Linear Differential Equations: A differential equation of form $\frac{dy}{dx} + Py = Q$, where P and Q are constants of functions of 'x' only, is known as a first order linear differential equation.

Another form of first order linear differential equation is

$$\frac{dx}{dy} + P_1 x = Q_1$$

where P_1 and Q_1 are constants or functions of y only.

Working Rule to Solve first order Linear differential Equation:

(i) Write the given differential equation in the form $\frac{dy}{dx} + Py = Q$ where P, Q are constants or functions of x only.

(ii) Find the Integrating Factor (I.F) $= e^{\int P dx}$.

(iii) Write the solution of given differential equation as

$$y(\text{I.F}) = \int (Q \times \text{I.F})\, dx + C$$

ie $$y e^{\int P dx} = \int \left(Q \cdot e^{\int P dx}\right) dx + C$$

In case, the first order linear differential equation is in the form $\frac{dx}{dy} + P_1 x = Q_1$, where P_1, Q_1 are constants of functions of y only. Then I.f $= e^{\int P_1 dy}$ and solution of differential equation is given by

$$x \cdot e^{\int P_1 dy} = \int \left(Q_1 e^{\int P_1 dy}\right) dy + C.$$

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For each of differential equations given in exercises 1 to 12 find general solution;

1. $\frac{dy}{dx} + 2y = \sin x.$

Soln: Given differential equation is $\frac{dy}{dx} + 2y = \sin x$

which is of form $\frac{dy}{dx} + Py = Q$ i.e Linear diff eqn.

Where $P = 2$ and $Q = \sin x$.

Then $I.F = e^{\int P dx} = e^{\int 2 dx} = e^{2\int dx} = e^{2x}$

$\therefore$ Solution of given differential equation is

$$y \times (I.F) = \int (Q \times (I.F))\, dx + C$$

ie $$ye^{2x} = \int e^{2x} \cdot \sin x\, dx + C \quad \ldots\ldots (1)$$

Now Let $I = \int e^{2x} \sin x\, dx$

$$= e^{2x} \cdot (-\cos x) - \int e^{2x} \cdot 2(-\cos x) dx \quad \begin{bmatrix}\text{Integrating}\\ \text{by parts}\end{bmatrix}$$

$$= -e^{2x} \cos x + 2\left[e^{2x} \cdot \sin x - \int 2e^{2x} \sin x\, dx\right]$$

$$= -e^{2x} \cos x + 2e^{2x} \sin x - 4I.$$

$\Rightarrow$ $$5I = e^{2x}\left[2\sin x - \cos x\right]$$

$$I = \frac{e^{2x}}{5}(2\sin x - \cos x)$$

$\therefore$ (1) becomes $$ye^{2x} = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$$

or $$y = \frac{1}{5}(2\sin x - \cos x) + Ce^{-2x}$$

Which is required general sol. of given diff. eqn.

(NOTE: We can directly use the formula.

$$\int e^{ax} \sin bx \cdot dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b\cos bx)\ )$$

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QNo 2. $\frac{dy}{dx} + 3y = e^{-2x}$

Soln: The given diff. eqn $\frac{dy}{dx} + 3y = e^{-2x}$ which is Linear i.e of form $\frac{dy}{dx} + Py = Q$ Where $P = 3$ and $Q = e^{-2x}$.

$$I.F = e^{\int P dx} = e^{\int 3 dx} = e^{3x}.$$

$$\therefore \quad y e^{3x} = \int e^{-2x} . e^{3x} dx + C = \int e^{x} dx + C$$

or $$y e^{3x} = e^{x} + C$$

$$\therefore \quad y = e^{-3x}(e^{x} + c) = e^{-3x}.e^{x} + c e^{-3x} = e^{-2x} + c e^{-3x}.$$

Which is required general solution.

QNo 3. $\frac{dy}{dx} + \frac{y}{x} = x^2$.

