

Chapter

Sequences and Series



Topic-1: Arithmetic Progression



1 MCQs with One Correct Answer

- Let $b_i > 1$ for $i = 1, 2, \dots, 101$. Suppose $\log_e b_1, \log_e b_2, \dots, \log_e b_{101}$ are in Arithmetic Progression (A.P.) with the common difference $\log_e 2$. Suppose a_1, a_2, \dots, a_{101} are in A.P. such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + \dots + b_{51}$ and $s = a_1 + a_2 + \dots + a_{51}$, then [Adv. 2016]
 - $s > t$ and $a_{101} > b_{101}$
 - $s > t$ and $a_{101} < b_{101}$
 - $s < t$ and $a_{101} > b_{101}$
 - $s < t$ and $a_{101} < b_{101}$
- In the sum of first n terms of an A.P. is cn^2 , then the sum of squares of these n terms is [2009]
 - $\frac{n(4n^2 - 1)c^2}{6}$
 - $\frac{n(4n^2 + 1)c^2}{3}$
 - $\frac{n(4n^2 - 1)c^2}{3}$
 - $\frac{n(4n^2 + 1)c^2}{6}$
- If the sum of the first $2n$ terms of the A.P. 2, 5, 8, ..., is equal to the sum of the first n terms of the A.P. 57, 59, 61, ..., then n equals [2001S]
 - 10
 - 12
 - 11
 - 13



2 Integer Value Answer/Non-Negative Integer

- Suppose that all the terms of an arithmetic progression (A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is [Adv. 2015]
- A pack contains n cards numbered from 1 to n . Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k , then $k - 20 =$ [Adv. 2013]
- Let $a_1, a_2, a_3, \dots, a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^p a_i, 1 \leq p \leq 100$. For any integer n with $1 \leq n \leq 20$, let $m = 5n$. If $\frac{S_m}{S_n}$ does not depend on n , then

- a_2 is [2011]
Let $S_k, k = 1, 2, \dots, 100$, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then the value of $\frac{100^2}{100!} + \sum_{k=1}^{100} (k^2 - 3k + 1)S_k$ is [2010]
- Let $a_1, a_2, a_3, \dots, a_{11}$ be real numbers satisfying $a_1 = 15, 27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4, \dots, 11$. If $\frac{a_1^2 + a_2^2 + \dots + a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + \dots + a_{11}}{11}$ is equal to [2010]



3 Numeric/New Stem Based Questions

- Let l_1, l_2, \dots, l_{100} be consecutive terms of an arithmetic progression with common difference d_1 , and let w_1, w_2, \dots, w_{100} be consecutive terms of another arithmetic progression with common difference d_2 , where $d_1 d_2 = 10$. For each $i = 1, 2, \dots, 100$, let R_i be a rectangle with length l_i , width w_i and area A_i . If $A_{51} - A_{50} = 1000$, then the value of $A_{100} - A_{90}$ is [Adv. 2022]
- Let $AP(a; d)$ denote the set of all the terms of an infinite arithmetic progression with first term a and common difference $d > 0$. If $AP(1; 3) \cap AP(2; 5) \cap AP(3; 7) = AP(a; d)$ then $a + d$ equals [Adv. 2019]
- Let X be the set consisting of the first 2018 terms of the arithmetic progression 1, 6, 11, ..., and Y be the set consisting of the first 2018 terms of the arithmetic progression 9, 16, 23, Then, the number of elements in the set $X \cup Y$ is [Adv. 2018]
- The sides of a right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side? [Adv. 2018]



4 Fill in the Blanks

13. Let p and q be roots of the equation $x^2 - 2x + A = 0$ and let r and s be the roots of the equation $x^2 - 18x + B = 0$. If $p < q < r < s$ are in arithmetic progression, then $A = \dots\dots\dots$ and $B = \dots\dots\dots$ [1997 - 2 Marks]
14. The sum of integers from 1 to 100 that are divisible by 2 or 5 is $\dots\dots\dots$ [1984 - 2 Marks]



6 MCQs with One or More than One Correct Answer

15. Let a_1, a_2, a_3, \dots be an arithmetic progression with $a_1 = 7$ and common difference 8. Let T_1, T_2, T_3, \dots be such that $T_1 = 3$ and $T_{n+1} - T_n = a_n$ for $n \geq 1$. Then, which of the following is/are TRUE? [Adv. 2022]
- (a) $T_{20} = 1604$ (b) $\sum_{k=1}^{20} T_k = 10510$
 (c) $T_{30} = 3454$ (d) $\sum_{k=1}^{30} T_k = 35610$
16. Let $S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2$. Then S_n can take value(s) [Adv. 2013]
- (a) 1056 (b) 1088 (c) 1120 (d) 1332
17. Let T_r be the r^{th} term of an A.P., for $r = 1, 2, 3, \dots$. If for some positive integers m, n we have

$T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then T_{mn} equals [1998 - 2 Marks]

- (a) $\frac{1}{mn}$ (b) $\frac{1}{m} + \frac{1}{n}$
 (c) 1 (d) 0



8 Comprehension/Passage Based Questions

Let V_r denote the sum of first r terms of an arithmetic progression (A.P.) whose first term is r and the common difference is $(2r-1)$. Let $T_r = V_{r+1} - V_r - 2$ and $Q_r = T_{r+1} - T_r$ for $r = 1, 2, \dots$

18. The sum $V_1 + V_2 + \dots + V_n$ is [2007 - 4 marks]

(a) $\frac{1}{12} n(n+1)(3n^2 - n + 1)$

(b) $\frac{1}{12} n(n+1)(3n^2 + n + 2)$

(c) $\frac{1}{2} n(2n^2 - n + 1)$

(d) $\frac{1}{3} (2n^3 - 2n + 3)$

19. T_r is always
 (a) an odd number (b) an even number
 (c) a prime number (d) a composite number
20. Which one of the following is a correct statement?
 (a) Q_1, Q_2, Q_3, \dots are in A.P. with common difference 5
 (b) Q_1, Q_2, Q_3, \dots are in A.P. with common difference 6
 (c) Q_1, Q_2, Q_3, \dots are in A.P. with common difference 11
 (d) $Q_1 = Q_2 = Q_3 = \dots$



10 Subjective Problems

21. The fourth power of the common difference of an arithmetic progression with integer entries is added to the product of any four consecutive terms of it. Prove that the resulting sum is the square of an integer. [2000 - 4 Marks]
22. The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in AP. Find the intervals in which β and γ lie. [1996 - 3 Marks]
23. If $\log_3 2, \log_3(2^x - 5)$, and $\log_3\left(2^x - \frac{7}{2}\right)$ are in arithmetic progression, determine the value of x . [1990 - 4 Marks]
24. The interior angles of a polygon are in arithmetic progression. The smallest angle is 120° , and the common difference is 5° . Find the number of sides of the polygon. [1980]



Topic-2: Geometric Progression



1 MCQs with One Correct Answer

- In the quadratic equation $ax^2 + bx + c = 0$, $\Delta = b^2 - 4ac$ and $\alpha + \beta$, $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$, are in G.P. where α, β are the root of $ax^2 + bx + c = 0$, then [2005S]
(a) $\Delta \neq 0$ (b) $b\Delta = 0$ (c) $c\Delta = 0$ (d) $\Delta = 0$
- An infinite G.P. has first term 'x' and sum '5', then x belongs to [2004S]
(a) $x < -10$ (b) $-10 < x < 0$
(c) $0 < x < 10$ (d) $x > 10$
- Suppose a, b, c are in A.P. and a^2, b^2, c^2 are in G.P. if $a < b < c$ and $a + b + c = \frac{3}{2}$, then the value of a is [2002S]
(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2\sqrt{3}}$ (c) $\frac{1}{2} - \frac{1}{\sqrt{3}}$ (d) $\frac{1}{2} - \frac{1}{\sqrt{2}}$
- Let α, β be the roots of $x^2 - x + p = 0$ and γ, δ be the roots of $x^2 - 4x + q = 0$. If $\alpha, \beta, \gamma, \delta$ are in G.P., then the integral values of p and q respectively, are [2001S]
(a) $-2, -32$ (b) $-2, 3$ (c) $-6, 3$ (d) $-6, -32$
- Consider an infinite geometric series with first term a and common ratio r . If its sum is 4 and the second term is $\frac{3}{4}$, then [2000S]
(a) $a = \frac{4}{7}, r = \frac{3}{7}$ (b) $a = 2, r = \frac{3}{8}$
(c) $a = \frac{3}{2}, r = \frac{1}{2}$ (d) $a = 3, r = \frac{1}{4}$
- Sum of the first n terms of the series $\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \dots$ is equal to [1988 - 2 Marks]
(a) $2^n - n - 1$ (b) $1 - 2^{-n}$
(c) $n + 2^{-n} - 1$ (d) $2^n + 1$
- If a, b, c are in G.P., then the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in — [1985 - 2 Marks]
(a) A.P. (b) G.P.
(c) H.P. (d) none of these
- The rational number, which equals the number $2.\overline{357}$ with recurring decimal is [1983 - 1 Mark]
(a) $\frac{2355}{1001}$ (b) $\frac{2379}{997}$
(c) $\frac{2355}{999}$ (d) none of these
- The third term of a geometric progression is 4. The product of the first five terms is [1982 - 2 Marks]
(a) 4^3 (b) 4^5
(c) 4^4 (d) none of these



2 Integer Value Answer/Non-Negative Integer

- Let a_1, a_2, a_3, \dots be a sequence of positive integers in arithmetic progression with common difference 2. Also, let b_1, b_2, b_3, \dots be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c , for which the equality

$$2(a_1 + a_2 + \dots + a_n) = b_1 + b_2 + \dots + b_n$$

holds for some positive integer n , is ____ [Adv. 2020]



6 MCQs with One or More than One Correct Answer

- Let n be an odd integer. If $\sin n\theta = \sum_{r=0}^n b_r \sin^r \theta$, for every value of θ , then [1998 - 2 Marks]
(a) $b_0 = 1, b_1 = 3$ (b) $b_0 = 0, b_1 = n$
(c) $b_0 = -1, b_1 = n$ (d) $b_0 = 0, b_1 = n^2 - 3n + 3$
- For $0 < \phi < \pi/2$, if
$$x = \sum_{n=0}^{\infty} \cos^{2n} \phi, y = \sum_{n=0}^{\infty} \sin^{2n} \phi, z = \sum_{n=0}^{\infty} \cos^{2n} \phi \sin^{2n} \phi$$

then: [1993 - 2 Marks]
(a) $xyz = xz + y$
(b) $xyz = xy + z$
(c) $xyz = x + y + z$
(d) $xyz = yz + x$



10 Subjective Problems

- If $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \dots + (-1)^{n-1} \left(\frac{3}{4}\right)^n$ and $b_n = 1 - a_n$, then find the least natural number n_0 such that $b_n > a_n \forall n \geq n_0$. [2006 - 6M]
- Let a, b, c, d be real numbers in G.P. If u, v, w , satisfy the system of equations [1999 - 10 Marks]
$$\begin{aligned} u + 2v + 3w &= 6 \\ 4u + 5v + 6w &= 12 \\ 6u + 9v &= 4 \end{aligned}$$

then show that the roots of the equation
$$\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right)x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2]x + u + v + w = 0$$

and $20x^2 + 10(a-d)^2x - 9 = 0$ are reciprocals of each other.

