Mechanical Oscillations (Part -1)

Q. 1. A point oscillates along the x axis according to the law x = a cos (ωt — n/4). Draw the approximate plots

(a) of displacement x, velocity projection v_x, and acceleration projection w_x as functions of time t;

(b) velocity projection v_x and acceleration projection w_x as functions of the coordinate x.

Ans. 1. (a) Given, x = $a \cos\left(\omega t - \frac{\pi}{4}\right)$

So $v_x = \dot{x} = -a\,\omega \sin\left(\omega t - \frac{\pi}{4}\right)$ and $w_x = \ddot{x} = -a\,\omega^2 \cos\left(\omega t - \frac{\pi}{4}\right)$ (1)

On-the basis of obtained expressions plots x(t) , v_x(t) and w_x(t) can be drawn as shown in the answersheet, (of the problem book).

(b) From Eqn (1)

$v_x = -a\,\omega \sin\left(\omega t - \frac{\pi}{4}\right)$ So, $v_x^2 = a^2 \omega^2 \sin^2\left(\omega t - \frac{\pi}{4}\right)$ (2)

But from the law x = a cos (ωt - π/4), so, $x^2 = a^2 \cos^2 (\omega t - \pi/4)$

or, cos (ωt - π/4) - X^2/ = 2 or $\sin^2$ (ωt - 3t/4) = $= 1 - \frac{x^2}{a^2}$ (3)

Using (3) in (2), $v_x^2 = a^2 \omega^2 \left(1 - \frac{x^2}{a^2}\right)$ or $v_x^2 = \omega^2 (a^2 - x^2)$ (4)

Again from Eqn (4), $w_x = -a\omega^2 \cos (\omega t - \pi/4) = -\omega^2 x$

Q. 2. A point moves along the x axis according to the law x = a $\sin^2$(ωt — π/4). Find: (a) the amplitude and period of oscillations; draw the plot x (t); (b) the velocity projection v_x as a function of the coordinate x; draw the plot v_x (x).

Ans. 2. (a) From the motion law of the particle

$$x = a\sin^2(\omega t - \pi/4) = \frac{a}{2}\left[1-\cos\left(2\omega t - \frac{\pi}{2}\right)\right]$$

or, $$x - \frac{a}{2} = -\frac{a}{2}\cos\left(2\omega t - \frac{\pi}{2}\right) = -\frac{a}{2}\sin 2\omega t = \frac{a}{2}\sin(2\omega t + \pi)$$

i.e. $$x - \frac{a}{2} = \frac{a}{2}\sin(2\omega t + \pi).$$

(1)

Now compairing this equation with the general equation of harmonic oscillations: X - A sin(ωot + a) Amplitude, A = a/2 and angular frequency, ω_0 - 2ω.

$$T = \frac{2\pi}{\omega_0} = \frac{\pi}{\omega}$$

Thus the period of one full oscillation,

(b) Differentiating Eqn (1) w.r.t. time

$$v_x = a\omega\cos(2\omega t + \pi) \text{ or } v_x^2 = a^2\omega^2\cos^2(2\omega t + \pi) = a^2\omega^2\left[1 - \sin^2(2\omega t + \pi)\right] \quad (2)$$

From Eqn (1) $$\left(x - \frac{a}{2}\right)^2 = \frac{a^2}{4}\sin^2(2\omega t + \pi)$$

or, $$4\frac{x^2}{a^2} + 1 - \frac{4x}{a} = \sin^2(2\omega t + \pi) \text{ or } 1 - \sin^2(2\omega t + \pi) = \frac{4x}{a}\left(1 - \frac{x}{a}\right) \quad (3)$$

From Eqns (2) and (3), $$v_x = a^2\omega^2\frac{4x}{a}\left(1 - \frac{x}{a}\right) = 4\omega^2 x(a - x)$$

Plot of v_x (x) is as shown in the answersheet.

Q. 3.A particle performs harmonic oscillations along the x axis about the equilibrium position x = 0. The oscillation frequency is ω = 4.00 s^{-1} . At a certain moment of time the particle has a coor- dinate x_o = 25.0 cm and its velocity is equal to v_{x0} = 100 cm/s. Find the coordinate x and the velocity v_x of the particle t = 2.40s after that moment.

Ans. 3 Let the general equation of S.H.M. be

h = a cos (ωt + α)

So, v_x = - a **ω** sin (ωt + α)

Let us assume that at t = 0 , x = h_0 and $v_x = v_{x_0}$.

Thus from Eqns (1) and (2) for t = 0, h_0 = a cos α, and v_{x_0} = - a **ω**sin α

Therefore $$\tan\alpha = -\frac{v_{x_0}}{\omega x_0} \quad \text{and} \quad a = \sqrt{x_0^2 + \left(\frac{v_{x_0}}{\omega}\right)^2} = 35.35 \text{ cm}$$

Under our assumption Eqns (1) and (2) give the sought x and v_x if

$$t = t = 2{\cdot}40\,\text{s},\ a = \sqrt{x_0^2 + \left(v_{x_0}/\omega\right)^2} \quad \text{and} \quad \alpha = \tan^{-1}\left(-\frac{v_x}{\omega x_0}\right) = -\frac{\pi}{4}$$

Putting all the given numerical values, we get :

x = - 29 cm and v_x = - 81 cm/s

Q. 4.Find the angular frequency and the amplitude of harmonic oscillations of a particle if at distances x_1 and x_2 from the equilibrium position its velocity equals v_1 a nd v_2 respectively.

Ans. From the Eqn

$v_x^2 = \omega^2 (a^2 - x^2)$ (see Eqn. 4 of 4.1)

$v_1^2 = \omega^2 (a^2 - x_1^2)$ and $v_2^2 = \omega^2 (a^2 - x_2^2)$

Solving these Eqns simultaneously, we get

$$\omega = \sqrt{(v_1^2 - v_2^2)/(x_2^2 - x_1^2)}, \quad a = \sqrt{(v_1 x_2^2 - v_2^2 x_1^2)/(v_1^2 - v_2^2)}$$

Q. 5. A point performs harmonic oscillations along a straight line with a period T = 0.60 s and an amplitude a = 10.0 cm. Find the mean velocity of the point averaged over the time interval during which it travels a distance a/2, starting from

(a) the extreme position;

(b) the equilibrium position.

