CLASS : XII CHAPTER – 5 (CONTINUITY & DIFFERENTIABILITY)

MISCELLANEOUS EXERCISE

Differentiate w.r.t x. the functions in exercise 1 to 11.

Q No 1. $\frac{d}{dx}(3x^2-9x+5)^9 = 9.(3x^2-9x+5)^{9-1} \cdot \frac{d}{dx}(3x^2-9x+5)$

$= 9(3x^2-9x+5)^9(6x-9) = 27(2x-3)(3x^2-9x+5)^9$

Q No.2 $\frac{d}{dx}(\sin^3 x+\cos^6 x) = \frac{d}{dx}(\sin^3 x)+\frac{d}{dx}(\cos^6 x)$

$= 3\sin^2 x \cdot \cos x + 6\cos^5 x \cdot (-\sin x)$

$= 3\sin^2 x\cos x - 6\sin x \cdot \cos^5 x.$

Q No 3. $\frac{d}{dx}(5x)^{3\cos 2x}$.

Let $y = (5x)^{3\cos 2x}$

Taking log on both sides

$\log y = 3\cos 2x \log 5x.$

$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = 3\cos 2x \cdot \frac{1}{5x}\cdot 5 + 3\log 5x \cdot(-\sin 2x)\cdot 2$

$\Rightarrow \frac{dy}{dx} = y\left[\frac{3\cos 2x}{x} - 6(\sin 2x)(\log 5x)\right] = (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6(\sin 2x)(\log 5x)\right]$

Q No 4 $\frac{d}{dx}(\sin^{-1}(x\sqrt{x}))$; $0 \le x \le 1$.

Sol: $\frac{d}{dx}(\sin^{-1} x\sqrt{x}) = \frac{d}{dx}\sin^{-1}(x^{3/2}) = \frac{1}{\sqrt{1-(x^{3/2})^2}} \cdot \frac{3}{2}x^{\frac{3}{2}-1}$

$= \frac{1}{\sqrt{1-x^3}} \cdot \frac{3}{2}(x)^{1/2} = \frac{3}{2}\left(\frac{\sqrt{x}}{\sqrt{1-x^3}}\right) = \frac{3}{2}\left(\sqrt{\frac{x}{1-x^3}}\right)$

Q No.5 $\frac{d}{dx}\frac{\left(\cos^{-1}\frac{x}{2}\right)}{\sqrt{2x+7}}$, $-2<x<2$.

$= \frac{\sqrt{(2x+7)}\cdot\frac{-1}{\sqrt{1-x^2/4}}\cdot\frac{1}{2} - \cos^{-1}\left(\frac{x}{2}\right)\cdot\frac{1}{2}(2x+7)^{-1/2}}{(2x+7)} = \frac{\frac{-\sqrt{2x+7}}{\sqrt{4-x^2}} - \frac{\cos^{-1}(x/2)}{\sqrt{2x+7}}}{2x+7}$

$$= -\frac{\left(2x+7+\cos^{-1}(x/2)\sqrt{4-x^2}\right)}{\sqrt{4-x^2}\,(2x+7)^{3/2}}$$

Q No 6 $y = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \cot^{-1}\left(\frac{\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}+\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}-\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}\right)$

$$= \cot^{-1}\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\sin\frac{x}{2}+\cos\frac{x}{2}-\cos\frac{x}{2}+\sin x/2}\right) = \cot^{-1}\left(\frac{2\cos x/2}{2\sin x/2}\right)$$

$$= \cot^{-1}\left(\cot x/2\right) = \frac{x}{2}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}.$$

Q No 7 $(\log x)^{\log x}$

Sol : $y = (\log x)^{\log x} \Rightarrow \log y = \log x \cdot \log(\log x)$

$$\therefore \frac{1}{y}\cdot\frac{dy}{dx} = \log x\cdot\frac{1}{\log x}\cdot\frac{1}{x} + \log(\log x)\cdot\frac{1}{x} = \frac{1}{x}\cdot\left(1+\log(\log x)\right)$$

$$\therefore \frac{dy}{dx} = y\cdot\frac{1}{x}\left(1+\log(\log x)\right) = \frac{y}{x}\left(1+\log(\log x)\right)$$