15. If $S_1, S_2, S_3, \dots, S_n$ are the sums of infinite geometric series whose first terms are 1, 2, 3, \dots, n and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n+1}$ respectively, then find the values of $S_1^2 + S_2^2 + S_3^2 + \dots + S_{2n-1}^2$ [1991 - 4 Marks]

16. Find three numbers a, b, c , between 2 and 18 such that
 (i) their sum is 25
 (ii) the numbers 2, a, b are consecutive terms of an A.P. and
 (iii) the numbers $b, c, 18$ are consecutive terms of a G.P. [1983 - 2 Marks]
17. Does there exist a geometric progression containing 27, 8 and 12 as three of its terms? If it exists, how many such progressions are possible? [1982 - 3 Marks]

Topic-3: Harmonic Progression, Relation Between A. M., G. M. and H. M. of two Positive Numbers



1 MCQs with One Correct Answer

1. Let a_1, a_2, a_3, \dots be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is [2012]
 (a) 22 (b) 23 (c) 24 (d) 25
2. Let the positive numbers a, b, c, d be in A.P. Then abc, abd, acd, bcd are [2001S]
 (a) NOT in A.P./G.P./H.P. (b) in A.P.
 (c) in G.P. (d) in H.P.
3. The harmonic mean of the roots of the equation $(5 + \sqrt{2})x^2 - (4 + \sqrt{5})x + 8 + 2\sqrt{5} = 0$ is [1999 - 2 Marks]
 (a) 2 (b) 4 (c) 6 (d) 8
4. Let a_1, a_2, \dots, a_{10} be in A.P. and h_1, h_2, \dots, h_{10} be in H.P. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then $a_4 h_7$ is [1999 - 2 Marks]
 (a) 2 (b) 3 (c) 5 (d) 6
5. If $\ln(a+c), \ln(a-c), \ln(a-2b+c)$ are in A.P., then [1994]
 (a) a, b, c are in A.P. (b) a^2, b^2, c^2 are in A.P.
 (c) a, b, c are in G.P. (d) a, b, c are in H.P.



2 Integer Value Answer/Non-Negative Integer

6. Let m be the minimum possible value of $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is [Adv. 2020]



4 Fill in the Blanks

7. Let the harmonic mean and geometric mean of two positive numbers be the ratio 4 : 5. Then the two numbers are in the ratio [1992 - 2 Marks]



6 MCQs with One or More than One Correct Answer

8. A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then [2008]

$$(a) \frac{1}{PS} + \frac{1}{ST} < \frac{2}{\sqrt{QS \times SR}} \quad (b) \frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$$

$$(c) \frac{1}{PS} + \frac{1}{ST} < \frac{4}{QR} \quad (d) \frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$$

9. If $x > 1, y > 1, z > 1$ are in G.P., then $\frac{1}{1 + \ln x}, \frac{1}{1 + \ln y}, \frac{1}{1 + \ln z}$ are in [1998 - 2 Marks]
 (a) A.P. (b) H.P.
 (c) G.P. (d) None of these
10. If the first and the $(2n-1)$ st terms of an A.P., a G.P. and an H.P. are equal and their n -th terms are a, b and c respectively, then [1988 - 2 Marks]
 (a) $a = b = c$ (b) $a \geq b \geq c$
 (c) $a + c = b$ (d) $ac - b^2 = 0$.



8 Comprehension/Passage Based Questions

Let A_1, G_1, H_1 denote the arithmetic, geometric and harmonic means, respectively, of two distinct positive numbers. For $n \geq 2$, Let A_{n-1} and H_{n-1} have arithmetic, geometric and harmonic means as A_n, G_n, H_n respectively. [2007 - 4 marks]

11. Which one of the following statements is correct?
 (a) $G_1 > G_2 > G_3 > \dots$
 (b) $G_1 < G_2 < G_3 < \dots$
 (c) $G_1 = G_2 = G_3 = \dots$
 (d) $G_1 < G_3 < G_5 < \dots$ and $G_2 > G_4 > G_6 > \dots$
12. Which one of the following statements is correct?
 (a) $A_1 > A_2 > A_3 > \dots$
 (b) $A_1 < A_2 < A_3 < \dots$
 (c) $A_1 > A_3 > A_5 > \dots$ and $A_2 < A_4 < A_6 < \dots$
 (d) $A_1 < A_3 < A_5 < \dots$ and $A_2 > A_4 > A_6 > \dots$
13. Which one of the following statements is correct?
 (a) $H_1 > H_2 > H_3 > \dots$
 (b) $H_1 < H_2 < H_3 < \dots$
 (c) $H_1 > H_3 > H_5 > \dots$ and $H_2 < H_4 < H_6 < \dots$
 (d) $H_1 < H_3 < H_5 < \dots$ and $H_2 > H_4 > H_6 > \dots$



9 Assertion and Reason/Statement Type Questions

14. Suppose four distinct positive numbers a_1, a_2, a_3, a_4 are in G.P. Let $b_1 = a_1, b_2 = b_1 + a_2, b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.

STATEMENT - 1 : The numbers b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P. and

STATEMENT - 2 : The numbers b_1, b_2, b_3, b_4 are in H.P. [2008]

- (a) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a correct explanation for STATEMENT - 1
 (b) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is **NOT** a correct explanation for STATEMENT - 1
 (c) STATEMENT - 1 is True, STATEMENT - 2 is False
 (d) STATEMENT - 1 is False, STATEMENT - 2 is True



10 Subjective Problems

15. If a, b, c are in A.P., a^2, b^2, c^2 are in H.P., then prove that either $a = b = c$ or $a, b, -\frac{c}{2}$ form a G.P. [2003 - 4 Marks]
 16. Let a, b be positive real numbers. If a, A_1, A_2, b are in arithmetic progression, a, G_1, G_2, b are in geometric progression and a, H_1, H_2, b are in harmonic progression, show that $\frac{G_1 G_2}{H_1 H_2} = \frac{A_1 + A_2}{H_1 + H_2} = \frac{(2a + b)(a + 2b)}{9ab}$. [2002 - 5 Marks]



Topic-4: Arithmetic-Geometric Sequence (A.G.S.), Some Special Sequences



2 Integer Value Answer/Non-Negative Integer

1. Let $\overbrace{75 \dots 57}^{r+2}$ denote the $(r+2)$ digit number where the first and the last digits are 7 and the remaining r digits are 5. Consider the sum $S = 77 + 757 + 7557 + \dots + \overbrace{75 \dots 57}^{98}$. If $S = \frac{\overbrace{75 \dots 57}^{99} + m}{n}$, where m and n are natural numbers less than 3000, then the value of $m + n$ is [Adv. 2023]



4 Fill in the Blanks

2. For any odd integer $n \geq 1$, $n^3 - (n-1)^3 + \dots + (-1)^{n-1} 1^3 = \dots$ [1996 - 1 Mark]
 3. The sum of the first n terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \dots$ is $n(n+1)^2/2$, when n is even. When n is odd, the sum is \dots [1988 - 2 Marks]



6 MCQs with One or More than One Correct Answer

4. Let α and β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers n , define $a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, n \geq 1$

17. Let a_1, a_2, \dots, a_n be positive real numbers in geometric progression. For each n , let A_n, G_n, H_n be respectively, the arithmetic mean, geometric mean, and harmonic mean of a_1, a_2, \dots, a_n . Find an expression for the geometric mean of G_1, G_2, \dots, G_n in terms of $A_1, A_2, \dots, A_n, H_1, H_2, \dots, H_n$. [2001 - 5 Marks]

18. Let p be the first of the n arithmetic means between two numbers and q the first of n harmonic means between the same numbers. Show that q does not lie between p and

$$\left(\frac{n+1}{n-1}\right)^2 p. \quad [1991 - 4 Marks]$$

19. If $a > 0, b > 0$ and $c > 0$, prove that

$$(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9 \quad [1984 - 2 Marks]$$

20. The harmonic mean of two numbers is 4. Their arithmetic mean A and the geometric mean G satisfy the relation. $2A + G^2 = 27$. Find the two numbers. [1979]

$$b_1 = 1 \text{ and } b_n = a_{n-1} + a_{n+1}, n \geq 2$$

Then which of the following options is/are correct?

[Adv. 2019]

- (a) $\sum_{n=1}^{\infty} \frac{\alpha_n}{10^n} = \frac{10}{89}$
 (b) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$
 (c) $a_1 + a_2 + a_3 + \dots + a_n = a_{n+2} - 1$ for all $n \geq 1$
 (d) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

5. For a positive integer n , let

$$a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{(2^n) - 1}. \text{ Then}$$

[1999 - 3 Marks]

- (a) $a(100) \leq 100$ (b) $a(100) > 100$
 (c) $a(200) \leq 100$ (d) $a(200) > 100$



10 Subjective Problems

6. Find the sum of the series :

$$\sum_{r=0}^n (-1)^r {}^n C_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} \dots \text{up to } m \text{ terms} \right]$$

[1985 - 5 Marks]

7. If n is a natural number such that

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \dots p_k^{\alpha_k} \text{ and } p_1, p_2, \dots, p_k \text{ are distinct primes, then show that } \ln n \geq k \ln 2$$

[1984 - 2 Marks]



Answer Key

Topic-1 : Arithmetic Progression

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|-----------|------------|---------|--------------|------------|------------|------------|---------|------------|
| 1. (b) | 2. (c) | 3. (c) | 4. (9) | 5. (5) | 6. (9) | 7. (3) | 8. (0) | 9. (18900) |
| 10. (157) | 11. (3748) | 12. (6) | 13. (-3, 77) | 14. (3050) | 15. (b, c) | 16. (a, d) | 17. (c) | 18. (b) |
| 19. (d) | 20. (b) | | | | | | | |

Topic-2 : Geometric Progression

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|---------|------------|--------|--------|--------|--------|--------|--------|--------|---------|
| 1. (c) | 2. (c) | 3. (d) | 4. (a) | 5. (d) | 6. (c) | 7. (a) | 8. (c) | 9. (b) | 10. (1) |
| 11. (b) | 12. (b, c) | | | | | | | | |

Topic-3 : Harmonic Progression, Relation Between A. M., G. M. and H.M. of two Positive Numbers

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|---------------|--------|---------|---------|---------|---------|-----------------|-----------|--------|
| 1. (d) | 2. (d) | 3. (b) | 4. (d) | 5. (d) | 6. (8) | 7. (4:1 or 1:4) | 8. (b, d) | 9. (b) |
| 10. (a, b, d) | | 11. (c) | 12. (a) | 13. (b) | 14. (c) | | | |

Topic-4 : Arithmetic-Geometric Sequence (A.G.S.), Some Special Sequences

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|-----------|-------------------------------|-------------------------|--------------|-----------|
| 1. (1219) | 2. $\frac{1}{4}(n+1)^2(2n-1)$ | 3. $\frac{n^2(n+1)}{2}$ | 4. (a, b, c) | 5. (a, d) |
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Hints & Solutions



Topic-1: Arithmetic Progression

- (b) $\log_e b_1, \log_e b_2, \dots, \log_e b_{101}$ are in A.P.
 $\Rightarrow b_1, b_2, \dots, b_{101}$ are in G.P.
 Also a_1, a_2, \dots, a_{101} are in A.P., where $a_1 = b_1$ and $a_{51} = b_{51}$.
 $\therefore b_2, b_3, \dots, b_{50}$ are GM's and a_2, a_3, \dots, a_{50} are AM's between b_1 and b_{51} .
 $\therefore \text{GM} < \text{AM} \Rightarrow b_2 < a_2, b_3 < a_3, \dots, b_{50} < a_{50}$
 $\therefore b_1 + b_2 + \dots + b_{51} < a_1 + a_2 + \dots + a_{51}$
 $\Rightarrow t < s$
 Also a_1, a_{51}, a_{101} are in AP and b_1, b_{51}, b_{101} are in GP
 $\therefore a_1 = b_1$ and $a_{51} = b_{51}, \therefore b_{101} > a_{101}$
- (c) Given : For an A.P, $S_n = cn^2$
 Then $T_n = S_n - S_{n-1} = cn^2 - c(n-1)^2$
 $= (2n-1)c$
 \therefore Sum of squares of n terms of this A.P.
 $= \sum T_n^2 = \sum (2n-1)^2 \cdot c^2$
 $= c^2 \left[4 \sum n^2 - 4 \sum n + n \right]$
 $= c^2 \left[\frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n \right]$
 $= c^2 n \left[\frac{2(2n^2 + 3n + 1) - 6(n+1) + 3}{3} \right]$
 $= c^2 n \left[\frac{4n^2 - 1}{3} \right] = \frac{n(4n^2 - 1)c^2}{3}$
- (c) Given $2 + 5 + 8 + \dots$ $2n$ terms $= 57 + 59 + 61 + \dots$ n terms
 $\Rightarrow \frac{2n}{2} [4 + (2n-1)3] = \frac{n}{2} [114 + (n-1)2]$
 $\Rightarrow 6n + 1 = n + 56 \Rightarrow 5n = 55 \Rightarrow n = 11$
- (9) $\frac{S_7}{S_{11}} = \frac{6}{11} \Rightarrow \frac{\frac{7}{2}[2a+6d]}{\frac{11}{2}[2a+10d]} = \frac{6}{11} \Rightarrow a = 9d$
 $a_7 = a + 6d = 15d$
 Given $130 < a_7 < 140$
 $\Rightarrow 130 < 15d < 140 \Rightarrow d = 9$
 [Since d is a natural number because all terms are natural numbers.]
- (5) Let $k, k+1$ be removed from pack.
 $\therefore (1 + 2 + 3 + \dots + n) - (k + k+1) = 1224$
 $\frac{n(n+1)}{2} - 2k = 1225 \Rightarrow k = \frac{n(n+1) - 2450}{4}$
 for $n = 50, k = 25, \therefore k - 20 = 5$
- (9) $\frac{S_m}{S_n} = \frac{S_{5n}}{S_n} = \frac{\frac{5n}{2}[2 \times 3 + (5n-1)d]}{\frac{n}{2}[6 + (n-1)d]}$ [$\because m = 5n$]