Ans. (a) When a particle starts from an extreme position, it is useful to write the motion law as x = a cos ωt (1)

(However x is the displacement from the equlibrium position)

It t_x be the time to cover the distence a/2 then from (1)

$$a - \frac{a}{2} = \frac{a}{2} = a \cos \omega t_1 \quad \text{or} \quad \cos \omega t_1 = \frac{1}{2} = \cos \frac{\pi}{3} \ (\text{as } t_1 < T/4)$$

Thus $$t_1 = \frac{\pi}{3\omega} = \frac{\pi}{3(2\pi/T)} = \frac{T}{6}$$

As $$x = a \cos \omega t, \quad \text{so}, \ v_x = -a\omega \sin \omega t$$

Thus $$v = |v_x| = -v_x = a\omega \sin \omega t, \quad \text{for } t \le t_1 = T/6$$

Hence sought mean velocity

$$<v> = \frac{\int v\, dt}{\int dt} = \int_0^{T/6} a(2\pi/T) \sin \omega t\, dt \Big/ T/6 = \frac{3a}{T} = 0{\cdot}5 \text{ m/s}$$

(b) In this case, it is easier to write the motion law in the form:

x = a sin **ωt** (2)

If t_2 be the time to cover the distance a/2, then from Eqn (2)

$$a/2 = a \sin \frac{2\pi}{T} t_2 \quad \text{or} \quad \sin \frac{2\pi}{T} t_2 = \frac{1}{2} = \sin \frac{\pi}{6} \quad (\text{as } t_2 < T/4)$$

Thus $$\frac{2\pi}{T} t_2 = \frac{\pi}{6} \quad \text{or}, \; t_2 = \frac{T}{12}$$

Differentiating Eqn (2) w.r.t time, we get

$$v_x = a\,\omega \cos \omega t = a \frac{2\pi}{T} \cos \frac{2\pi}{T} t$$

So, $$v = |v_x| = a \frac{2\pi}{T} \cos \frac{2\pi}{T} t, \quad \text{for} \quad t \le t_2 = T/12$$

Hence the sought mean velocity

$$< v > = \frac{\int v\, dt}{\int dt} = \frac{1}{(T/12)} \int_0^{T/12} a \frac{2\pi}{T} \cos \frac{2\pi}{T} t\, dt = \frac{6a}{T} = 1 \text{ m/s}$$

Q. 6. At the moment t = 0 a point starts oscillating along the x axis according to the law x = a sin ωt. Find:
(a) the mean value of its velocity vector projection (v_s);
(b) the modulus of the mean velocity vector |(v)| ;
(c) the mean value of the velocity modulus (v) averaged over 3/8 of the period after the start.

Ans. (a) As x = a sin **ωt** **so,** **v**$_x$ = a**ω** cos **ωt**

Thus $$< v_x > = \int v_x\, dt / \int dt = \frac{\int_0^{\frac{3}{8}T} a\,\omega \cos (2\pi/T)\, t\, dt}{\frac{3}{8}T} = \frac{2\sqrt{2}\, a\, \omega}{3\pi} \left(\text{using } T = \frac{2\pi}{\omega} \right)$$

(b) In accordance with the problem

$$\vec{v} = v_x \vec{i}, \quad \text{so}, \; |<\vec{v}>| = |< v_x >|$$

Hence, using part (a), $$|<\vec{v}>| = \left| \frac{2\sqrt{2}\, a\, \omega}{3\pi} \right| = \frac{2\sqrt{2}\, a\, \omega}{3\pi}$$

(c) We have got, v_x = a **ω** cos **ωt**

$$\left. \begin{aligned} \text{So, } v = |v_x| &= a\,\omega \cos \omega t, \quad \text{for } t \le T/4 \\ &= -a\,\omega \cos \omega t, \quad \text{for } T/4 \le t \le \frac{3}{8} T \end{aligned} \right]$$

Hence, $$\langle v \rangle = \frac{\int v\, dt}{\int dt} = \frac{\int_0^{T/4} a\,\omega \cos \omega t\, dt + \int_{T/4}^{3T/8} -a\,\omega \cos \omega t\, dt}{3T/8}$$

Using **ω** = 2 ω/T, and on evaluating the integral we get

$$\langle v \rangle = \frac{2(4-\sqrt{2})\,a\,\omega}{3\pi}$$

Q. 7. A particle moves along the x axis according to the law x = a cos ωt. Find the distance that the particle covers during the time interval from t = 0 to t.

Ans. From the motion law, x = a cos **ωt**,, it is obvious that the time taken to cover the distance equal to the amplitude (a), starting from extreme position equals T/4.
Now one can write

$$t = n\frac{T}{4} + t_0, \quad \left(\text{where } t_0 < \frac{T}{4} \text{ and } n = 0, 1, 2, \ldots\right)$$

As the particle moves according to the law, x .= a cos **ωt**, so at n = 1,3,5 ... or for odd n values it passes through the mean positon and for even numbers of n it comes to an extreme position (if t_0 = 0).

Case (1) when n is an odd number: In this case, from the equation
x = x a sin **ω**t, if the t is counted from nT/4 and the distance covered in the time interval to be comes

$$s_1 = a \sin \omega t_0 = a \sin \omega\left(t - n\frac{T}{4}\right) = a \sin\left(\omega t - \frac{n\pi}{2}\right)$$

Thus the sought distance covered for odd n is

$$s = n a + s_1 = n a + a \sin\left(\omega t - \frac{n\pi}{2}\right) = a\left[n + \sin\left(\omega t - \frac{n\pi}{2}\right)\right]$$

Case (2), when n is even, In this case from the equation x = a cos **ω**t, the distance covered (s_2) in the interval t_0, is given by

$$a - s_2 = a \cos \omega t_0 = a \cos \omega\left(t - n\frac{T}{4}\right) = a \cos\left(\omega t - n\frac{\pi}{2}\right)$$

or $$s_2 = a\left[1 - \cos\left(\omega t - \frac{n\pi}{2}\right)\right]$$

Hence the sought distance for n is even

$$s = n\,a + s_2 = n\,a + a\left[1 - \cos\left(\omega\,t - \frac{n\,\pi}{2}\right)\right] = a\left[n + 1 - \cos\left(\omega\,t - \frac{n\,\pi}{2}\right)\right]$$

In general

$$s = \begin{cases} a\left[n + 1 - \cos\left(\omega\,t - \dfrac{n\,\pi}{2}\right)\right], & n \text{ is even} \\ a\left[n + \sin\left(\omega\,t - \dfrac{n\,\pi}{2}\right)\right], & n \text{ is odd} \end{cases}$$

Q. 8. At the moment t = 0 a particle starts moving along the x axis so that its velocity projection varies as v_x = 35 cos πt cm/s, where t is expressed in seconds. Find the distance that this particle covers during t = 2.80 s after the start.