Q No 8 $\frac{d}{dx}\left(\cos(a\cos x + b\sin x)\right)$

$$= -\sin(a\cos x + b\sin x)\cdot\frac{d}{dx}(a\cos x + b\sin x)$$

$$= -\sin(a\cos x + b\sin x)(-a\sin x + b\cos x)$$

Q No 9 $(\sin x - \cos x)^{(\sin x - \cos x)}$ $\quad \frac{\pi}{4} < x < \frac{3\pi}{4}$

Sol Let $y = (\sin x - \cos x)^{(\sin x - \cos x)}$

$$\Rightarrow \log y = (\sin x - \cos x)\log(\sin x - \cos x)$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = (\sin x-\cos x)\frac{\cos x+\sin x}{\sin x-\cos x} + \log(\sin x-\cos x)(\cos x+\sin x)$$

$$\Rightarrow \frac{dy}{dx} = y(\cos x+\sin x)\left[1+\log(\sin x-\cos x)\right]$$

$$= (\sin x-\cos x)^{(\sin x-\cos x)}(\cos x+\sin x)(1+\log(\sin x-\cos x))$$

QNo.10 $\frac{d}{dx}(x^x + x^a + a^x + a^a)$; $x>0,\ a>0.$

$$= \frac{d}{dx}(x^x) + \frac{d}{dx}(x^a) + \frac{d}{dx}(a^x) + \frac{d}{dx}(a^a)$$

$$= x^x\cdot\frac{d}{dx}(x\log x) + ax^{a-1} + a^x\log a + 0 \quad \left[\frac{d}{dx}(u)^v = u^v\cdot\frac{d}{dx}(v\log u)\right]$$

$$= x^x\left[x\cdot\frac{1}{x}+\log x\cdot 1\right] + ax^{a-1} + a^x\log a. = x^x(1+\log x) + ax^{a-1} + a^x\log a.$$

QNo11. $\frac{d}{dx}\left(x^{x^2-3} + (x-3)^{x^2}\right) = \frac{d}{dx}\left(x^{x^2-3}\right) + \frac{d}{dx}(x-3)^{x^2}$

$$= x^{x^2-3}\frac{d}{dx}\left[(x^2-3)\log x\right] + (x-3)^{x^2}\frac{d}{dx}\left(x^2\cdot\log(x-3)\right)$$

$$= x^{x^2-3}\left[(x^2-3)\cdot\frac{1}{x} + \log x(2x-0)\right] + (x-3)^{x^2}\left[x^2\cdot\frac{1}{x-3}(1-0) + \log(x-3)\cdot 2x\right] \quad \left[\because \frac{d}{dx}(u^v) = u^v\frac{d}{dx}(v\log u)\right]$$

$$= x^{x^2-3}\left[\frac{x^2-3}{x} + 2x\log x\right] + (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$$

QNo12. Find $\frac{dy}{dx}$ if $y = 12(1-\cos t)$; $x = 10(t-\sin t)$; $-\frac{\pi}{2} < t < \frac{\pi}{2}$

Sol $\frac{dy}{dt} = 12(0+\sin t) = 12\sin t$ and $\frac{dx}{dt} = 10(1-\cos t)$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12\sin t}{10(1-\cos t)} = \frac{24\sin t/2\cos t/2}{10\times 2\sin^2 t/2} = \frac{6}{5}(\cot t/2)$$

Q No 13 : Find $\frac{dy}{dx}$ if $y = \sin^{-1}x + \sin^{-1}\left(\sqrt{1-x^2}\right)$ $0<x<1$

Sol.
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-1+x^2}} \frac{-x}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}\left(1 - \frac{x}{|x|}\right)$$

$$= \frac{1}{\sqrt{1-x^2}}(1-1) = 0 \quad \because \text{ for } 0<x<1 \quad |x| = x$$

Q No 14. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ for $-1<x<1$ prove $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

$\Rightarrow$ $x\sqrt{1+y} = -y\sqrt{1+x}$ ...(1)

$\Rightarrow$ $x^2(1+y) = y^2(1+x)$

$\Rightarrow$ $x^2+x^2y = y^2+y^2x$

$\Rightarrow$ $(x^2-y^2)+xy(x-y) = 0 \Rightarrow (x-y)(x+y+xy) = 0$

$\Rightarrow$ $x-y=0$ or $x+y+xy=0$

$\Rightarrow$ $y = x$ or. $y(1+x) = -x \Rightarrow y = x$ or $y = \frac{-x}{1+x}$.

