# CHAPTER XXIX.

## SUMMATION OF SERIES.

380. Examples of summation of certain series have occurred in previous chapters; it will be convenient here to give a synopsis of the methods of summation which have already been explained.

(i) Arithmetical Progression, Chap. IV.

(ii) Geometrical Progression, Chap. V.

(iii) Series which are partly arithmetical and partly geometrical, Art. 60.

(iv) Sums of the powers of the Natural Numbers and allied Series, Arts. 68 to 75.

(v) Summation by means of Undetermined Coefficients, Art. 312.

(vi) Recurring Series, Chap. XXIV.

We now proceed to discuss methods of greater generality; but in the course of the present chapter it will be seen that some of the foregoing methods may still be usefully employed.

381. If the $r^{\text{th}}$ term of a series can be expressed as the difference of two quantities one of which is the same function of $r$ that the other is of $r-1$, the sum of the series may be readily found.

For let the series be denoted by

$$u_1 + u_2 + u_3 + \dots + u_n,$$

and its sum by $S_n$, and suppose that any term $u_r$ can be put in the form $v_r - v_{r-1}$; then

$$\begin{aligned} S_n &= (v_1 - v_0) + (v_2 - v_1) + (v_3 - v_2) + \dots + (v_{n-1} - v_{n-2}) + (v_n - v_{n-1}) \\ &= v_n - v_0. \end{aligned}$$

*Example.* Sum to $n$ terms the series

$$\frac{1}{(1+x)(1+2x)}+\frac{1}{(1+2x)(1+3x)}+\frac{1}{(1+3x)(1+4x)}+\ldots\ldots$$

If we denote the series by

$$u_1+u_2+u_3+\ldots\ldots+u_n,$$

we have
$$u_1=\frac{1}{x}\left(\frac{1}{1+x}-\frac{1}{1+2x}\right),$$

$$u_2=\frac{1}{x}\left(\frac{1}{1+2x}-\frac{1}{1+3x}\right),$$

$$u_3=\frac{1}{x}\left(\frac{1}{1+3x}-\frac{1}{1+4x}\right),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u_n=\frac{1}{x}\left(\frac{1}{1+nx}-\frac{1}{1+\overline{n+1}\,.\,x}\right),$$

$\therefore$ by addition,
$$S_n=\frac{1}{x}\left(\frac{1}{1+x}-\frac{1}{1+\overline{n+1}\,.\,x}\right)$$

$$=\frac{n}{(1+x)(1+\overline{n+1}\,.\,x)}.$$

382. Sometimes a suitable transformation may be obtained by separating $u_n$ into partial fractions by the methods explained in Chap. XXIII.

*Example.* Find the sum of

$$\frac{1}{(1+x)(1+ax)}+\frac{a}{(1+ax)(1+a^2x)}+\frac{a^2}{(1+a^2x)(1+a^3x)}+\ldots \text{ to } n \text{ terms.}$$

The $n^{\text{th}}$ term $=\dfrac{a^{n-1}}{(1+a^{n-1}x)(1+a^nx)}=\dfrac{A}{1+a^{n-1}x}+\dfrac{B}{1+a^nx}$ suppose;

$$\therefore\ a^{n-1}=A(1+a^nx)+B(1+a^{n-1}x).$$

By putting $1+a^{n-1}x$, $1+a^nx$ equal to zero in succession, we obtain

$$A=\frac{a^{n-1}}{1-a},\quad B=-\frac{a^n}{1-a}.$$

Hence
$$u_1=\frac{1}{1-a}\left(\frac{1}{1+x}-\frac{a}{1+ax}\right),$$

similarly,
$$u_2=\frac{1}{1-a}\left(\frac{a}{1+ax}-\frac{a^2}{1+a^2x}\right),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u_n=\frac{1}{1-a}\left(\frac{a^{n-1}}{1+a^{n-1}x}-\frac{a^n}{1+a^nx}\right),$$

$$\therefore\ S_n=\frac{1}{1-a}\left(\frac{1}{1+x}-\frac{a^n}{1+a^nx}\right).$$

383. *To find the sum of* n *terms of a series each term of which is composed of* r *factors in arithmetical progression, the first factors of the several terms being in the same arithmetical progression.*

Let the series be denoted by $u_1 + u_2 + u_3 + \ldots\ldots + u_n$, where

$$u_n = (a + nb)(a + \overline{n+1} \,.\, b)(a + \overline{n+2} \,.\, b) \ldots (a + \overline{n+r-1} \,.\, b).$$

Replacing $n$ by $n-1$, we have

$$u_{n-1} = (a + \overline{n-1} \,.\, b)(a + nb)(a + \overline{n+1} \,.\, b) \ldots (a + \overline{n+r-2} \,.\, b);$$

$$\therefore\ (a + \overline{n-1} \,.\, b)\, u_n = (a + \overline{n+r-1} \,.\, b)\, u_{n-1} = v_n, \text{ say.}$$

Replacing $n$ by $n+1$ we have

$$(a + \overline{n+r} \,.\, b)\, u_n = v_{n+1};$$

therefore, by subtraction,

$$(r+1)\, b \,.\, u_n = v_{n+1} - v_n.$$

Similarly, $$(r+1)\, b \,.\, u_{n-1} = v_n - v_{n-1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(r+1)\, b \,.\, u_2 = v_3 - v_2,$$

$$(r+1)\, b \,.\, u_1 = v_2 - v_1.$$

By addition, $$(r+1)\, b \,.\, S_n = v_{n+1} - v_1;$$

that is, $$S_n = \frac{v_{n+1} - v_1}{(r+1)\, b}$$

$$= \frac{(a + \overline{n+r} \,.\, b)\, u_n}{(r+1)\, b} + C, \text{ say;}$$

where $C$ is a quantity independent of $n$, which may be found by ascribing to $n$ some particular value.

The above result gives us the following convenient rule:

*Write down the* n<sup></sup>th *term, affix the next factor at the end, divide by the number of factors thus increased and by the common difference, and add a constant.*

It may be noticed that $C = -\dfrac{v_1}{(r+1)\, b} = -\dfrac{a}{(r+1)\, b}\, u_1$; it is however better not to quote this result, but to obtain $C$ as above indicated.

*Example.* Find the sum of $n$ terms of the series

$$1 \cdot 3 \cdot 5 + 3 \cdot 5 \cdot 7 + 5 \cdot 7 \cdot 9 + \ldots\ldots$$

The $n^{\text{th}}$ term is $(2n-1)(2n+1)(2n+3)$; hence by the rule

$$S_n = \frac{(2n-1)(2n+1)(2n+3)(2n+5)}{4 \cdot 2} + C.$$

To determine $C$, put $n=1$; then the series reduces to its first term, and we have

$$15 = \frac{1 \cdot 3 \cdot 5 \cdot 7}{8} + C; \text{ whence } C = \frac{15}{8};$$

$$\therefore S_n = \frac{(2n-1)(2n+1)(2n+3)(2n+5)}{8} + \frac{15}{8}$$

$$= n(2n^3 + 8n^2 + 7n - 2), \text{ after reduction.}$$

384. The sum of the series in the preceding article may also be found either by the method of Undetermined Coefficients [Art. 312] or in the following manner.

We have $u_n = (2n-1)(2n+1)(2n+3) = 8n^3 + 12n^2 - 2n - 3$;

$$\therefore S_n = 8\Sigma n^3 + 12\Sigma n^2 - 2\Sigma n - 3n,$$

using the notation of Art. 70;

$$\therefore S_n = 2n^2(n+1)^2 + 2n(n+1)(2n+1) - n(n+1) - 3n$$

$$= n(2n^3 + 8n^2 + 7n - 2).$$

385. It should be noticed that the rule given in Art. 383 is only applicable to cases in which the factors of each term form an arithmetical progression, and the first factors of the several terms are in the *same* arithmetical progression.

Thus the sum of the series

$$1 \cdot 3 \cdot 5 + 2 \cdot 4 \cdot 6 + 3 \cdot 5 \cdot 7 + \ldots\ldots \text{ to } n \text{ terms,}$$

may be found by either of the methods suggested in the preceding article, but not directly by the rule of Art. 383. Here

$$u_n = n(n+2)(n+4) = n(\overline{n+1} + 1)(\overline{n+2} + 2)$$

$$= n(n+1)(n+2) + 2n(n+1) + n(n+2) + 2n$$

$$= n(n+1)(n+2) + 3n(n+1) + 3n.$$

The rule can now be applied to each term; thus

$$S_n = \tfrac{1}{4}n(n+1)(n+2)(n+3) + n(n+1)(n+2) + \tfrac{3}{2}n(n+1) + C$$

$$= \tfrac{1}{4}n(n+1)(n+4)(n+5), \text{ the constant being zero.}$$

386. *To find the sum of* n *terms of a series each term of which is composed of the reciprocal of the product of* r *factors in arithmetical progression, the first factors of the several terms being in the same arithmetical progression.*

Let the series be denoted by $u_1 + u_2 + u_3 + \ldots\ldots + u_n$, where

$$\frac{1}{u_n} = (a+nb)(a+\overline{n+1}\,.\,b)(a+\overline{n+2}\,.\,b)\ldots\ldots(a+\overline{n+r-1}\,.\,b).$$

Replacing $n$ by $n-1$,

$$\frac{1}{u_{n-1}} = (a+\overline{n-1}\,.\,b)\,(a+nb)\,(a+\overline{n+1}\,.\,b)\ldots(a+\overline{n+r-2}\,.\,b);$$

$$\therefore\ (a+\overline{n+r-1}\,.\,b)\,u_n = (a+\overline{n-1}\,.\,b)\,u_{n-1} = v_n,\ \text{say.}$$

Replacing $n$ by $n+1$, we have

$$(a+nb)\,u_n = v_{n+1};$$

therefore, by subtraction,

$$(r-1)\,b\,.\,u_n = v_n - v_{n+1}.$$

Similarly $\qquad (r-1)\,b\,.\,u_{n-1} = v_{n-1} - v_n,$

.........................

$$(r-1)\,b\,.\,u_2 = v_2 - v_3,$$

$$(r-1)\,b\,.\,u_1 = v_1 - v_2.$$

By addition, $\qquad (r-1)\,b\,.\,S_n = v_1 - v_{n+1};$

that is $\qquad S_n = \dfrac{v_1 - v_{n+1}}{(r-1)\,b} = C - \dfrac{(a+nb)\,u_n}{(r-1)\,b},$

where $C$ is a quantity independent of $n$, which may be found by ascribing to $n$ some particular value.

Thus $S_n = C - \dfrac{1}{(r-1)\,b}\,.\,\dfrac{1}{(a+\overline{n+1}\,.\,b)\ldots(a+\overline{n+r-1}\,.\,b)}.$

Hence the sum may be found by the following rule:

*Write down the* n[th] *term, strike off a factor from the beginning, divide by the number of factors so diminished and by the common difference, change the sign and add a constant.*

The value of $C = \dfrac{v_1}{(r-1)\,b} = \dfrac{a+rb}{(r-1)\,b}\,u_1$; but it is advisable in each case to determine $C$ by ascribing to $n$ some particular value.

