, S are co-normal points. The co- normal points are also called the feet of the normals.

If the normal passes through point $P(x_1, y_1)$ which is not on parabola, then $y_1 = mx_1 - 2am - am^3 \Rightarrow am^3 + (2a - x_1)m + y_1 = 0$ ......(i)

Which gives three values of m**.** Let three values of m are m_1, m_2 and m_3, which are the slopes of the normals at Q, R and S respectively, then the coordinates of Q, R and S are $(am_1^2, -2am_1), (am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$ respectively. These three points are called the feet of the normals.

Now $m_1 + m_2 + m_3 = 0$, $m_1 m_2 + m_2 m_3 + m_3 m_1 = \frac{(2a - x_1)}{a}$ and $m_1 m_2 m_3 = \frac{-y_1}{a}$

In general, three normals can be drawn from a point to a parabola.

(1) The algebraic sum of the slopes of three concurrent normals is zero.

(2) The sum of the ordinates of the co-normal points is zero.

(3) The centroid of the triangle formed by the co-normal points lies on the axis of the parabola.

(4) The centroid of a triangle formed by joining the foots of the normal of the parabola lies on its axis and is given by $\left(\frac{am_1^2+am_2^2+am_3^2}{3}, \frac{2am_1+2am_2+2am_3}{3}\right)=\left(\frac{am_1^2+am_2^2+am_3^2}{3}, 0\right)$

(5) If three normals drawn to any parabola $y^2=4ax$ from a given point (h, k) be real, then $h>2a$ for $a=1$, normals drawn to the parabola $y^2=4x$ from any point (h, k) are real, if $h>2$.

(6) Out of these three at least one is real, as imaginary normals will always occur in pairs.

5.1.14 Circle through Co-normal points

Equation of the circle passing through the three (co-normal) points on the parabola $y^2=4ax$, normal at which pass through a given point (α, β); is $x^2+y^2-(2a+\alpha)x-\frac{\beta}{2}y=0$

(1) The algebraic sum of the ordinates of the four points of intersection of a circle and a parabola is zero.

(2) The common chords of a circle and a parabola are in pairs, equally inclined to the axis of parabola.

(3) The circle through co-normal points passes through the vertex of the parabola.

(4) The centroid of four points; in which a circle intersects a parabola, lies on the axis of the parabola.

Example: 31 The normals at three points P, Q, R of the parabola $y^2=4ax$ meet in (h, k), the centroid of triangle PQR lies on **[MP PET 1999]**

(a) $x=0$ (b) $y=0$ (c) $x=-a$ (d) $y=a$

Solution: (b) Since the centroid of the triangle formed by the co-normal points lies on the axis of the parabola.

Example: 32 If two of the three feet of normals drawn from a point to the parabola $y^2=4x$ be (1, 2) and (1, – 2)then the third foot is

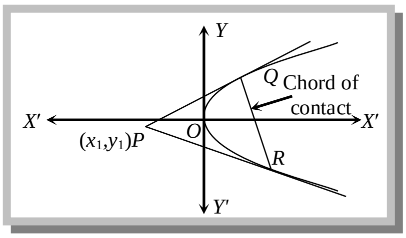
(a) $(2, 2\sqrt{2})$ (b) $(2, -2\sqrt{2})$ (c) $(0, 0)$ (d) None of these

Solution: (c) The sum of the ordinates of the foot $= y_1+y_2+y_3=0$

$\therefore 2+(-2)+y_3=0 \Rightarrow y_3=0$

5.1.15 Equation of the Chord of contact of Tangents to a Parabola

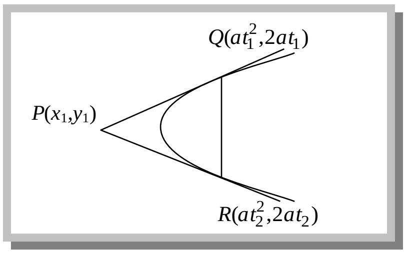
Let PQ and PR be tangents to the parabola $y^2=4ax$ drawn from any external point $P(x_1, y_1)$ then QR is called the 'Chord of contact' of the parabola $y^2=4ax$.