Now. (1) is satisfied $x=y=0$

$\therefore$ $y = \frac{-x}{1+x}$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+x)(-1)-(-x)(0+1)}{(1+x)^2} = \frac{-1}{(1+x)^2}$$

Q No 15: If $(x-a)^2+(y-b)^2 = c^2$ for some $c>0$, prove that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$ is a constant independent of a and b.

Sol. Given $(x-a)^2+(y-b)^2=c^2$ ---(1)

Differentiating w.r.t x we get

$$2(x-a)+2(y-b)\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{(x-a)}{(y-b)} \quad \text{---(2)}$$

$$\Rightarrow \frac{d^2y}{dx^2}=-\frac{(y-b)(1-0)-(x-a)\cdot\frac{dy}{dx}}{(y-b)^2}=\frac{(y-b)-(x-a)\left(-\frac{x-a}{y-b}\right)}{(y-b)^2} \quad (\text{from (2)})$$

$$=\frac{(y-b)^2+(x-a)^2}{(y-b)^3}=-\frac{c^2}{(y-b)^3} \quad (\text{using (1)})$$

$$\therefore \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}=\frac{\left[1+\left(\frac{x-a}{y-b}\right)^2\right]^{3/2}}{-\frac{c^2}{(y-b)^3}}=\frac{\left[(y-b)^2+(x-a)^2\right]^{3/2}}{\left[(y-b)^2\right]^{3/2}\left[-\frac{c^2}{(y-b)^3}\right]}$$

$$=\frac{\left[c^2\right]^{3/2}}{-c^2}=\frac{c^3}{-c^2}=-c$$ which is constant

and independent of a and b.

QNo16 If $\cos y = x\cos(a+y)$ with $\cos a \neq \pm 1$, prove that

$$\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}.$$

Sol Given $\cos y = x\cos(a+y)$

$$\Rightarrow x=\frac{\cos y}{\cos(a+y)}$$

$$\therefore \frac{dx}{dy}=\frac{\cos(a+y)(-\sin y)-\cos y(-\sin(a+y))}{\cos^2(a+y)}=\frac{\sin(a+y)\cos y-\cos(a+y)\sin y}{\cos^2(a+y)}$$

$$=\frac{\sin(a+y-y)}{\cos^2(a+y)}=\frac{\sin a}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$ Note that $\sin a \neq 0$ as $\cos a \neq \pm 1$

QNo.17 · If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$\frac{dx}{dt} = a\left[-\sin t + t\cos t + \sin t\right]$ and $\frac{dy}{dt} = a\left(\cos t - t(-\sin t) - \cos t \cdot 1\right)$

$= at\cos t$ $\qquad = +at\sin t$

$$\therefore \frac{dy}{dx} = \frac{at\sin t}{at\cos t} = \tan t$$

$$\therefore \frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx} = \frac{\sec^2 t}{at\cos t} = \frac{1}{at}\sec^3 t$$ $\left[\because \frac{dx}{dt} = at\cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at\cos t}\right.$

QNo.18 If $f(x) = |x|^3$ show that $f''(x)$ exists for all x, find it.