*Example* 1. Find the sum of $n$ terms of the series

$$\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+\ldots\ldots$$

The $n^{\text{th}}$ term is $\dfrac{1}{n(n+1)(n+2)(n+3)}$;

hence, by the rule, we have

$$S_n=C-\frac{1}{3(n+1)(n+2)(n+3)}.$$

Put $n=1$, then $\dfrac{1}{1.2.3.4}=C-\dfrac{1}{3.2.3.4}$; whence $C=\dfrac{1}{18}$;

$$\therefore\ S_n=\frac{1}{18}-\frac{1}{3(n+1)(n+2)(n+3)}.$$

By making $n$ indefinitely great, we obtain $S_\infty=\dfrac{1}{18}$.

*Example* 2. Find the sum to $n$ terms of the series

$$\frac{3}{1.2.4}+\frac{4}{2.3.5}+\frac{5}{3.4.6}+\ldots\ldots$$

Here the rule is not directly applicable, because although 1, 2, 3, ......, the first factors of the several denominators, are in arithmetical progression, the factors of any one denominator are not. In this example we may proceed as follows:

$$u_n=\frac{n+2}{n(n+1)(n+3)}=\frac{(n+2)^2}{n(n+1)(n+2)(n+3)}$$

$$=\frac{n(n+1)+3n+4}{n(n+1)(n+2)(n+3)}$$

$$=\frac{1}{(n+2)(n+3)}+\frac{3}{(n+1)(n+2)(n+3)}+\frac{4}{n(n+1)(n+2)(n+3)}.$$

Each of these expressions may now be taken as the $n^{\text{th}}$ term of a series to which the rule is applicable.

$$\therefore\ S_n=C-\frac{1}{n+3}-\frac{3}{2(n+2)(n+3)}-\frac{4}{3(n+1)(n+2)(n+3)};$$

put $n=1$, then

$$\frac{3}{1.2.4}=C-\frac{1}{4}-\frac{3}{2.3.4}-\frac{4}{3.2.3.4};\ \text{whence } C=\frac{29}{36};$$

$$\therefore\ S_n=\frac{29}{36}-\frac{1}{n+3}-\frac{3}{2(n+2)(n+3)}-\frac{4}{3(n+1)(n+2)(n+3)}.$$

387. In cases where the methods of Arts. 383, 386 are *directly* applicable, instead of quoting the rules we may always effect the summation in the following way, which is sometimes called 'the Method of Subtraction.'

*Example.* Find the sum of $n$ terms of the series

$$2 \cdot 5 + 5 \cdot 8 + 8 \cdot 11 + 11 \cdot 14 + \ldots\ldots$$

The arithmetical progression in this case is

$$2, 5, 8, 11, 14, \ldots\ldots$$

In each term of the given series introduce as a new factor the next term of the arithmetical progression; denote this series by $S'$, and the given series by $S$; then

$$S' = 2 \cdot 5 \cdot 8 + 5 \cdot 8 \cdot 11 + 8 \cdot 11 \cdot 14 + \ldots\ldots + (3n-1)(3n+2)(3n+5);$$

$$\therefore S' - 2 \cdot 5 \cdot 8 = 5 \cdot 8 \cdot 11 + 8 \cdot 11 \cdot 14 + 11 \cdot 14 \cdot 17 + \ldots \text{ to } (n-1) \text{ terms.}$$

By subtraction,

$$-2 \cdot 5 \cdot 8 = 9[5 \cdot 8 + 8 \cdot 11 + 11 \cdot 14 + \ldots \text{to} (n-1) \text{ terms}] - (3n-1)(3n+2)(3n+5),$$

$$-2 \cdot 5 \cdot 8 = 9[S - 2 \cdot 5] - (3n-1)(3n+2)(3n+5),$$

$$9S = (3n-1)(3n+2)(3n+5) - 2 \cdot 5 \cdot 8 + 2 \cdot 5 \cdot 9,$$

$$S = n(3n^2 + 6n + 1).$$

388. When the $n^{\text{th}}$ term of a series is a rational integral function of $n$ it can be expressed in a form which will enable us readily to apply the method given in Art. 383.

For suppose $\phi(n)$ is a rational integral function of $n$ of $p$ dimensions, and assume

$$\phi(n) = A + Bn + Cn(n+1) + Dn(n+1)(n+2) + \ldots\ldots,$$

where $A, B, C, D, \ldots\ldots$ are undetermined constants $p+1$ in number.

This identity being true for all values of $n$, we may equate the coefficients of like powers of $n$; we thus obtain $p+1$ simple equations to determine the $p+1$ constants.

*Example.* Find the sum of $n$ terms of the series whose general term is

$$n^4 + 6n^3 + 5n^2.$$

Assume

$$n^4 + 6n^3 + 5n^2 = A + Bn + Cn(n+1) + Dn(n+1)(n+2) + En(n+1)(n+2)(n+3);$$

it is at once obvious that $A=0$, $B=0$, $E=1$; and by putting $n=-2$, $n=-3$ successively, we obtain $C=-6$, $D=0$. Thus

$$n^4 + 6n^3 + 5n^2 = n(n+1)(n+2)(n+3) - 6n(n+1).$$

Hence $$S_n=\frac{1}{5}n(n+1)(n+2)(n+3)(n+4)-2n(n+1)(n+2)$$
$$=\frac{1}{5}n(n+1)(n+2)(n^2+7n+2).$$

## Polygonal and Figurate Numbers.

389. If in the expression $n+\frac{1}{2}n(n-1)b$, which is the sum of $n$ terms of an arithmetical progression whose first term is 1 and common difference $b$, we give to $b$ the values 0, 1, 2, 3, ..., we get

$$n,\ \tfrac{1}{2}n(n+1),\ n^2,\ \tfrac{1}{2}n(3n-1),\ldots\ldots,$$

which are the $n^{\text{th}}$ terms of the **Polygonal Numbers** of the second, third, fourth, fifth,......orders; the first order being that in which each term is unity. The polygonal numbers of the second, third, fourth, fifth,......orders are sometimes called *linear*, *triangular*, *square*, *pentagonal*,......

390. *To find the sum of the first* n *terms of the* r$^{\text{th}}$ *order of polygonal numbers.*

The $n^{\text{th}}$ term of the $r^{\text{th}}$ order is $n+\frac{1}{2}n(n-1)(r-2)$;

$$\therefore\ S_n=\Sigma n+\tfrac{1}{2}(r-2)\,\Sigma(n-1)n$$
$$=\tfrac{1}{2}n(n+1)+\tfrac{1}{6}(r-2)(n-1)n(n+1)\ \text{[Art. 383]}$$
$$=\tfrac{1}{6}n(n+1)\{(r-2)(n-1)+3\}.$$

391. If the sum of $n$ terms of the series

$$1,\ 1,\ 1,\ 1,\ 1,\ \ldots\ldots,$$

be taken as the $n^{\text{th}}$ term of a new series, we obtain

$$1,\ 2,\ 3,\ 4,\ 5,\ \ldots\ldots.$$

If again we take $\dfrac{n(n+1)}{2}$, which is the sum of $n$ terms of the last series, as the $n^{\text{th}}$ term of a new series, we obtain

$$1,\ 3,\ 6,\ 10,\ 15,\ \ldots\ldots.$$

By proceeding in this way, we obtain a succession of series such that in any one, the $n^{\text{th}}$ term is the sum of $n$ terms of the preceding series. The successive series thus formed are known as **Figurate Numbers** of the first, second, third, ... orders.

392. *To find the* n$^{th}$ *term and the sum of* n *terms of the* r$^{th}$ *order of figurate numbers.*

The $n^{th}$ term of the first order is 1; the $n^{th}$ term of the second order is $n$; the $n^{th}$ term of the third order is $\Sigma n$, that is $\frac{1}{2}n(n+1)$; the $n^{th}$ term of the fourth order is $\Sigma \frac{n(n+1)}{1 \cdot 2}$, that is $\frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}$; the $n^{th}$ term of the fifth order is $\Sigma \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}$, that is $\frac{n(n+1)(n+2)(n+3)}{\underline{|4}}$; and so on.

Thus it is easy to see that the $n^{th}$ term of the $r^{th}$ order is

$$\frac{n(n+1)(n+2)\dots(n+r-2)}{\underline{|r-1}}, \quad \text{or} \quad \frac{\underline{|n+r-2}}{\underline{|n-1}\,\underline{|r-1}}.$$

Again, the sum of $n$ terms of the $r^{th}$ order is

$$\frac{n(n+1)(n+2)\dots(n+r-1)}{\underline{|r}},$$

which is the $n^{th}$ term of the $(r+1)^{th}$ order.

NOTE. In applying the rule of Art. 383 to find the sum of $n$ terms of any order of figurate numbers, it will be found that the constant is always zero.

393. The properties of *figurate numbers* are historically interesting on account of the use made of them by Pascal in his *Traité du triangle arithmétique*, published in 1665.

The following table exhibits the *Arithmetical Triangle* in its simplest form

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 ... |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 ... | |
| 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 ... | | |
| 1 | 4 | 10 | 20 | 35 | 56 | 84 ... | | | |
| 1 | 5 | 15 | 35 | 70 | 126 ... | | | | |
| 1 | 6 | 21 | 56 | 126 ... | | | | | |
| 1 | 7 | 28 | 84 ... | | | | | | |
| 1 | 8 | 36 ... | | | | | | | |
| 1 | 9 ... | | | | | | | | |
| 1 ... | | | | | | | | | |

Pascal constructed the numbers in the triangle by the following rule:

*Each number is the sum of that immediately above it and that immediately to the left of it;*

thus $15 = 5 + 10,\ 28 = 7 + 21,\ 126 = 56 + 70.$

From the mode of construction, it follows that the numbers in the successive horizontal rows, or vertical columns, are the figurate numbers of the first, second, third, ... orders.

A line drawn so as to cut off an equal number of units from the top row and the left-hand column is called a *base*, and the bases are numbered beginning from the top left-hand corner. Thus the 6th base is a line drawn through the numbers 1, 5, 10, 10, 5, 1; and it will be observed that there are six of these numbers, and that they are the coefficients of the terms in the expansion of $(1 + x)^5$.

The properties of these numbers were discussed by Pascal with great skill: in particular he used his *Arithmetical Triangle* to develop the theory of Combinations, and to establish some interesting propositions in Probability. The subject is fully treated in Todhunter's *History of Probability*, Chapter II.

394. Where no ambiguity exists as to the number of terms in a series, we have used the symbol $\Sigma$ to indicate summation; but in some cases the following modified notation, which indicates the limits between which the summation is to be effected, will be found more convenient.

Let $\phi(x)$ be any function of $x$, then $\sum\limits_{x=l}^{x=m} \phi(x)$ denotes the sum of the series of terms obtained from $\phi(x)$ by giving to $x$ all positive integral values from $l$ to $m$ inclusive.

For instance, suppose it is required to find the sum of all the terms of the series obtained from the expression

$$\frac{(p-1)(p-2)\dots(p-r)}{\underline{|r}}$$

by giving to $p$ all integral values from $r + 1$ to $p$ inclusive.