The chord of contact of tangents drawn from a point (x_1, y_1) to the parabola $y^2=4ax$ is $yy_1=2a(x+x_1)$

The equation is same as equation of the tangents at the point (x_1, y_1).

Note :
- ❑ The chord of contact joining the point of contact of two perpendicular tangents always passes through focus.
- ❑ If tangents are drawn from the point (x_1, y_1) to the parabola $y^2=4ax$, then the length of their chord of contact is

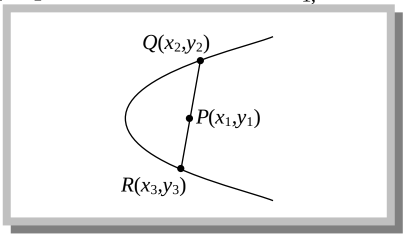


$\frac{1}{|a|}\sqrt{(y_1^2-4ax_1)(y_1^2+4a^2)}$

- The area of the triangle formed by the tangents drawn from (x_1, y_1) to $y^2 = 4ax$ and their chord of contact is $\frac{(y_1^2-4ax_1)^{3/2}}{2a}$.

5.1.16 Equation of the Chord of the Parabola which is bisected at a given point

The equation of the chord at the parabola $y^2 = 4ax$ bisected at the point (x_1, y_1) is given by $T = S_1$, where $T = yy_1 - 2a(x + x_1)$ and $S_1 = y_1^2 - 4ax_1$. *i.e.*, $yy_1 - 2a(x+x_1) = y_1^2 - 4ax_1$



5.1.17 Equation of the Chord joining any two points on the Parabola

Let $P(at_1^2, 2at_1), Q(at_2^2, 2at_2)$ be any two points on the parabola $y^2 = 4ax$. Then, the equation of the chord joining these points is, $y - 2at_1 = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2)$ or $y - 2at_1 = \frac{2}{t_1 + t_2}\left(x - at_1^2\right)$ or $y(t_1 + t_2) = 2x + 2at_1t_2$

(1) **Condition for the chord joining points having parameters t_1 and t_2 to be a focal chord:** If the chord joining points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ on the parabola passes through its focus, then $(a, 0)$ satisfies the equation $y(t_1 + t_2) = 2x + 2at_1t_2 \Rightarrow 0 = 2a + 2at_1t_2 \Rightarrow t_1t_2 = -1$ or $t_2 = -\frac{1}{t_1}$

(2) **Length of the focal chord:** The length of a focal chord having parameters t_1 and t_2 for its end points is $a(t_2 - t_1)^2$.

Note:
- If one extremity of a focal chord is $(at_1^2, 2at_1)$, then the other extremity $(at_2^2, 2at_2)$ becomes $\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$ by virtue of relation $t_1t_2 = -1$.
- If one end of the focal chord of parabola is $(at^2, 2at)$, then other end will be $\left(\frac{a}{t^2}, -2at\right)$ and length of chord $= a\left(t + \frac{1}{t}\right)^2$.
- The length of the chord joining two point 't_1' and 't_2' on the parabola $y^2 = 4ax$ is $a(t_1 - t_2)\sqrt{(t_1 + t_2)^2 + 4}$
- The length of intercept made by line $y = mx + c$ between the parabola $y^2 = 4ax$ is $\frac{4}{m^2}\sqrt{a(1+m^2)(a - mc)}$.

Important Tips

- ☞ *The focal chord of parabola $y^2 = 4ax$ making an angle α with the x-axis is of length $4a\operatorname{cosec}^2\alpha$.*
- ☞ *The length of a focal chord of a parabola varies inversely as the square of its distance from the vertex.*

☞ *If l_1 and l_2 are the length of segments of a focal chord of a parabola, then its latus-rectum is* $\frac{4l_1l_2}{l_1+l_2}$

☞ *The semi latus rectum of the parabola $y^2=4ax$ is the harmonic mean between the segments of any focal chord of the parabola.*