$$= \frac{5[(6-d)+5nd]}{(6-d)+nd}, \text{ which will be independent of } n$$

if $d = 6$ or $d = 0$

For a proper A.P., we take $d = 6$

$$\therefore a_2 = a_1 + d = 3 + 6 = 9$$

(3) We know that $S_\infty = \frac{a}{1-r}$

$$\therefore S_k = \begin{cases} \frac{k-1}{k}, & k \neq 1 \\ 0, & k = 1 \\ \frac{1}{(k-1)!}, & k \geq 2 \end{cases}$$

Now $\sum_{k=1}^{100} |(k^2 - 3k + 1)S_k| = \sum_{k=2}^{100} |(k^2 - 3k + 1)| \frac{1}{(k-1)!}$
 $= |-1| + \sum_{k=3}^{100} \frac{(k^2 - 1) + 1 - 3(k-1) - 2}{(k-1)!}$
 since $k^2 - 3k + 1 > 0 \quad \forall k \geq 3$
 $= 1 + \sum_{k=3}^{100} \left(\frac{1}{(k-3)!} - \frac{1}{(k-1)!} \right)$
 $= 1 + \left(1 - \frac{1}{2!} \right) + \left(\frac{1}{1!} - \frac{1}{3!} \right) + \left(\frac{1}{2!} - \frac{1}{4!} \right) + \dots$
 $\dots + \left(\frac{1}{96!} - \frac{1}{98!} \right) + \left(\frac{1}{97!} - \frac{1}{99!} \right)$
 $= 3 - \frac{1}{98!} - \frac{1}{99!} = 3 - \frac{9900}{100!} - \frac{100}{100!} = 3 - \frac{10000}{100!} = 3 - \frac{(100)^2}{100!}$
 $\Rightarrow \frac{100^2}{100!} + \sum_{k=1}^{100} |(k^2 - 3k + 1)S_k| = 3.$

- (0) Given : $a_k = 2a_{k-1} - a_{k-2}$
 $\Rightarrow \frac{a_{k-2} + a_k}{2} = a_{k-1}, 3 \leq k \leq 11$
 $\Rightarrow a_1, a_2, a_3, \dots, a_{11}$ are in AP.
 If a is the first term and D the common difference then
 $a_1^2 + a_2^2 + \dots + a_{11}^2 = 990$
 $\Rightarrow 11a^2 + d^2(1^2 + 2^2 + \dots + 10^2) + 2ad(1 + 2 + \dots + 10)$
 $= 990$
 $\Rightarrow 11a^2 + \frac{10 \times 11 \times 21}{6} d^2 + 2ad \times \frac{10 \times 11}{2} = 990$

$$\Rightarrow a^2 + 35d^2 + 10d = 90$$

$$\text{Since } a = a_1 = 15$$

$$\therefore 35d^2 + 150d + 135 = 0 \Rightarrow 7d^2 + 30d + 27 = 0$$

$$\Rightarrow (d+3)(7d+9) = 0 \Rightarrow d = -3 \text{ or } -9/7$$

$$\text{then } a_2 = 15 - 3 = 12 \text{ or } 15 - \frac{9}{7} = \frac{96}{7} > \frac{27}{2}$$

$$\therefore d \neq -9/7$$

$$\text{Hence } \frac{a_1 + a_2 + \dots + a_{11}}{11} = \frac{\frac{11}{2}[2 \times 15 + 10(-3)]}{11} = 0$$

9. (18900) For A.P. l_1, l_2, \dots, l_{100}

Let $T_1 = a$ and common difference $= d_1$

and similarly now for A.P. w_1, w_2, \dots, w_{100}

$T_1 = b$ and common difference $= d_2$

$$A_{51} - A_{50} = l_{51}w_{51} - l_{50}w_{50}$$

$$= (a + 50d_1)(b + 50d_2) - (a + 49d_1)(b + 49d_2)$$

$$= 50bd_1 + 50ad_2 + 2500d_1d_2 - 49bd_1 - 49ad_2 - 2401d_1d_2$$

$$= bd_1 + ad_2 + 99d_1d_2 = 1000$$

$$\Rightarrow bd_1 + ad_2 = 1000 - 990 = 10 \quad \dots(i) \text{ (As } d_1d_2 = 10)$$

$$\therefore A_{100} - A_{90} = l_{100}w_{100} - l_{90}w_{90}$$

$$= (a + 99d_1)(b + 99d_2) - (a + 89d_1)(b + 89d_2)$$

$$= 99bd_1 + 99ad_2 + 99^2d_1d_2 - 89bd_1 - 89ad_2 - 89^2d_1d_2$$

$$= 10(bd_1 + ad_2) + 1880d_1d_2$$

$$\Rightarrow 10(10) + 1880(10) = 18900$$

10. (157)

$$AP(1, 3): 1, 4, 7, 10, 13, \dots$$

$$AP(2, 5): 2, 7, 12, 17, 22, \dots$$

$$AP(3, 7): 3, 10, 17, 24, 31, \dots$$

$$\text{For } AP(1, 3) \cap AP(2, 5) \cap AP(3, 7)$$

first term will be the minimum common value of a term.

\therefore we need to find that minimum number which.

when divided by 7 leaves remainder 3 $\rightarrow (7m + 3)$

and when divided by 5 leaves remainder 2 $\rightarrow (5p + 2)$

and when divided by 3 leaves remainder 1 $\rightarrow (3q + 1)$

By hit and trial 52 is such number $(7 \times 7 + 3)$

\therefore first term 'a' of intersection $AP = 52$

Also common difference 'd' of intersection AP

$$= \text{LCM}(7, 5, 3) = 105$$

$$\therefore a + d = 52 + 105 = 157$$

11. (3748) The given sequences upto 2018 terms are

$$1, 6, 11, 16, \dots, 10086$$

$$\text{and } 9, 16, 23, \dots, 14128$$

The common terms are

$$16, 15, 86, \dots \text{ upto } n \text{ terms, where } T_n \leq 10086$$

$$\Rightarrow 16 + (n-1)35 \leq 10086$$

$$\Rightarrow 35n - 19 \leq 10086$$

$$\Rightarrow n \leq \frac{10105}{35} = 288.7$$

$$\therefore n = 288$$

$$\therefore n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$$

$$= 2018 + 2018 - 288 = 3748$$

12. (6) Let the sides be $a - d, a, a + d$ where d is positive. Using

Pythagoras theorem,

$$(a + d)^2 = (a - d)^2 + a^2 \Rightarrow a = 4d$$

$$\therefore \text{Sides are } 3d, 4d, 5d$$

$$\text{Area} = 24 \Rightarrow \frac{1}{2} \times 3d \times 4d = 24 \Rightarrow d^2 = 4 \Rightarrow d = 2$$

$$\therefore \text{Smallest side} = 3d = 6.$$

13. Since p and q are the roots of $x^2 - 2x + A = 0$

$$\therefore p + q = 2, pq = A$$

Also r and s are the roots of equation $x^2 - 18x + B = 0$

$$\therefore r + s = 18, rs = B$$

Since, p, q, r, s are in A.P.

Therefore, let $p = a - 3d, q = a - d, r = a + d$ and $s = a + 3d$.

As $p < q < r < s$, we have $d > 0$

$$\text{Now, } 2 = p + q = a - 3d + a - d = 2a - 4d$$

$$\Rightarrow a - 2d = 1 \quad \dots(i)$$

$$\text{Also } 18 = r + s = a + d + a + 3d$$

$$\Rightarrow 18 = 2a + 4d \Rightarrow 9 = a + 2d \quad \dots(ii)$$

On subtracting (i) from (ii), we get

$$\Rightarrow 8 = 4d \Rightarrow 2 = d$$

On putting in (ii) we obtain $a = 5$

$$\therefore p = a - 3d = 5 - 6 = -1, \quad q = a - d = 5 - 2 = 3$$

$$r = a + d = 5 + 2 = 7, \quad s = a + 3d = 5 + 6 = 11$$

$$\Rightarrow A = pq = -3 \text{ and } B = rs = 77.$$

14. The sum of integers from 1 to 100 that are divisible by 2 or 5
 $= (\text{sum of integers from 1 to 100 divisible by 2}) + (\text{sum of integers from 1 to 100 divisible by 5}) - (\text{sum of integers from 1 to 100 divisible by 10})$

$$= (2 + 4 + 6 + \dots + 100) + (5 + 10 + 15 + \dots + 100) - (10 + 20 + \dots + 100)$$

$$= \frac{50}{2} [2 \times 2 + 49 \times 2] + \frac{20}{2} [2 \times 5 + 19 \times 5]$$

$$= \frac{10}{2} [2 \times 10 + 9 \times 10] = 2550 + 1050 - 550 = 3050$$

15. (b, c) Given, $a_1 = 7, d = 8$

$$\text{Hence, } a_n = 7 + (n-1)8 \text{ and } T_1 = 3$$

$$\text{Also } T_{n+1} = T_n + a_n$$

$$T_n = T_{n-1} + a_{n-1}$$

$$\vdots$$

$$T_2 = T_1 + a_1$$

$$\therefore T_{n+1} = (T_{n-1} + a_{n-1}) + a_n$$

$$= T_{n-2} + a_{n-2} + a_{n-1} + a_n$$

$$\vdots$$

$$\Rightarrow T_{n+1} = T_1 + a_1 + a_2 + \dots + a_n$$

$$\Rightarrow T_{n+1} = T_1 + \frac{n}{2} [2(7) + (n-1)8]$$

$$\Rightarrow T_{n+1} = 3 + n(4n+3) \quad \dots(i)$$

$$\text{Hence, for } n = 19; T_{20} = 3 + (19)(79) = 1504$$

$$\text{For } n = 29; T_{30} = 3 + (29)(119) = 3454 \rightarrow (C)$$

$$\sum_{k=1}^{20} T_k = 3 + \sum_{k=2}^{20} T_k = 3 + \sum_{k=1}^{19} (3 + 4n^2 + 3n)$$

$$= 3 + 3(19) + \frac{3(19)(20)}{2} + \frac{4(19)(20)(39)}{6}$$

$$= 3 + 10507 = 10510 \rightarrow (b)$$

$$\text{And similarly } \sum_{k=1}^{30} T_k = 3 + \sum_{k=1}^{29} (4n^2 + 3n + 3) = 35615$$