Writing the factors of the numerator in ascending order,

$$\text{the required sum} = \sum_{p=r+1}^{p=p} \frac{(p-r)(p-r+1)\ldots(p-1)}{\underline{|r}}$$

$$= \frac{1}{\underline{|r}}\{1.2.3\ldots r + 2.3.4\ldots(r+1) + \ldots + (p-r)(p-r+1)\ldots(p-1)\}$$

$$= \frac{1}{\underline{|r}} \cdot \frac{(p-r)(p-r+1)\ldots(p-1)\,p}{r+1}, \qquad \text{[Art. 383.]}$$

$$= \frac{p(-1)(p-2)\ldots(p-r)}{\underline{|r+1}}.$$

Since the given expression is zero for all values of $p$ from 1 to $r$ inclusive, we may write the result in the form

$$\sum_{p=1}^{p=p} \frac{(p-1)(p-2)\ldots(p-r)}{\underline{|r}} = \frac{p(p-1)(p-2)\ldots(p-r)}{\underline{|r+1}}.$$

## EXAMPLES. XXIX. a.

Sum the following series to $n$ terms:

1. $1.2.3 + 2.3.4 + 3.4.5 + \ldots\ldots$
2. $1.2.3.4 + 2.3.4.5 + 3.4.5.6 + \ldots\ldots$
3. $1.4.7 + 4.7.10 + 7.10.13 + \ldots\ldots$
4. $1.4.7 + 2.5.8 + 3.6.9 + \ldots\ldots$
5. $1.5.9 + 2.6.10 + 3.7.11 + \ldots\ldots$

Sum the following series to $n$ terms and to infinity:

6. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots\ldots$
7. $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots\ldots$
8. $\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + \ldots\ldots$
9. $\frac{1}{1.4.7} + \frac{1}{4.7.10} + \frac{1}{7.10.13} + \ldots\ldots$
10. $\frac{4}{1.2.3} + \frac{5}{2.3.4} + \frac{6}{3.4.5} + \ldots\ldots$
11. $\frac{1}{3.4.5} + \frac{2}{4.5.6} + \frac{3}{5.6.7} + \ldots\ldots$
12. $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \frac{7}{4.5.6} + \ldots\ldots$

Find the sum of $n$ terms of the series:

13. $1 . 3 . 2^2 + 2 . 4 . 3^2 + 3 . 5 . 4^2 + \ldots\ldots$

14. $(n^2 - 1^2) + 2(n^2 - 2^2) + 3(n^2 - 3^2) + \ldots\ldots$

Find the sum of $n$ terms of the series whose $n^{\text{th}}$ term is

15. $n^2(n^2 - 1)$.

16. $(n^2 + 5n + 4)(n^2 + 5n + 8)$.

17. $\dfrac{n^2(n^2-1)}{4n^2-1}$.

18. $\dfrac{n^4 + 2n^3 + n^2 - 1}{n^2 + n}$.

19. $\dfrac{n^3 + 3n^2 + 2n + 2}{n^2 + 2n}$.

20. $\dfrac{n^4 + n^2 + 1}{n^4 + n}$.

21. Shew that the $n^{\text{th}}$ term of the $r^{\text{th}}$ order of figurate numbers is equal to the $r^{\text{th}}$ term of the $n^{\text{th}}$ order.

22. If the $n^{\text{th}}$ term of the $r^{\text{th}}$ order of figurate numbers is equal to the $(n+2)^{\text{th}}$ term of the $(r-2)^{\text{th}}$ order, shew that $r = n + 2$.

23. Shew that the sum of the first $n$ of all the sets of polygonal numbers from the linear to that of the $r^{\text{th}}$ order inclusive is

$$\frac{(r-1)n(n+1)}{12}(rn - 2n - r + 8).$$

## Summation by the Method of Differences.

395. Let $u_n$ denote some rational integral function of $n$, and let $u_1, u_2, u_3, u_4, \ldots$ denote the values of $u_n$ when for $n$ the values 1, 2, 3, 4,... are written successively.

We proceed to investigate a method of finding $u_n$ when a certain number of the terms $u_1, u_2, u_3, u_4, \ldots$ are given.

From the series $u_1, u_2, u_3, u_4, u_5, \ldots$ obtain a second series by subtracting each term from the term which immediately follows it.

The series

$$u_2 - u_1,\quad u_3 - u_2,\quad u_4 - u_3,\quad u_5 - u_4, \ldots$$

thus found is called the *series of the first order of differences*, and may be conveniently denoted by

$$\Delta u_1,\quad \Delta u_2,\quad \Delta u_3,\quad \Delta u_4, \ldots$$

By subtracting each term of this series from the term that immediately follows it, we have

$$\Delta u_2 - \Delta u_1,\quad \Delta u_3 - \Delta u_2,\quad \Delta u_4 - \Delta u_3, \ldots$$

which may be called the *series of the second order of differences*, and denoted by

$$\Delta_2 u_1,\quad \Delta_2 u_2,\quad \Delta_3 u_3, \ldots$$

From this series we may proceed to form the *series of the third, fourth, fifth,... orders of differences*, the general terms of these series being $\Delta_3 u_r$, $\Delta_4 u_r$, $\Delta_5 u_r$,... respectively.

From the law of formation of the series

$$\begin{array}{cccccc} u_1, & u_2, & u_3, & u_4, & u_5, & u_6, \ldots\ldots\ldots \\ & \Delta u_1, & \Delta u_2, & \Delta u_3, & \Delta u_4, & \Delta u_5, \ldots\ldots\ldots \\ & & \Delta_2 u_1, & \Delta_2 u_2, & \Delta_2 u_3, & \Delta_2 u_4, \ldots\ldots\ldots \\ & & & \Delta_3 u_1, & \Delta_3 u_2, & \Delta_3 u_3, \ldots\ldots\ldots \\ & & & \ldots\ldots\ldots & \ldots\ldots\ldots & \end{array}$$

it appears that any term in any series is equal to the term immediately preceding it added to the term below it on the left.

Thus $\qquad u_2 = u_1 + \Delta u_1$, and $\Delta u_2 = \Delta u_1 + \Delta_2 u_1$.

By addition, since $u_2 + \Delta u_2 = u_3$ we have

$$u_3 = u_1 + 2\Delta u_1 + \Delta_2 u_1.$$

In an exactly similar manner by using the second, third, and fourth series in place of the first, second, and third, we obtain

$$\Delta u_3 = \Delta u_1 + 2\Delta_2 u_1 + \Delta_3 u_1.$$

By addition, since $u_3 + \Delta u_3 = u_4$, we have

$$u_4 = u_1 + 3\Delta u_1 + 3\Delta_2 u_1 + \Delta_3 u_1.$$

So far as we have proceeded, the numerical coefficients follow the same law as those of the Binomial theorem. We shall now prove by induction that this will always be the case. For suppose that

$$u_{n+1} = u_1 + n\Delta u_1 + \frac{n(n-1)}{1 \,.\, 2}\Delta_2 u_1 + \ldots + {}^nC_r \Delta_r u_1 + \ldots\ldots + \Delta_n u_1 ;$$

then by using the second to the $(n+2)^{\text{th}}$ series in the place of the first to the $(n+1)^{\text{th}}$ series, we have

$$\Delta u_{n+1} = \Delta u_1 + n\Delta_2 u_1 + \frac{n(n-1)}{1 \,.\, 2}\Delta_3 u_1 + \ldots + {}^nC_{r-1}\Delta_r u_1 + \ldots + \Delta_{n+1} u_1 .$$

By addition, since $u_{n+1} + \Delta u_{n+1} = u_{n+2}$, we obtain

$$u_{n+2} = u_1 + (n+1)\Delta u_1 + \ldots + ({}^nC_r + {}^nC_{r-1})\Delta_r u_1 + \ldots + \Delta_{n+1} u_1 .$$

But $^nC_r + {^nC_{r-1}} = \left(\frac{n-r+1}{r} + 1\right) \times {^nC_{r-1}} = \frac{n+1}{r} \times {^nC_{r-1}}$

$$= \frac{(n+1)\,n\,(n-1)\ldots(\overline{n+1}-r+1)}{1\,.\,2\,.\,3\ldots(r-1)\,r} = {^{n+1}C_r}.$$

Hence if the law of formation holds for $u_{n+1}$ it also holds for $u_{n+2}$, but it is true in the case of $u_4$, therefore it holds for $u_5$, and therefore universally. Hence

$$u_n = u_1 + (n-1)\,\Delta u_1 + \frac{(n-1)(n-2)}{1\,.\,2}\,\Delta_2 u_1 + \ldots + \Delta_{n-1} u_1.$$

396. To find the sum of $n$ terms of the series

$$u_1, \quad u_2, \quad u_3, \quad u_4, \ldots\ldots$$

in terms of the differences of $u_1$.

Suppose the series $u_1, u_2, u_3, \ldots$ is the first order of differences of the series

$$v_1, \quad v_2, \quad v_3, \quad v_4, \ldots,$$

then $v_{n+1} = (v_{n+1} - v_n) + (v_n - v_{n-1}) + \ldots + (v_2 - v_1) + v_1$ identically;

$$\therefore\ v_{n+1} = u_n + u_{n-1} + \ldots + u_2 + u_1 + v_1.$$

Hence in the series

$$0, \quad v_2, \quad v_3, \quad v_4, \quad v_5 \ldots\ldots$$

$$u_1, \quad u_2, \quad u_3, \quad u_4, \ldots\ldots$$

$$\Delta u_1, \quad \Delta u_2, \quad \Delta u_3 \ldots\ldots$$

the law of formation is the same as in the preceding article;

$$\therefore\ v_{n+1} = 0 + nu_1 + \frac{n(n-1)}{1\,.\,2}\,\Delta u_1 + \ldots + \Delta_n u_1;$$

that is, $u_1 + u_2 + u_3 + \ldots + u_n$

$$= nu_1 + \frac{n(n-1)}{\lfloor 2}\,\Delta u_1 + \frac{n(n-1)(n-2)}{\lfloor 3}\,\Delta_2 u_1 + \ldots + \Delta_n u_1.$$

The formulæ of this and the preceding article may be expressed in a slightly different form, as follows: if $a$ is the first term of a given series, $d_1, d_2, d_3, \ldots$ the first terms of the successive orders of differences, the $n^{\text{th}}$ term of the given series is obtained from the formula

$$a + (n-1)\,d_1 + \frac{(n-1)(n-2)}{\lfloor 2}\,d_2 + \frac{(n-1)(n-2)(n-3)}{\lfloor 3}\,d_3 + \ldots;$$

and the sum of $n$ terms is

$$na + \frac{n(n-1)}{\lfloor 2} d_1 + \frac{n(n-1)(n-2)}{\lfloor 3} d_2 + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4} d_3 + \ldots .$$

*Example.* Find the general term and the sum of $n$ terms of the series

12, 40, 90, 168, 280, 432,......

The successive orders of difference are

28, 50, 78, 112, 152,......

22, 28, 34, 40,......

6, 6, 6,......

0, 0,...

