CBSE Test Paper 02

Chapter 8 Application of Integrals

1. The area bounded by the curves $y = \sqrt{5-x^2}$ and $y = \ |x-1|$ is
   a. $\frac{5\pi-5}{4}$ sq. units
   b. $\frac{5\pi+2}{4}$ sq. units
   c. $\frac{5\pi-2}{4}$ sq. units
   d. $\frac{\pi-5}{4}$ sq. units
2. The area of the loop between the curve y=a sin x and the x – axis and x= 0, x=π is
   a. 3a
   b. 2a square units
   c. none of these
   d. a
3. The area bounded by the curves $x^2 = 4y \ and \ x = 4y - 2$ is
   a. $\frac{9}{2} \ sq. units$
   b. $\frac{9}{8} \ sq. units$
   c. $\frac{9}{4} \ sq. units$
   d. 9 sq. units
4. The area bounded by the ellipse $x^2 + 9y^2 \ = 9$ and the straight line x + 3y = 3 is
   a. 4π
   b. $\Rightarrow \frac{3}{4}(\pi - 2)$
   c. 6π
   d. 9π
5. The area lying in the first quadrant and bounded by the curve y = x^3, the x - axis and the ordinates at x = - 2 and x = 1 is
   a. 2
   b. $\frac{15}{4}$

c. 3
d. 6

6. Find the area of the region bounded by the curves y = |x - 2|, x = 1, x = 3 and the x-axis.

7. Find the value of c for which the area of figure bounded by the curve y = 3, the straight lines x=1 and x=c and the x-axis is equal to $\frac{16}{3}$.

8. Find the area of the region enclosed by the lines y=x, x=e, and the curve $y = \frac{1}{x}$ and the positive x-axis.

9. Integrate the following function $\frac{1}{\sqrt{(x-1)(x-2)}}$.

10. Integrate the function $\frac{(1+\log x)^2}{x}$.

11. Integrate the functions $(4x + 2)\sqrt{x^2 + x + 1}$.

12. $\int_0^{\pi/4} \log(1 + \tan x) dx$.

13. Find $\int \frac{x^3}{x^4+3x^2+2} dx$.

14. $\int \frac{dx}{\sqrt{\sin^3 x \sin(x+\alpha)}}$.

15. Using integration, find the area of the region {(x, y): $x^2 + y^2 \leq 16$, $x^2 \leq 6y$}.

Solution

1. (c) $\frac{5\pi-2}{4}$ sq. units
   Explanation: Required area:
   $\int\limits_{-1}^{2}\sqrt{5-x^2}dx-\int\limits_{-1}^{1}(1-x)dx-\int\limits_{1}^{2}(x-1)dx$
   $=\left[\frac{x\sqrt{5-x^2}}{2}+\frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}\right]_{-1}^{2}$
   $-\left[x-\frac{x^2}{2}\right]_{-1}^{1}-\left[\frac{x^2}{2}-x\right]_{1}^{2}$
   $=\left(\frac{5\pi-2}{4}\right)sq. units$
2. (b) 2a square units
   Explanation: Required area =$\int\limits_{0}^{\pi}a\,\text{sinx}dx$
   $=a[-\cos x]_0^{\pi}$
   $=a(-\cos\pi+\cos 0)=a(1+1)=2a$
3. (b) $\frac{9}{8}\ sq. units$
   Explanation: Eliminating y, we get :
   $x^2-x-2=0\Rightarrow x=-1,2$
   Required area :$=\int\limits_{-1}^{2}\left(\frac{x}{4}+\frac{1}{2}-\frac{x^2}{4}\right)dx=\frac{1}{8}(4-1)+\frac{3}{2}-\frac{1}{12}(8+1)$
   $=\frac{3}{8}+\frac{3}{2}-\frac{3}{4}=\frac{9}{8}sq. units$
4. (b) $\Rightarrow\frac{3}{4}(\pi-2)$
   Explanation: Required area: $=\int\limits_{0}^{3}\left(\frac{1}{3}\sqrt{3^2-x^2}-\frac{1}{3}(3-x)\right)dx$
   $=\frac{1}{3}\left[\frac{x\sqrt{3^2-x^2}}{2}+\frac{3^2}{2}\sin^{-1}\frac{x}{3}-3x+\frac{x^2}{2}\right]_0^3$
   $=\frac{1}{3}\left[0+\frac{3^2}{2}\sin^{-1}1-3^2+\frac{3^2}{2}\right]=\frac{1}{3}\left[0+\frac{3^2}{2}\sin^{-1}0-0+0\right]$
   $=\frac{1}{3}\left[\frac{3^2}{2}.\frac{\pi}{2}-\frac{3^2}{2}-0\right]=3/4\,(\pi-2)$