16. (a, d)

$$S_n = -1^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 + \dots$$

$$= (3^2 + 7^2 + 11^2 + \dots) + (4^2 + 8^2 + 12^2 + \dots) - (1^2 + 5^2 + 9^2 + \dots) - (2^2 + 6^2 + 10^2 + \dots)$$

$$= \sum_{r=1}^n (4r-1)^2 + \sum_{r=1}^n (4r)^2 - \sum_{r=1}^n (4r-3)^2 - \sum_{r=1}^n (4r-2)^2$$

$$= \left[\sum_{r=1}^n (4r-1)^2 - (4r-3)^2 \right] + 4 \left[\sum_{r=1}^n (2r)^2 - (2r-1)^2 \right]$$

$$= 8 \sum_{r=1}^n (2r-1) + 4 \sum_{r=1}^n (4r-1)$$

$$= 8 \left[2 \frac{n(n+1)}{2} - n \right] + 4 \left[4 \frac{n(n+1)}{2} - n \right]$$

$$= 8n^2 + 8n^2 + 4n = 16n^2 + 4n$$

$$\text{For } n = 8, 16n^2 + 4n = 1056$$

$$\text{and for } n = 9, 16n^2 + 4n = 1332$$

$$17. (c) T_m = a + (m-1)d = 1/n$$

$$\text{and } t_n = a + (n-1)d = 1/m$$

$$\Rightarrow (m-n)d = 1/n - 1/m = (m-n)/mn \Rightarrow d = 1/mn$$

$$\therefore a = \frac{1}{mn}$$

$$\text{Now, } t_{mn} = a + (mn-1)d = \frac{1}{mn} + (mn-1)\frac{1}{mn}$$

$$= \frac{1}{mn} + 1 - \frac{1}{mn} = 1$$

$$18. (b) V_1 + V_2 + \dots + V_n = \sum_{r=1}^n V_r = \sum_{r=1}^n \left(r^3 - \frac{r^2}{2} + \frac{r}{2} \right)$$

$$= \sum_{r=1}^n r^3 - \frac{\sum_{r=1}^n r^2}{2} + \frac{\sum_{r=1}^n r}{2}$$

$$= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}$$

$$= \frac{n(n+1)}{4} \left[n(n+1) - \frac{2n+1}{3} + 1 \right]$$

$$= \frac{n(n+1)(3n^2 + n + 2)}{12}$$

$$19. (d) T_r = V_{r+1} - V_r - 2$$

$$= \left[(r+1)^3 - \frac{(r+1)^2}{2} + \frac{r+1}{2} \right] - \left[r^3 - \frac{r^2}{2} + \frac{r}{2} \right] - 2$$

$$= 3r^2 + 2r + 1 = (r+1)(3r-1)$$

For each r , T_r has two different factors other than 1 and itself.

$\therefore T_r$ is always a composite number.

$$20. (b) Q_{r+1} - Q_r = T_{r+2} - T_{r+1} - (T_{r+1} - T_r)$$

$$= T_{r+2} - 2T_{r+1} + T_r$$

$$= (r+3)(3r+5) - 2(r+2)(3r+2) + (r+1)(3r-1)$$

$$\therefore Q_{r+1} - Q_r = 6(r+1) + 5 - 6r - 5 = 6 \text{ (constant)}$$

$$\therefore Q_1, Q_2, Q_3, \dots \text{ are in AP with common difference 6.}$$

$$21. \text{ Let } a-3d, a-d, a+d \text{ and } a+3d \text{ be any four consecutive terms}$$

of an A.P. with common difference $2d$.

Since, terms of A.P. are integers, $2d$ is also an integer.

$$\text{Now, } (2d)^4 + (a-3d)(a-d)(a+d)(a+3d)$$

$$= 16d^4 + (a^2 - 9d^2)(a^2 - d^2) = (a^2 - 5d^2)^2$$

$$\therefore a^2 - 5d^2 = a^2 - 9d^2 + 4d^2$$

$$= (a-3d)(a+3d) + (2d)^2 = \text{some integer}$$

Hence, the resulting sum is the square of an integer.

$$22. \text{ Since } x_1, x_2, x_3 \text{ are in A.P., hence we can suppose}$$

$$x_1 = a-d, x_2 = a \text{ and } x_3 = a+d \text{ and } x_1, x_2, x_3 \text{ are the roots of}$$

$$x^3 - x^2 + \beta x + \gamma = 0 \quad \dots(i)$$

$$\text{Since } x_1, x_2, x_3 \text{ i.e., } (a-d), a, (a+d) \text{ are the roots of}$$

$$x^3 - x^2 + \beta x + \gamma = 0 \quad \dots(ii)$$

$$\therefore a-d+a+a+d=1 \quad \dots(iii)$$

$$= (a-d)a+a(a+d)+(a-d)(a+d)=\beta$$

$$\text{and } (a-d)a(a+d)=-\gamma$$

$$\text{From (i), } 3a=1 \Rightarrow a=1/3$$

$$\text{From (ii), } 3a^2-d^2=\beta$$

$$\Rightarrow 3(1/3)^2-d^2=\beta \Rightarrow 1/3-\beta=d^2$$

$$\therefore d^2 \geq 0 \quad \forall d \in \mathbb{R}$$

$$\therefore \frac{1}{3}-\beta \geq 0$$

$$\Rightarrow \beta \leq \frac{1}{3} \Rightarrow \beta \in (-\infty, 1/3]$$

$$\text{From (iii), } a(a^2-d^2)=-\gamma$$

$$\Rightarrow \frac{1}{3} \left(\frac{1}{9} - d^2 \right) = -\gamma \Rightarrow \frac{1}{27} - \frac{1}{3} d^2 = -\gamma$$

$$\Rightarrow \gamma + \frac{1}{27} = \frac{1}{3} d^2 \Rightarrow \gamma + \frac{1}{27} \geq 0$$

$$\Rightarrow \gamma \geq -\frac{1}{27} \Rightarrow \gamma \in \left[-\frac{1}{27}, \infty \right)$$

$$\therefore \beta \in (-\infty, 1/3) \text{ and } \gamma \in [-1/27, \infty]$$

$$23. \text{ Given : } \log_3 2, \log_3 (2^x - 5), \log_3 (2^x - 7/2) \text{ are in A.P.}$$

$$\therefore 2 \log_3 (2^x - 5) = \log_3 2 + \log_3 (2^x - 7/2)$$

$$\Rightarrow (2^x - 5)^2 = 2 \left(2^x - \frac{7}{2} \right)$$

$$\Rightarrow (2^x)^2 - 10 \cdot 2^x + 25 - 2 \cdot 2^x + 7 = 0$$

$$\Rightarrow (2^x)^2 - 12 \cdot 2^x + 32 = 0$$

Let $2^x = y$, then we get,

$$y^2 - 12y + 32 = 0 \Rightarrow (y-4)(y-8) = 0$$

$$\Rightarrow y = 4 \text{ or } 8 \Rightarrow 2^x = 2^2 \text{ or } 2^3 \Rightarrow x = 2 \text{ or } 3$$

But for $\log_3 (2^x - 5)$ and $\log_3 (2^x - 7/2)$ to be defined

$$2^x - 5 > 0 \quad \text{and } 2^x - 7/2 > 0$$

$$\Rightarrow 2^x > 5 \quad \text{and } 2^x > 7/2$$

$$\therefore 2^x > 5$$

$$\Rightarrow x \neq 2 \text{ and therefore } x = 3.$$

$$24. \text{ Let there be } n \text{ sides in the polygon.}$$

\therefore Sum of all n interior angles of polygon $= (n-2) \times 180^\circ$
Since the angles are in A.P. with the smallest angle 120° and common difference 5° .

\therefore Sum of all interior angles of polygon

$$= \frac{n}{2} [2 \times 120 + (n-1) \times 5]$$

$$\therefore \frac{n}{2} [2 \times 120 + (n-1) \times 5] = (n-2) \times 180$$

$$\Rightarrow \frac{n}{2} [5n + 235] = (n-2) \times 180$$

$$\Rightarrow 5n^2 + 235n = 360n - 720$$

$$\Rightarrow n^2 - 25n + 144 = 0 \therefore n = 16, 9$$

Also if $n = 16$, then 16th angle $= 120 + 15 \times 5 = 195^\circ > 180^\circ$, which is not possible. $\therefore n = 9$.



Topic-2: Geometric Progression

$$1. (c) \text{ In the quadratic equation } ax^2 + bx + c = 0$$

$$\Delta = b^2 - 4ac \text{ and } \alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$$

$$\text{and } \alpha^3 + \beta^3 = -\frac{b^3}{a^3} - \frac{3c}{a} \left(-\frac{b}{a} \right) = -\left(\frac{b^3 - 3abc}{a^3} \right)$$

Since $\alpha + \beta$, $\alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$ are in G.P.

$$\therefore \frac{b}{a}, \frac{b^2 - 2ac}{a^2}, -\frac{(b^3 - 3abc)}{a^3} \text{ are in G.P.}$$

$$\Rightarrow \left(\frac{b^2 - 2ac}{a^2} \right)^2 = \frac{b}{a} \left(\frac{b^3 - 3abc}{a^3} \right)$$

$$\Rightarrow b^4 + 4a^2c^2 - 4ab^2c = b^4 - 3ab^2c$$

$$\Rightarrow 4a^2c^2 - ab^2c = 0 \Rightarrow ac\Delta = 0$$

$$\Rightarrow c\Delta = 0 \quad (\because \text{In quadratic Eq. } a \neq 0)$$

2. (c) $\frac{x}{1-r} = 5 \Rightarrow r = 1 - \frac{x}{5}$

Since G.P. contains infinite terms
 $\therefore -1 < r < 1$

$$\Rightarrow -1 < 1 - \frac{x}{5} < 1 \Rightarrow -2 < -\frac{x}{5} < 0$$

$$\Rightarrow -10 < x < 0 \Rightarrow 0 < \frac{x}{5} < 2$$

$$\therefore 0 < x < 10.$$

3. (d) Since a, b, c are in A.P.

$$\therefore 2b = a + c$$

But given $a + b + c = 3/2 \Rightarrow b = 1/2$ and then $a + c = 1$

Also a^2, b^2, c^2 are in G.P. $\Rightarrow b^4 = a^2c^2$

$$\Rightarrow b^2 = \pm ac \Rightarrow ac = \frac{1}{4} \text{ or } -\frac{1}{4}$$

and $a + c = 1$

Considering $a + c = 1$ and $ac = 1/4$

$$(a-c)^2 = 1 - 1 = 0 \Rightarrow a = c \text{ but } a \neq c$$

as given that $a < b < c$

$$\therefore a + c = 1 \text{ and } ac = -1/4$$

$$\Rightarrow (a-c)^2 = 1 + 1 = 2 \Rightarrow a - c = \pm\sqrt{2}$$

but $a < c \Rightarrow a - c = -\sqrt{2}$ (ii)

Solving (i) and (ii), we get $a = \frac{1}{2} - \frac{1}{\sqrt{2}}$

4. (a) α, β are the roots of $x^2 - x + p = 0$

$$\therefore \alpha + \beta = 1$$

$$\alpha\beta = p$$

γ, δ are the roots of $x^2 - 4x + q = 0$

$$\therefore \gamma + \delta = 4$$

$$\gamma\delta = q$$

$\alpha, \beta, \gamma, \delta$ are in G.P.

$$\therefore \text{Let } \alpha = a; \beta = ar; \gamma = ar^2; \delta = ar^3.$$

Substituting these values in equations (i), (ii), (iii) and (iv), we get

$$a + ar = 1 \quad \text{....(v)}$$

$$a^2r = p \quad \text{....(vi)}$$

$$ar^2 + ar^3 = 4 \quad \text{....(vii)}$$

$$a^2r^5 = q \quad \text{....(viii)}$$

On dividing (vii) by (v), we get

$$\frac{ar^2(1+r)}{a(1+r)} = \frac{4}{1} \Rightarrow r^2 = 4 \Rightarrow r = 2, -2$$

From (v), $a = \frac{1}{1+r} = \frac{1}{1+2}$ or $\frac{1}{1-2} = \frac{1}{3}$ or -1

Since p is an integer (given), r is also an integer (2 or -2).