Example: 33 If the points $(au^2, 2au)$ and $(av^2, 2av)$ are the extremities of a focal chord of the parabola $y^2=4ax$, then **[MP PET 1998, 93]**

(a) $uv-1=0$ (b) $uv+1=0$ (c) $u+v=0$ (d) $u-v=0$

Solution: (b) Equation of focal chord for the parabola $y^2=4ax$ passes through the point $(au^2, 2au)$ and $(av^2, 2av)$

$\Rightarrow\ y-2au=\frac{2av-2au}{av^2-au^2}(x-au^2) \Rightarrow y-2au=\frac{2a(v-u)}{a(v-u)(v+u)}(x-au^2) \Rightarrow y-2au=\frac{2}{v+u}(x-au^2)$

It this is focal chord, so it would passes through focus $(a, 0)$

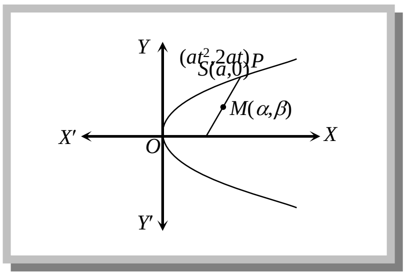
$\Rightarrow\ 0-2au=\frac{2}{v+u}(a-au^2) \Rightarrow -uv-u^2=1-u^2$, $\therefore\ uv+1=0$

Trick : Given points $(au^2, 2au)$ and $(av^2, 2av)$, then $t_1=u$ and $t_2=v$, we know that $t_1t_2=-1$. Hence $uv+1=0$.

Example: 34 The locus of the midpoint of the line segment joining the focus to a moving point on the parabola $y^2=4ax$ is another parabola with the directrix **[IIT Screening 2002]**

(a) $x=-a$ (b) $x=-\frac{a}{2}$ (c) $x=0$ (d) $x=\frac{a}{2}$

Solution: (c) Let $M(\alpha, \beta)$ be the mid point of PS.



$\alpha=\frac{at^2+a}{2}, \beta=\frac{2at+0}{2} \Rightarrow 2\alpha=at^2+a,\ at=\beta$

$\therefore\ 2\alpha=a.\frac{\beta^2}{a^2}+a$ or $2a\alpha=\beta^2+a^2$

$\therefore$ The locus is $y^2=\frac{4a}{2}(x-\frac{a}{2})=4b(x-b), \left\{b=\frac{a}{2}\right\}$

$\therefore$ Directrix is $(x-b)+b=0$ or $x=0$.

Example: 35 The length of chord of contact of the tangents drawn from the point (2, 5) to the parabola $y^2=8x$, is **[MNR 1976]**

(a) $\frac{1}{2}\sqrt{41}$ (b) $\sqrt{41}$ (c) $\frac{3}{2}\sqrt{41}$ (d) $2\sqrt{41}$

Solution: (c) Equation of chord of contact of tangents drawn from a point (x_1, y_1) to parabola $y^2=4ax$ is $yy_1=2a(x+x_1)$. So that $5y=2\times 2(x+2) \Rightarrow 5y=4x+8$.

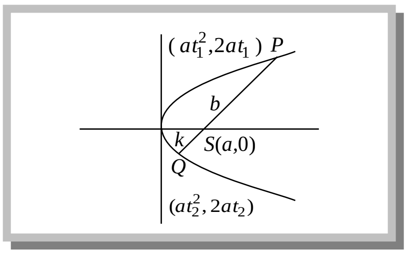
Point of intersection of chord of contact with parabola $y^2=8x$ are $\left(\frac{1}{2}, 2\right)$, (8, 8), So the length of chord is $\frac{3}{2}\sqrt{41}$.

