CHAPTER 10

HEIGHT AND DISTANCE

Suppose you want to find out the length and breadth of the playground in your school. How will you measure it? You will have to use a measuring instrument like a scale (ruler) or a measuring tape. Can you use a ruler to easily measure the length of the playground? What difficulties will you face?

Rajesh said – If I use a small ruler then it will take a long time to measure the field because it is very long. I'll have to lift and place the ruler again and again and this can introduce errors in measurement. So, I should use a long measuring tape instead.

Zahida said – To find out the length and breadth of the field, I will have to take the measuring tape from one end to the other. At one corner of the field, one student should stand holding one end of the tape. Another student should take the measuring tape to the second corner and note the reading. This will give us the length of the field.

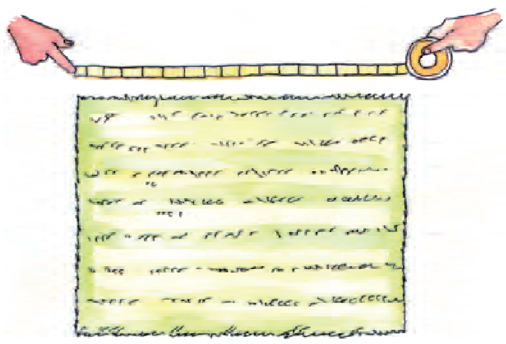

Figure - 1

Jamuna asked – Can we use this method to find out the height of a palm (*Khajur*) tree or the height of volleyball poles? We will have to find the distance between the ground and the top of the tree (or pole) which is difficult. How will we reach the top? Who will climb to the top of the tree?

Aslam – Then what should we do?

Can we use any mathematical technique in these situations?

Can we use trigonometry to find out heights and distances?

Let us see –

Suppose your school has a flagstaff (pole) and you want to find its height. We know that trigonometric ratios are relations between the sides and angles of a triangle. Can you visualize a right angled triangle where the flagstaff is one side of the triangle? What values should we know if we want to find the height of the pole?

Imagine that the pole is one side BC of a right angled triangle. C is the top of the pole and B is the other end. Take any point A on the playground which is 10 m away from B (*figure*–2). Suppose that the line joining A to the top of the pole C forms an angle of 60^o with the ground.

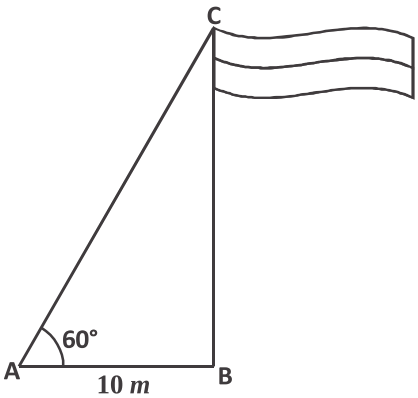

Figure - 2

Then in DABC

$\angle CAB = 60°$

AB = 10 meter

$$\tan A = \frac{BC}{AB}$$

$$\tan 60° = \frac{BC}{10}$$

$$BC = 10 \tan 60°$$

$$BC = 10\sqrt{3} \text{ meter}$$

In this way, we can find out the height of the flagstaff using trigonometry.

Angle of Elevation

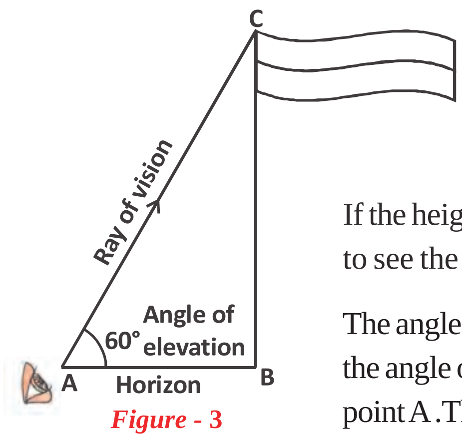

Figure - 3

Let us again examine *figure*–2. Suppose we stand in the field and look at the top of the flag. The line AC drawn from our eye at point A to the point C on the top of the flagpole is called the *line of sight*.