From (vi), $a \neq \frac{1}{3}$. Hence $a = -1$ and $r = -2$.

$$\therefore p = \frac{(-1)^2 \times (-2)}{(-1)^2 \times (-2)^5} = -\frac{2}{-32}$$

5. (d) Sum = 4 and second term = $3/4$,

$$\Rightarrow \frac{a}{1-r} = 4 \text{ and } ar = 3/4 \Rightarrow r = \frac{3}{4a}$$

$$\therefore \frac{a}{1-\frac{3}{4a}} = 4 \Rightarrow \frac{4a^2}{4a-3} = 4$$

$$\Rightarrow a^2 - 4a + 3 = 0 \quad \therefore a = 1 \text{ or } 3$$

When $a = 1$, $r = 3/4$ and when $a = 3$, $r = 1/4$

6. (c) Sum of first n terms $= \frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \dots + n\text{th term}$

$$= \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{8}\right) + \left(1 - \frac{1}{16}\right) + \dots + n\text{th term}$$

$$= (1 + 1 + 1 + \dots + n\text{th term}) - \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots + \frac{1}{2^n}\right)$$

$$= n - \left[\frac{\frac{1}{2} \left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} \right] = n - 1 + 2^{-n}$$

7. (a) If a, b, c are in G.P., then

$$b^2 = ac$$

Now $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root

Let it be α , then $a\alpha^2 + 2b\alpha + c = 0$

$$d\alpha^2 + 2e\alpha + f = 0$$

$$\Rightarrow \frac{\alpha^2}{2(bf - ec)} = \frac{\alpha}{cd - af} = \frac{1}{2(ae - bd)}$$

$$\Rightarrow \alpha^2 = \frac{bf - ce}{ae - bd}; \alpha = \frac{cd - af}{2(ae - bd)}$$

Putting the value of α , in $\alpha^2 = \frac{bf - ce}{ae - bd}$, we get

$$\frac{(cd - af)^2}{4(ae - bd)^2} = \frac{bf - ce}{ae - bd}$$

$$\Rightarrow (cd - af)^2 = 4(ae - bd)(bf - ce)$$

On dividing both sides by a^2c^2 , we get

$$\left(\frac{d}{a} - \frac{f}{c}\right)^2 = 4\left(\frac{e}{c} - \frac{bd}{ac}\right)\left(\frac{bf}{ac} - \frac{e}{a}\right)$$

$$\left(\frac{d}{a} - \frac{f}{c}\right)^2 = 4\left(\frac{e}{c} - \frac{d}{b}\right)\left(\frac{f}{b} - \frac{e}{a}\right) \quad [\text{using eq. (i)}]$$

$$\Rightarrow \frac{d^2}{a^2} + \frac{f^2}{c^2} - \frac{2df}{ac} = \frac{4ef}{cb} - \frac{4e^2}{ac} - \frac{4df}{b^2} + \frac{4de}{ab}$$

$$\Rightarrow \frac{d^2}{a^2} + \frac{f^2}{c^2} + \frac{4e^2}{b^2} + 2\frac{d}{a} \cdot \frac{f}{c} - 4\frac{e}{b} \cdot \frac{f}{c} - 4\frac{d}{a} \cdot \frac{e}{b} = 0$$

[using eq. (i)]

$$\Rightarrow \left(\frac{d}{a} + \frac{f}{c} - 2\frac{e}{b}\right)^2 = 0 \Rightarrow \frac{d}{a} + \frac{f}{c} = \frac{2e}{b}$$

$$\therefore \frac{d}{a}, \frac{e}{b}, \frac{f}{c} \text{ are in A.P.}$$

8. (c) $2.357 = 2 + [.357 + 0.000357 + \dots]$

$$= 2 + \left[\frac{357}{10^3} + \frac{357}{10^6} + \dots \right] = 2 + \frac{357}{1 - \frac{1}{10^3}}$$

$$= 2 + \frac{357}{999} = \frac{2355}{999}$$

[Sum of a G.P. with infinite term $= \frac{a}{1-r}$]

9. (b) Given : $ar^2 = 4$

Now product of first five terms of a G.P.
 $= a \cdot ar \cdot ar^2 \cdot ar^3 \cdot ar^4 = a^5 r^{10} = (ar^2)^5 = 4^5$

10. (1) It is given that

$$2(a_1 + a_2 + \dots + a_n) = b_1 + b_2 + \dots + b_n$$

$$\Rightarrow 2 \times \frac{n}{2} (2c + (n-2)x_2) = c \left(\frac{2^n - 1}{2 - 1} \right) [\because a_1 = c, b_1 = c]$$

$$\Rightarrow c(2^n - 1 - 2n) = 2n^2 - 2n$$

$$\Rightarrow c = \frac{2n^2 - 2n}{2^n - 1 - 2n} \in \mathbb{N}$$

$$\text{So, } 2n^2 - 2n \geq 2^n - 1 - 2n$$

$$\Rightarrow 2n^2 + 1 \geq 2^n \Rightarrow n < 7$$

$$\therefore c \in \mathbb{N} \Rightarrow c > 0 \Rightarrow n > 2$$

$$\Rightarrow n \text{ can be } 3, 4, 5 \text{ or } 6.$$

Checking c against these values of n

$$\text{When } n = 3, c = 14$$

$$\text{When } n = 4, c = \frac{24}{7} \text{ which is not possible}$$

$$\text{When } n = 5, c = \frac{40}{21} \text{ which is not possible}$$

$$\text{When } n = 6, c = \frac{60}{51} \text{ which is not possible}$$

$$\therefore \text{ we get } c = 12 \text{ when } n = 3$$

Hence, there exists only one value of c which holds the inequality.

11. (b) Putting $\theta = 0$, we get $b_0 = 0$

$$\therefore \sin n\theta = \sum_{r=1}^n b_r \sin^r \theta \Rightarrow \frac{\sin n\theta}{\sin \theta} = \sum_{r=1}^n b_r (\sin \theta)^{r-1}$$

$$= b_1 + b_2 \sin \theta + b_3 \sin^2 \theta + \dots + b_n \sin^{n-1} \theta$$

$$\therefore \lim_{\theta \rightarrow 0} \frac{\sin n\theta}{\sin \theta} = b_1 \Rightarrow b_1 = n.$$

12. (b, c) For $0 < \phi < \frac{\pi}{2}$

$$x = \sum_{n=0}^{\infty} \cos^{2n} \phi = 1 + \cos^2 \phi + \cos^4 \phi + \dots \infty$$

$$\Rightarrow x = \frac{1}{1 - \cos^2 \phi} = \frac{1}{\sin^2 \phi} \quad \dots(i)$$

[using sum of infinite G.P., $\cos^2 \alpha$ being < 1]

$$y = \sum_{n=0}^{\infty} \sin^{2n} \phi = 1 + \sin^2 \phi + \sin^4 \phi + \dots \infty$$

$$\Rightarrow y = \frac{1}{1 - \sin^2 \phi} = \frac{1}{\cos^2 \phi} \quad \dots(ii)$$

$$\text{and } z = \sum_{n=0}^{\infty} \cos^{2n} \phi \sin^{2n} \phi$$

$$\Rightarrow z = 1 + \cos^2 \phi \sin^2 \phi + \cos^4 \phi \sin^4 \phi + \dots \infty$$

$$\Rightarrow z = \frac{1}{1 - \cos^2 \phi \sin^2 \phi} \quad \dots(iii)$$

On substituting the values of $\cos^2 \phi$ and $\sin^2 \phi$ in (iii), from (i) and (ii), we get

$$z = \frac{1}{1 - \frac{1}{x} \cdot \frac{1}{y}} \Rightarrow z = \frac{xy}{xy - 1} \Rightarrow xyz - z = xy \Rightarrow xyz = xy + z.$$

$$\text{Now, } x + y + z = \frac{1}{\cos^2 \phi} + \frac{1}{\sin^2 \phi} + \frac{1}{1 - \cos^2 \phi \sin^2 \phi}$$

$$\begin{aligned} & [\sin^2 \phi (1 - \cos^2 \phi \sin^2 \phi) + \cos^2 \phi (1 - \cos^2 \phi \sin^2 \phi) \\ & + \cos^2 \phi \sin^2 \phi] \\ & = \frac{\cos^2 \phi \sin^2 \phi (1 - \cos^2 \phi \sin^2 \phi)}{\cos^2 \phi \sin^2 \phi (1 - \cos^2 \phi \sin^2 \phi)} \\ & = \frac{(\sin^2 \phi + \cos^2 \phi) (1 - \cos^2 \phi \sin^2 \phi) + \cos^2 \phi \sin^2 \phi}{\cos^2 \phi \sin^2 \phi (1 - \cos^2 \phi \sin^2 \phi)} \\ & = \frac{1}{\cos^2 \phi \sin^2 \phi (1 - \cos^2 \phi \sin^2 \phi)} = xyz \end{aligned}$$

\therefore (b) and (c) both are correct.

$$13. a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \dots + (-1)^{n-1} \left(\frac{3}{4}\right)^n$$

$$= \frac{\frac{3}{4} \left(1 - \left(-\frac{3}{4}\right)^n \right)}{1 + \frac{3}{4}} = \frac{3}{7} \left(1 - \left(-\frac{3}{4}\right)^n \right)$$

$$b_n = 1 - a_n \text{ and } b_n > a_n \quad \forall n \geq n_0$$

$$\therefore 1 - a_n > a_n \Rightarrow 2a_n < 1$$

$$\Rightarrow \frac{6}{7} \left[1 - \left(-\frac{3}{4}\right)^n \right] < 1 \Rightarrow -\left(-\frac{3}{4}\right)^n < \frac{1}{6}$$

$$\Rightarrow (-3)^{n+1} < 2^{2n-1}$$

For n to be even, inequality always holds. For n to be odd, it holds for $n \geq 7$.

\therefore The least natural number, for which it holds is 6 [it holds for every even natural number]

14. On solving the system of equations, $u + 2v + 3w = 6$, $4u + 5v + 6w = 12$ and $6u + 9v = 4$ we get $u = -1/3$, $v = 2/3$, $w = 5/3$

$$\therefore u + v + w = 2, \frac{1}{u} + \frac{1}{v} + \frac{1}{w} = -\frac{9}{10}$$

Let r be the common ratio of the G.P., a, b, c, d . Then $b = ar$, $c = ar^2$, $d = ar^3$.

Then the first equation

$$\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w} \right) x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2]x + (u+v+w) = 0 \text{ becomes}$$

$$-\frac{9}{10}x^2 + [(ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar^2)^2]x + 2 = 0$$

$$\Rightarrow 9x^2 - 10a^2(1-r)^2[r^2 + (r+1)^2 + r^2(r+1)^2]x - 20 = 0$$

$$\Rightarrow 9x^2 - 10a^2(1-r)^2(r^4 + 2r^3 + 3r^2 + 2r + 1)x - 20 = 0$$

$$\Rightarrow 9x^2 - 10a^2(1-r)^2(1+r+r^2)^2x - 20 = 0, \quad \dots(i)$$

$$\text{and } 20x^2 + 10(a-d)^2x - 9 = 0 \text{ becomes}$$

$$20x^2 + 10(a - ar^3)^2x - 9 = 0$$

$$\Rightarrow 20x^2 + 10a^2(1-r^3)^2x - 9 = 0 \quad \dots(ii)$$

Since Eq. (ii) can be obtained by the substitution $x = 1/x$ in Eq. (i), therefore Eqs. (i) and (ii) have reciprocal roots.

$$15. S_1 = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots \infty$$

$$S_2 = 2 + 2 \cdot \frac{1}{3} + 2 \left(\frac{1}{3}\right)^2 + \dots \infty$$

$$S_3 = 3 + 3 \cdot \frac{1}{4} + 3 \left(\frac{1}{4}\right)^2 + \dots \infty$$

$$S_n = n + n \cdot \frac{1}{n+1} + n \left(\frac{1}{n+1}\right)^2 + \dots \infty$$

$$\Rightarrow S_1 = \frac{1}{1-\frac{1}{2}} = 2 \quad \left[\because S_\infty = \frac{a}{1-r} \right]$$

$$S_2 = \frac{2}{1-\frac{1}{3}} = 3, \quad S_3 = \frac{3}{1-\frac{1}{4}} = 4,$$

$$S_n = \frac{n}{1-\frac{1}{n+1}} = (n+1) \quad \therefore S_1^2 + S_2^2 + S_3^2 + \dots + S_{2n-1}^2$$

$$= 2^2 + 3^2 + 4^2 + \dots + (n+1)^2 + \dots + (2n)^2$$

$$= [1^2 + 2^2 + 3^2 + 4^2 + \dots + (2n)^2] - 1$$

$$= \frac{2n(2n+1)(4n+1)}{6} - 1$$

$$\left[\because 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \right]$$

$$= \frac{n(2n+1)(4n+1) - 3}{3}$$

16. Given : $2 < a, b, c < 18$ $a + b + c = 25$ (i)
 $\therefore 2, a, b$ are in AP, $\therefore 2a = b + 2$
 $\therefore 2a - b = 2$ (ii)
 $\therefore b, c, 18$ are in GP, $\therefore c^2 = 18b$ (iii)