Sol. Comparing with Linear diff. eqn $\frac{dy}{dx} + Py = Q$

$P = \frac{1}{x}$ and $Q = x^2$.

$$\therefore \quad I.F = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

$\therefore$ Solution of given diff. eqn is given by

$$y x = \int x Q dx + C = \int x . x^2 dx + C$$

$$= \int x^3 dx + C$$

or $$xy = \frac{x^4}{4} + C$$

Which is required sol. of given diff. eqn.

QNo. 4. $\frac{dy}{dx} + (\sec x) y = \tan x \quad (0 \leq x \leq \pi/2)$

Sol. Comparing with $\frac{dy}{dx} + Py = Q$ We get

$P = \sec x \quad Q = \tan x.$

$$I.F = e^{\int P dx} = e^{\int \sec x . dx} = e^{\log|\sec x + \tan x|}$$

$$= e^{\log(\sec x + \tan x)} = \sec x + \tan x.$$

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$\therefore$ Sol. of given diff. eqn is

$$y\cdot(\sec x+\tan x) = \int \tan x\cdot(\sec x+\tan x)\,dx + C$$

$$= \int(\sec x\cdot\tan x + \tan^2 x)\,dx + C$$

$$= \int \sec x\cdot\tan x\,dx + \int(\sec^2 x - 1)\,dx + C$$

or $\quad y(\sec x+\tan x) = (\sec x + \tan x) - x + C$

or $\quad (y-1)(\sec x+\tan x) + x = C$

which is required general solution.

QNo. 5 $\quad \cos^2 x\dfrac{dy}{dx} + y = \tan x \quad (0 \leq x \leq \pi/2)$

Sol. Given diff. eqn is $\cos^2 x\cdot\dfrac{dy}{dx} + y = \tan x$

Dividing by $\cos^2 x$, we get

$$\frac{dy}{dx} + \frac{1}{\cos^2 x}y = \frac{\tan x}{\cos^2 x}.$$

or $\quad \dfrac{dy}{dx} + \sec^2 x\, y = \tan x\cdot\sec^2 x.$

comparing with $\dfrac{dy}{dx} + Py = Q.$

$P = \sec^2 x, \quad Q = \tan x\cdot\sec^2 x.$

$$I.f = e^{\int P dx} = e^{\int\sec^2 x\cdot dx} = e^{\tan x}.$$

$\therefore$ Sol. of given diff. eqn is given by.

$$y e^{\tan x} = \int e^{\tan x}(\tan x\cdot\sec^2 x)\,dx + C \quad \text{---(1)}$$

Now for solving $I = \int e^{\tan x}(\tan x\cdot\sec^2 x)\,dx.$

Put $\quad \tan x = t$

$\Rightarrow \quad \sec^2 x\,dx = dt.$

$\therefore \quad I = \int e^t\cdot t\cdot dt.$

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$= t.e^t - \int 1.e^t\,dt = te^t - e^t = \tan x\, e^{\tan x} - e^{\tan x}$

$\therefore$ from (1)

$$y\, e^{\tan x} = e^{\tan x}(\tan x - 1) + c$$

or $\quad y = \tan x - 1 + c\,e^{-\tan x}$

Which is reqd. sol. of given diff. eqn.

Q No. 6 $\quad x\frac{dy}{dx} + 2y = x^2 \log x.$

Sol. Given diff. eqn is $x\frac{dy}{dx} + 2y = x^2 \log x$

Dividing by x

$$\frac{dy}{dx} + \frac{2}{x}y = x\log x.$$

Comparing with $\frac{dy}{dx} + Py = Q.$

$$P = \frac{2}{x} \qquad Q = x\log x.$$

$$I.F = e^{\int P dx} = e^{\int \frac{2}{x}dx} = e^{2\log|x|} = e^{\log|x|^2} = e^{\log x^2} = x^2$$

$\therefore$ Sol. of given diff. eqn. is given by.

$$y.x^2 = \int x^2.x.\log x\,dx + C$$

$$= \int \log x \cdot x^3\,dx + c$$

$$= \log x.\frac{x^4}{4} - \int \frac{1}{x}.\frac{x^4}{4}\,dx + c.$$

$$= \frac{x^4}{4}\log x - \frac{1}{4}.\frac{x^4}{4} + c$$

$$= \frac{x^4}{4}\log x - \frac{1}{16}x^4 + C$$

or $\quad y = \frac{x^2}{4}\log x - \frac{1}{16}x^2 + Cx^{-2}$

Which is required general solution of given diff. equation.