Hence the $n^{\text{th}}$ term $= 12 + 28(n-1) + \dfrac{22(n-1)(n-2)}{\lfloor 2} + \dfrac{6(n-1)(n-2)(n-3)}{\lfloor 3}$

$$= n^3 + 5n^2 + 6n.$$

The sum of $n$ terms may now be found by writing down the value of $\Sigma n^3 + 5\Sigma n^2 + 6\Sigma n$. Or we may use the formula of the present article and obtain $S_n = 12n + \dfrac{28n(n-1)}{\lfloor 2} + \dfrac{22n(n-1)(n-2)}{\lfloor 3} + \dfrac{6n(n-1)(n-2)(n-3)}{\lfloor 4}$

$$= \frac{n}{12}(3n^2 + 26n + 69n + 46),$$

$$= \frac{1}{12} n(n+1)(3n^2 + 23n + 46).$$

397. It will be seen that this method of summation will only succeed when the series is such that in forming the orders of differences we eventually come to a series in which all the terms are equal. This will always be the case if the $n^{\text{th}}$ term of the series is a rational integral function of $n$.

For simplicity we will consider a function of three dimensions; the method of proof, however, is perfectly general.

Let the series be

$$u_1 + u_2 + u_3 + \ldots\ldots + u_n + u_{n+1} + u_{n+2} + u_{n+3} + \ldots\ldots$$

where $$u_n = An^3 + Bn^2 + Cn + D,$$

and let $v_n$, $w_n$, $z_n$ denote the $n^{\text{th}}$ term of the first, second, third orders of differences;

then $\quad v_n = u_{n+1} - u_n = A(3n^2 + 3n + 1) + B(2n + 1) + C;$

that is, $\quad v_n = 3An^2 + (3A + 2B)n + A + B + C;$

Similarly $\quad w_n = v_{n+1} - v_n = 3A(2n+1) + 3A + 2B$

and $\quad z_n = w_{n+1} - w_n = 6A.$

Thus the terms in the third order of differences are equal; and generally, if the $n^{\text{th}}$ term of the given series is of $p$ dimensions, the terms in the $p^{\text{th}}$ order of differences will be equal.

Conversely, if the terms in the $p^{\text{th}}$ order of differences are equal, the $n^{\text{th}}$ term of the series is a rational integral function of $n$ of $p$ dimensions.

*Example.* Find the $n^{\text{th}}$ term of the series $-1, -3, 3, 23, 63, 129, \ldots\ldots$

The successive orders of differences are

$$-2,\ 6,\ 20,\ 40,\ 66,\ldots\ldots$$
$$8,\ 14,\ 20,\ 26,\ldots\ldots$$
$$6,\ 6,\ 6,\ldots\ldots$$

Thus the terms in the third order of differences are equal; hence we may assume $\quad u_n = A + Bn + Cn^2 + Dn^3,$

where $A$, $B$, $C$, $D$ have to be determined.

Putting 1, 2, 3, 4 for $n$ in succession, we have four simultaneous equations, from which we obtain $A=3$, $B=-3$, $C=-2$, $D=1$;

hence the general term of the series is $3-3n-2n^2+n^3$.

398. *If* $a_n$ *is a rational integral function of* p *dimensions in* n, *the series*

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

*is a recurring series, whose scale of relation is* $(1-x)^{p+1}$.

Let $S$ denote the sum of the series; then

$$S(1-x) = a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 + \ldots + (a_n - a_{n-1})x^n - a_nx^{n+1}$$
$$= a_0 + b_1x + b_2x^2 + \ldots + b_nx^n - a_nx^{n+1}, \text{ say;}$$

here $b_n = a_n - a_{n-1}$, so that $b_n$ is of $p-1$ dimensions in $n$.

Multiplying this last series by $1-x$, we have

$S(1-x)^2$

$$= a_0 + (b_1 - a_0)x + (b_2 - b_1)x^2 + \ldots + (b_n - b_{n-1})x^n - (a_n + b_n)x^{n+1} + a_nx^{n+2}$$
$$= a_0 + (b_1 - a_0)x + c_2x^2 + c_3x^3 + \ldots + c_nx^n - (a_n + b_n)x^{n+1} + a_nx^{n+2}, \text{ say;}$$

here $c_n = b_n - b_{n-1}$, so that $c_n$ is of $p-2$ dimensions in $n$.

Hence it follows that after the successive multiplications by $1-x$, the coefficients of $x^n$ in the first, second, third, ... products are general terms in the first, second, third, ... orders of differences of the coefficients.

By hypothesis $a_n$ is a rational integral function of $n$ of $p$ dimensions; therefore after $p$ multiplications by $1-x$ we shall arrive at a series the terms of which, with the exception of $p$ terms at the beginning, and $p$ terms at the end of the series, form a geometrical progression, each of whose coefficients is the same. [Art. 397.]

Thus $$S(1-x)^p = k(x^p + x^{p+1} + \ldots + x^n) + f(x),$$

where $k$ is a constant, and $f(x)$ stands for the $p$ terms at the beginning and $p$ terms at the end of the product.

$$\therefore\ S(1-x)^p = \frac{k(x^p - x^{n+1})}{1-x} + f(x);$$

that is, $$S = \frac{kx^p(1-x^{n-p+1}) + (1-x)f(x)}{(1-x)^{p+1}};$$

thus the series is a recurring series whose scale of relation is $(1-x)^{p+1}$. [Art. 325.]

If the general term is not given, the dimensions of $a_n$ are readily found by the method explained in Art. 397.

*Example.* Find the generating function of the series

$$3 + 5x + 9x^2 + 15x^3 + 23x^4 + 33x^5 + \ldots\ldots$$

Forming the successive orders of differences of the coefficients, we have the series

$$2,\ 4,\ 6,\ 8,\ 10, \ldots\ldots$$
$$2,\ 2,\ 2,\ 2, \ldots\ldots;$$

thus the terms in the second order of differences are equal; hence $a_n$ is a rational integral function of $n$ of two dimensions; and therefore the scale of relation is $(1-x)^3$. We have

$$\begin{aligned} S &= 3 + 5x + 9x^2 + 15x^3 + 23x^4 + 33x^5 + \ldots\ldots \\ -3xS &= \quad\ -9x - 15x^2 - 27x^3 - 45x^4 - 69x^5 - \ldots\ldots \\ 3x^2S &= \qquad\qquad 9x^2 + 15x^3 + 27x^4 + 45x^5 + \ldots\ldots \\ -x^3S &= \qquad\qquad\qquad -3x^3 - 5x^4 - 9x^5 - \ldots\ldots \end{aligned}$$

By addition, $$(1-x)^3 S = 3 - 4x + 3x^2;$$

$$\therefore\ S = \frac{3 - 4x + 3x^2}{(1-x)^3}.$$

399. We have seen in Chap. XXIV. that the generating function of a recurring series is a rational fraction whose denominator is the scale of relation. Suppose that this denominator can be resolved into the factors $(1-ax)(1-bx)(1-cx)$......; then the generating function can be separated into partial fractions of the form

$$\frac{A}{1-ax}+\frac{B}{1-bx}+\frac{C}{1-cx}+\ldots\ldots$$

Each of these fractions can be expanded by the Binomial Theorem in the form of a geometrical series; hence in this case the recurring series can be expressed as the sum of a number of geometrical series.

If however the scale of relation contains any factor $1-ax$ more than once, corresponding to this repeated factor there will be partial fractions of the form $\frac{A_2}{(1-ax)^2}$, $\frac{A_3}{(1-ax)^3}$,......; which when expanded by the Binomial Theorem do not form geometrical series; hence in this case the recurring series cannot be expressed as the sum of a number of geometrical series.

400. The successive orders of differences of the geometrical progression

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ \ldots\ldots$$

are

$$a(r-1),\ a(r-1)r,\ a(r-1)r^2,\ a(r-1)r^3,\ldots\ldots$$
$$a(r-1)^2,\ a(r-1)^2r,\ a(r-1)^2r^2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

which are themselves geometrical progressions having the same common ratio $r$ as the original series.

401. Let us consider the series in which

$$u_n = ar^{n-1} + \phi(n),$$

where $\phi(n)$ is a rational integral function of $n$ of $p$ dimensions, and from this series let us form the successive orders of differences. Each term in any of these orders is the sum of two parts, one arising from terms of the form $ar^{n-1}$, and the other from terms of the form $\phi(n)$ in the original series. Now since $\phi(n)$ is of $p$ dimensions, the part arising from $\phi(n)$ will be zero in the $(p+1)^{\text{th}}$ and succeeding orders of differences, and therefore these series will be geometrical progressions whose common ratio is $r$.

[Art. 400.]

Hence if the first few terms of a series are given, and if the $p^{\text{th}}$ order of differences of these terms form a geometrical progression whose common ratio is $r$, then we may assume that the general term of the given series is $ar^{n-1}+f(n)$, where $f(n)$ is a rational integral function of $n$ of $p-1$ dimensions.

*Example.* Find the $n^{\text{th}}$ term of the series

$$10,\ 23,\ 60,\ 169,\ 494, \ldots\ldots$$

The successive orders of differences are

$$13,\ 37,\ 109,\ 335, \ldots\ldots$$

$$24,\ 72,\ 216, \ldots\ldots$$

Thus the second order of differences is a geometrical progression in which the common ratio is 3; hence we may assume for the general term

$$u_n = a \cdot 3^{n-1} + bn + c.$$

To determine the constants $a$, $b$, $c$, make $n$ equal to 1, 2, 3 successively; then $\qquad a+b+c=10,\ 3a+2b+c=23,\ 9a+3b+c=60;$

whence $\qquad a=6,\ b=1,\ c=3.$

Thus $\qquad u_n = 6 \cdot 3^{n-1}+n+3 = 2 \cdot 3^n + n + 3.$

402. In each of the examples on recurring series that we have just given, on forming the successive orders of differences we have obtained a series the law of which is obvious on inspection, and we have thus been enabled to find a general expression for the $n^{\text{th}}$ term of the original series.

If, however, the recurring series is equal to the sum of a number of geometrical progressions whose common ratios are $a$, $b$, $c$, ..., its general term is of the form $Aa^{n-1}+Bb^{n-1}+Cc^{n-1}$, and therefore the general term in the successive orders of differences is of the same form; that is, all the orders of differences follow the same law as the original series. In this case to find the general term of the series we must have recourse to the more general method explained in Chap. XXIV. But when the coefficients are large the scale of relation is not found without considerable arithmetical labour; hence it is generally worth while to write down a few of the orders of differences to see whether we shall arrive at a series the law of whose terms is evident.