If the height of the pole is more than our height then we have to look upwards to see the top of the pole.

The angle formed between the line of sight AC and the horizontal AB is called the angle of elevation (*figure*-3). Here, we have supposed that our eye is at the point A .Thus, the angle of elevation is the angle between the line of sight and the horizontal line at point A.

If the height of pole is more then we will have to raise our heads even more to see its top-most point.

In this situation will the angle of elevation be more or less? That is, will the value of ϕ be more than θ.

The angle of elevation increases when the height of the pole is increased that is the angle between the horizon and line of sight increases (*figure*-4).

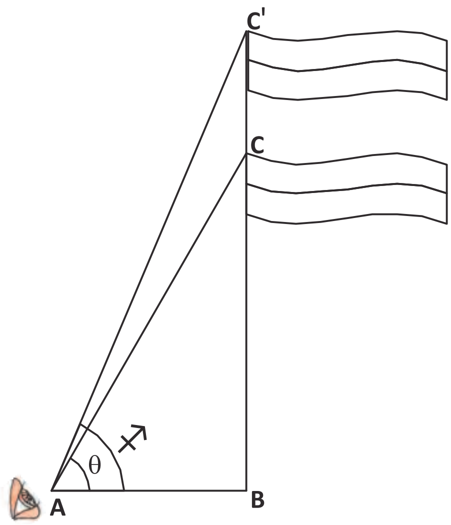

Figure - 4

Think and Discuss

How will the angle of elevation be affected if the height of pole is decreased?

Let us consider another situation where we are looking at the top of the pole not from point A but from the point A' which is farther away from the foot of the pole (*figure*-5).

We find that if we look at the top of the pole from the point A' then the angle between the line of sight and the horizon decreases that is the value of the angle of elevation becomes less.

Thus, we observe that the angle of elevation increases with increase in the height of the object but decreases with increase in distance between the observer and the object.

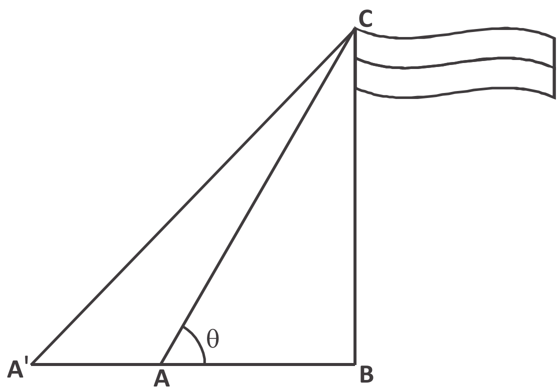

Figure - 5

We can find the height of mountains, distance between planets, depth of oceans, distance between the earth and sun using trigonometry. Astronomers have used it to calculate distances from the Earth to the planets and stars.

We use trigonometry to solve problems in our daily life as well. Let us see some examples.

Example-1. From a point on the ground, which is 15 m away from the foot of a building, the angle of elevation of the top of the building is found to be 45°. Find the height of the building.

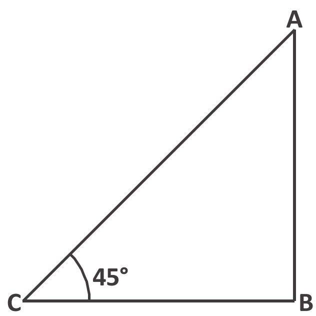

Figure - 6

Solution : In the figure, AB is the height of the building. From a point C on the ground, which is 15 m away from the foot B of the building the angle of elevation of the top A of the building, $\angle ACB = 45°$

Let the height of the building be h.

Then, $\tan 45° = \frac{AB}{BC}$ in ΔABC

Or $\tan 45° = \frac{h}{15}$

Or $1 = \frac{h}{15}$ $[\because \tan 45° = 1]$

$\therefore$ $h = 15$ meter

Therefore, height of the building is 15 *m*.

Example-2. A ladder is placed against a vertical wall so that it forms an angle of 60° with the ground. If the foot of the ladder is 4 meters away from the base of the wall then find the length of the ladder.