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Q No. 7. $x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x.$

Sol. Given diff. eqn is $(x \log x) \frac{dy}{dx} + y = \frac{2}{x} \log x$

or $\frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2}{x^2}.$

comparing with $\frac{dy}{dx} + Py = Q.$

$$P = \frac{1}{x \log x} \qquad Q = \frac{2}{x^2}.$$

$$I.F = e^{\int P dx} = e^{\int \frac{1}{x \log x} dx} = e^{\int \frac{1/x}{\log x} dx}$$

$$= e^{\int \frac{d/dx(\log x)}{\log x} dx} = e^{\log(\log x)} = \log x.$$

$\therefore$ Sol. of given diff. eqn. is

$$y \cdot \log x = \int \log x \cdot \frac{2}{x^2} dx + C$$

$$= 2 \int (\log x) x^{-2} dx + c$$

$$= 2\left[\log x \cdot \frac{x^{-1}}{-1} - \int \frac{1}{x} \cdot \frac{x^{-1}}{-1} dx\right] + c$$

$$= -\frac{2 \log x}{x} + 2\int x^{-2} dx + C$$

$$= -\frac{2 \log x}{x} + 2\frac{x^{-1}}{-1} + C$$

$$= -2\frac{\log x}{x} - \frac{2}{x} + C$$

or $x \log x \, y = -2 \log x - 2 + Cx$

Which is required sol. of given diff eqn.

Q No 8. $(1+x^2) dy + 2xy \, dx = \cot x \, dx \quad (x \neq 0)$

Sol. Given diff. eqn is $(1+x^2)dy + 2xy \, dx = \cot x \, dx.$

or. $\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right) y = \frac{\cot x}{1+x^2}$

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Comparing with $\frac{dy}{dx} + Py = Q$

$$P = \frac{2x}{1+x^2} \quad \text{and} \quad Q = \frac{\cot x}{1+x^2}$$

$$I.F = e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\int \frac{\frac{d}{dx}(1+x^2)}{1+x^2} dx}$$

$$= e^{\log(1+x^2)} = 1+x^2.$$

$\therefore$ The sol. of given diff. eqn is

$$y \cdot (1+x^2) = \int (1+x^2) \cdot \frac{\cot x}{1+x^2} dx + C$$

$$= \int \cot x \, dx + C$$

or $$y(1+x^2) = \log|\sin x| + C$$

which is required general eqn. of given diff. eqn.

Q No. 9. $x\frac{dy}{dx} + y - x + xy \cot x = 0 \quad (x \neq 0)$

Sol. given diff. eqn is $x\frac{dy}{dx} + y - x + xy \cot x = 0$.

or $$\frac{dy}{dx} + \frac{y}{x}(1 + x\cot x) = 1 \quad \cdots\cdots$$

Comparing with $\frac{dy}{dx} + Py = Q$

$$P = \frac{1 + x\cot x}{x} = \frac{1}{x} + \cot x \qquad Q = 1.$$

$$IF = e^{\int P dx} = e^{\int (\frac{1}{x} + \cot x) dx} = e^{\log x + \log \sin x}$$

$$= e^{\log(x \sin x)} \quad [\text{Assuming } x \sin x \text{ to be +ve}]$$

$$= x \sin x.$$

$\therefore$ Sol. of given diff. eqn is

$$y \cdot x \sin x = \int x \sin x \cdot 1 \, dx$$

$$= x \cdot (-\cos x) - \int 1 \cdot (-\cos x) dx + C$$

$$= -x \cos x + \int \cos x + C$$

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or $\quad y \cdot x \sin x = -x\cos x + \sin x + C$

or $\quad y = -\cot x + \frac{1}{x} + \frac{C}{x\sin x}$.