5. (b) $\frac{15}{4}$

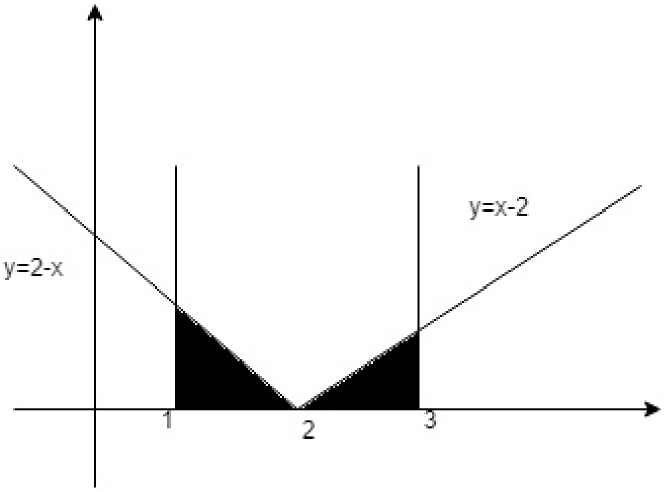
Explanation: Required area $= \int\limits_{-2}^{1} x^3dx = \left[\frac{x^4}{4}\right]_{-2}^{1} = \frac{15}{4}$

6. We have to find the area of the region bounded by the curves y = |x - 2|, x = 1, x = 3 and the x-axis.

Required area =$\int_1^2 (2-x)dx + \int_2^3 (x-2)dx$

= $\left[2x - x^2\right]_1^2 + \left[x^2 - 2x\right]_2^3$

= [(4 - 4) - (2 - 1)] + [(9 - 6) - (4 - 4)] = -1 + 3 = 2



7. we have, $\int_0^c 3dx = \frac{16}{3}$

$3(x)_0^c = \frac{16}{3}$

$3c = \frac{16}{3}$

$c = \frac{16}{9}$

8. Required area = the area of the region enclosed by the lines y=x, x=e, and the curve $y = \frac{1}{x}$ and the positive x-axis

=$\int_0^1 xdx + \int_1^e \frac{1}{x}dx$

$= \frac{1}{2} + 1$

$= \frac{3}{2} sq\ units$

9. $\int \frac{1}{\sqrt{(x-1)(x-2)}}dx$

$= \int \frac{1}{\sqrt{x^2-2x-x+2}}dx$

$= \int \frac{1}{\sqrt{x^2-3x+2}}dx$

$= \int \frac{1}{\sqrt{x^2-3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+2}}dx$

$= \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$

$= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+c$

$= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+c$

10. Let $I=\int\frac{(1+\log x)^2}{x}dx$ ...(i)

Putting $1+\log x=t$

$\Rightarrow \frac{1}{x}=\frac{dt}{dx}$

$\Rightarrow \frac{dx}{x}=dt$

$\therefore$ From eq. (i), $I=\int t^2dt$

$=\frac{t^3}{3}+c$

$=\frac{1}{3}(1+\log x)^3+c$

11. Let $I=\int(4x+2)\sqrt{x^2+x+1}dx$

$=\int 2(2x+1)\sqrt{x^2+x+1}dx$

$=\int 2\sqrt{x^2+x+1}\,(2x+1)dx$ ...(i)