Sol. Given $f(x) = |x|^3 = |x|^2 \cdot |x| = x^2|x|$

for $x \neq 0$ $f'(x) = x^2 \cdot \frac{d}{dx}|x| + |x|\frac{d}{dx}(x^2)$

$= x^2 \cdot \frac{x}{|x|} + |x| \cdot 2x.$ $\qquad \because \frac{d}{dx}(|x|) = \frac{x}{|x|}$

$= x|x| + 2x|x| = 3x|x|$

Also. $f''(x) = \frac{d}{dx}(3x|x|) = 3\left[x \cdot \frac{d}{dx}(|x|) + |x|\frac{d}{dx}(x)\right]$

$= 3\left[x \cdot \frac{x}{|x|} + |x| \cdot 1\right] = 3\left[\frac{x^2}{|x|} + |x|\right] = 3\left[|x| + |x|\right]$

$= 6|x|.$

for $x = 0$ $\quad Lf'(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^2|h| - 0}{h}$

$= \lim_{h \to 0^-} (h|h|) = \lim_{h \to 0^-} h(-h) = 0$

$Rf'(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^2|h| - 0}{h} = \lim_{h \to 0^+} h(|h|)$

$= \lim_{h \to 0^+} h(h) = 0.$

$\therefore Lf'(0) = Rf'(0) = 0$

and $Lf''(0) = \underset{h\to 0^-}{Lt} \frac{f'(0+h) - f'(0)}{h} = \underset{h\to 0}{Lt} \frac{3h|h| - 0}{h} = \underset{h\to 0^-}{Lt} 3(-h) = 0$

and $Rf''(0) = \underset{h\to 0^+}{Lt} \frac{f'(0+h) - f'(0)}{h} = \underset{h\to 0^+}{Lt} \frac{3h(|h|) - 0}{h} = \underset{h\to 0^+}{Lt} 3(+h) = 0$

$\therefore \; Lf''(0) = Rf''(0) = 0$

Thus We find that $f''(x) = \begin{cases} 6|x| & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

$\Rightarrow \; f''(x) = 6|x| \;\; \forall\, x \in R$

Q No. 19 Using Mathematical Induction prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integer n.

Sol:

Let $P(n)$ is $\frac{d}{dx}(x^n) = n\,x^{n-1}$

Now $P(1)$ is $\frac{d}{dx}(x^1) = 1 \cdot x^{1-1}$

i.e. $1 = 1$, Which is true

$\therefore$ $P(1)$ is true for $n=1$.

Let $P(m)$ be true for $m \in N$.

$\Rightarrow \; \frac{d}{dx}(x^m) = m\,x^{m-1} \quad \ldots (1)$

Now $\frac{d}{dx}(x^{m+1}) = \frac{d}{dx}(x^m \cdot x) = x \cdot \frac{d}{dx}(x^m) + x^m \frac{d}{dx}(x)$

$= x \cdot m x^{m-1} + x^m = x^m \cdot m + x^m = (m+1)\,x^m = (m+1)\,x^{(m+1)-1}$

$\therefore$ $P(m+1)$ is true.

Thus $P(1)$ is true and $P(m) \Rightarrow P(m+1)$, $n \in N$)

Hence by principle of mathematical induction

$P(n)$ is true $\forall\, n \in N$

$\therefore \; \frac{d}{dx}(x^n) = n\,x^{n-1} \quad \forall\, n \in N.$

Q.No.20 Using the fact that $\sin(A+B) = \sin A\cos B + \cos A\sin B$, and differentiation, obtain sum formula for cosines.

Sol:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B.$$

Differentiate both sides w.r.t x taking B as constant

$$\Rightarrow \cos(A+B)\cdot\frac{dA}{dx} = \cos B\cdot\cos A\frac{dA}{dx} + \sin B(-\sin A)\cdot\frac{dA}{dx}$$

$$\Rightarrow \cos(A+B)\frac{dA}{dx} = [\cos A\cos B - \sin A\sin B]\frac{dA}{dx}$$

$$\Rightarrow \cos(A+B) = \cos A\cos B - \sin A\sin B.$$

Q.No.21: Does there exist a function which is continuous everywhere but not differentiable at exactly two points. Justify your answer.

Sol: Yes. eg. Take $f(x) = |x-1| + |x-2|$

Then $f(x)$ is continuous $\forall x\in\mathbb{R}$ and differentiable at all x except 1 and 2.