Solution : Let AC be the ladder which is x meters long, i.e. AC = x m. It is given that the foot of the ladder, A is 4 meters away from the wall.

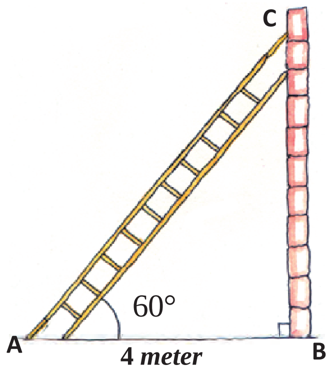

Figure - 7

Thus, in Δ ABC $\quad AB = 4\ m$

And $\quad \angle BAC = 60°$

Then $\quad \cos 60° = \dfrac{AB}{AC}$

Or $\dfrac{1}{2} = \dfrac{4}{x}$

Or $x = 8$ *meter*

Thus, the length of the ladder is 8 meters.

Example-3. A tree breaks due to storm and the broken part bends so that the top of the tree touches the ground making an angle 60° with it. The distance between the foot of the tree to the point where the top touches the ground is 6 m. Find the height of the tree.

Solution : Let AC be the broken part of the tree (figure-8).

It is given that the distance between the foot of the tree to the point where the top touches the ground is 6 m.

Thus, in right triangle ΔABC

$$\tan 60° = \frac{BC}{AB}$$

$$\sqrt{3} = \frac{BC}{6}$$

$$BC = 6\sqrt{3}\ m$$

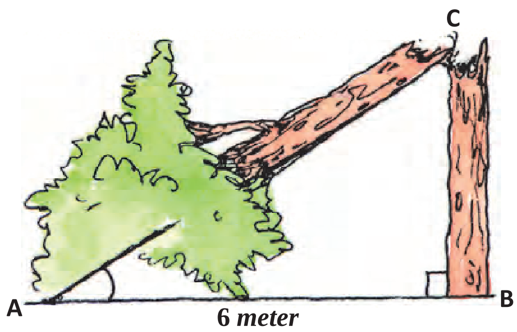

Figure - 8

Again, in right triangle ΔABC

$$\sin 60° = \frac{BC}{AC}$$

$$\text{Or } \frac{\sqrt{3}}{2} = \frac{6\sqrt{3}}{AC}$$

$$\text{Or } AC = \frac{6\cancel{\sqrt{3}} \times 2}{\cancel{\sqrt{3}}}$$

$$AC = 12\ m$$

Thus, height of the tree $= BC + AC$

$$= 6\sqrt{3}+12$$

$$= 6\left(\sqrt{3}+2\right)\ m$$

Example-4. An observer 1.4 m tall is 25.6 m away from a tower. The angle of elevation of the top of the tower from her eyes is 45°. What is the height of the tower?

Solution : Here, BC is the tower, AE the observer and $\angle CED$ is the angle of elevation.

And $AB = ED = 25.6\ m$

$AE = BD = 1.4\ m$

In right-angle triangle ΔCDE

$$\tan 45° = \frac{DC}{ED}$$

$$1 = \frac{DC}{25.6}$$

DC = 25.6 मी.

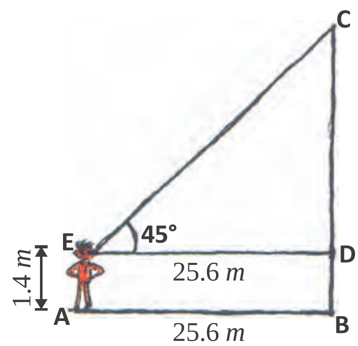

Figure - 9

Thus, height of tower $= BD + DC$

$= 1.4 + 25.6$

= 27 मी.

Note:- If height of the observer is not given then the observer is assumed to be a point.

There is a flag in front of Farida's house (Figure-10). She wants to know the height of the flag. Can we find the height without taking the pole out of the ground?