Example: 36 If b, k are the intercept of a focal chord of the parabola $y^2=4ax$, then K is equal to **[Rajasthan PET 1999]**

(a) $\frac{ab}{b-a}$ (b) $\frac{b}{b-a}$ (c) $\frac{a}{b-a}$ (d) $\frac{ab}{a-b}$

Solution: (a) Let 't_1' 't_2' be the ends of focal chords

$\therefore$. $t_1t_2=-1$. If S is the focus and P, Q are the ends of the focal chord, then



$SP=\sqrt{(at_1^2-a)^2+(2at_1-0)^2}=a(t_1^2+1)=b$ (Given).... (i)

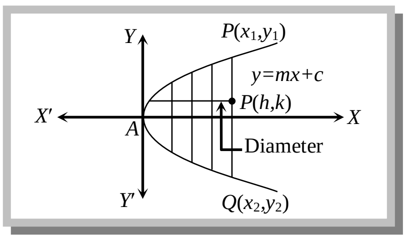
$\therefore\ SQ=a(t_2^2+1)=a\left(\frac{1}{t_1^2}+1\right)$ (Given) $\left[\because t_2=-\frac{1}{t_1} \Rightarrow t_2^2=\frac{1}{t_1^2}\right]$

$=\frac{a(t_1^2+1)}{t_1^2}=k$ ....(ii), $\therefore\ \frac{b}{k}=t_1^2$ [Divide (i) by (ii)]

Putting in (1), we get $a\left(\frac{b}{k}+1\right)=b \Rightarrow \frac{ab}{k}+a=b \Rightarrow k=\frac{ab}{b-a}$

5.1.18 Diameter of a Parabola

The locus of the middle points of a system of parallel chords is called a diameter and in case of a parabola this diameter is shown to be a straight line which is parallel to the axis of the parabola.



The equation of the diameter bisecting chords of the parabola $y^2 = 4ax$ of slope m is $y = \frac{2a}{m}$

Note : ❑ Every diameter of a parabola is parallel to its axis.

❑ The tangent at the end point of a diameter is parallel to corresponding system of parallel chords.

❑ The tangents at the ends of any chord meet on the diameter which bisects the chord.

Example: 37 Equation of diameter of parabola $y^2 = x$ corresponding to the chord $x - y + 1 = 0$ is **[Rajasthan PET 2003]**

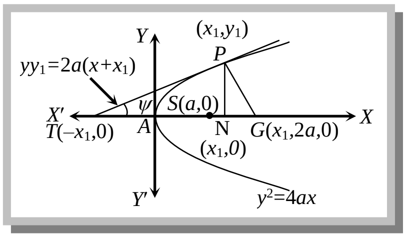
(a) $2y = 3$ (b) $2y = 1$ (c) $2y = 5$ (d) $y = 1$

Solution: (b) Equation of diameter of parabola is $y = \frac{2a}{m}$, Here $a = \frac{1}{4}, m = 1 \Rightarrow y = \frac{2.\frac{1}{4}}{1} \Rightarrow 2y = 1$

5.1.19 Length of Tangent, Subtangent ,Normal and Subnormal

Let the parabola $y^2 = 4ax$. Let the tangent and normal at $P(x_1, y_1)$ meet the axis of parabola at T and G respectively, and tangent at $P(x_1, y_1)$ makes angle ψ with the positive direction of x-axis.

$A(0,0)$ is the vertex of the parabola and $PN = y$. Then,



(1) Length of tangent $= PT = PN \text{ cosec } \psi = y_1 \text{ cosec } \psi$

(2) Length of normal $= PG = PN \text{cosec}(90^o - \psi) = y_1 \sec\psi$

(3) Length of subtangent $= TN = PN \cot\psi = y_1 \cot\psi$

(4) Length of subnormal $= NG = PN \cot(90^o - \psi) = y_1 \tan\psi$

where , $\tan\psi = \frac{2a}{y_1} = m$, [slope of tangent at $P(x, y)$]

Length of tangent, subtangent, normal and subnormal to $y^2 = 4ax$ at $(at^2, 2at)$

(1) Length of tangent at $(at^2, 2at) = 2at \text{ cosec } \psi = 2at\sqrt{(1 + \cot^2\psi)} = 2at\sqrt{1 + t^2}$

(2) Length of normal at $(at^2, 2at) = 2at \sec\psi = 2at\sqrt{(1 + \tan^2\psi)} = 2a\sqrt{t^2 + t^2\tan^2\psi} = 2a\sqrt{(t^2 + 1)}$

(3) Length of subtangent at $(at^2, 2at) = 2at\cot\psi = 2at^2$

(4) Length of subnormal at $(at^2, 2at) = 2at\tan\psi = 2a$

Example: 38 The length of the subtangent to the parabola $y^2 = 16x$ at the point whose abscissa is 4, is

(a) 2 (b) 4 (c) 8 (d) None of these

Solution: (c) Since the length of the subtangent at a point to the parabola is twice the abscissa of the point. Therefore, the required length is 8.