Putting x^2 + x + 1 = t

$\Rightarrow (2x+1)=\frac{dt}{dx}$

$\Rightarrow (2x+1)\,dx=dt$

$\therefore$ From eq. (i), $I=\int 2\sqrt{t}dt$

$=2\int t^{\frac{1}{2}}\,dt$

$=2\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+c$

$=\frac{4}{3}t^{\frac{3}{2}}+c$

$=\frac{4}{3}\left(x^2+x+1\right)^{\frac{3}{2}}+c$

12. $I=\int_0^{\pi/4}\log(1+\tan x)dx$ ...(1)

$I=\int_0^{\pi/4}\log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$

$=\int_0^{\pi/4}\log\left[1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}.\tan x}\right]dx$

$=\int_0^{\pi/4}\log\left[1+\frac{1-\tan x}{1+\tan x}\right)dx$

$=\int_0^{\pi/4}\log\left[\frac{1+\tan x+1-\tan x}{1+\tan x}\right]dx$

$=\int_0^{\pi/4}\log\left[\frac{2}{1+\tan x}\right]dx$

$=\int_0^{\pi/4}\log 2dx-\int_0^{\pi/4}\log(1+\tan x)dx$

$=\log 2[x]_0^{\pi/4}-I$

$2I = \log 2\left[\frac{\pi}{4} - 0\right]$
$I = \log 2.\frac{\pi}{8}$

13. According to the question, $I = \int \frac{x^3}{x^4+3x^2+2}dx$
$I = \int \frac{x^2.x}{x^4+3x^2+2}dx$
Let $x^2 = t \Rightarrow 2xdx = dt$
$\Rightarrow \quad xdx = \frac{dt}{2}$
$\therefore \quad I = \frac{1}{2}\int \frac{t}{t^2+3t+2}dt$
$= \frac{1}{2}\int \frac{t}{(t+2)(t+1)}dt$
By using partial fractions,
$\frac{t}{(t+2)(t+1)} = \frac{A}{t+2} + \frac{B}{t+1}$
$I = \frac{1}{2}\int \frac{t}{(t+2)(t+1)}dt = \frac{1}{2}\int \frac{A}{t+2} + \frac{B}{t+1}dt$...(i)
$t = A(t+1) + B(t+2)$
if $t = -2 \Rightarrow -2 = A(-1), \therefore A = 2$
if $t = -1 \Rightarrow -1 = B(1), \therefore B = -1$
put values of A and B in (i)
$I = \frac{1}{2}\left[\int \frac{2}{t+2}dt - \int \frac{1}{t+1}dt\right]$
$= \frac{1}{2}[2\log|t+2| - \log|t+1|] + C$
$= \log|t+2| - \frac{1}{2}\log|t+1| + C$
$= \log|t+2| - \log\sqrt{t+1} + C$
$= \log\left|\frac{t+2}{\sqrt{t+1}}\right| + C$
$put\ t = x^2$
$I = \log\left|\frac{x^2+2}{\sqrt{x^2+1}}\right| + C$

14. $\int \frac{dx}{\sqrt{\sin^3 x \sin(x+\alpha)}}$
$\int \frac{dx}{\sqrt{\sin^4 x.\frac{\sin(x+\alpha)}{\sin x}}}$
$\int \frac{dx}{\sin^2 x\sqrt{\frac{\sin(x+\alpha)}{\sin x}}} = \int \frac{cosec^2 x dx}{\sqrt{\frac{\sin(x+\alpha)}{\sin x}}}$
$= \int \frac{cosec^2 x dx}{\sqrt{\frac{\sin x.\cos\alpha + \cos x.\sin\alpha}{\sin x}}}$
$= \int \frac{cosec^2 x dx}{\sqrt{\cos\alpha + \cot x.\sin\alpha}}$

put $\cos\alpha + \cot x.\sin\alpha = t$

$0 - \cos ec^2 x.\sin\alpha dx = dt$

$= \int -\frac{1}{sin\alpha}\frac{dt}{\sqrt{t}} = -\frac{1}{\sin\alpha}.\frac{t^{1/2}}{1/2} + C$