Let us see-

Example-5. From a point, P on the ground the angle of elevation of a building is 30^{O}. A flag is placed on the top of the building and the angle of elevation from point P to the flag is 45^{O}. Find the height of the flagstaff and the distance between P and the building.

Solution : In figure-10, AB is the height of the building, BD is the length of the flagstaff and P is the observation point. Notice, that here we have two right angle triangles, ΔPAB and ΔPAD. We have to find the length of the flagstaff and the distance between P and the building.

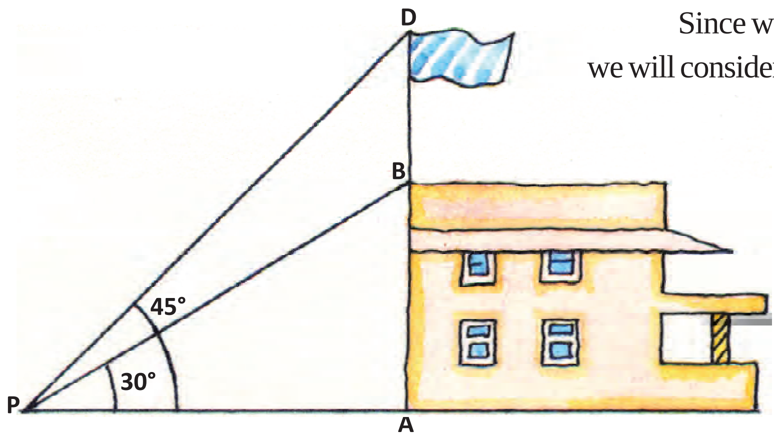

Figure - 10

Since we know the height AB of the building, first we will consider right angle triangle ΔPAB.

Here, $\tan 30° = \dfrac{AB}{PA}$

That is, $\dfrac{1}{\sqrt{3}} = \dfrac{10}{PA}$

Therefore, $PA = 10\sqrt{3}\ m$

$\therefore$ Distance between P and the building is $10\sqrt{3}\ m$

Now, let us suppose that BD = x *meter*

And AD = AB + BD = $(10 + x)\ m$

Then in right triangle ΔPAD

$$\tan 45° = \frac{AD}{PA}$$

$$= \frac{10+x}{10\sqrt{3}}$$

$$\Rightarrow \quad 1 = \frac{10+x}{10\sqrt{3}}$$

$$\Rightarrow 10\sqrt{3} = 10 + x$$

$$x = 10\left(\sqrt{3}-1\right) m$$

Thus, height of the flagpole is $10\left(\sqrt{3}-1\right) m$

We have already seen the relation between angle of elevation and height and distance. We saw that value of the angle of elevation increases with increase in the height of the object but decreases with increase in distance between the observer and the object.

We will now solve some examples based on the statement given above.

Example-6. A boy is standing at some distance from a 30 m tall building. The angle of elevation from his eyes to the top of the building increases from 30° to 60° as he walks towards the building. Find the distance he walked towards the building.

Solution : Suppose that BC represents the building and the boy is standing at point A.