Which is required soln. of given diff. eqn.

QNo. 10 $\quad (x+y)\frac{dy}{dx} = 1$

Sol. Given diff. eqn. is $(x+y)\frac{dy}{dx} = 1$

or. $\quad x+y = \frac{dx}{dy}$. $\quad$ or. $\quad \frac{dx}{dy} - x = y$.

Comparing with $\frac{dx}{dy} + P_1 x = Q_1$

$P_1 = -1 \qquad Q_1 = y$.

If $= e^{\int P dy} = e^{\int -1 dy} = e^{-y}$.

$\therefore$ The soln. of given diff. eqn is

$$x e^{-y} = \int e^{-y} \cdot y \, dy + C$$

$$= y\frac{e^{-y}}{-1} - \int 1 \times \frac{e^{-y}}{-1} dy + C$$

$$= -y e^{-y} + \frac{e^{-y}}{-1} + C$$

$\Rightarrow \quad x e^{-y} = -y e^{-y} - e^{-y} + C$

or $\quad x = -y - 1 + Ce^{y}$.

Which is required general sol. of given diff. eqn.

QNo 11. $\quad y\,dx + (x-y^2)dy = 0$

Sol. Given diff. eqn. is. $y\,dx + (x-y^2)dy = 0$

or $\quad y\frac{dx}{dy} + x - y^2 = 0$

$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{y}x = y$.

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Comparing with $\frac{dx}{dy} + P_1 x = Q_1$

$P_1 = \frac{1}{y}$ $\quad Q_1 = y.$

$$\text{IF} = e^{\int P_1 dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y.$$

$\therefore$ Sol. of given diff. eqn is

$$x \cdot y = \int y \cdot y \, dy + c$$

$$xy = \frac{y^3}{3} + c \implies x = \frac{y^2}{3} + c y^{-1}$$

which is reqd soln of given diff. eqn.

Q No. 12. $(x+3y^2)\frac{dy}{dx} = y.$

Sol. Given diff. eqn is $(x+3y^2)\frac{dy}{dx} = y \quad (y>0)$

$$\implies x + 3y^2 = y \cdot \frac{dx}{dy}$$

or $$\frac{dx}{dy} + \left(-\frac{1}{y}\right)x = 3y.$$

Comparing with $\frac{dx}{dy} + P_1 x = Q_1.$

$P_1 = -\frac{1}{y}$ $\quad Q_1 = 3y.$

$$\text{IF} = e^{\int P_1 dy} = e^{\int -\frac{1}{y} dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1} = 1/y.$$

$\therefore$ Sol of given diff. eqn is

$$x \cdot \frac{1}{y} = \int \frac{1}{y} \cdot 3y \, dy + C$$

$$= 3\int dy + c$$

$$\implies \frac{x}{y} = 3y + c.$$

or $$x = 3y^2 + cy.$$

which is required sol. of given diff. eqn.

For each of the differential equation given in Exercises 13 to 15 find a particular solution satisfying given condition.

Q No. 13. $\frac{dy}{dx} + 2y \tan x = \sin x$; $y = 0$ when $x = \pi/3$

Sol.. The given diff. eqn is $\frac{dy}{dx} + 2y\tan x = \sin x$

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Which is of form $\frac{dy}{dx} + Py = Q$.

where $P = 2\tan x$ $\quad Q = \sin x$.