Example: 39 If P is a point on the parabola $y^2 = 4ax$ such that the subtangent and subnormal at P are equal, then the coordinates of P are

(a) $(a, 2a)$ or $(a, -2a)$ (b) $(2a, 2\sqrt{2}a)$ or $(2a, -2\sqrt{2}a)$

(c) $(4a, -4a)$ or $(4a, 4a)$ (d) None of these

Solution: (a) Since the length of the subtangent at a point on the parabola is twice the abscissa of the point and the length of the subnormal is equal to semi-latus-rectum. Therefore if $P(x, y)$ is the required point, then $2x = 2a \Rightarrow x = a$

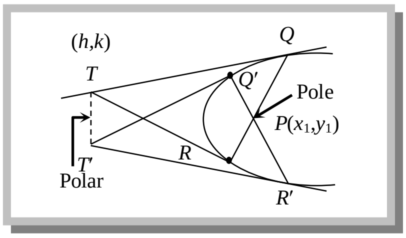
Now (x, y) lies on the parabola $y^2 = 4ax \Rightarrow 4a^2 = y^2 \Rightarrow y = \pm 2a$

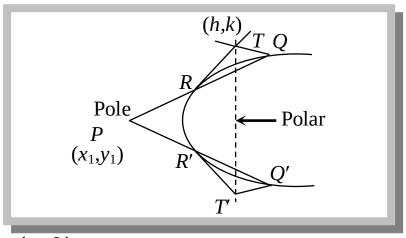
Thus the required points are $(a, 2a)$ and $(a, -2a)$.

5.1.20 Pole and Polar

The locus of the point of intersection of the tangents to the parabola at the ends of a chord drawn from a fixed point P is called the polar of point P and the point P is called the pole of the polar.

Equation of polar: Equation of polar of the point (x_1, y_1) with respect to parabola $y^2 = 4ax$ is same as chord of contact and is given by $yy_1 = 2a(x + x_1)$

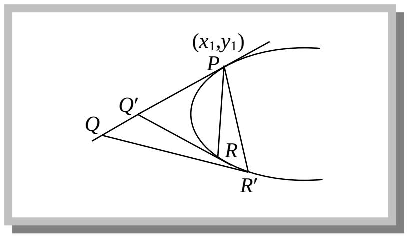




(1) **Polar of the focus is directrix:** Since the focus is $(a, 0)$

$\therefore$ Equation of polar of $y^2 = 4ax$ is $y.0 = 2a(x + a) \Rightarrow x + a = 0$, which is the directrix of the parabola $y^2 = 4ax$.

(2) **Any tangent is the polar of its point of contact:** If the point $P(x_1 y_1)$ be on the parabola. Its polar and tangent at P are identical. Hence the tangent is the polar of its own point of contact.



Coordinates of pole: The pole of the line $lx + my + n = 0$ with respect to the parabola $y^2 = 4ax$ is $\left(\frac{n}{l}, \frac{-2am}{l}\right)$.

(i) Pole of the chord joining (x_1, y_1) and (x_2, y_2) is $\left(\frac{y_1 y_2}{4a}, \frac{y_1 + y_2}{2}\right)$ which is the same as the point of intersection of tangents at (x_1, y_1) and (x_2, y_2).

(ii) The point of intersection of the polar of two points Q and R is the pole of QR.

5.1.21 Characterstics of Pole and Polar

(1) **Conjugate points:** If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of $Q(x_2, y_2)$ goes through $P(x_1, y_1)$ and such points are said to be conjugate points.