BC = 30 m

In right angle ΔABC

$$\tan 30° = \frac{BC}{AC}$$

$$\frac{1}{\sqrt{3}} = \frac{30}{AC}$$

$$AC = 30\sqrt{3}\ m$$

Again, in right angle ΔBCD

$$\tan 60° = \frac{BC}{CD}$$

$$\tan 60° = \frac{30}{CD}$$

$$\sqrt{3} = \frac{30}{CD}$$

$$CD = \frac{30}{\sqrt{3}}$$

$$= \frac{10 \times 3}{\sqrt{3}}$$

$$CD = 10\sqrt{3}\ m$$

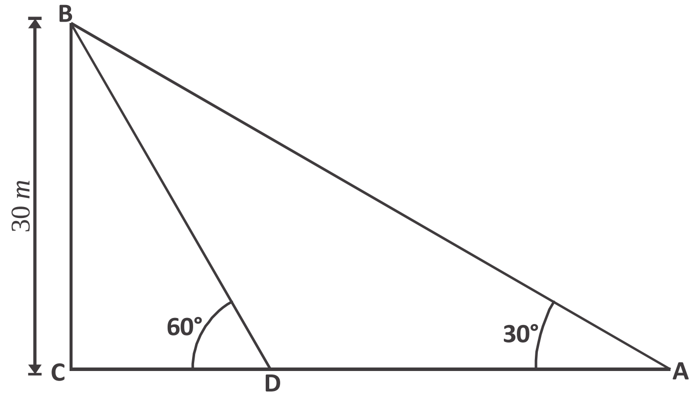

Figure - 11

Thus, distance walked by the boy towards the building $AD = AC - CD$

$$= 30\sqrt{3} - 10\sqrt{3}$$

$20\sqrt{3}\ m$ Thus, the boy walked $20\sqrt{3}\ m$ towards the building.

Angle of Depression

Let us discuss another situation:-

Rama is standing on the balcony of her house watching a car come towards her. The angle so formed by the line of sight with the horizontal is called the angle of depression (*figure*-12).

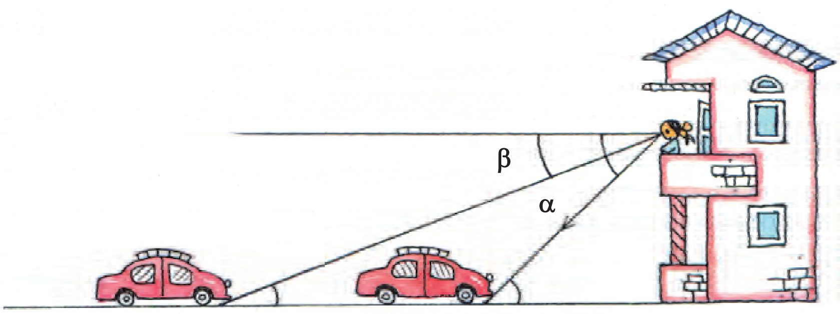

Figure - 12

Now, if the car comes closer to the house (figure-12) then what will be the change in the angle of depression?

What will be the relation between angles α and β?

Will $\alpha > \beta$

$\alpha < \beta$

Or $\alpha = \beta$

You can see that when the distance between the car and the house decreases the value of the angle of depression increases.

That is, $\alpha > \beta$

Think and Discuss

Suppose the car is right below Rama in figure-12 then what will be the angle of depression?

Example-7. The angle of depression of the top of a tower to a flowerpot on the ground, which is 30 m away from the foot of the tower, is 30°. Find the height of the tower.

Solution : Let AB be the tower and point O be the flowerpot.

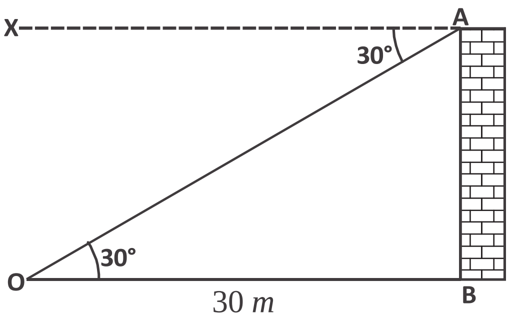

Figure - 13

Angle of depression $\angle XAO = 30°$ and

$OB = 30$ *meter*

$\angle XAO = \angle AOB = 30°$

(alternate angles)

In triangle ΔOAB

$$\tan 30° = \frac{AB}{OB}$$

$$AB = OB \tan 30°$$

$$= 30 \times \frac{1}{\sqrt{3}}$$

$$= \frac{30}{\sqrt{3}}$$

$$= 10\sqrt{3} \text{ meter}$$

Therefore, height of the tower is $10\sqrt{3}$ *meter*

Example-8. The angles of depressions from the top of a light pole to the top and foot of a building are 45^{O} and 60^{O} respectively. If the height of the building is 12 m then find the height of the light pole and its distance from the building.

Solution : Let PQ be the light pole. AB is a 12 meter high building which is at a distance of x meters from the light pole.