$$I.F = e^{\int P dx} = e^{\int 2\tan x\, dx} = e^{-2\log|\cos x|} = e^{\log|\cos x|^{-2}}$$

$$= |\cos x|^{-2} = \frac{1}{|\cos x|^2} = \frac{1}{\cos^2 x}.$$

$\therefore$ Soln of given diff. eqn. is

$$y\left(\frac{1}{\cos^2 x}\right) = \int \sin x \cdot \left(\frac{1}{\cos^2 x}\right) dx + C$$

$$= \int\left(\frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\right) dx + C$$

$$= \int \sec x \cdot \tan x \, dx + C$$

or $\quad y \cdot \sec^2 x = \sec x + C \quad \ldots\ldots (i)$

When $x = \pi/3$; $y = 0$

$\therefore \quad 0 = \sec \pi/3 + C \Rightarrow 0 = 2 + C \Rightarrow C = -2$

$\therefore$ (i) becomes $\quad y \sec^2 x = \sec x - 2$

or $\quad y = \frac{\sec x}{\sec^2 x} - \frac{2}{\sec^2 x} = \frac{1}{\sec x} - \frac{2}{\sec^2 x}$

$$y = \cos x - 2\cos^2 x$$

Which is required particular solution.

Q.No 14 $\quad (1+x^2)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2}$; $y = 0$ when $x = 1$

Sol. Given diff. eqn is $(1+x^2)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2}$

or $\quad \frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right) y = \frac{1}{(1+x^2)^2}$

Comparing with $\frac{dy}{dx} + Py = Q$.

$$P = \frac{2x}{1+x^2} \qquad Q = \frac{1}{(1+x^2)^2}$$

$$If = e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1+x^2)} = 1+x^2$$

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$\therefore$ Sol. of given diff. eqn is

$$y(1+x^2) = \int (1+x^2)\frac{1}{(1+x^2)^2}\,dx + c$$

$$= \int \frac{1}{1+x^2}\,dx + c$$

or $\quad y(1+x^2) = \tan^{-1}x + c \quad \cdots\cdots (1)$

when $x=1$, $y=0$

$$0 = \tan^{-1}(1) + c$$

$$\Rightarrow 0 = \pi/4 + c \quad \Rightarrow \quad c = -\pi/4$$

$\therefore$ (1) becomes $y(1+x^2) = \tan^{-1}x - \pi/4$.

which is required particular sol. of given diff. eqn.

Q No. 15. $\frac{dy}{dx} - 3y\cot x = \sin 2x$; $y = 2$ when $x = \pi/2$.

Sol. The given diff. eqn is $\frac{dy}{dx} - 3y\cot x = \sin 2x$

Comparing with $\frac{dy}{dx} + Py = Q$.

$$P = -3\cot x \; ; \; Q = \sin 2x.$$

$$IF = e^{\int P dx} = e^{\int -3\cot x\, dx} = e^{-3\log\sin x} = e^{\log(\sin x)^{-3}} = (\sin x)^{-3}.$$

$\therefore$ Soln. of given diff. eqn is

$$y(\sin x)^{-3} = \int (\sin x)^{-3}\sin 2x\,dx + c$$

$$= \int (\sin x)^{-3}\cdot 2\cdot\sin x\cdot\cos x\,dx + c$$

$$= 2\cdot\frac{(\sin x)^{-1}}{-1} + c$$

or $\quad y = -2\sin^2 x + c\sin^3 x \quad \cdots\cdots (1)$

When $x = \pi/2$, $y = 2$.

$$\therefore \quad 2 = -2\sin^2(\pi/2) + c\sin^3(\pi/2)$$

$$\Rightarrow \quad 2 = -2 + c \quad \Rightarrow \quad c = 4$$

$\therefore$ from (1), the required particular soln. is

$$y = -2\sin^2 x + 4\sin^3 x$$

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Q.No16. Find the equation of the curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