403. We add some examples in further illustration of the preceding principles.

*Example* 1. Find the sum of $n$ terms of the series

$$\frac{5}{1 \cdot 2} \cdot \frac{1}{3} + \frac{7}{2 \cdot 3} \cdot \frac{1}{3^2} + \frac{9}{3 \cdot 4} \cdot \frac{1}{3^3} + \frac{11}{4 \cdot 5} \cdot \frac{1}{3^4} + \ldots\ldots$$

Here $$u_n = \frac{2n+3}{n(n+1)} \cdot \frac{1}{3^n}.$$

Assuming $$\frac{2n+3}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1},$$

we find $$A = 3, \ B = -1.$$

Hence $$u_n = \left(\frac{3}{n} - \frac{1}{n+1}\right) \frac{1}{3^n} = \frac{1}{n} \cdot \frac{1}{3^{n-1}} - \frac{1}{n+1} \cdot \frac{1}{3^n},$$

and therefore $$S_n = 1 - \frac{1}{n+1} \cdot \frac{1}{3^n}.$$

*Example* 2. Find the sum of $n$ terms of the series

$$\frac{1}{3} + \frac{3}{3 \cdot 7} + \frac{5}{3 \cdot 7 \cdot 11} + \frac{7}{3 \cdot 7 \cdot 11 \cdot 15} + \ldots\ldots$$

The $n^{\text{th}}$ term is $$\frac{2n-1}{3 \cdot 7 \cdot 11 \ldots\ldots (4n-5)(4n-1)}.$$

Assume $$\frac{2n-1}{3 \cdot 7 \ldots\ldots (4n-5)(4n-1)} = \frac{A(n+1)+B}{3 \cdot 7 \ldots\ldots 4n-1} - \frac{An+B}{3 \cdot 7 \ldots\ldots (4n-5)}.$$

$$\therefore \ 2n - 1 = An + (A+B) - (An+B)(4n-1).$$

On equating coefficients we have three equations involving the two unknowns $A$ and $B$, and our assumption will be correct if values of $A$ and $B$ can be found to satisfy all three.

Equating coefficients of $n^2$, we obtain $A = 0$.

Equating the absolute terms, $-1 = 2B$; that is $B = -\frac{1}{2}$; and it will be found that these values of $A$ and $B$ satisfy the third equation.

$$\therefore \ u_n = \frac{1}{2} \cdot \frac{1}{3 \cdot 7 \ldots\ldots (4n-5)} - \frac{1}{2} \cdot \frac{1}{3 \cdot 7 \ldots\ldots (4n-5)(4n-1)};$$

hence $$S_n = \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{3 \cdot 7 \cdot 11 \ldots\ldots (4n-1)}.$$

*Example* 3. Sum to $n$ terms the series

$$6 \cdot 9 + 12 \cdot 21 + 20 \cdot 37 + 30 \cdot 57 + 42 \cdot 81 + \ldots\ldots$$

By the method of Art. 396, or that of Art. 397, we find that the $n^{\text{th}}$ term of the series $\quad 6, 12, 20, 30, 42, \ldots\ldots$ is $n^2 + 3n + 2$,

and the $n^{\text{th}}$ term of the series

$$9, 21, 37, 57, 81, \ldots\ldots \text{ is } 2n^2 + 6n + 1.$$

Hence $$u_n=(n+1)(n+2)\{2n(n+3)+1\}$$
$$=2n(n+1)(n+2)(n+3)+(n+1)(n+2);$$

$$\therefore\ S_n=\frac{2}{5}n(n+1)(n+2)(n+3)(n+4)+\frac{1}{3}(n+1)(n+2)(n+3)-2.$$

*Example* 4. Find the sum of $n$ terms of the series

$$2\,.\,2+6\,.\,4+12\,.\,8+20\,.\,16+30\,.\,32+\ldots\ldots$$

In the series 2, 6, 12, 20, 30, ...... the $n^{\text{th}}$ term is $n^2+n$;

hence $$u_n=(n^2+n)\,2^n.$$

Assume $(n^2+n)\,2^n=(An^2+Bn+C)\,2^n-\{A(n-1)^2+B(n-1)+C\}\,2^{n-1}$; dividing out by $2^{n-1}$ and equating coefficients of like powers of $n$, we have

$$2=A,\quad 2=2A+B,\quad 0=C-A+B;$$

whence $$A=2,\quad B=-2,\quad C=4.$$

$$\therefore\ u_n=(2n^2-2n+4)\,2^n-\{2(n-1)^2-2(n-1)+4\}\,2^{n-1};$$

and $$S_n=(2n^2-2n+4)\,2^n-4=(n^2-n+2)\,2^{n+1}-4.$$

## EXAMPLES. XXIX. b.

Find the $n^{\text{th}}$ term and the sum of $n$ terms of the series:

1. 4, 14, 30, 52, 80, 114, ......
2. 8, 26, 54, 92, 140, 198, ......
3. 2, 12, 36, 80, 150, 252, ......
4. 8, 16, 0, $-64$, $-200$, $-432$, ......
5. 30, 144, 420, 960, 1890, 3360, ......

Find the generating functions of the series:

6. $1+3x+7x^2+13x^3+21x^4+31x^5+\ldots\ldots$
7. $1+2x+9x^2+20x^3+35x^4+54x^5+\ldots\ldots$
8. $2+5x+10x^2+17x^3+26x^4+37x^5+\ldots\ldots$
9. $1-3x+5x^2-7x^3+9x^4-11x^5+\ldots\ldots$
10. $1^4+2^4x+3^4x^2+4^4x^3+5^4x^4+\ldots\ldots$

Find the sum of the infinite series:

11. $\dfrac{1\,.\,2}{3}+\dfrac{2\,.\,3}{3^2}+\dfrac{3\,.\,4}{3^3}+\dfrac{4\,.\,5}{3^4}+\ldots\ldots$
12. $1^2-\dfrac{2^2}{5}+\dfrac{3^2}{5^2}-\dfrac{4^2}{5^3}+\dfrac{5^2}{5^4}-\dfrac{6^2}{5^5}+\ldots\ldots$

Find the general term and the sum of $n$ terms of the series:

13. 9, 16, 29, 54, 103,......

14. −3, −1, 11, 39, 89, 167,......

15. 2, 5, 12, 31, 86,......

16. 1, 0, 1, 8, 29, 80, 193,......

17. 4, 13, 35, 94, 262, 755......

Find the sum of $n$ terms of the series:

18. $1+2x+3x^2+4x^3+5x^4+\dots\dots$

19. $1+3x+6x^2+10x^3+15x^4+\dots\dots$

20. $\frac{3}{1 \cdot 2} \cdot \frac{1}{2}+\frac{4}{2 \cdot 3} \cdot \frac{1}{2^2}+\frac{5}{3 \cdot 4} \cdot \frac{1}{2^3}+\frac{6}{4 \cdot 5} \cdot \frac{1}{2^4}+\dots\dots$

21. $\frac{1^2}{2 \cdot 3} \cdot 4+\frac{2^2}{3 \cdot 4} \cdot 4^2+\frac{3^2}{4 \cdot 5} \cdot 4^3+\frac{4^2}{5 \cdot 6} \cdot 4^4+\dots\dots$

22. $3 \cdot 4+8 \cdot 11+15 \cdot 20+24 \cdot 31+35 \cdot 44+\dots\dots$

23. $1 \cdot 3+4 \cdot 7+9 \cdot 13+16 \cdot 21+25 \cdot 31+\dots\dots$

24. $1 \cdot 5+2 \cdot 15+3 \cdot 31+4 \cdot 53+5 \cdot 81+\dots\dots$

25. $\frac{1}{1 \cdot 3}+\frac{2}{1 \cdot 3 \cdot 5}+\frac{3}{1 \cdot 3 \cdot 5 \cdot 7}+\frac{4}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}+\dots\dots$

26. $\frac{1 \cdot 2}{\lfloor 3}+\frac{2 \cdot 2^2}{\lfloor 4}+\frac{3 \cdot 2^3}{\lfloor 5}+\frac{4 \cdot 2^4}{\lfloor 6}+\dots\dots$

27. $2 \cdot 2+4 \cdot 4+7 \cdot 8+11 \cdot 16+16 \cdot 32+\dots\dots$

28. $1 \cdot 3+3 \cdot 3^2+5 \cdot 3^3+7 \cdot 3^4+9 \cdot 3^5+\dots\dots$

29. $\frac{1}{2 \cdot 4}+\frac{1 \cdot 3}{2 \cdot 4 \cdot 6}+\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}+\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}+\dots\dots$

30. $\frac{2}{1 \cdot 2}+\frac{5}{2 \cdot 3} \cdot 2+\frac{10}{3 \cdot 4} \cdot 2^2+\frac{17}{4 \cdot 5} \cdot 2^3+\dots\dots$

31. $\frac{4}{1 \cdot 2 \cdot 3} \cdot \frac{1}{3}+\frac{5}{2 \cdot 3 \cdot 4} \cdot \frac{1}{3^2}+\frac{6}{3 \cdot 4 \cdot 5} \cdot \frac{1}{3^3}+\dots\dots$

32. $\frac{1}{\lfloor 3}+\frac{5}{\lfloor 4}+\frac{11}{\lfloor 5}+\frac{19}{\lfloor 6}+\dots\dots$

33. $\frac{19}{1 \cdot 2 \cdot 3} \cdot \frac{1}{4}+\frac{28}{2 \cdot 3 \cdot 4} \cdot \frac{1}{8}+\frac{39}{3 \cdot 4 \cdot 5} \cdot \frac{1}{16}+\frac{52}{4 \cdot 5 \cdot 6} \cdot \frac{1}{32}+\dots\dots$

404. There are many series the summation of which can be brought under no general rule. In some cases a skilful modification of the foregoing methods may be necessary; in others it will be found that the summation depends on the properties of certain known expansions, such as those obtained by the Binomial, Logarithmic, and Exponential Theorems.

*Example* 1. Find the sum of the infinite series

$$\frac{2}{\underline{|1}}+\frac{12}{\underline{|2}}+\frac{28}{\underline{|3}}+\frac{50}{\underline{|4}}+\frac{78}{\underline{|5}}+\ldots\ldots$$

th term of the series 2, 12, 28, 50, 78...... is $3n^2+n-2$; hence

$$u_n=\frac{3n^2+n-2}{\underline{|n}}=\frac{3n(n-1)+4n-2}{\underline{|n}}$$

$$=\frac{3}{\underline{|n-2}}+\frac{4}{\underline{|n-1}}-\frac{2}{\underline{|n}}.$$

Put $n$ equal to 1, 2, 3, 4,... in succession; then we have

$$u_1=4-\frac{2}{\underline{|1}};\quad u_2=3+\frac{4}{\underline{|1}}-\frac{2}{\underline{|2}};\quad u_3=\frac{3}{\underline{|1}}+\frac{4}{\underline{|2}}-\frac{2}{\underline{|3}};$$

and so on.

Whence $S_\infty=3e+4e-2(e-1)=5e+2$.

*Example* 2. If $(1+x)^n=c_0+c_1x+c_2x^2+\ldots+c_nx^n$, find the value of

$$1^2c_1+2^2c_2+3^2c_3+\ldots+n^2c_n.$$

As in Art. 398 we may easily shew that

$$1^2+2^2x+3^2x^2+4^2x^3+\ldots+n^2x^{n-1}+\ldots=\frac{1+x}{(1-x)^3}.$$

Also $c_n+c_{n-1}x+\ldots c_2x^{n-2}+c_1x^{n-1}+c_0x^n=(1+x)^n$.

Multiply together these two results; then the given series is equal to the coefficient of $x^{n-1}$ in $\dfrac{(1+x)^{n+1}}{(1-x)^3}$, that is, in $\dfrac{(2-\overline{1-x})^{n+1}}{(1-x)^3}$.