From (ii), $a = \frac{b+2}{2}$

From (i), $\frac{b+2}{2} + b + c = 25 \Rightarrow 3b = 48 - 2c$

From (iii), $c^2 = 6(48 - 2c) \Rightarrow c^2 + 12c - 288 = 0$
 $\Rightarrow c = 12, -24$ (rejected) $\Rightarrow a = 5, b = 8, c = 12$

17. Consider a G.P. with first term A and common ratio R
 $T_p = 27 = AR^{p-1}$ (i)
 $T_q = 8 = AR^{q-1}$ (ii)
 $T^r = 12 = AR^{r-1}$ (iii)

From (i) and (ii),

$$R^{p-q} = \frac{27}{8} \Rightarrow R^{p-q} = (3/2)^3 \quad \text{.....(iv)}$$

From (ii) and (iii),

$$R^{q-r} = \frac{8}{12} \Rightarrow R^{q-r} = (3/2)^{-1} \quad \text{.....(v)}$$

From (iv) and (v),

$$R = 3/2; p - q = 3; q - r = -1$$

$$p - 2q + r = 4; p, q, r \in \mathbb{N} \quad \text{.....(vi)}$$

As there can be infinite natural numbers for p, q and r to satisfy equation (vi)

\therefore There can be infinite GP's.

Topic-3: Harmonic Progression, Relation Between A. M., G. M. and H. M. of two Positive Numbers

1. (d) $\therefore a_1, a_2, a_3, \dots$ are in H.P.

$$\therefore \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \dots \text{ are in A.P.}$$

$$\therefore \frac{1}{a_1} = \frac{1}{5} \text{ and } \frac{1}{a_{20}} = \frac{1}{25}$$

$$\frac{1}{a_1} + 19d = \frac{1}{a_{20}} \Rightarrow \frac{1}{5} + 19d = \frac{1}{25} \Rightarrow d = \frac{-4}{475}$$

$$\text{Now } \frac{1}{a_n} = \frac{1}{5} + (n-1)\left(\frac{-4}{475}\right)$$

$$\text{Clearly } a_n < 0, \text{ if } \frac{1}{a_n} < 0 \Rightarrow \frac{1}{5} - \frac{4n}{475} + \frac{4}{475} < 0$$

$$\Rightarrow -4n < -99 \text{ or } n > \frac{99}{4} = 24\frac{3}{4} \therefore n \geq 25$$

\therefore Least value of n is 25.

2. (d) a, b, c, d are in A.P. $\therefore d, c, b, a$ are also in A.P.

$$\Rightarrow \frac{d}{abcd}, \frac{c}{abcd}, \frac{b}{abcd}, \frac{a}{abcd} \text{ are also in A.P.}$$

$$\Rightarrow \frac{1}{abc}, \frac{1}{abd}, \frac{1}{acd}, \frac{1}{bcd} \text{ are in A.P.}$$

$\therefore abc, abd, acd, bcd$ are in H.P.

3. (b) $\frac{1}{H} = \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \frac{\alpha + \beta}{2\alpha\beta} = \frac{(4 + \sqrt{5})(5 + \sqrt{2})}{(5 + \sqrt{2})(8 + 2\sqrt{5})} \cdot \frac{1}{2} = \frac{1}{4}$

4. (d) $a_1 = h_1 = 2, a_{10} = h_{10} = 3$
 $3 = a_{10} = 2 + 9d \Rightarrow d = 1/9 \therefore a_4 = 2 + 3d = 7/3$

$$\text{Now, } 3 = h_{10} \Rightarrow \frac{1}{3} = \frac{1}{h_{10}} = \frac{1}{2} + 9D \therefore D = -\frac{1}{54}$$

$$\text{and } \frac{1}{h_7} = \frac{1}{2} + 6D = \frac{1}{2} - \frac{1}{9} = \frac{7}{18} \therefore a_4 h_7 = \frac{7}{3} \times \frac{18}{7} = 6.$$

5. (d) Since, $\ell_n(a+c), \ell_n(a-c), \ell_n(a-2b+c)$ are in A.P.
 $\therefore \ell_n(a-2b+c) - \ell_n(a-c) = \ell_n(a-c) - \ell_n(a+c)$

$$\Rightarrow \ell_n \frac{a-2b+c}{a-c} = \ell_n \frac{a-c}{a+c} \Rightarrow \frac{a-2b+c}{a-c} = \frac{a-c}{a+c}$$

$$\Rightarrow a+c, a-c, a-2b+c \text{ are in GP.}$$

$$\Rightarrow (c-a)^2 = (a+c)(a-2b+c)$$

$$\Rightarrow (c-a)^2 = (a+c)^2 - 2b(a+c)$$

$$\Rightarrow 2b(a+c) = (a+c)^2 - (c-a)^2$$

$$\Rightarrow 2b(a+c) = 4ac \Rightarrow b = \frac{2ac}{a+c}$$

$\therefore a, b, c$ are in H.P.

6. (8) By AM - GM inequality
 $AM \geq GM$

$$\therefore \frac{3^{y_1} + 3^{y_2} + 3^{y_3}}{3} \geq \left[3^{(y_1+y_2+y_3)} \right]^{\frac{1}{3}}$$

$$\Rightarrow 3^{y_1} + 3^{y_2} + 3^{y_3} \geq 3^4$$

$$\Rightarrow \log_3 (3^{y_1} + 3^{y_2} + 3^{y_3}) \geq 4 \Rightarrow m = 4$$

$$\therefore \log_3 x_1 + \log_3 x_2 + \log_3 x_3 = \log_3 (x_1 x_2 x_3)$$

Again by AM - GM inequality

$AM \geq GM$

$$\frac{x_1 + x_2 + x_3}{3} \geq \sqrt[3]{x_1 x_2 x_3} \Rightarrow x_1 x_2 x_3 \leq 27$$

$$\Rightarrow \log_3 (x_1 x_2 x_3) \leq \log_3 (3^3)$$

$$\Rightarrow \log_3 x_1 + \log_3 x_2 + \log_3 x_3 \leq 3 \Rightarrow M = 3$$

$$\text{Now, } \log_2 (m^3) + \log_3 (M^2) = 6 + 2 = 8$$

7. Let a and b be two positive numbers.

$$\text{Their, H.M.} = \frac{2ab}{a+b} \text{ and G.M.} = \sqrt{ab}$$

$$\therefore HM : GM = 4 : 5 \therefore \frac{2ab}{(a+b)\sqrt{ab}} = \frac{4}{5}$$

$$\Rightarrow \frac{2\sqrt{ab}}{a+b} = \frac{4}{5} \Rightarrow \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{5+4}{5-4}$$

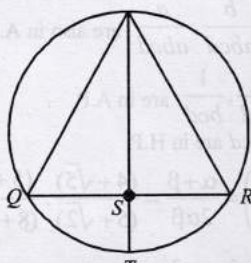
[By Componendo and Dividendo]

$$\Rightarrow \frac{(\sqrt{a} + \sqrt{b})^2}{(\sqrt{a} - \sqrt{b})^2} = \frac{9}{1} \Rightarrow \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}} = 3, -3$$

$$\Rightarrow \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{3+1}{3-1}, \frac{-3+1}{-3-1} \Rightarrow \frac{a}{b} = 4, \frac{1}{4}$$

$$\therefore a : b = 4 : 1 \text{ or } 1 : 4$$

8. (b, d)



We know by geometry $PS \times ST = QS \times SR$
 $\therefore S$ is not the centre of circle,
 $PS \neq ST$

And we know that for two unequal real numbers,
 H.M. < G.M.

$$\Rightarrow \frac{2}{\frac{1}{PS} + \frac{1}{ST}} < \sqrt{PS \times ST} \Rightarrow \frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{PS \times ST}} \quad \dots(i)$$

$$\Rightarrow \frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}} \quad [\text{using eqn (i)}]$$

\therefore (b) is the correct option.

$$\text{Also } \sqrt{QS \times SR} < \frac{QS + SR}{2} \quad (\because \text{GM} < \text{AM})$$

$$\Rightarrow \frac{1}{\sqrt{QS \times SR}} > \frac{2}{QR} \Rightarrow \frac{1}{\sqrt{QS \times SR}} > \frac{4}{QR} \quad \dots(ii)$$

$$\text{From equations (ii) and (iii), } \frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$$

\therefore (d) is also the correct option.

9. (b) If x, y, z are in G.P. ($x, y, z > 1$); then $\log x, \log y, \log z$ will be in A.P.

$$\Rightarrow 1 + \log x, 1 + \log y, 1 + \log z \text{ will also be in A.P.}$$

$$\therefore \frac{1}{1 + \log x}, \frac{1}{1 + \log y}, \frac{1}{1 + \log z} \text{ will be in H.P.}$$

10. (a, b, d) Let x be the first term and y the $(2n-1)$ th terms of AP, GP and HP whose n th terms are a, b, c respectively.For AP, $y = x + (2n-2)d$

$$\Rightarrow d = \frac{y-x}{2(n-1)}$$

$$\therefore a = x + (n-1)d = x + \frac{1}{2}(y-x) = \frac{1}{2}(x+y) \quad \dots(i)$$

$$\text{For G.P., } y = xr^{2n-2} \Rightarrow r = \left(\frac{y}{x}\right)^{\frac{1}{2n-2}}$$

$$\therefore b = xr^{n-1} = x \cdot \left(\frac{y}{x}\right)^{\frac{1}{2}} = \sqrt{xy} \quad \dots(ii)$$

$$\text{For H.P., } \frac{1}{y} = \frac{1}{x} + (2n-2)d_1 \Rightarrow d_1 = \frac{x-y}{2xy(n-1)}$$

$$\frac{1}{c} = \frac{1}{x} + (n-1)d_1 = \frac{1}{x} + \frac{x-y}{2xy}$$

$$\therefore \frac{1}{c} = \frac{x+y}{2xy} \Rightarrow c = \frac{2xy}{x+y} \quad \dots(iii)$$

Thus from (i), (ii) and (iii), a, b, c are A.M., G.M. and H.M. respectively of x and y .

$$11. (c) \text{ Given } A_1 = \frac{a+b}{2}, G_1 = \sqrt{ab}, H_1 = \frac{2ab}{a+b}$$

$$\text{Also } A_n = \frac{A_{n-1} + H_{n-1}}{2}, G_n = \sqrt{A_{n-1}H_{n-1}}$$

$$H_n = \frac{2A_{n-1}H_{n-1}}{A_{n-1} + H_{n-1}}$$

$$\Rightarrow G_n^2 = A_n H_n \Rightarrow A_n H_n = A_{n-1} H_{n-1}$$

Similarly we can prove

$$A_n H_n = A_{n-1} H_{n-1} = A_{n-2} H_{n-2} = \dots = A_1 H_1$$

$$\Rightarrow A_n H_n = ab$$

$$\therefore G_1^2 = G_2^2 = G_3^2 = \dots = ab$$

$$\Rightarrow G_1 = G_2 = G_3 = \dots = \sqrt{ab}$$

$$12. (a) A_n = \frac{A_{n-1} + H_{n-1}}{2}$$

$$\therefore A_n - A_{n-1} = \frac{A_{n-1} + H_{n-1}}{2} - A_{n-1}$$

$$= \frac{H_{n-1} - A_{n-1}}{2} < 0 \quad (\because A_{n-1} > H_{n-1})$$

$$\Rightarrow A_n < A_{n-1} \text{ or } A_{n-1} > A_n$$

$$\therefore A_1 > A_2 > A_3 > \dots$$

$$13. (b) A_n H_n = ab \Rightarrow H_n = \frac{ab}{A_n}$$

$$\therefore \frac{1}{A_{n-1}} < \frac{1}{A_n} \Rightarrow H_{n-1} < H_n \quad \therefore H_1 < H_2 < H_3 < \dots$$

14. (c) Given : a_1, a_2, a_3, a_4 are in G.P.