Sol. Here we are given that slope of tangent = sum of coordinates of point

i.e $\frac{dy}{dx} = x+y.$ or $\frac{dy}{dx} + (-1)y = x.$

Which is of form $\frac{dy}{dx} + Py = Q.$

with $P = -1$ $Q = x.$

$If = e^{\int P dx} = e^{-\int dx} = e^{-x}.$

$\therefore$ Soln. of given difl eqn. is

$$ye^{-x} = \int e^{-x} . x \, dx$$

$$= x\frac{e^{-x}}{-1} - \int 1 . \frac{e^{-x}}{-1} dx + c$$

$$ye^{-x} = -xe^{-x} + \int e^{-x} dx + c$$

$$ye^{-x} = -xe^{-x} + \frac{e^{-x}}{-1} + c$$

or $y = -x - 1 + ce^{x}$ .....(1)

Since (1) passes through origin $(0,0)$

$\therefore$ $0 = -0 - 1 + ce^{0}$ $\Rightarrow$ $c = 1.$

$\therefore$ (1) becomes $y = -x - 1 + e^{x}.$

Which is required curve.

Q.No17. Find the equation of a curve passing through the point $(0, 2)$ given that sum of the coordinates of any point on the curve exceeds the magnitude of the slope of tangent to the curve at any point by 5.

Soln. According to the question.

$$x + y = \frac{dy}{dx} + 5.$$

or $\frac{dy}{dx} + (-1)y = x - 5.$

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Comparing with $\frac{dy}{dx} + Py = Q$.

$P = -1 \qquad Q = x-5$.

$IF = e^{\int P dx} = e^{-\int dx} = e^{-x}$.

$\therefore$ Sol of given diff eqn is given by.

$$ye^{-x} = \int (x-5)e^{-x} dx + c$$

$$= (x-5)\frac{e^{-x}}{-1} - \int \frac{e^{-x}}{-1} dx + c$$

$$= (5-x)e^{-x} + \int e^{-x} dx + c$$

or $ye^{-x} = (5-x)e^{-x} - e^{-x} + c \quad ----(1)$

As $(0,2)$ lies on (i)

$$2e^{-0} = (5-0)e^{-0} - e^{-0} + c$$

$\Rightarrow \quad 2 = 5-1+c \qquad \Rightarrow C = -2$

$\therefore$ (1) becomes

$$ye^{-x} = (5-x)e^{-x} - e^{-x} - 2$$

or $y = 4 - x - 2e^{x}$

Which is required eqn of the curve.

QNo 18. The Integrating factor of the diff. eqn $x\frac{dy}{dx} - y = 2x^2$ is

(A) e^{-x} (B) e^{-y} (C) $\frac{1}{x}$ (D) x.

Sol. Given diff. eqn is $x\frac{dy}{dx} - y = 2x^2$

or $\frac{dy}{dx} - \frac{1}{x}y = 2x$

Comparing with $\frac{dy}{dx} + Py = Q$.

$P = -1/x$.

$\therefore IF = e^{\int P dx} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}}$

$= e^{\log \frac{1}{x}} = \frac{1}{x}$.

$\therefore$ (C) is the right option.

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Q No. 19 The integrating factor of the differential equation $(1-y^2)\frac{dx}{dy} + yx = ay \, (-1<y<1)$ is

(A) $\frac{1}{y^2-1}$ (B) $\frac{1}{\sqrt{y^2-1}}$ (C) $\frac{1}{1-y^2}$ (D) $\frac{1}{\sqrt{1-y^2}}$.

Sol.

$$(1-y^2)\frac{dx}{dy} + yx = ay \; ; \; -1<y<1.$$

or $$\frac{dx}{dy} + \left(\frac{y}{1-y^2}\right)x = \frac{ay}{1-y^2}$$

$$\text{If} = e^{\int \frac{y}{1-y^2} dy}$$

$$= e^{-\frac{1}{2}\int \frac{-2y}{1-y^2} dy} = e^{-\frac{1}{2}\log(1-y^2)}$$

$$= e^{\log(1-y^2)^{-1/2}}$$

$$= (1-y^2)^{-1/2} = \frac{1}{\sqrt{1-y^2}}$$

$\therefore$ (D) is the correct option.

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