Now Here $f(x) = |x-1| + |x-2|$

$$= \begin{cases} -(x-1)+(-(x-2)) & ; x<1 \\ x-1-(x-2) & ; 1\le x\le 2 \\ x-1+x-2 & ; x>2 \end{cases}$$

ie $$f(x) = \begin{cases} -2x+3 & ; x<1 \\ 1 & ; 1\le x\le 2 \\ 2x-3 & ; x>2 \end{cases}$$

Clearly $f(x)$ is continuous at all x except possibly at 1, 2.

At $x=1$

$f(1) = 1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(-2x+3) = -2+3 = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(1) = 1$$

$$\therefore \lim_{x\to 1^-} f(x) = f(1) = \lim_{x\to 1^+} f(x) \Rightarrow f \text{ is continuous at } x=1$$

At $x=2$: $f(2)=1$

$$\underset{x\to 2^-}{Lt} f(x) = \underset{x\to 2^-}{Lt} (1) = 1$$

$$\underset{x\to 2^+}{Lt} f(x) = \underset{x\to 2^+}{Lt} 2x-3 = 4-3=1$$

$$\therefore \underset{x\to 2^-}{Lt} f(x) = f(2) = \underset{x\to 2^+}{Lt} f(x) \Rightarrow f \text{ is conti at. } x=2$$

$\therefore$ f is continuous $\forall\, x\in R$

Also $$f'(x) = \begin{cases} -2 & ; x<1 \\ 0 & ; 1<x<2 \\ 2 & ; x>2. \end{cases}$$

Now At $x=1$, $$Lf'(1) = \underset{h\to 0^-}{Lt} \frac{f(1+h)-f(1)}{h} = \underset{h\to 0^-}{Lt} \frac{2(1+h)+3-1}{h} = -2$$

and $$Rf'(1) = \underset{h\to 0^+}{Lt} \frac{f(1+h)-f(1)}{h} = \underset{h\to 0^+}{Lt} \frac{1-1}{h} = 0$$

$$\therefore Lf'(1) \neq Rf'(1)$$

$\therefore$ f is not differentiable at $x=1$

At $x=2$ $$Lf'(2) = \underset{h\to 0^-}{Lt} \frac{f(2+h)-f(2)}{h} = \underset{h\to 0^-}{Lt} \frac{1-1}{h} = 0$$

and $$Rf'(2) = \underset{h\to 0^+}{Lt} \frac{f(2+h)-f(2)}{h} = \underset{h\to 0^+}{Lt} \frac{2(2+h)-3-1}{h} = 2$$

$$\therefore Lf'(2) \neq Rf'(2)$$

$\therefore$ f is not differentiable at $x=2$.

Thus we see that $f(x) = |x-1| + |x-2|$ is continuous everywhere and differentiable also at all $x \in R$ except at 1, 2.

Q No 22 If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Sol. $\Rightarrow y = f(x)(mc - nb) + (na - lc)\, g(x) + (lb - ma)\, h(x)$

$\Rightarrow \frac{dy}{dx} = f'(x)(mc - nb) + (na - lc)\, g'(x) + (lb - ma)\, h'(x)$

$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Q No.23 If $y = e^{a\cos^{-1}x}$; $-1 \le x \le 1$ show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

Sol. $y = e^{a\cos^{-1}x}$. $-1 \le x \le 1$ ---(1)

$\Rightarrow \frac{dy}{dx} = e^{a\cos^{-1}x} \cdot a \cdot \frac{-1}{\sqrt{1-x^2}} = \frac{-ay}{\sqrt{1-x^2}}$ $\left[\because e^{a\cos^{-1}x} = y\right]$

$\Rightarrow \sqrt{1-x^2}\,\frac{dy}{dx} = -ay$ ---(2)

Again differentiating, we get

$\sqrt{1-x^2}\,\frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{-2x}{2\sqrt{1-x^2}} = -a\frac{dy}{dx}$.

or $\sqrt{1-x^2}\,\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}} \cdot \frac{dy}{dx} = -a\left(\frac{-ay}{\sqrt{1-x^2}}\right)$ [using (2)]

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx^2} - a^2y = 0$.

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