Ans. Obviously the motion law is of the from, x = a shoot, and v_x = **ω** a cos **ω**t. Comparing v_x - **ω** a cos **ω**t with v_x - 35 cos πt , we get

$$\omega = \pi,\ a = \frac{35}{\pi}, \text{thus } T = \frac{2\pi}{\omega} = 2 \text{ and } T/4 = 0{\cdot}5\text{ s}$$

Now we can write

$$t = 2{\cdot}8\text{ s} = 5 \times \frac{T}{4} + 0{\cdot}3 \quad \left(\text{where } \frac{T}{4} = 0{\cdot}5\text{ s}\right)$$

As n = 5 is odd, like (4 = 7), we have to basically find the distance covered by the particle starting from the extreme position in the time interval 0 = 3 s.
Thus from the Eqn.

$$x = a\cos\omega\,t = \frac{35}{\pi}\cos\pi\,(0{\cdot}3)$$

$$\frac{35}{\pi} - s_1 = \frac{35}{\pi}\cos\pi\,(0{\cdot}3) \quad \text{or} \quad s_1 = \frac{35}{\pi}\{1 - \cos 0{\cdot}3\,\pi\}$$

Hence the sought distance

$$s = 5 \times \frac{35}{\pi} + \frac{35}{\pi}\{1 - \cos 0{\cdot}3\,\pi\}$$

$$= \frac{35}{\pi}\{6 - \cos 0{\cdot}3\,\pi\} = \frac{35}{22} \times 7\,(6 - \cos 54°) = 60\text{ cm}$$

Q. 9. A particle performs harmonic oscillations along the x axis according to the law x = a cos ωt. Assuming the probability P of the particle to fall within an interval from —a to +a to be equal to unity, find how the probability density dP/dx depends on x. Here dP denotes the probability of the particle falling within an interval from x to x dx. Plot dP/dx as a function of x.

Ans. As the motion is periodic the particle repeatedly passes through any given region in the range - a ≤ x ≤ a.

The probability that it lies in the range (x, x + dx) is defined as the fraction

$\frac{\Delta t}{t}$ (as $t \to \infty$) where Δt is the time that the particle lies in the range (x, x + dx) out of the total time t. Because of periodicity this is

$$dP = \frac{dP}{dx}dx = \frac{dt}{T} = \frac{2\,dx}{v\,T}$$

where the factor 2 is needed to take account of the fact that the particle is in the range (x , x + d x) during both up and down phases of its motion. Now in a harmonic oscillator.

$$v = \dot{x} = \omega a \cos \omega t = \omega\sqrt{a^2 - x^2}$$

Thus since **ω**T = 2 π (T is the time period)

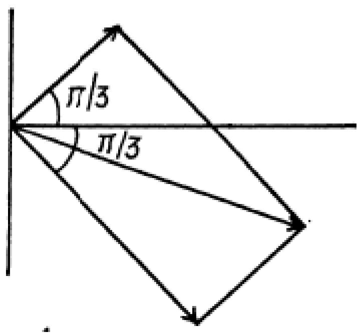
We get $$dP = \frac{dP}{dx}dx = \frac{1}{\pi}\frac{dx}{\sqrt{a^2 - x^2}}$$

Note that $$\int_{-a}^{+a} \frac{dP}{dx}\,dx = 1$$

so $$\frac{dP}{dx} = \frac{1}{\pi}\frac{1}{\sqrt{a^2 - x^2}}$$

Q. 10. Using graphical means, find an amplitude a of oscillations resulting from the superposition of the following oscillations of the same direction:
(a) x_1 =3 .0 cos (ωt -1- π/3), x_2 =8 .0 sin (ωt + π/6);
(b) x_1 = 3.0 cos ωt, x_2 = 5.0 cos (ωt+ π/4), x_3 =6 .0 sin ωt.

Ans. (a) We take a graph paper and choose an axis (X - axis) and an origin. Draw a vector of magnitude 3 inclined at an angle **π/3** with the X -axis. Draw another vector of magnitude 8 inclined at an angle **- π/3** (Since sin (**ωt**+ **π**/ 6) » cos (**ωt-** **π**/ 3)) with the X - axis. The magnitude of the resultant of both these vectors (drawn from the origin) obtained using parallelogram law is the resultant, amplitude.



Clearly $$R^2 = 3^2 + 8^2 + 2\cdot 3\cdot 8\cdot \cos\frac{2\pi}{3} = 9 + 64 - 48 \times \frac{1}{2}$$

Thus R = R7 units

(b) One can follow the same graphical method here but the result can be obtained more quickly by breaking into sines and cosines and adding:

Resultant

$$x = \left(3 + \frac{5}{\sqrt{2}}\right)\cos\omega t + \left(6 - \frac{5}{\sqrt{2}}\right)\sin\omega t$$
$$= A\cos(\omega t + \alpha)$$

Then

$$A^2 = \left(3 + \frac{5}{\sqrt{2}}\right)^2 + \left(6 - \frac{5}{\sqrt{2}}\right)^2 = 9 + 25 + \frac{30 - 60}{\sqrt{2}} + 36$$
$$= 70 - 15\sqrt{2} = 70 - 21\cdot 2$$

So, A = 6-985 = 7 units

Note- In using graphical method convert all oscillations to either sines or cosines but do not use both.

Mechanical Oscillations (Part -2)

Q. 11. A point participates simultaneously in two harmonic oscillations of the same direction: x_1 = a cos ωt and
x_2 = a cos 2ωt.
Find the maximum velocity of the point.

Ans. Given , x_1 = a cos ωt and x 2 = a cos 2 ωt

so, the net displacement,

$$x = x_1 + x_2 = a\{\cos\omega t + \cos 2\omega t\} = a\{\cos\omega t + 2\cos^2\omega t - 1\}$$

and $$v_x = \dot{x} = a\{-\omega\sin\omega t - 4\omega\cos\omega t\sin\omega t\}$$

For $\dot{x}$ to be maximum,

$$\ddot{x} = a\omega^2\cos\omega t - 4a\omega^2\cos^2\omega t + 4a\omega^2\sin^2\omega t = 0$$

Solving for acceptable value

cos **ω**t = 0.644

thus sin **ωt** = 0.765

And $$v_{max} = |v_{x_{max}}| = +a\omega[0{\cdot}765 + 4\times 0{\cdot}765\times 0{\cdot}644] = +2{\cdot}73\ a\omega$$

Q. 12. The superposition of two harmonic oscillations of the same direction results in the oscillation of a point according to the law x = a cos 2.1t cos 50.0t, where t is expressed in seconds. Find the angular frequencies of the constituent oscillations and the period with which they beat.