Two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are conjugate points with respect to the parabola $y^2 = 4ax$, if $y_1 y_2 = 2a(x_1 + x_2)$.

(2) **Conjugate lines:** If the pole of a line $ax + by + c = 0$ lies on the another line $a_1 x + b_1 y + c_1 = 0$, then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

Two lines $l_1 x + m_1 y + n_1 = 0$ and $l_2 x + m_2 y + n_2 = 0$ are conjugate lines with respect to parabola $y^2 = 4ax$, if $(l_1 n_2 + l_2 n_1) = 2am_1 m_2$

Note : ❑ The chord of contact and polar of any point on the directrix always passes through focus.

❑ The pole of a focal chord lies on directrix and locus of poles of focal chord is the directrix.

❑ The polars of all points on directrix always pass through a fixed point and this fixed point is focus.

Example: 40 The pole of the line $2x = y$ with respect to the parabola $y^2 = 2x$ is

(a) $\left(0, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, 0\right)$ (c) $\left(0, -\frac{1}{2}\right)$ (d) None of these

Solution: (a) Let (x_1, y_1) be the pole of line $2x = y$ *w.r.t.* parabola $y^2 = 2x$ its polar is $yy_1 = x + x_1$

Also polar is $y = 2x$, $\therefore \frac{y_1}{1} = \frac{1}{2} = \frac{x_1}{0}$,$\therefore x_1 = 0, y_1 = \frac{1}{2}$. So Pole is $\left(0, \frac{1}{2}\right)$

Example: 41 If the polar of a point with respect to the circle $x^2 + y^2 = r^2$ touches the parabola $y^2 = 4ax$, the locus of the pole is **[EAMCET 1995]**

(a) $y^2 = -\frac{r^2}{a}x$ (b) $x^2 = \frac{-r^2}{a}y$ (c) $y^2 = \frac{r^2}{a}x$ (d) $x^2 = \frac{r^2}{a}y$

Solution: (a) Polar of a point (x_1, y_1) *w.r.t.* $x^2 + y^2 = r^2$ is $xx_1 + yy_1 = r^2$ *i.e.* $yy_1 = -xx_1 + r^2$

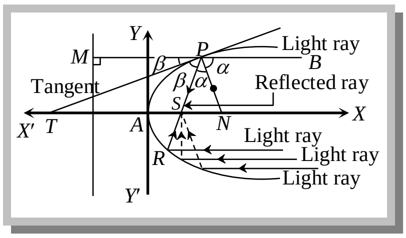
$\Rightarrow y = -\frac{x_1}{y_1}x + \frac{r^2}{y_1} \Rightarrow y = mx + c$, where $m = -\frac{x_1}{y_1}; c = \frac{r^2}{y_1}$

This touches the parabola $y^2 = 4ax$, If $c = \frac{a}{m} \Rightarrow \frac{r^2}{y_1} = \frac{a}{-x_1/y_1} = -\frac{ay_1}{x_1}$

$\therefore$ Required locus of pole (x_1, y_1) is $\frac{r^2}{y} = \frac{-ay}{x}$ *i.e.*, $y^2 = \frac{-r^2}{a}x$

5.1.22 Reflection property of a Parabola

The tangent (PT) and normal (PN) of the parabola $y^2 = 4ax$ at. P are the internal and external bisectors of $\angle SPM$ and BP is parallel to the axis of the parabola and $\angle BPN = \angle SPN$



Note : ❑ When the incident ray is parallel to the axis of the parabola, the reflected ray will always pass through the focus.

Example: 42 A ray of light moving parallel to the *x*-axis gets reflected from a parabolic mirror whose equation is $(y - 2)^2 = 4(x + 1)$. After reflection, the ray must pass through the point

(a) (0, 2) (b) (2, 0) (c) (0, –2) (d) (–1, 2)

Solution: (a) The equation of the axis of the parabola is $y - 2 = 0$, which is parallel to the *x*-axis. So, a ray parallel to *x*-axis is parallel to the axis of the parabola. We know that any ray parallel to the axis of a parabola passes through the focus after reflection. Here (0, 2) is the focus.

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