The only terms containing $x^{n-1}$ in this expansion arise from

$$2^{n+1}(1-x)^{-3}-(n+1)\,2^n(1-x)^{-2}+\frac{(n+1)\,n}{\underline{|2}}\,2^{n-1}(1-x)^{-1};$$

$\therefore$ the given series $=\dfrac{n(n+1)}{\underline{|2}}\,2^{n+1}-n(n+1)\,2^n+\dfrac{n(n+1)}{\underline{|2}}\,2^{n-1}$

$$=n(n+1)\,2^{n-2}.$$

*Example* 3. If $b=a+1$, and $n$ is a positive integer, find the value of

$$b^n-(n-1)\,ab^{n-2}+\frac{(n-2)(n-3)}{\lfloor 2}\,a^2b^{n-4}-\frac{(n-3)(n-4)(n-5)}{\lfloor 3}\,a^3b^{n-6}+\ldots$$

By the Binomial Theorem, we see that

$$1,\ n-1,\ \frac{(n-3)(n-2)}{\lfloor 2},\ \frac{(n-5)(n-4)(n-3)}{\lfloor 3},\ \ldots\ldots$$

are the coefficients of $x^n$, $x^{n-2}$, $x^{n-4}$, $x^{n-6}$,...... in the expansions of $(1-x)^{-1}$, $(1-x)^{-2}$, $(1-x)^{-3}$, $(1-x)^{-4}$,...... respectively. Hence the sum required is equal to the coefficient of $x^n$ in the expansion of the series

$$\frac{1}{1-bx}-\frac{ax^2}{(1-bx)^2}+\frac{a^2x^4}{(1-bx)^3}-\frac{a^3x^6}{(1-bx)^4}+\ldots\ldots,$$

and although the given expression consists only of a finite number of terms, this series may be considered to extend to infinity.

But the sum of the series $=\dfrac{1}{1-bx}\div\left(1+\dfrac{ax^2}{1-bx}\right)=\dfrac{1}{1-bx+ax^2}$

$$=\frac{1}{1-(a+1)\,x+ax^2},\text{ since } b=a+1.$$

Hence the given series $=$ coefficient of $x^n$ in $\dfrac{1}{(1-x)(1-ax)}$

$$=\text{coefficient of } x^n \text{ in } \frac{1}{a-1}\left(\frac{a}{1-ax}-\frac{1}{1-x}\right)$$

$$=\frac{a^{n+1}-1}{a-1}.$$

*Example* 4. If the series

$$1+\frac{x^3}{\lfloor 3}+\frac{x^6}{\lfloor 6}+\ldots\ldots,\quad x+\frac{x^4}{\lfloor 4}+\frac{x^7}{\lfloor 7}+\ldots\ldots,\quad \frac{x^2}{\lfloor 2}+\frac{x^5}{\lfloor 5}+\frac{x^8}{\lfloor 8}+\ldots\ldots$$

are denoted by $a$, $b$, $c$ respectively, shew that $a^3+b^3+c^3-3abc=1$.

If $\omega$ is an imaginary cube root of unity,

$$a^3+b^3+c^3-3abc=(a+b+c)(a+\omega b+\omega^2c)(a+\omega^2b+\omega c).$$

Now $\quad a+b+c=1+x+\dfrac{x^2}{\lfloor 2}+\dfrac{x^3}{\lfloor 3}+\dfrac{x^4}{\lfloor 4}+\dfrac{x^5}{\lfloor 5}+\ldots\ldots$

$$=e^x;$$

and $\quad a+\omega b+\omega^2c=1+\omega x+\dfrac{\omega^2x^2}{\lfloor 2}+\dfrac{\omega^3x^3}{\lfloor 3}+\dfrac{\omega^4x^4}{\lfloor 4}+\dfrac{\omega^5x^5}{\lfloor 5}+\ldots\ldots$

$$=e^{\omega x};$$

similarly $\quad a+\omega^2b+\omega c=e^{\omega^2x}.$

$$\therefore\ a^3+b^3+c^3-3abc=e^x\cdot e^{\omega x}\cdot e^{\omega^2x}=e^{(1+\omega+\omega^2)x}$$

$$=1,\text{ since } 1+\omega+\omega^2=0.$$

405. *To find the sum of the* $r^{th}$ *powers of the first* n *natural numbers.*

Let the sum be denoted by $S_n$; then

$$S_n = 1^r + 2^r + 3^r + \dots + n^r.$$

Assume that

$$S_n = A_0 n^{r+1} + A_1 n^r + A_2 n^{r-1} + A_3 n^{r-2} + \dots + A_r n + A_{r+1} \dots\dots\dots (1),$$

where $A_0$, $A_1$, $A_2$, $A_3$, ... are quantities whose values have to be determined.

Write $n+1$ in the place of $n$ and subtract; thus

$$(n+1)^r = A_0 \{(n+1)^{r+1} - n^{r+1}\} + A_1 \{(n+1)^r - n^r\}$$
$$+ A_2 \{(n+1)^{r-1} - n^{r-1}\} + A_3 \{(n+1)^{r-2} - n^{r-2}\} + \dots + A_r \dots (2).$$

Expand $(n+1)^{r+1}$, $(n+1)^r$, $(n+1)^{r-1}$, ... and equate the coefficients of like powers of $n$. By equating the coefficients of $n^r$, we have

$$1 = A_0 (r+1), \text{ so that } A_0 = \frac{1}{r+1}.$$

By equating the coefficients of $n^{r-1}$, we have

$$r = \frac{A_0 (r+1) r}{2} + A_1 r; \text{ whence } A_1 = \frac{1}{2}.$$

Equate the coefficients of $n^{r-p}$, substitute for $A_0$ and $A_1$, and multiply both sides of the equation by

$$\frac{\lfloor p}{r(r-1)(r-2)\dots(r-p+1)};$$

we thus obtain

$$1 = \frac{1}{p+1} + \frac{1}{2} + A_2 \frac{p}{r} + A_3 \frac{p(p-1)}{r(r-1)} + A_4 \frac{p(p-1)(p-2)}{r(r-1)(r-2)} + \dots (3).$$

In (1) write $n-1$ in the place of $n$ and subtract; thus

$$n^r = A_0 \{n^{r+1} - (n-1)^{r+1}\} + A_1 \{n^r - (n-1)^r\} + A_2 \{n^{r-1} - (n-1)^{r-1}\} + \dots$$

Equate the coefficients of $n^{r-p}$, and substitute for $A_0$, $A_1$; thus

$$0 = \frac{1}{p+1} - \frac{1}{2} + A_2 \frac{p}{r} - A_3 \frac{p(p-1)}{r(r-1)} + A_4 \frac{p(p-1)(p-2)}{r(r-1)(r-2)} - \dots (4)$$

From (3) and (4), by addition and subtraction,

$$\frac{1}{2}-\frac{1}{p+1}=A_2\frac{p}{r}+A_4\frac{p(p-1)(p-2)}{r(r-1)(r-2)}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5).$$

$$0=A_3\frac{p(p-1)}{r(r-1)}+A_5\frac{p(p-1)(p-2)(p-3)}{r(r-1)(r-2)(r-3)}+\ldots\ldots\ldots \quad (6).$$

By ascribing to $p$ in succession the values 2, 4, 6,..., we see from (6) that each of the coefficients $A_3$, $A_5$, $A_7$,... is equal to zero; and from (5) we obtain

$$A_2=\frac{1}{6}\cdot\frac{r}{\underline{|2}};\quad A_4=-\frac{1}{30}\cdot\frac{r(r-1)(r-2)}{\underline{|4}};$$

$$A_6=\frac{1}{42}\cdot\frac{r(r-1)(r-2)(r-3)(r-4)}{\underline{|6}};\ \ldots\ldots$$

By equating the absolute terms in (2), we obtain

$$1=A_0+A_1+A_2+A_3+\ \ldots\ldots\ +A_r;$$

and by putting $n=1$ in equation (1), we have

$$1=A_0+A_1+A_2+A_3+\ \ldots\ldots\ +A_r+A_{r+1};$$

thus $$A_{r+1}=0.$$

406. The result of the preceding article is most conveniently expressed by the formula,

$$S_n \doteq \frac{n^{r+1}}{r+1}+\frac{1}{2}n^r+B_1\frac{r}{\underline{|2}}n^{r-1}-B_3\frac{r(r-1)(r-2)}{\underline{|4}}n^{r-3}$$
$$+B_5\frac{r(r-1)(r-2)(r-3)(r-4)}{\underline{|6}}n^{r-5}+\ldots,$$

where $B_1=\frac{1}{6}$, $B_3=\frac{1}{30}$, $B_5=\frac{1}{42}$, $B_7=\frac{1}{30}$, $B_9=\frac{5}{66}$, ...

The quantities $B_1$, $B_3$, $B_5$, ... are known as *Bernoulli's Numbers*; for examples of their application to the summation of other series the advanced student may consult Boole's *Finite Differences*.

*Example.* Find the value of $1^5+2^5+3^5+\ldots\ldots+n^5$.

We have $$S_n=\frac{n^6}{6}+\frac{n^5}{2}+B_1\frac{5}{\underline{|2}}n^4-B_3\frac{5\,.\,4\,.\,3}{\underline{|4}}n^2+C,$$

$$=\frac{n^6}{6}+\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12},$$

the constant being zero.

## EXAMPLES. XXIX. c.

Find the sum of the following series:

1. $\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\dfrac{x^7}{7!}+\ldots\ldots$

2. $\dfrac{x}{1.2}+\dfrac{x^2}{2.3}+\dfrac{x^3}{3.4}+\ldots\ldots$

3. $x+\dfrac{x^5}{5!}+\dfrac{x^9}{9!}+\ldots\ldots$

4. $\dfrac{1}{r!}+\dfrac{2!}{(r+1)!}+\dfrac{3!}{(r+2)!}+\ldots\ldots$

5. $1+2x+\dfrac{2^2-1}{2!}\cdot\dfrac{x^2}{1}+\dfrac{3^2-1}{3!}\cdot\dfrac{x^3}{2}+\dfrac{4^2-1}{4!}\cdot\dfrac{x^4}{3}+\ldots\ldots$

6. $\dfrac{p^r}{r!}+\dfrac{p^{r-1}}{(r-1)!}\cdot\dfrac{q}{1}+\dfrac{p^{r-2}}{(r-2)!}\cdot\dfrac{q^2}{2!}+\dfrac{p^{r-3}}{(r-3)!}\cdot\dfrac{q^3}{3!}+\ldots\ldots$ to $r+1$ terms.

7. $\dfrac{n(1+x)}{1+nx}-\dfrac{n(n-1)}{2!}\cdot\dfrac{1+2x}{(1+nx)^2}$
$+\dfrac{n(n-1)(n-2)}{3!}\cdot\dfrac{1+3x}{(1+nx)^3}-\ldots\ldots$ to $n$ terms.

8. $1+3\,\dfrac{2n+1}{2n-1}+5\left(\dfrac{2n+1}{2n-1}\right)^2+\ldots\ldots$ to $n$ terms.