Ans. We write:

$$a\cos 2{\cdot}1\,t\cos 50{\cdot}0\,t = \frac{a}{2}\{\cos 52{\cdot}1\,t + \cos 47{\cdot}9\,t\}$$

Thus the angular frequencies of constituent oscillations are
52.1 s^{-1} and 47.9 s^{-1}

To get the beat period note that the variable amplitude a cos 2.1t becomes maximum (positive or negative), when

2.1t = nπ Thus the interval between two maxima is

$\frac{\pi}{2 \cdot 1} = 1 \cdot 5$ s nearly.

Q. 13. A point A oscillates according to a certain harmonic law in the reference frame K' which in its turn performs harmonic oscillations relative to the reference frame K. Both oscillations occur along the same direction. When the K' frame oscillates at the frequency 20 or 24 Hz, the beat frequency of the point A in the K frame turns 167 out to be equal to v. At what frequency of oscillation of the frame K' will the beat frequency of the point A become equal to 2v?

Ans. If the frequency of A with respect to K' is v_0 and K ' oscillates with frequency $\bar{v}$ with respect to K, the beat frequency of the point A in the .K-frame will be v when

$\bar{v} = v_0 \pm v$

In the present case $\bar{v}$ = 20 or 24.

This means v_0 - 22. & v - 2

Thus beats of 2v = 4 will be heard when $\bar{v}$ = 26 or 18 .

Q. 14. A point moves in the plane xy according to the law x = a sin ωt, y = b cos ωt, where a, b, and ω are positive constants. Find:
(a) the trajectory equation y (x) of the point and the direction of its motion along this trajectory;
(b) the acceleration ω of the point as a function of its radius vector r relative to the origin of coordinates.

Ans. (a) From the Eqn : x = a sin **ω**t

$\sin^2 \omega t = x^2/a^2$ or $\cos^2 \omega t = 1 - \frac{x^2}{a^2}$ (1)

And from the equation :

$y = b \cos \omega t$

$\cos^2 \omega t = y^2/b^2$ (2)

From Eqns (1) and (2), we get :

$1 - \frac{x^2}{a^2} = \frac{y^2}{b^2}$ or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

which is the standard equation of the ellipse shown in the figure, we observe that,

at t = 0 , x = 0 and y = b
and at

$$t = \frac{\pi}{2\omega}, \; x = +a \text{ and } y = 0$$

Thus we observe that at t = 0, the point is at point 1 (Fig.) and at the following moments, the co-ordinate y diminishes and x becomes positive. Consequently the motion is clockwise.

(b) As x = d sin **ω**t and y=b cos **ω**t

So we may write $\vec{r} = a \sin \omega t \, \vec{i} + b \cos \omega t \, \vec{j}$

Thus $\ddot{\vec{r}} = \vec{w} = -\omega^2 \vec{r}$

Q. 15. Find the trajectory equation y (x) of a point if it moves according to the following laws:
(a) x = a sin ωt, y = a sin 2ωt;
(b) x = a sin cot, y = a cos 2ωt.
Plot these trajectories.

Ans. (a) From the Eqn. : x = a sin **ω**t , we have

$$\cos \omega t = \sqrt{1 - (x^2/a^2)}$$

and from the Eqn. :y = a sin 2 **ω**t

$$y = 2a \sin \omega t \cos \omega t = 2x\sqrt{1-(x^2/a^2)} \quad \text{or} \quad y^2 = 4x^2\left(1-\frac{x^2}{a^2}\right)$$

(b) From the Eqn. : x = a sin **ωt**

$\sin^2$ **ωt** = x^2/a^2

Fromy = a cos 2 **ωt**

$$y = a(1-2\sin^2 \omega t) = a\left(1-2\frac{x^2}{a^2}\right)$$

For tiie plots see the plots of answer sheet of tlve problem book.

Q. 16. A particle of mass m is located in a unidimensional potential field where the potential energy of the particle depends on the coordinate x as U (x) = U_0 (1 — cos ax); U_0 and a are constants. Find the period of small oscillations that the particle performs about the equilibrium position.

Ans. As U (x) = U_0 (1 - cos ax)

So, $F_x = -\frac{dU}{dx} = -U_0 a \sin a x$ (1)

or, $F_x = -U_0 a a x$ (because for small angle of oscillations $\sin a x = a x$)

or, $F_x = -U_0 a^2 x$ (1)

But we know $F_x = -m \omega_0^2 x$, for small oscillation

Thus $\omega_0^2 = \frac{U_0 a^2}{m}$ or $\omega_0 = a \sqrt{\frac{U_0}{m}}$

Hence the sought time period

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{a} \sqrt{\frac{m}{U_0}} = 2\pi \sqrt{\frac{m}{a^2 U_0}}$$

Q. 17. Solve the foregoing problem if the potential energy has the form U (x) = a/x^2 — b/x, where a and b are positive constants.

Ans. $U(x) = \frac{a}{x^2} - \frac{b}{x}$

then the equilibrium position is $x = x_0$ when $U''(x_0) = 0$

or, $-\frac{2a}{x_0^3} + \frac{b}{x_0^2} = 0 \Rightarrow x_0 = \frac{2a}{b}$

Now write: $x = x_0 + y$

Then $U(x) = \frac{a}{x_0^2} - \frac{b}{x_0} + (x - x_0) U'(x_0) + \frac{1}{2}(x - x_0)^2 U''(x_0)$

But $U''(x_0) = \frac{6a}{x_0^4} - \frac{2b}{x_0^3} = (2a/b)^{-3}(3b - 2b) = b^4/8a^3$

So finally: $U(x) = U(x_0) + \frac{1}{2}\left(\frac{b^4}{8a^3}\right) y^2 + \ldots$

We neglect remaining terms for small oscillations and compare with the P.E. for a harmonic, oscillator:

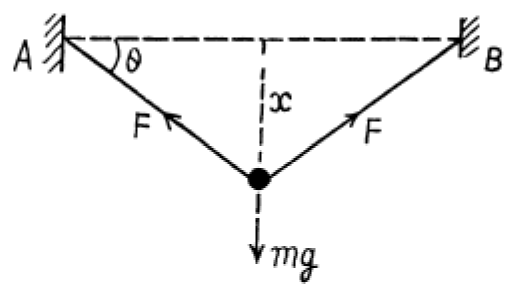
$$\frac{1}{2} m \omega^2 y^2 = \frac{1}{2}\left(\frac{b^4}{8a^3}\right) y^2, \text{ so } \omega = \frac{b^2}{\sqrt{8a^3 m}}$$

Thus $$T = 2\pi \frac{\sqrt{8 m a^3}}{b^2}$$

Note: Equilibrium position is generally a minimum of the potential energy. Then U' (x_o) - 0 , U'' (x_0) > 0 . The equilibrium position can in principle be a maxim um but then U'' (x_o) < 0 and the frequency of oscillations about this equilibrium position will be imaginary.
The answer given in the book is incorrect both numerically and dimensionally.