Thus, QB = x m, AB = 12 m

The angles of depressions from the top of a light pole to the top and foot of a building are 45^{O} and 60^{O} respectively.

$\angle APX = 45°$ and $\angle BPX = 60°$ and let $PR = h$

In right angle triangle PRA

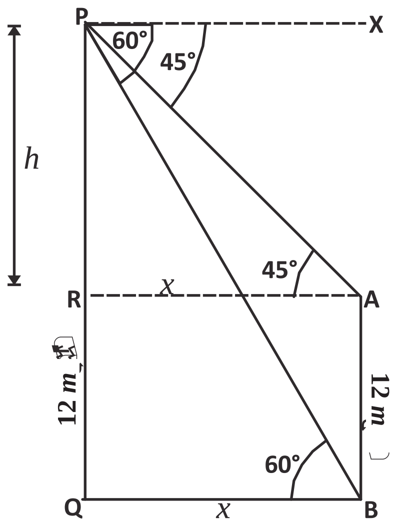

Figure - 14

$$\tan 45° = \frac{PR}{RA}$$

$$\Rightarrow \quad 1 = \frac{PR}{x}$$

$$\Rightarrow \quad PR = x$$

$$\therefore h = x$$

In right angle triangle ΔPQB

$$\Rightarrow \quad \tan 60° = \frac{PQ}{QB}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h+12}{x}$$

$$\Rightarrow \quad \sqrt{3}x = h + 12$$

On putting the value of x

$$\Rightarrow \quad \sqrt{3}h = h + 12$$

$$\Rightarrow \quad \sqrt{3}h - h = 12$$

$$\Rightarrow \quad h\left(\sqrt{3}-1\right) = 12$$

$$\Rightarrow \quad h = \frac{12}{\sqrt{3}-1}$$

$$\Rightarrow \quad h = \frac{12}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$ (on rationalizing the denominator)

$$\Rightarrow \qquad h = \frac{12(\sqrt{3}+1)}{\left(\sqrt{3}\right)^2 - 1^2}$$

$$\Rightarrow \qquad h = \frac{12\left(\sqrt{3}+1\right)}{3-1}$$

$$\Rightarrow \qquad h = 6\left(\sqrt{3}+1\right)m.$$

Height of light pole $= PR + RQ$

$$= 6\left(\sqrt{3}+1\right)+12$$

$$= 6\sqrt{3}+6+12$$

$$= 6\sqrt{3}+18$$

$$= 6\left(\sqrt{3}+3\right)m$$

Since $x = h$ therefore $x = 6\left(\sqrt{3}+1\right)\ m$

The height of the light pole will be $6\left(\sqrt{3}+3\right)m$ and distance from the building is $6\left(\sqrt{3}+1\right)m$

Example-9. The angles of depression from the peak of a hill to two houses on opposite sides of the hill are 30^{O} and 60^{O} respectively. Find the distance between the houses given that the height of the hill is 60m.

Solution : Let PQ be the hill and A and B be the two houses on opposite sides of the hill.

Given that $PQ = 60\ m$

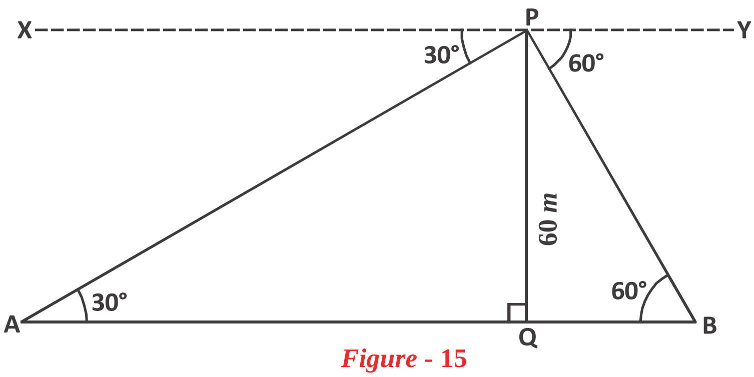

Figure - 15

$\angle XPA = 30°$

$\therefore\ \angle PAQ = 30°$ (alternate angles)