9. $1-\dfrac{n^2}{1^2}+\dfrac{n^2(n^2-1^2)}{1^2.2^2}-\dfrac{n^2(n^2-1^2)(n^2-2^2)}{1^2.2^2.3^2}+\ldots\ldots$ to $n+1$ terms.

10. $(1+2)\log_e 2+\dfrac{1+2^2}{2!}(\log_e 2)^2+\dfrac{1+2^3}{3!}(\log_e 2)^3+\ldots\ldots$

11. $\dfrac{1}{1.2.3}+\dfrac{1}{3.4.5}+\dfrac{1}{5.6.7}+\ldots\ldots$

12. $\dfrac{2}{1!}+\dfrac{3}{2!}+\dfrac{6}{3!}+\dfrac{11}{4!}+\dfrac{18}{5!}+\ldots\ldots$

13. $1+\dfrac{2x^2}{2!}-\dfrac{x^3}{3!}+\dfrac{7x^4}{4!}-\dfrac{23x^5}{5!}+\dfrac{121x^6}{6!}-\ldots\ldots$

14. Without assuming the formula, find the sum of the series:

(1) $1^6+2^6+3^6+\ldots\ldots+n^6$. (2) $1^7+2^7+3^7+\ldots\ldots+n^7$.

15. Find the sum of $1^3+2^3+\frac{3^3}{\lfloor 2}+\frac{4^3}{\lfloor 3}+\frac{5^3}{\lfloor 4}+\ldots\ldots$

16. Shew that the coefficient of $x^n$ in the expansion of $\frac{x}{(1-x)^2-cx}$ is

$$n\left\{1+\frac{n^2-1}{\lfloor 3}c+\frac{(n^2-1)(n^2-4)}{\lfloor 5}c^2+\frac{(n^2-1)(n^2-4)(n^2-9)}{\lfloor 7}c^3+\ldots\ldots\right\}.$$

17. If $n$ is a positive integer, find the value of

$$2^n-(n-1)2^{n-2}+\frac{(n-2)(n-3)}{\lfloor 2}2^{n-4}-\frac{(n-3)(n-4)(n-5)}{\lfloor 3}2^{n-6}+\ldots\ldots;$$

and if $n$ is a multiple of 3, shew that

$$1-(n-1)+\frac{(n-2)(n-3)}{\lfloor 2}-\frac{(n-3)(n-4)(n-5)}{\lfloor 3}+\ldots\ldots=(-1)^n.$$

18. If $n$ is a positive integer greater than 3, shew that

$$n^3+\frac{n(n-1)}{\lfloor 2}(n-2)^3+\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}(n-4)^3+\ldots$$
$$=n^2(n+3)\,2^{n-4}.$$

19. Find the sum of $n$ terms of the series :

(1) $\frac{1}{1+1^2+1^4}+\frac{2}{1+2^2+2^4}+\frac{3}{1+3^2+3^4}+\ldots\ldots$

(2) $\frac{5}{1.2}-\frac{3}{2.3}+\frac{9}{3.4}-\frac{7}{4.5}+\frac{13}{5.6}-\frac{11}{6.7}+\frac{17}{7.8}-\ldots\ldots$

20. Sum to infinity the series whose $n^{\text{th}}$ term is $\frac{(-1)^{n+1}x^n}{n(n+1)(n+2)}$.

21. If $(1+x)^n=c_0+c_1x+c_2x^2+c_3x^3+\ldots\ldots+c_nx^n$, $n$ being a positive integer, find the value of

$$(n-1)^2c_1+(n-3)^2c_3+(n-5)^2c_5+\ldots\ldots$$

22. Find the sum of $n$ terms of the series :

(1) $\frac{2}{1.5}-\frac{4}{5.7}+\frac{8}{7.17}-\frac{16}{17.31}+\frac{32}{31.65}-\ldots\ldots$

(2) $\frac{7}{1.2.3}-\frac{17}{2.3.4}+\frac{31}{3.4.5}-\frac{49}{4.5.6}+\frac{71}{5.6.7}-\ldots\ldots$

23. Prove that, if $a<1$, $(1+ax)(1+a^3x)(1+a^5x)\ldots\ldots$

$$=1+\frac{ax}{1-a^2}+\frac{a^4x^2}{(1-a^2)(1-a^4)}+\frac{a^9x^3}{(1-a^2)(1-a^4)(1-a^6)}+\ldots\ldots$$

**24.** If $A_r$ is the coefficient of $x^r$ in the expansion of

$$(1+x)^2\left(1+\frac{x}{2}\right)^2\left(1+\frac{x}{2^2}\right)^2\left(1+\frac{x}{2^3}\right)^2 \ldots\ldots,$$

prove that $$A_r = \frac{2^2}{2^r-1}(A_{r-1}+A_{r-2}), \text{ and } A_4 = \frac{1072}{315}.$$

**25.** If $n$ is a multiple of 6, shew that each of the series

$$n - \frac{n(n-1)(n-2)}{\underline{|3}} \cdot 3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{\underline{|5}} \cdot 3^2 - \ldots\ldots,$$

$$n - \frac{n(n-1)(n-2)}{\underline{|3}} \cdot \frac{1}{3} + \frac{n(n-1)(n-2)(n-3)(n-4)}{\underline{|5}} \cdot \frac{1}{3^2} - \ldots\ldots,$$

is equal to zero.

**26.** If $n$ is a positive integer, shew that

$$(p+q)^n - (n-1)pq(p+q)^{n-2} + \frac{(n-2)(n-3)}{\underline{|2}} p^2q^2(p+q)^{n-4} - \ldots\ldots$$

is equal to $\dfrac{p^{n+1}-q^{n+1}}{p-q}$.

**27.** If $P_r = (n-r)(n-r+1)(n-r+2) \ldots\ldots (n-r+p-1)$,

$Q_r = r(r+1)(r+2) \ldots\ldots (r+q-1)$,

shew that

$$P_1Q_1 + P_2Q_2 + P_3Q_3 + \ldots\ldots + P_{n-1}Q_{n-1} = \frac{\underline{|p}\ \underline{|q}\ \underline{|n-1+p+q}}{\underline{|p+q+1}\ \underline{|n-2}}.$$

**28.** If $n$ is a multiple of 3, shew that

$$1 - \frac{n-3}{2} + \frac{(n-4)(n-5)}{\underline{|3}} - \frac{(n-5)(n-6)(n-7)}{\underline{|4}} + \ldots\ldots$$
$$+ (-1)^{r-1}\frac{(n-r-1)(n-r-2)\ldots(n-2r+1)}{\underline{|r}} + \ldots,$$

is equal to $\dfrac{3}{n}$ or $-\dfrac{1}{n}$, according as $n$ is odd or even.

**29.** If $x$ is a proper fraction, shew that

$$\frac{x}{1-x^2} - \frac{x^3}{1-x^6} + \frac{x^5}{1-x^{10}} - \ldots\ldots = \frac{x}{1+x^2} + \frac{x^3}{1+x^6} + \frac{x^5}{1+x^{10}} + \ldots\ldots$$

# CHAPTER XXX.

## Theory of Numbers.

407. In this chapter we shall use the word *number* as equivalent in meaning to *positive integer*.

A number which is not exactly divisible by any number except itself and unity is called a *prime number*, or a *prime;* a number which is divisible by other numbers besides itself and unity is called a *composite number*; thus 53 is a prime number, and 35 is a composite number. Two numbers which have no common factor except unity are said to be prime to each other; thus 24 is prime to 77.

408. We shall make frequent use of the following elementary propositions, some of which arise so naturally out of the definition of a prime that they may be regarded as axioms.

(i) If a number $a$ divides a product $bc$ and is prime to one factor $b$, it must divide the other factor $c$.

For since $a$ divides $bc$, every factor of $a$ is found in $bc$; but since $a$ is prime to $b$, no factor of $a$ is found in $b$; therefore all the factors of $a$ are found in $c$; that is, $a$ divides $c$.

(ii) If a prime number $a$ divides a product $bcd...$, it must divide one of the factors of that product; and therefore if a prime number $a$ divides $b^n$, where $n$ is any positive integer, it must divide $b$.

(iii) If $a$ is prime to each of the numbers $b$ and $c$, it is prime to the product $bc$. For no factor of $a$ can divide $b$ or $c$; therefore the product $bc$ is not divisible by any factor of $a$, that is, $a$ is prime to $bc$. Conversely if $a$ is prime to $bc$, it is prime to each of the numbers $b$ and $c$.

Also if $a$ is prime to each of the numbers $b$, $c$, $d$, ..., it is prime to the product $bcd...$; and conversely if $a$ is prime to any number, it is prime to every factor of that number.

(iv) If $a$ and $b$ are prime to each other, every positive integral power of $a$ is prime to every positive integral power of $b$. This follows at once from (iii).

(v) If $a$ is prime to $b$, the fractions $\frac{a}{b}$ and $\frac{a^n}{b^m}$ are in their lowest terms, $n$ and $m$ being any positive integers. Also if $\frac{a}{b}$ and $\frac{c}{d}$ are any two equal fractions, and $\frac{a}{b}$ is in its lowest terms, then $c$ and $d$ must be equimultiples of $a$ and $b$ respectively.

409. *The number of primes is infinite.*

For if not, let $p$ be the greatest prime number; then the product $2 . 3 . 5 . 7 . 11 \ldots p$, in which each factor is a prime number, is divisible by each of the factors $2, 3, 5, \ldots p$; and therefore the number formed by adding unity to their product is not divisible by any of these factors; hence it is either a prime number itself or is divisible by some prime number greater than $p$: in either case $p$ is not the *greatest* prime number, and therefore the number of primes is not limited.

410. *No rational algebraical formula can represent prime numbers only.*

If possible, let the formula $a + bx + cx^2 + dx^3 + \ldots$ represent prime numbers only, and suppose that when $x = m$ the value of the expression is $p$, so that

$$p = a + bm + cm^2 + dm^3 + \ldots\ldots;$$

when $x = m + np$ the expression becomes

$$a + b(m + np) + c(m + np)^2 + d(m + np)^3 + \ldots,$$

that is, $\quad a + bm + cm^2 + dm^3 + \ldots + \text{a multiple of } p,$

or $\quad p + \text{a multiple of } p,$

thus the expression is divisible by $p$, and is therefore not a prime number.

411. *A number can be resolved into prime factors in only one way.*

Let $N$ denote the number; suppose $N = abcd\ldots$, where $a, b, c, d, \ldots$ are prime numbers. Suppose also that $N = \alpha\beta\gamma\delta\ldots$, where $\alpha, \beta, \gamma, \delta, \ldots$ are other prime numbers. Then

$$abcd\ldots = \alpha\beta\gamma\delta\ldots;$$

hence $\alpha$ must divide $abcd\ldots$; but each of the factors of this product is a prime, therefore $\alpha$ must divide one of them, $a$ suppose. But $\alpha$ and $a$ are both prime; therefore $\alpha$ must be equal to $a$. Hence $bcd\ldots=\beta\gamma\delta\ldots$; and as before, $\beta$ must be equal to one of the factors of $bcd\ldots$; and so on. Hence the factors in $\alpha\beta\gamma\delta\ldots$ are equal to those in $abcd\ldots$, and therefore $N$ can only be resolved into prime factors in one way.