Q. 18. Find the period of small oscillations in a vertical plane performed by a ball of mass m = 40 g fixed at the middle of a horizontally stretched string l = 1.0 m in length. The tension of the string is assumed to be constant and equal to F = 10 N.

Ans. Let us locate and depict the forces acting on the ball at the position when it is at a distance x down from the unreformed position of the string.
At this position, the unbalanced downward force on the ball
= mg - 2.F sinθ



By Newton's law, $m\ddot{x} = m g - 2F \sin\theta$

$$= m g - 2F\theta \quad (\text{when } \theta \text{is small})$$

$$= m g - 2F\frac{x}{l/2} = m g - \frac{4F}{l}x$$

Thus $$\ddot{x} = g - \frac{4F}{m l}x = -\frac{4F}{m l}\left(x - \frac{m g l}{4F}\right)$$

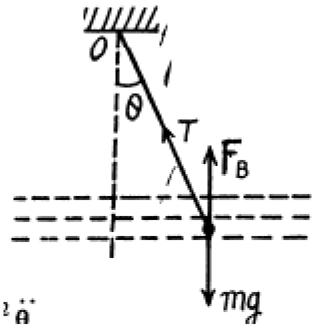
putting $x' = x - \frac{m g l}{T}$, we get

$$\ddot{x}' = -\frac{4T}{m l}x'$$

Thus $$T = \pi\sqrt{\frac{m l}{F}} = 0{\cdot}2\text{ s}$$

Q. 19. Determine the period of small oscillations of a mathematical pendulum, that is a ball suspended by a thread l = 20 cm in length, if it is located in a liquid whose density is η = 3.0 times less than that of the ball. The resistance of the liquid is to be neglected.

Ans. Let us depict the forces acting on the oscillating ball at an arbitrary angular position 0. (Fig.), relative to equilibrium position where F_B is the force of buoyancy. For the ball from the equation:



Nz - Iβ_z (where we have taken the positive sense of Z axis in the direction of angular velocity i.e. $\dot{\theta}$ of the ball and passes through the point of suspension of the pendulum O), we get :

$$-m\,g\,l\sin\theta + F_B\,l\sin\theta = m\,l^2\ddot{\theta} \qquad (1)$$

Using $m = \frac{4}{3}\pi r^3 \sigma,\ F_B = \frac{4}{3}\pi r^3 \rho$ and sin θ = θ for small θ, in Eqn (1), we get:

$$\ddot{\theta} = -\frac{g}{l}\left(1 - \frac{\rho}{\sigma}\right)\theta$$

Thus the sought time period

$$T = 2\pi \frac{1}{\sqrt{\frac{g}{l}\left(1 - \frac{\rho}{\sigma}\right)}} = 2\alpha\sqrt{\frac{l/g}{1 - \frac{1}{\eta}}}$$

Hence $T = 2\alpha\sqrt{\frac{\eta l}{g(\eta - 1)}} = 1{\cdot}1\text{s}$

Q. 20. A ball is suspended by a thread of length l at the point 0 on the wall, forming a small angle α with the vertical (Fig. 4.1). Then

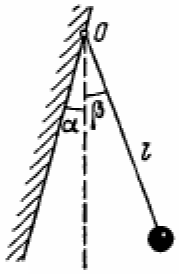


Fig. 4.1.

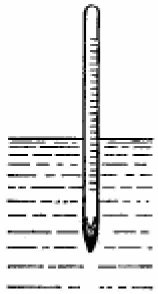


Fig. 4.2.

the thread with the ball was deviated through a small angle β(β > α) and set free. Assuming the collision of the ball against the wall to be perfectly elastic, find the oscillation period of such a pendulum.

Ans. Obviously for small p the ball execute part of S.H.M. Due to the perfectly elastic collision the velocity of ball simply reversed. As the ball is in S.H.M. (|θ| < α on the left) its motion law in differential from can be written as

$$\ddot{\theta} = -\frac{g}{l}\theta = -\omega_0^2\theta \qquad (1)$$

If we assume that the ball is released from the extreme position, θ = β at t = 0, the solution of differential equation would be taken in the form

$$\theta = \beta\cos\omega_0 t = \beta\cos\sqrt{\frac{g}{l}}\,t \qquad (2)$$

If t ' be the time taken by the ball to go from the extreme position θ = β to the wall i.e. θ = - α, then Eqn. (2) can be rewritten as

$$-\alpha = \beta\cos\sqrt{\frac{g}{l}}\,t'$$

or $$t' = \sqrt{\frac{l}{g}}\cos^{-1}\left(-\frac{\alpha}{\beta}\right) = \sqrt{\frac{l}{g}}\left(\pi - \cos^{-1}\frac{\alpha}{\beta}\right)$$

Thus the sought time $T = 2t' = 2\sqrt{\frac{l}{g}}\left(\pi - \cos^{-1}\frac{\alpha}{\beta}\right)$

$$= 2\sqrt{\frac{l}{g}}\left(\frac{\pi}{2} + \sin^{-1}\frac{\alpha}{\beta}\right), \quad [\text{because } \sin^{-1}x + \cos^{-1}x = \pi/2]$$

Mechanical Oscillations (Part -3)

Q.21. A pendulum clock is mounted in an elevator car which starts going up with a constant acceleration w, with w < g. At a height h the acceleration of the car reverses, its magnitude remaining constant. How soon after the start of the motion will the clock show the right time again?

Ans. Let the downward acceleration of the elevator car has continued for time t then the sought time

$t = \sqrt{\frac{2h}{w}} + t'$, where obviously $\sqrt{\frac{2h}{w}}$ is the time of upward acceleration of the elevator.