Similarly, $\angle YPB = \angle PBQ = 60°$

In right angle triangle ΔPQA

$$\tan 30° = \frac{PQ}{AQ}$$

$$\frac{1}{\sqrt{3}} = \frac{60}{AQ}$$

$$AQ = 60\sqrt{3}\ m$$

Again in triangle PQB

$$\tan 60° = \frac{PQ}{BQ}$$

$$\sqrt{3} = \frac{60}{BQ}$$

$$BQ = \frac{60}{\sqrt{3}}$$

$$BQ = 20\sqrt{3}\ m$$

Thus, distance between the two houses $AB = AQ + BQ$

$$= 60\sqrt{3} + 20\sqrt{3}$$

$$AB = 80\sqrt{3}\ m$$

Example-10. A straight road goes straight up till the base of the building. A man standing on the top of the building sees a car at 30^{O} angle of depression. The car is moving towards the building at a uniform speed. After the car has covered a distance of 30 m the angle of depression becomes 60^{O}. If the time taken by the car to reach the building from this point is 10 seconds then find the speed of the car and the height of the building.

Solution : Let AB be the building and its height be h meters. Also,

BC = x *meter*

Given,

CD = 30 *m*

$\angle ADB = 30°$

$\angle ACB = 60°$

Then in right angle triangle ABD

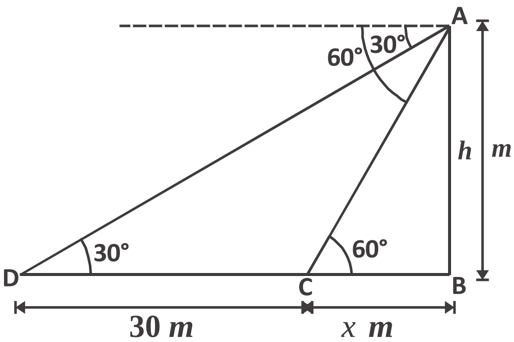

Figure - 16

$$\tan 30° = \frac{AB}{DB}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{30+x}$$

$$h = \frac{30+x}{\sqrt{3}} \quad ....(1)$$

Again in right angle triangle ABC,

$$\tan 60° = \frac{AB}{BC}$$

$$\sqrt{3} = \frac{h}{x}$$

$$h = x\sqrt{3} \qquad \text{..................(2)}$$

From equations (1) and (2),

$$\frac{30+x}{\sqrt{3}} = x\sqrt{3}$$

$\Rightarrow \quad 30 + x = 3x$

$\Rightarrow \quad 2x = 30$

$\Rightarrow \quad x = 15\ m$

Thus, height of the building $h = x\sqrt{3} = 15\sqrt{3}\ \ m$

According to the question,

The time taken to reach the building from point C is 10 seconds.

$$\therefore \quad \text{Speed of car} = \frac{\text{Distance}}{\text{Time}}$$

$$= \frac{15}{10}$$

$$= 1.5\ \ m/s$$

Exercise - 1

1. The angle of elevation of the top of a tower from a point on the ground, which is 90 m away from the foot of the tower, is 30°. Find the height of the tower.
2. There is a vertical column which is 3h meter tall. Find the angle of elevation from a point $\sqrt{3}h$ meters away from the base of the vertical column.
3. A kite is flying at a height of 60 m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string, assuming that there is no slack in the string.
4. The angle of elevation of the top of a building from the foot of a tower is 30° and the angle of elevation of the top of the tower from the foot of the building is 60°. If the building is 15 m high, find the height of the tower.

5. Two towers are at a distance of 120 meters from each other. The angle of depression from the top of the second tower to the top of the first tower is 30°. If the second tower is 40 m high then find the height of the first tower.

6. If the angles of elevation of the top of the tower from two points at distances of a and b cm from the base of the tower and in the same straight line with it, are complementary, then show that the height of the tower is $\sqrt{ab}$.

7. The angle of elevation is 60^{O} from the top of a 15 m high building to the top of a tower and the angle of depression to the bottom of the tower is 30^{O}. Find the height of the tower and the distance between the tower and the building.