412. *To find the number of divisors of a composite number.*

Let $N$ denote the number, and suppose $N=a^p b^q c^r\ldots$, where $a$, $b$, $c$, ... are different prime numbers and $p$, $q$, $r$, ... are positive integers. Then it is clear that each term of the product

$$(1+a+a^2+\ldots+a^p)(1+b+b^2+\ldots+b^q)(1+c+c^2+\ldots+c^r)\ldots$$

is a divisor of the given number, and that no other number is a divisor; hence the number of divisors is the number of terms in the product, that is,

$$(p+1)(q+1)(r+1)\ldots\ldots$$

This includes as divisors, both unity and the number itself.

413. *To find the number of ways in which a composite number can be resolved into two factors.*

Let $N$ denote the number, and suppose $N=a^p b^q c^r\ldots$, where $a$, $b$, $c\ldots$ are different prime numbers and $p$, $q$, $r\ldots$ are positive integers. Then each term of the product

$$(1+a+a^2+\ldots+a^p)(1+b+b^2+\ldots+b^q)(1+c+c^2+\ldots+c^r)\ldots$$

is a divisor of $N$; but there are *two* divisors corresponding to each way in which $N$ can be resolved into two factors; hence the required number is

$$\frac{1}{2}(p+1)(q+1)(r+1)\ldots\ldots$$

This supposes $N$ not a perfect square, so that one at least of the quantities $p$, $q$, $r$, ... is an odd number.

If $N$ is a perfect square, one way of resolution into factors is $\sqrt{N}\times\sqrt{N}$, and to this way there corresponds only *one* divisor $\sqrt{N}$. If we exclude this, the number of ways of resolution is

$$\frac{1}{2}\left\{(p+1)(q+1)(r+1)\ldots-1\right\},$$

and to this we must add the one way $\sqrt{N}\times\sqrt{N}$; thus we obtain for the required number

$$\frac{1}{2}\left\{(p+1)(q+1)(r+1)\ldots+1\right\}.$$

414. *To find the number of ways in which a composite number can be resolved into two factors which are prime to each other.*

As before, let the number $N = a^p b^q c^r \ldots$. Of the two factors one must contain $a^p$, for otherwise there would be some power of $a$ in one factor and some power of $a$ in the other factor, and thus the two factors would not be prime to each other. Similarly $b^q$ must occur in one of the factors only; and so on. Hence the required number is equal to the number of ways in which the product $abc\ldots$ can be resolved into two factors; that is, the number of ways is $\frac{1}{2}(1+1)(1+1)(1+1)\ldots$ or $2^{n-1}$, where $n$ is the number of *different* prime factors in $N$.

415. *To find the sum of the divisors of a number.*

Let the number be denoted by $a^p b^q c^r \ldots$, as before. Then each term of the product

$$(1 + a + a^2 + \ldots + a^p)(1 + b + b^2 + \ldots + b^q)(1 + c + c^2 + \ldots + c^r)\ldots$$

is a divisor, and therefore the *sum* of the divisors is equal to this product; that is,

$$\text{the sum required} = \frac{a^{p+1}-1}{a-1} \cdot \frac{b^{q+1}-1}{b-1} \cdot \frac{c^{r+1}-1}{c-1} \cdots.$$

*Example* 1. Consider the number 21600.

Since $21600 = 6^3 \cdot 10^2 = 2^3 \cdot 3^3 \cdot 2^2 \cdot 5^2 = 2^5 \cdot 3^3 \cdot 5^2$,

the number of divisors $= (5+1)(3+1)(2+1) = 72$;

$$\text{the sum of the divisors} = \frac{2^6-1}{2-1} \cdot \frac{3^4-1}{3-1} \cdot \frac{5^3-1}{5-1}$$

$$= 63 \times 40 \times 31$$

$$= 78120.$$

Also 21600 can be resolved into two factors prime to each other in $2^{3-1}$, or 4 ways.

*Example* 2. If $n$ is odd shew that $n(n^2-1)$ is divisible by 24.

We have $n(n^2-1) = n(n-1)(n+1)$.

Since $n$ is odd, $n-1$ and $n+1$ are two consecutive even numbers; hence one of them is divisible by 2 and the other by 4.

Again $n-1$, $n$, $n+1$ are three consecutive numbers; hence one of them is divisible by 3. Thus the given expression is divisible by the product of 2, 3, and 4, that is, by 24.

*Example* 3. Find the highest power of 3 which is contained in $\underline{|100}$.

Of the first 100 integers, as many are divisible by 3 as the number of times that 3 is contained in 100, that is, 33; and the integers are 3, 6, 9,...99. Of these, some contain the factor 3 again, namely 9, 18, 27,...99, and their number is the quotient of 100 divided by 9. Some again of these last integers contain the factor 3 a third time, namely 27, 54, 81, the number of them being the quotient of 100 by 27. One number only, 81, contains the factor 3 four times.

Hence the highest power required $= 33 + 11 + 3 + 1 = 48$.

This example is a particular case of the theorem investigated in the next article.

416. *To find the highest power of a prime number* a *which is contained in* $\underline{|n}$.

Let the greatest integer contained in $\frac{n}{a}, \frac{n}{a^2}, \frac{n}{a^3}, \ldots$ respectively be denoted by $I\left(\frac{n}{a}\right), I\left(\frac{n}{a^2}\right), I\left(\frac{n}{a^3}\right), \ldots$ Then among the numbers $1, 2, 3, \ldots n$, there are $I\left(\frac{n}{a}\right)$ which contain $a$ at least once, namely the numbers $a, 2a, 3a, 4a, \ldots$ Similarly there are $I\left(\frac{n}{a^2}\right)$ which contain $a^2$ at least once, and $I\left(\frac{n}{a^3}\right)$ which contain $a^3$ at least once; and so on. Hence the highest power of $a$ contained in $\underline{|n}$ is

$$I\left(\frac{n}{a}\right) + I\left(\frac{n}{a^2}\right) + I\left(\frac{n}{a^3}\right) + \ldots$$

417. In the remainder of this chapter we shall find it convenient to express a multiple of $n$ by the symbol $M(n)$.

418. *To prove that the product of* r *consecutive integers is divisible by* $\underline{|r}$.

Let $P_n$ stand for the product of $r$ consecutive integers, the least of which is $n$; then

$$P_n = n(n+1)(n+2)\ldots(n+r-1),$$

and

$$P_{n+1} = (n+1)(n+2)(n+3)\ldots(n+r);$$

$$\therefore\ nP_{n+1} = (n+r)P_n = nP_n + rP_n;$$

$$\therefore\ P_{n+1} - P_n = \frac{P_n}{n} \times r$$

$= r$ times the product of $r-1$ consecutive integers.

Hence if the product of $r-1$ consecutive integers is divisible by $\underline{|r-1}$, we have

$$P_{n+1} - P_n = rM(\underline{|r-1})$$
$$= M(\underline{|r}).$$

Now $P_1 = \underline{|r}$, and therefore $P_2$ is a multiple of $\underline{|r}$; therefore also $P_3$, $P_4$, ... are multiples of $\underline{|r}$. We have thus proved that if the product of $r-1$ consecutive integers is divisible by $\underline{|r-1}$, the product of $r$ consecutive integers is divisible by $\underline{|r}$; but the product of every two consecutive integers is divisible by $\underline{|2}$; therefore the product of every three consecutive integers is divisible by $\underline{|3}$; and so on generally.

This proposition may also be proved thus:

By means of Art. 416, we can shew that every prime factor is contained in $\underline{|n+r}$ as often *at least* as it is contained in $\underline{|n}\,\underline{|r}$. This we leave as an exercise to the student.

419. *If* p *is a prime number, the coefficient of every term in the expansion of* $(a+b)^p$, *except the first and last, is divisible by* p.

With the exception of the first and last, every term has a coefficient of the form

$$\frac{p(p-1)(p-2)\dots(p-r+1)}{\underline{|r}},$$

where $r$ may have any integral value not exceeding $p-1$. Now this expression is an integer; also since $p$ is prime no factor of $\underline{|r}$ is a divisor of it, and since $p$ is greater than $r$ it cannot divide any factor of $\underline{|r}$; that is, $(p-1)(p-2)\dots(p-r+1)$ must be divisible by $\underline{|r}$. Hence every coefficient except the first and the last is divisible by $p$.

420. *If* p *is a prime number, to prove that*

$$(a+b+c+d+\dots)^p = a^p + b^p + c^p + d^p + \dots + M(p).$$

Write $\beta$ for $b+c+\dots$; then by the preceding article

$$(a+\beta)^p = a^p + \beta^p + M(p).$$

Again $\quad \beta^p = (b+c+d+\dots)^p = (b+\gamma)^p$ suppose;

$$= b^p + \gamma^p + M(p).$$

By proceeding in this way we may establish the required result.

421. [Fermat's Theorem.] *If* p *is a prime number and* N *is prime to* p, *then* $N^{p-1}-1$ *is a multiple of* p.

We have proved that

$$(a+b+c+d+\dots)^p = a^p+b^p+c^p+d^p+\dots+M(p);$$

let each of the quantities $a, b, c, d, \dots$ be equal to unity, and suppose they are $N$ in number; then

$$N^p = N + M(p);$$

that is,

$$N(N^{p-1}-1) = M(p).$$

But $N$ is prime to $p$, and therefore $N^{p-1}-1$ is a multiple of $p$.

COR. Since $p$ is prime, $p-1$ is an even number except when $p=2$. Therefore

$$\left(N^{\frac{p-1}{2}}+1\right)\left(N^{\frac{p-1}{2}}-1\right) = M(p).$$

Hence either $N^{\frac{p-1}{2}}+1$ or $N^{\frac{p-1}{2}}-1$ is a multiple of $p$,

that is $N^{\frac{p-1}{2}} = Kp \pm 1$, where $K$ is some positive integer.

422. It should be noticed that in the course of Art. 421 it was shewn that $N^p - N = M(p)$ *whether* N *is prime to* p *or not*; this result is sometimes more useful than Fermat's theorem.

*Example* 1. Shew that $n^7-n$ is divisible by 42.

Since 7 is a prime, $n^7-n=M(7)$;

also $n^7-n = n(n^6-1) = n(n+1)(n-1)(n^4+n^2+1)$.

Now $(n-1)n(n+1)$ is divisible by $\lfloor 3$; hence $n^7-n$ is divisible by $6\times 7$, or 42.

*Example* 2. If $p$ is a prime number, shew that the difference of the $p^{\text{th}}$ powers of any two numbers exceeds the difference of the numbers by a multiple of $p$.

Let $x, y$ be the numbers; then

$$x^p - x = M(p) \text{ and } y^p - y = M(p),$$

that is,

$$x^p - y^p - (x-y) = M(p);$$

whence we obtain the required result.

*Example* 3. Prove that every square number is of the form $5n$ or $5n\pm1$.

If $N$ is not prime to 5, we have $N^2=5n$ where $n$ is some positive integer. If $N$ is prime to 5 then $N^4-1$ is a multiple of 5 by Fermat's theorem; thus either $N^2-1$ or $N^2+1$ is a multiple of 5; that is, $N^2=5n\pm1$.