Then b_1, b_2, b_3, b_4 are the numbers

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4$$

$$\text{or } a, a + ar, a + ar + ar^2, a + ar + ar^2 + ar^3$$

Since, above numbers are neither in A.P. nor in G.P. Therefore, statement 1 is true.

$$\text{Also } \frac{1}{a}, \frac{1}{a+ar}, \frac{1}{a+ar+ar^2}, \frac{1}{a+ar+ar^2+ar^3} \text{ are not in A.P.}$$

$\therefore b_1, b_2, b_3, b_4$ are not in H.P.

\therefore Statement 2 is false.

15. Given : a, b, c are in A.P.

$$\therefore 2b = a + c$$

Now a^2, b^2, c^2 are in H.P.

$$\Rightarrow \frac{1}{b^2} - \frac{1}{a^2} = \frac{1}{c^2} - \frac{1}{b^2}$$

$$\Rightarrow \frac{(a-b)(a+b)}{b^2 a^2} = \frac{(b-c)(b+c)}{b^2 c^2}$$

$$\Rightarrow ac^2 + bc^2 = a^2 b + a^2 c \quad (\because a-b = b-c)$$

$$\Rightarrow ac(c-a) + b(c-a)(c+a) = 0$$

$$\Rightarrow (c-a)(ab+bc+ca) = 0$$

$$\Rightarrow \text{Either } c-a=0 \text{ or } ab+bc+ca=0$$

$$\Rightarrow \text{Either } c=a \text{ or } (a+c)b+ca=0$$

and hence from (i), $2b^2 + ca = 0$

$$\text{Either } a=b=c \text{ or } b^2 = a\left(\frac{-c}{2}\right)$$

$\therefore a, b, -c/2$ are in G.P.

16. Clearly $A_1 + A_2 = a + b$

$$\frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{a} + \frac{1}{b}$$

$$\Rightarrow \frac{H_1 + H_2}{H_1 H_2} = \frac{a+b}{ab} = \frac{A_1 + A_2}{G_1 G_2}$$

$$\Rightarrow \frac{G_1 G_2}{H_1 H_2} = \frac{A_1 + A_2}{H_1 + H_2}$$

$$\text{Also } \frac{1}{H_1} = \frac{1}{a} + \frac{1}{3} \left(\frac{1}{b} - \frac{1}{a} \right) \Rightarrow H_1 = \frac{3ab}{2b+a}$$

$$\frac{1}{H_2} = \frac{1}{a} + \frac{2}{3} \left(\frac{1}{b} - \frac{1}{a} \right) \Rightarrow H_2 = \frac{3ab}{2a+b}$$

$$\Rightarrow \frac{A_1 + A_2}{H_1 + H_2} = \frac{a+b}{3ab \left(\frac{1}{2b+a} + \frac{1}{2a+b} \right)} = \frac{(2b+a)(2a+b)}{9ab}$$

17. Given that a_1, a_2, \dots, a_n are positive real numbers in G.P.

$$\therefore \begin{cases} a_1 = a \\ a_2 = ar \\ a_3 = ar^2 \\ \vdots \\ a_n = ar^{n-1} \end{cases} \quad \left\{ \begin{array}{l} \text{As } a_1, a_2, \dots, a_n \text{ are positive} \\ \therefore r > 0 \end{array} \right.$$

A_n is A.M. of a_1, a_2, \dots, a_n

$$\therefore A_n = \frac{a_1 + a_2 + \dots + a_n}{n} = \frac{a + ar + \dots + ar^{n-1}}{n}$$

$$A_n = \frac{a(1-r^n)}{n(1-r)} \quad \dots (i) \quad (r \neq 1)$$

G_n is G.M. of a_1, a_2, \dots, a_n

$$\therefore G_n = \sqrt[n]{a_1 a_2 \dots a_n} = \sqrt[n]{a \cdot ar \cdot ar^2 \dots ar^{n-1}} = \sqrt[n]{a^n r^{\frac{n(n-1)}{2}}}$$

$$G_n = ar^{\frac{n-1}{2}} \quad \dots (ii) \quad (r \neq 1)$$

H_n is H.M. of a_1, a_2, \dots, a_n

$$\therefore H_n = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} = \frac{n}{\frac{1}{a} + \frac{1}{ar} + \dots + \frac{1}{ar^{n-1}}} = \frac{n}{\frac{1}{a} \left(\frac{1}{r^n} - 1 \right)} = \frac{n}{a} \left(\frac{1-r^n}{1-r} \right) \cdot \frac{r}{1-r}$$

$$H_n = \frac{anr^{n-1}(1-r)}{(1-r)^2} \quad \dots (iii) \quad (r \neq 1)$$

We also observe that

$$A_n H_n = \frac{a(1-r^n)}{n(1-r)} \times \frac{anr^{n-1}(1-r)}{(1-r)^2} = a^n r^{n-1} = G_n^2$$

$$\therefore A_n H_n = G_n^2 \quad \dots (iv)$$

Now, G.M. of G_1, G_2, \dots, G_n is

$$G = \sqrt[n]{G_1 G_2 \dots G_n}$$

$$G = \sqrt[n]{A_1 H_1 \sqrt{A_2 H_2} \dots \sqrt{A_n H_n}}$$

[using equation (iv)]

$$G = (A_1 A_2 \dots A_n H_1 H_2 \dots H_n)^{1/2n} \quad \dots (v)$$

If $r = 1$, then

$$A_n = G_n = H_n = a$$

$$\text{Also } A_n H_n = G_n^2$$

\therefore For $r = 1$ also, equation (v) holds.

$$\therefore G = (A_1 A_2 \dots A_n H_1 H_2 \dots H_n)^{1/2n}$$

18. Let a and b be two numbers and $A_1, A_2, A_3, \dots, A_n$ be n A.M.'s between a and b .

$\therefore a, A_1, A_2, \dots, A_n, b$ are in A.P.

Since, there are $(n+2)$ terms in the series, therefore

$$a + (n+1)d = b \Rightarrow d = \frac{b-a}{n+1}$$

$$\therefore A_1 = a + \frac{b-a}{n+1} = \frac{an+b}{n+1}$$

$$\therefore p = \frac{an+b}{n+1} \quad \dots (i)$$

The first H.M. between a and b , when n H.M.'s are inserted between a and b can be obtained by replacing a by $\frac{1}{a}$ and b by

$\frac{1}{b}$ in Eq. (i) and then taking its reciprocal.

$$\therefore q = \frac{1}{\left(\frac{1}{a} \right) + \frac{1}{b}} = \frac{(n+1)ab}{bn+a}$$

$$\therefore q = \frac{(n+1)ab}{a+bn} \quad \dots (ii)$$

Now, we have to prove that q cannot lie between p and $\frac{(n+1)^2}{(n-1)^2} p$.

$$\text{Now, } n+1 > n-1 \Rightarrow \frac{n+1}{n-1} > 1$$

$$\Rightarrow \left(\frac{n+1}{n-1} \right)^2 > 1 \text{ or } p \left(\frac{n+1}{n-1} \right)^2 > p$$

$$\Rightarrow p < p \left(\frac{n+1}{n-1} \right)^2 \quad \dots (iii)$$

If q does not lie between p and q , then q is either less than p or

$$q > p \left(\frac{n+1}{n-1} \right)^2$$

$$\text{Now, } \frac{p}{q} = \frac{(na+b)(nb+a)}{(n+1)^2 ab}$$

$$\Rightarrow \frac{p}{q} - 1 = \frac{n(a^2 + b^2) + ab(n^2 + 1) - (n+1)^2 ab}{(n+1)^2 ab}$$

$$\Rightarrow \frac{p}{q} - 1 = \frac{n(a^2 + b^2 - 2ab)}{(n+1)^2 ab}$$

$$\Rightarrow \frac{p}{q} - 1 = \frac{n}{(n+1)^2} \left(\frac{a-b}{\sqrt{ab}} \right)^2 = \frac{n}{(n+1)^2} \left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}} \right)^2$$

$$\Rightarrow \frac{p}{q} - 1 > 0 \quad \therefore p > q \quad \dots (iv)$$

From (iii) and (iv), we get, $q < p < \left(\frac{n+1}{n-1}\right)^2 p$

$\therefore q$ can not lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$.

19. Given that $a, b, c > 0$
We know for positive numbers, A.M. \geq G.M.
 \therefore For positive numbers a, b, c , we get

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \quad \dots(i)$$

Also for positive numbers $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$;

$$\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{3} \geq \sqrt[3]{\frac{1}{abc}} \quad \dots(ii)$$

On multiplying eqs (i) and (ii), we get

$$\left(\frac{a+b+c}{3}\right) \left(\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{3}\right) \geq \sqrt[3]{abc} \times \frac{1}{\sqrt[3]{abc}}$$

$$\therefore (a+b+c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$$

20. Let the two numbers be a and b , then

$$\frac{a+b}{2} = A; \sqrt{ab} = G; \frac{2ab}{a+b} = 4 \quad \dots(i)$$

$$\text{Now, } 2A + G^2 = 27 \Rightarrow a+b+ab=27 \quad \dots(ii)$$

On putting $ab = 27 - (a+b)$ in Eq. (i),

$$\frac{54 - 2(a+b)}{a+b} = 4 \Rightarrow a+b=9, \text{ then } ab = 27 - 9 = 18$$

On solving (i) and (ii), we get $a=6, b=3$ or $a=3, b=6$, which are the required numbers.

Topic-4: Arithmetic-Geometric Sequence (A.G.S.), Some Special Sequences

1. (1219) Given $S = 77 + 757 + 7557 + \dots + \overbrace{75 \dots 57}^{98}$

$$10S = \overbrace{770 + 7570 + \dots + 75 \dots 570}^{98}$$

$$-9S = \overbrace{77 - 13 - 13 \dots - 13 - 75 \dots 570}^{98 \text{ times}}$$

$$9S = -77 + 13 \times 98 + \overbrace{75 \dots 57}^{99} + 13$$

$$S = \overbrace{75 \dots 57}^{99} + 1210$$

$$\Rightarrow m = 1210 \text{ and } n = 9 \Rightarrow m+n = 1219.$$

2. Since n is an odd integer, $(-1)^{n-1} = 1$ and $n-1, n-3, n-5, \dots$ are even integers.

$$\text{Now, } n^3 - (n-1)^3 + (n-2)^3 - (n-3)^3 + \dots + (-1)^{n-1} 1^3$$

$$= n^3 + (n-1)^3 + (n-2)^3 + \dots + 1^3$$

$$= [n^3 + (n-1)^3 + (n-2)^3 + \dots + 1^3] - 2 \times 2^3 \left[\left(\frac{n-1}{2}\right)^3 + \left(\frac{n-3}{2}\right)^3 + \dots + 1^3 \right] \quad \dots(i)$$

$$= \left[\frac{n(n+1)}{2} \right]^2 - 16 \left[\frac{1}{2} \left(\frac{n-1}{2} \right) \left(\frac{n-1}{2} + 1 \right) \right]^2$$

[Since in the equation (i), the first square bracket contain the sum of cubes of 1st n natural numbers. Whereas the second square bracket contains the sum of the cubes of natural numbers from 1

to $\left(\frac{n-1}{2}\right)$ as $(n-1), (n-3), \dots$ are even integers.]