One should i/ote that if the point of suspension of a mathematical pendulum moves with an acceleration $\vec{w}$, then the time period of the pendulum becomes

$2\pi\sqrt{\frac{l}{|\vec{g}-\vec{w}|}}$ (see 4.30)

In this problem the time period of the pendulum while it is moving upward with acceleration w becomes

$2\pi\sqrt{\frac{l}{g+w}}$ and its time period while the elevator moves downward with the same magnitude of acceleration becomes

$2\pi\sqrt{\frac{l}{g-w}}$

As the time of upward acceleration equals $\sqrt{\frac{2h}{w}}$, the total number of oscillations during this time equals

$$\frac{\sqrt{2h/w}}{2\pi\sqrt{l/(g+w)}}$$

Thus the indicated time $= \frac{\sqrt{2h/w}}{2\pi\sqrt{l/(g+w)}} \cdot 2\pi\sqrt{l/g} = \sqrt{2h/w}\sqrt{(g+w)/g}$

Similarly the indicated time for the time interval t ' we demand that

$$\sqrt{2h/w}\ \sqrt{(g+w)/g} + t'\sqrt{(g-w)/g} = \sqrt{2h/w} + t'$$

or, $$t' = \sqrt{2h/w}\ \frac{\sqrt{g+w} - \sqrt{g}}{\sqrt{g} - \sqrt{g-w}}$$

Hence the sought time

$$t = \sqrt{\frac{2h}{w}} + t' = \sqrt{\frac{2h}{w}}\ \frac{\sqrt{g+w} - \sqrt{g-w}}{\sqrt{g} - \sqrt{g-w}}$$

$$= \sqrt{\frac{2h}{w}}\ \frac{\sqrt{1+\beta} - \sqrt{1-\beta}}{1 - \sqrt{1-\beta}}, \text{ where } \beta = w/g$$

Q.22. Calculate the period of small oscillations of a hydrometer (Fig. 4.2) which was slightly pushed down in the vertical direction. The mass of the hydrometer is m = 50 g, the radius of its tube is r = 3.2 mm, the density of the liquid is ρ = 1.00 g/cm^3. The resistance of the liquid is assumed to be negligible.

Ans. If the hydrometer were in equlibrium or floating, its weight will be balanced by the buoyancy force acting on it by the fluid. During its small oscillation, let us locate the hydrometer when it is at a vertically downward distance x from its equilibrium position. Obviously the net unbalanced force on the hydrometer is the excess buoyancy force directed upward and equals $\pi r^2 x \rho g$. Hence for the hydrometer.

$$m\ddot{x} = -\pi r^2 \rho g x$$

Or $$\ddot{x} = -\frac{\pi r^2 \rho g}{m} x$$

Hence the sought time period

$$T = 2\alpha\sqrt{\frac{m}{\pi r^2 \rho g}} = 2{\cdot}5\ \text{s}.$$

Q.23. A non-deformed spring whose ends are fixed has a stiffness x = 13 N/m. A small body of mass m = 25 g is attached at the point removed from one of the ends by η = 1/3 of the spring's length. Neglecting the mass of the spring, find the period of small longitudinal oscillations of the body. The force of gravity is assumed to be absent.

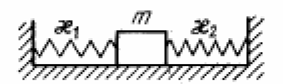


Fig. 4.3.

Ans. At first let us calculate the stiffness k_1 and k_2 of both the parts of the spring. If we subject the original spring of stiffness k having the natural length l_0 (say), under the deforming forces F - F (say) to elongate the spring by the amount x y then
F = k x (1)

Therefore the elongation per unit length of the spring is x / l_0 . Now let us subject one o f the parts of the spring of natural length η\ l_0 under the same deforming forces F - F. Then the elongation of the spring will be

$$\frac{x}{l_0}\eta l_0 = \eta x$$

Thus $F = \kappa_1 (\eta x)$ (2)

Hence from Eqns (1) and (2)

$$\kappa = \eta \kappa_1 \quad \text{or} \quad \kappa_1 = \kappa/\eta$$

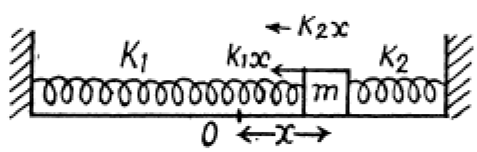
$$\kappa_2 = \frac{\kappa}{1-\eta}$$

$$F_x = m w_x$$

$$-\kappa_1 x - \kappa_2 x = m\ddot{x}$$

$$\text{or,} \quad -\left(\frac{\kappa}{\eta} + \frac{\kappa}{1-\eta}\right) x = m\ddot{x}$$

$$\text{Thus} \quad \ddot{x} = -\frac{\kappa}{m}\frac{1}{\eta(m)} x$$



Hence the sought time period

$$T = 2\pi\sqrt{\eta(1-\eta)m/\kappa} = 0{\cdot}13\ \text{s}$$

Q.24. Determine the period of small longitudinal oscillations of a body with mass m in the system shown in Fig. 4.3. The stiffness values of the springs are x_1 and x_2. The friction and the masses of the springs are negligible.

Ans. Similar to the Soln of 4.23, the net unbalanced force on the block m when it is at a small horizontal distance x from the equilibrium position becomes $(k_1 + k_2)x$.

From $F_x = m w_x$ for the block :

$$-(\kappa_1 + \kappa_2)x = m\ddot{x}$$

Thus $$\ddot{x} = -\left(\frac{\kappa_1 + \kappa_2}{m}\right)x$$

Hence the sought time period $T = 2\pi\sqrt{\dfrac{m}{\kappa_1 + \kappa_2}}$

Alternate : Let us set the block m in motion to perform small oscillation. Let us locate the block when it is at a distance x from its equilibrium position.

As the spring force is restoring conservative force and deformation of both the springs are same, so from the conservation of mechanical energy of oscillation of the spring-block system :

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\kappa_1 x^2 + \frac{1}{2}\kappa_2 x^2 = \text{Constant}$$

Differentiating with respect to time

$$\frac{1}{2}m\,2\dot{x}\ddot{x} + \frac{1}{2}(\kappa_1 + \kappa_2)\,2x\dot{x} = 0$$

or, $$\ddot{x} = -\frac{(\kappa_1 + \kappa_2)}{m}x$$

Hence the sought time period $T = 2\pi\sqrt{\dfrac{m}{\kappa_1 + \kappa_2}}$

Q.25. Find the period of small vertical oscillations of a body with mass m in the system illustrated in Fig. 4.4. The stiffness values of the springs are x_1 and x_2, their masses are negligible.