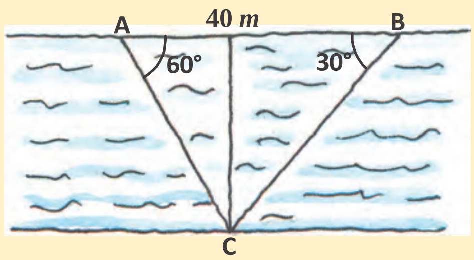

Figure - 17

8. The distance between two points A and B on the same side of a riverbank is 40 m. There is a point C on the opposite, parallel river bank such that $\angle BAC=60°$ and $\angle ABC=30°$ (*Figure*-17). Find the width of the river.

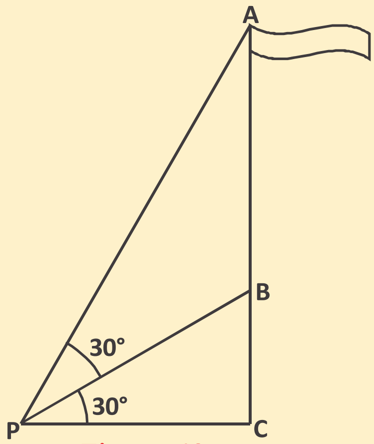

Figure - 18

9. The angles of depression on a point P on the ground made by the peak of a temple and the flag on top of it are 30^{O} and 60^{O} respectively. If the temple is 10 m high, find the height of the flag (*Figure*-18).

10. Two electricity poles of equal height are on the opposite sides of a 40 m wide road. The angles of depression formed by the two poles at a point on the road are 30^{O} and 60^{O} respectively. Find the height of the poles and their respective distance from the point.

11. An observer spots a balloon moving at a height of 90 m above the horizon. If at a certain time the angle of elevation made by the balloon with the observer's eyes is 45^{O} which reduces to 30^{O} after some time then find the distance covered by the balloon (figure-19).

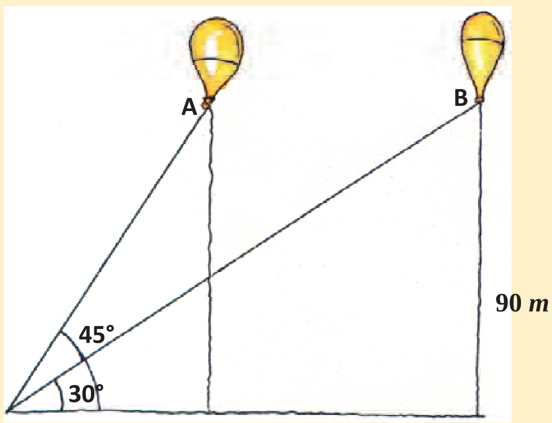

Figure - 19

What we have learnt

1. Trigonometric ratios can be used to find the relative distances and heights of trees, buildings, towers, planets, stars etc.
2. The ray joining the eye of the observer with the object being observed is called the ray of vision.
3. The angle of elevation is the angle formed between the ray of vision of the observed object and the horizon.
4. The angle of depression is the angle formed between the ray of vision of the observed object and the horizon, when the object is below the horizon.
5. The angle of elevation between a building, tower etc. and a point at some distance from the base of the building increases with increase in the height of the object.
6. The angle of depression between a building, tower etc. and a point at some distance from the base of the building decreases with increase in distance between the observer and the object.

ANSWER KEY

Exercise - 1

(1) $30\sqrt{3}$ *meter* (2) 60°

(3) 120 *meter* (4) 45 *meter*

(5) $40\left(\sqrt{3}+1\right)$ *meter* (7) 60 *meter*, $15\sqrt{3}$ *meter*

(8) $10\sqrt{3}$ *meter* (9) 20 *meter*

(10) $10\sqrt{3}$ *meter*, at a distance of 10 m from the first pole and 30 meter from the second pole

(11) $90\left(\sqrt{3}-1\right)$ *meter*