$= \frac{-2}{\sin\alpha}\sqrt{\cos\alpha + \cot x.\sin\alpha} + C$

15. According to the question , given region is (x, y) : $x^2 + y^2 \leq 16, x^2 \leq 6y$

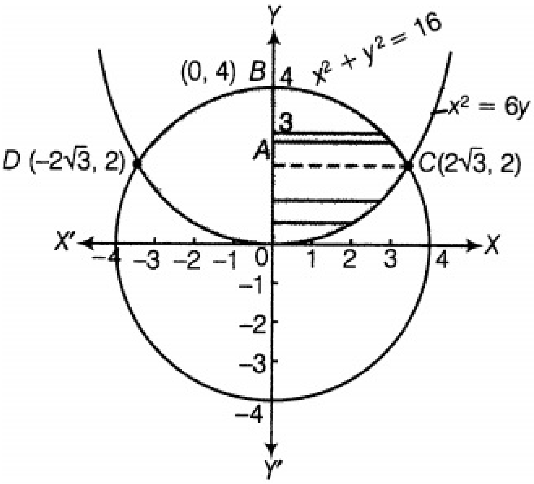
Above region consists a parabola whose vertex is (0, 0) and

Axis of parabola is along Y-axis.

Above region also consists a circle $x^2 + y^2 = 16$ whose centre is (0, 0) and

Radius of circle is = 4.

First, let us sketch the region, as shown below:



For finding the points of intersection of two curves, we have

$x^2 + y^2 = 16$ ....(i)

and $x^2 = 6y$ .... (ii)

On putting $x^2 = 6y$ from Eq.(ii) in Eq. (i), we get

$y^2 + 6y - 16 = 0$

$y^2 + 8y - 2y - 16 = 0$

y(y + 8) - 2 (y + 8) = 0

(y - 2) (y + 8) = 0

y = 2 or - 8

When y = 2, then from Eq. (ii), we get

x = $\pm\sqrt{12}$ = $\pm 2\sqrt{3}$

When y = - 8, then from Eq. (ii), we get

$x^2 = -48$ which is not possible [∵ square root of negative terms does not exist.]

So, y = -8 is rejected.

Here, we consider only one value of y i.e. 2

Thus, the two curves meet at points C($2\sqrt{3}$, 2) and D(-2$\sqrt{3}$, 2).

Now,

Required area = Area of shaded region OCBDO

= 2 [Area of region OACO + Area of region ABCA]

$$= 2\left[\int_0^2 x_{\text{ (parabola)}} dy + \int_2^4 x_{\text{ (circle)}} dy\right]$$
$$= 2\left[\int_0^2 \sqrt{6y} dy + \int_2^4 \sqrt{16 - y^2} dy\right]$$
$$= 2\left[\sqrt{6}\int_0^2 \sqrt{y} dy + \int_2^4 \sqrt{16 - y^2} dy\right]$$
$$= 2\left(\left[\sqrt{6} \cdot y^{3/2} \cdot \frac{2}{3}\right]_0^2 + \left[\frac{y}{2}\sqrt{16 - y^2} + \frac{16}{2}\sin^{-1}\frac{y}{4}\right]_2^4\right)$$
$$= 2\left\{\left[\frac{2\sqrt{6}}{3} y^{3/2}\right]_0^2 + \left[0 + 8\sin^{-1} 1 - \sqrt{12} - 8\sin^{-1}\frac{1}{2}\right]\right\}$$
$$= 2\left[\frac{2\times\sqrt{2}\times\sqrt{3}}{3} \times \left[(\sqrt{2})^2\right]^{3/2} + 8\sin^{-1}\left(\sin\frac{\pi}{2}\right) - 2\sqrt{3} - 8\sin^{-1}\left(\sin\frac{\pi}{6}\right)\right]$$
$$= 2\left[\frac{2\sqrt{2}\times\sqrt{3}}{3} \times 2\sqrt{2} + 4\pi - 2\sqrt{3} - \frac{8\pi}{6}\right]$$
$$= 2\left[\frac{8\sqrt{3}}{3} + 4\pi - 2\sqrt{3} - \frac{4\pi}{3}\right]$$
$$= 2\left[\frac{2\sqrt{3}}{3} + \frac{8\pi}{3}\right]$$
$$= \left(\frac{4\sqrt{3}}{3} + \frac{16\pi}{3}\right)$$

$\therefore I = \frac{4\sqrt{3}}{3} + \frac{16\pi}{3}$ sq units.