Ans. During the vertical oscillation let us locate the block at a vertical down distance x from its equilibrium position. At this moment if x_1 and x_2 are the additional or further elongation of the upper & lower springs relative to the equilibrium position, then the net unbalanced force on the block will be k_2x_2 directed in upward direction. Hence

$-\kappa_2 x_2 = m\ddot{x}$ (1)

We also have $x = x_1 + x_2$ (2)

As the springs are massless and initially the net force on the spring is also zero so for the spring

$k_1x_1 = k_2x_2$ (3)

Solving the Eqns (1), (2) and (3) simultaneously, we get

$$-\frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2} x = m \ddot{x}$$

Thus $\ddot{x} = -\frac{(\kappa_1 \kappa_2 / \kappa_1 + \kappa_2)}{m} x$

Hence the sought time period $T = 2\pi \sqrt{m \frac{(\kappa_1 \kappa_2)}{\kappa_1 \kappa_2}}$

Q.26. A small body of mass in is fixed to the middle of a stretched string of length 2/. In the equilibrium position the string tension is equal to T_0. Find the angular frequency of small oscillations of the body in the transverse direction. The mass of the string is negligible, the gravitational field is absent.

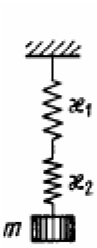


Fig. 4.4.

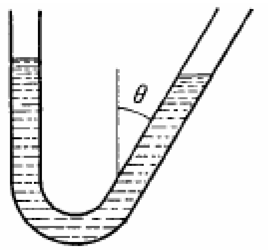


Fig. 4.5.

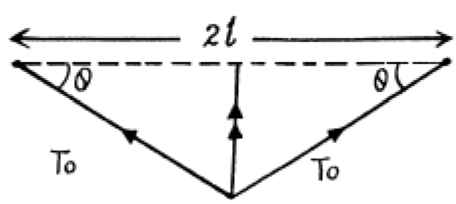
Ans. The force F, acting on the weight deflected from the position of equilibrium is 2 T_0 sin θ.

Since the angle θ is small, the net restoring force, $F = 2 T_0 \frac{x}{l}$

or, $F = kx$, where $k = \frac{2 T_0}{l}$

So, by using the formula,

$\omega_0 = \sqrt{\frac{k}{m}}$, $\omega_0 = \sqrt{\frac{2 T_0}{m l}}$

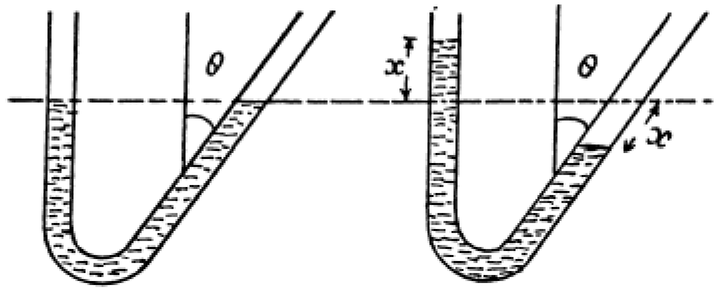


4.27. Determine the period of oscillations of mercury of mass m = 200 g poured into a bent tube (Fig. 4.5) whose right arm forms an angle θ = 30° with the vertical. The cross-sectional area of the tube is S = 0.50 cm^2. The viscosity of mercury is to be neglected.

Ans. If the mercury rises m the left arm by x it m ust fall by a slanting length equal to x in the other arm.

Total pressure difference in the two arms will then be ρg x + pgx cosθ = ρgx (1 + cosθ)
This will give rise to a restoring force - ρgSx (1+ cosθ)

This must equal mass times acceleration which can be obtained from work energy principle.



The K.E. of the mercury in the tube is clearly : $\frac{1}{2} m \dot{x}^2$

So mass times acceleration must be : $m\ddot{x}$

Hence $$m\ddot{x} + \rho g S(1+\cos\theta)x = 0$$

This is S.H.M. with a time period

$$T = 2\pi\sqrt{\frac{m}{\rho g S(1+\cos\theta)}}.$$

Q. 28. A uniform rod is placed on two spinning wheels as shown in Fig. 4.6. The axes of the wheels are separated by a distance l = 20 cm, the coefficient of friction between the rod and the wheels is k = 0.18. Demonstrate that in this case the rod performs harmonic oscillations. Find the period of these oscillations.

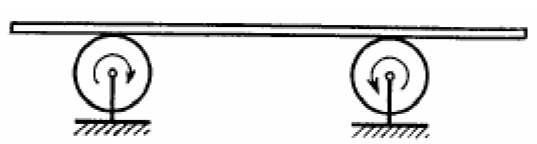
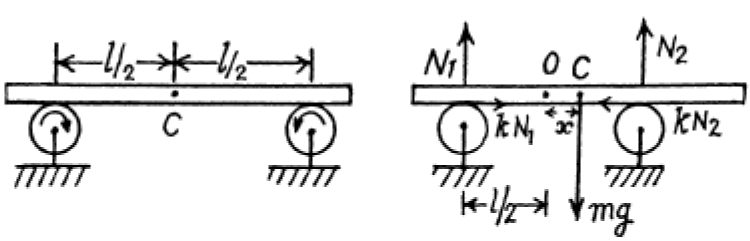


Fig. 4.6.

Ans. In the equilibrium position the C.M. of the rod lies nid way between the two rotating wheels. Let us displace the rod horizontally by some small distance and then release it Let us depict the forces acting on the rod when its C.M. is at distance x from its equilibrium position (Fig.). Since there is no net vertical force acting on the rod, Newton’s second law gives:



$$N_1 + N_2 = mg \qquad (1)$$

For the translational motion of the rod from the Eqn. : $F_x = m\, w_{cx}$

$$kN_1 - kN_2 = m\ddot{x} \qquad \textbf{(2)}$$

As the rod experiences no net torque about an axis perpendicular to the plane of the Fig. through the C.M. of the rod.

$$N_1\left(\frac{l+x}{2}\right) = N_2\left(\frac{l-x}{2}\right) \qquad (3)$$

Solving Eqns. (1), (2) and (3) simultaneously we get

$$\ddot{x} = -k\frac{2g}{l}x$$

Hence the sought time period

$$T = 2\pi\sqrt{\frac{l}{2kg}} = \pi\sqrt{\frac{2l}{kg}} = 1\cdot5\text{ s}$$

Q.29. Imagine a shaft going all the way through the Earth from pole to pole along its rotation axis. Assuming the Earth to be a homogeneous ball and neglecting the air drag, find:
(a) the equation of motion of a body falling down into the shaft;
(b) how long does it take the body to reach the other end of the shaft;
(c) the velocity of the body at the Earth's centre.

Ans. (a) The only force acting on the ball is the gravitational force $\vec{F}$ of magnitude $\gamma\frac{4}{3}\pi\rho m r,$ where y is the gravitational constant ρ, the density of the Earth and r is the distance of the body from the centre of the Earth.