$$= \frac{1}{4} n^2 (n+1)^2 - 16 \frac{(n-1)^2 (n+1)^2}{16 \times 4}$$

$$= \frac{1}{4} (n+1)^2 [n^2 - (n-1)^2] = \frac{1}{4} (n+1)^2 (2n-1)$$

3. For n to be odd, let $n = 2m+1$

$$\therefore \text{Required sum} = 1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + \dots + 2(2m)^2 + (2m+1)^2$$

$$= \Sigma (2m+1)^2 + 4[1^2 + 2^2 + 3^2 + \dots + m^2]$$

$$= \frac{(2m+1)(2m+2)(4m+2+1)}{6} + \frac{4m(m+1)(2m+1)}{6}$$

$$= \frac{(2m+1)(m+1)}{6} [2(4m+3) + 4m]$$

$$= \frac{(2m+1)(2m+2)(6m+3)}{6} = \frac{(2m+1)^2 (2m+2)}{2}$$

$$= \frac{n^2 (n+1)}{2} [\because 2m+1 = n]$$

4. (a, b, c) Since α, β are roots of $x^2 - x - 1 = 0$ with $\alpha > \beta$

$$\therefore \alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$$

$$\text{Also, } \alpha + \beta = 1, \alpha\beta = -1, \alpha - \beta = \sqrt{5}$$

$$\alpha^2 - \alpha - 1 = 0 \text{ and } \beta^2 - \beta - 1 = 0$$

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, n \geq 1; b_1 = 1 \text{ and } b_n = a_{n-1} + a_{n+1}, n \geq 2$$

Let us now check the given options, one by one

$$(a) \sum_{n=1}^{\infty} \frac{a_n}{10^n} = \sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{\sqrt{5}(10^n)} = \frac{1}{\sqrt{5}} \left[\sum_{n=1}^{\infty} \left(\frac{\alpha}{10} \right)^n - \sum_{n=1}^{\infty} \left(\frac{\beta}{10} \right)^n \right]$$

$$= \frac{1}{\sqrt{5}} \left[\frac{\frac{\alpha}{10}}{1 - \frac{\alpha}{10}} - \frac{\frac{\beta}{10}}{1 - \frac{\beta}{10}} \right] = \frac{1}{\sqrt{5}} \left[\frac{\alpha}{10 - \alpha} - \frac{\beta}{10 - \beta} \right]$$

$$= \frac{1}{\sqrt{5}} \left[\frac{10\alpha - \alpha\beta - 10\beta + \alpha\beta}{(10 - \alpha)(10 - \beta)} \right]$$

$$= \frac{1}{5} \left[\frac{10(\alpha - \beta)}{100 - 10(\alpha + \beta) + \alpha\beta} \right] = \frac{1}{5} \left[\frac{10\sqrt{5}}{100 - 10 - 1} \right] = \frac{10}{89}$$

Thus option (a) is correct.

$$(b) b_n = a_{n+1} + a_{n-1} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}$$

$$= \frac{(\alpha^{n+1} + \alpha^{n-1}) - (\beta^{n+1} + \beta^{n-1})}{\alpha - \beta}$$

$$= \frac{\alpha^{n-1}(\alpha^2 + 1) - \beta^{n-1}(\beta^2 + 1)}{\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}} [\alpha^{n-1}(\alpha + 2) - \beta^{n-1}(\beta + 2)]$$

[using $\alpha^2 = \alpha + 1, \beta^2 = \beta + 1$]

$$= \frac{1}{\sqrt{5}} \left[\alpha^{n-1} \left(\frac{1+\sqrt{5}}{2} + 2 \right) - \beta^{n-1} \left(\frac{1-\sqrt{5}}{2} + 2 \right) \right]$$

$$= \frac{1}{\sqrt{5}} \left[\alpha^{n-1} \left(\frac{5+\sqrt{5}}{2} \right) - \beta^{n-1} \left(\frac{5-\sqrt{5}}{2} \right) \right]$$

$$\begin{aligned}
 &= \frac{\sqrt{5}}{\sqrt{5}} \left[\alpha^{n-1} \left(\frac{\sqrt{5}+1}{2} \right) + \beta^{n-1} \left(\frac{1-\sqrt{5}}{2} \right) \right] \\
 &= \alpha^{n-1} \alpha + \beta^{n-1} \beta = \alpha^n + \beta^n \\
 &\text{Thus option (b) is correct.} \\
 (c) \quad a_1 + a_2 + a_3 + \dots + a_n \\
 &= \frac{\alpha^1 - \beta^1}{\alpha - \beta} + \frac{\alpha^2 - \beta^2}{\alpha - \beta} + \frac{\alpha^3 - \beta^3}{\alpha - \beta} + \dots + \frac{\alpha^n - \beta^n}{\alpha - \beta} \\
 &= \frac{1}{\sqrt{5}} [(\alpha + \alpha^2 + \alpha^3 + \dots + \alpha^n) - (\beta + \beta^2 + \beta^3 + \dots + \beta^n)] \\
 &= \frac{1}{\sqrt{5}} \left[\frac{\alpha(1 - \alpha^n)}{1 - \alpha} - \frac{\beta(1 - \beta^n)}{1 - \beta} \right] \\
 &= \frac{1}{\sqrt{5}} \left[\frac{\alpha(1 - \beta)(1 - \alpha^n) - \beta(1 - \alpha)(1 - \beta^n)}{(1 - \alpha)(1 - \beta)} \right] \\
 &= \frac{1}{\sqrt{5}} \left[\frac{(\alpha - \alpha\beta)(1 - \alpha^n) - (\beta - \alpha\beta)(1 - \beta^n)}{(1 - \alpha)(1 - \beta)} \right] \\
 &= \frac{1}{\sqrt{5}} \left[\frac{(\alpha + 1)(1 - \alpha^n) - (\beta + 1)(1 - \beta^n)}{1 - (\alpha + \beta) + \alpha\beta} \right] \\
 &= \frac{1}{\sqrt{5}} \left[\frac{\alpha^2(1 - \alpha^n) - \beta^2(1 - \beta^n)}{1 - 1 - 1} \right] \\
 &\quad \text{[using } \alpha^2 - \alpha - 1 = 0, \beta^2 - \beta - 1 = 0] \\
 &= \frac{1}{\sqrt{5}} \left[\frac{(\alpha^2 - \beta^2) - (\alpha^{n+2} - \beta^{n+2})}{-1} \right] \\
 &= \left[\frac{-(\alpha - \beta)(\alpha + \beta)}{\alpha - \beta} + \frac{(\alpha^{n+2} - \beta^{n+2})}{\alpha - \beta} \right] = -1 + a_{n+2}
 \end{aligned}$$

Thus option (c) is correct.

$$\begin{aligned}
 (d) \quad \sum_{n=1}^{\infty} \frac{b_n}{10^n} &= \sum_{n=1}^{\infty} \frac{\alpha^n + \beta^n}{10^n} \quad [\text{using } b_n = \alpha^n + \beta^n \text{ from (b)}] \\
 &= \sum_{n=1}^{\infty} \left(\frac{\alpha}{10} \right)^n + \left(\frac{\beta}{10} \right)^n = \frac{\frac{\alpha}{10}}{1 - \frac{\alpha}{10}} + \frac{\frac{\beta}{10}}{1 - \frac{\beta}{10}} \\
 &= \frac{\alpha}{10 - \alpha} + \frac{\beta}{10 - \beta} = \frac{\alpha(10 - \beta) + \beta(10 - \alpha)}{(10 - \alpha)(10 - \beta)} \\
 &= \frac{10(\alpha + \beta) - 2\alpha\beta}{100 - 10\alpha - 10\beta + \alpha\beta} = \frac{10 + 2}{100 - 10 - 1} = \frac{12}{89}
 \end{aligned}$$

Thus option (d) is incorrect.

5. (a, d) We have

$$\begin{aligned}
 a(n) &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{2^{n-1}} \\
 &= \left[1 + \left(\frac{1}{2} + \frac{1}{3} \right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \right) + \left(\frac{1}{8} + \dots + \frac{1}{15} \right) + \dots + \left(\frac{1}{2^{n-1}} + \dots + \frac{1}{2^n - 1} \right) \right] \\
 &< \left[1 + \left(\frac{1}{2} + \frac{1}{2} \right) + \left(\frac{1}{4} + \frac{1}{4} + \dots + \frac{1}{4} \right) + \left(\frac{1}{8} + \frac{1}{8} + \dots + \frac{1}{8} \right) + \dots + \left(\frac{1}{2^{n-1}} + \dots + \frac{1}{2^{n-1}} \right) \right]
 \end{aligned}$$

$$< \left[1 + \frac{2}{2} + \frac{4}{4} + \frac{8}{8} + \dots + \frac{2^{n-1}}{2^{n-1}} \right] = [1 + 1 + \dots + 1] = n$$

Also $a(100) < 100$

$$\begin{aligned}
 a(n) &= \left[1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4} \right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \dots + \left(\frac{1}{2^{n-1}} + \dots + \frac{1}{2^n} \right) - \frac{1}{2^n} \right] \\
 &> \left[1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4} \right) + \left(\frac{1}{8} + \frac{1}{8} + \dots + \frac{1}{8} \right) + \dots + \left(\frac{1}{2^n} + \dots + \frac{1}{2^n} \right) - \frac{1}{2^n} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \dots + \frac{2^{n-1}}{2^n} - \frac{1}{2^n} \\
 &= 1 + \underbrace{\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} \right)}_{n \text{ times}} - \frac{1}{2^n}
 \end{aligned}$$

$$= 1 + \frac{n}{2} - \frac{1}{2^n} = \left(1 - \frac{1}{2^n} \right) + \frac{n}{2}$$

$$\therefore a(200) > \left[\left(1 - \frac{1}{2^{200}} \right) + \frac{200}{2} \right] > 100,$$

$\therefore a(200) > 100$.

Given series

$$\begin{aligned}
 &= \sum_{r=0}^n (-1)^r {}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \dots \text{ up to } m \text{ terms} \right] \\
 &= \sum_{r=0}^n (-1)^r {}^nC_r \left[\left(\frac{1}{2} \right)^r + \left(\frac{3}{4} \right)^r + \left(\frac{7}{8} \right)^r + \left(\frac{15}{16} \right)^r + \dots \text{ to } m \text{ terms} \right]
 \end{aligned}$$

Now,

$$\begin{aligned}
 \sum_{r=0}^n (-1)^r {}^nC_r \left(\frac{1}{2} \right)^r &= 1 - {}^nC_1 \cdot \frac{1}{2} + {}^nC_2 \cdot \frac{1}{2^2} - {}^nC_3 \cdot \frac{1}{2^3} + \dots \\
 &= \left(1 - \frac{1}{2} \right)^n = \frac{1}{2^n}
 \end{aligned}$$

$$\text{Similarly, } \sum_{r=0}^n (-1)^r {}^nC_r \left(\frac{3}{4} \right)^r = \left(1 - \frac{3}{4} \right)^n = \frac{1}{4^n} \text{ etc.}$$

Hence the sum of the series

$$= \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{16^n} + \dots \text{ to } m \text{ terms}$$

$$= \frac{1}{2^n} \left(1 - \left(\frac{1}{2} \right)^m \right)$$

[Sum of G.P.]

$$\begin{aligned}
 &= \frac{1 - \frac{1}{2^m}}{1 - \frac{1}{2^n}} \\
 &= \frac{2^{mn} - 1}{2^{mn}(2^n - 1)}
 \end{aligned}$$

7.

$$\text{Given : } n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \dots p_k^{\alpha_k} \quad \dots (i)$$

Where $n \in \mathbb{N}$ and $p_1, p_2, p_3, \dots, p_k$ are distinct prime numbers.

Taking log on both sides of Eq. (i), we get

$$\log n = \alpha_1 \log p_1 + \alpha_2 \log p_2 + \dots + \alpha_k \log p_k \quad \dots (ii)$$

Since $p_1, p_2, p_3, \dots, p_k$ are distinct prime numbers, therefore maximum any one prime number out of $p_1, p_2, p_3, \dots, p_k$ will be 2 and the remaining prime numbers will be greater than 2.

$$\therefore \log n \geq \alpha_1 \log 2 + \alpha_2 \log 2 + \alpha_3 \log 2 + \dots + \alpha_k \log 2$$

$$\Rightarrow \log n \geq (\alpha_1 + \alpha_2 + \dots + \alpha_k) \log 2$$

$$\Rightarrow \log n \geq k \log 2$$