But, $g = \gamma\frac{4\pi}{3}\rho R,$ so the expression for $\vec{F}$ can be written as,

$\vec{F} = -mg\frac{\vec{r}}{R},$ here R is the radius of the Earth and the equation of motion in projection form has the form, or, $m\ddot{x} + \frac{mg}{R}x = 0$

(b) The equation, obtained above has the form of an equation of S.H.M. hawing the time period,

$$T = 2\pi\sqrt{\frac{R}{g}},$$

Hence the body will reach the other end of the shaft in the time,

$$t = \frac{T}{2} = \pi\sqrt{\frac{R}{g}} = 42 \text{ min}.$$

(c) From the conditions of the speed of the body at the centre of the Earth will be maximum, having the magnitude,

$$v = R\omega = R\sqrt{g/R} = \sqrt{gR} = 7{\cdot}9 \text{ km/s}.$$

Q.30. Find the period of small oscillations of a mathematical pendulum of length 1 if its point of suspension 0 moves relative to the Earth's surface in an arbitrary direction with a constant acceleration w (Fig. 4.7). Calculate that period if l = 21 cm, ω = g/2, and the angle between the vectors w and g equals β = 120°.

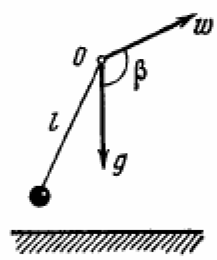


Fig. 4.7.

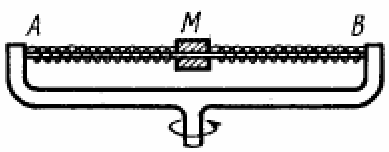


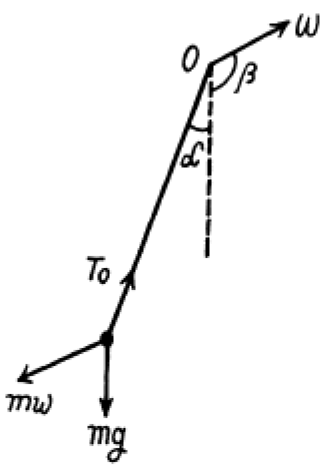
Fig. 4.8.

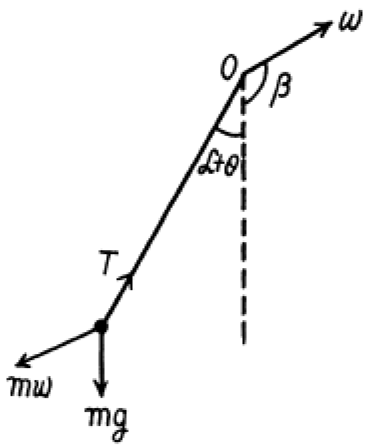
Ans. In the frame of point of suspension the mathematical pendulum of mass m (say) will oscillate. In this frame, the body m will experience the inertial force $m(-\vec{w})$ in addition to the real forces during its oscillations. Therefore in equilibrium position m is deviated by some angle say a. In equilibrium position

$$T_0\cos\alpha = mg + mw\cos(\pi-\beta) \quad \text{and} \quad T_0\sin\alpha = mw\sin(\pi-\beta)$$

So, from these two Eqns

$$\left.\begin{aligned} \tan\alpha &= \frac{g - w\cos\beta}{w\sin\beta} \\ \text{and } \cos\alpha &= \sqrt{\frac{m^2w^2\sin^2\beta + (mg - mw\cos\beta)^2}{mg - mw\cos\beta}} \end{aligned}\right] \qquad (1)$$





Let us displace the bob m from its equilibrium position by some small angle and then release it Now locate die ball at an angular position (α + θ) from vertical as shown in the figure.

From the Eqn :

$$N_{0z} = I\beta_z$$

$$-m\,g\,l\sin(\alpha+\theta) - m\,w\cos(\pi-\beta)\,l\sin(\alpha+\theta) + m\,w\sin(\pi-\beta)\,l\cos(\alpha+\theta) = m\,l^2\,\ddot{\theta}$$

$$\text{or,} - g(\sin\alpha\cos\theta + \cos\alpha\sin\theta) - w\cos(\pi-\beta)(\sin\alpha\cos\theta+\cos\alpha\sin\theta) + w\sin\beta\,(\cos\alpha\cos\theta - \sin\alpha\sin\theta) = l\ddot{\theta}$$

But for small θ, $\sin\theta \approx \theta$ $\cos\theta \approx 1$

So, $$-g(\sin\alpha + \cos\alpha\,\theta) - w\cos(\pi-\beta)(\sin\alpha+\cos\alpha\,\theta) + w\sin\beta\,(\cos\alpha - \sin\alpha\,\theta) = l\ddot{\theta}$$

or, $$(\tan\alpha + \theta)(w\cos\beta - g) + w\sin\beta\,(1 - \tan\alpha\,\theta) = \frac{l}{\cos\alpha}\ddot{\theta} \qquad (2)$$

Solving Eqns (1) and (2) simultaneously we get

$$-(g^2 - 2wg\cos\beta + w^2)\,\theta = l\sqrt{g^2 + w^2 - 2wg\cos\beta}\;\ddot{\theta}$$

Thus $$\ddot{\theta} = -\frac{|\vec{g} - \vec{w}|}{l}\theta$$

Hence the sought time period $$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{|\vec{g}-\vec{w}|}}$$

Q.31. In the arrangement shown in Fig. 4.8 the sleeve M of mass m = 0.20 kg is fixed between two identical springs whose combined stiffness is equal to x = 20 N/m. The sleeve can slide without friction over a horizontal bar AB. The arrangement rotates with a constant angular velocity ω = 4.4 rad/s about a vertical

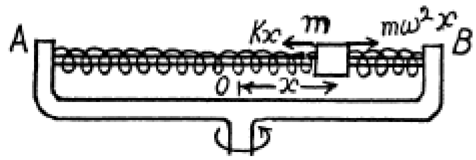
axis passing through the middle of the bar. Find the period of small oscillations of the sleeve. At what values of ω will there be no oscillations of the sleeve?

Ans. Obviously the sleeve performs small oscillations in the frame of rotating rod. In the rod's frame let us depict the forces acting on the sleeve along the length of the rod while the sleeve is at a small distance x towards right from its equilibrium position. The free body diagram of block does not contain Coriolis force, because it is perpendicular to the length of the rod.

From F_x - mw_x for the sleeve in the frame of rod

$$-\kappa x + m\omega^2 x = m\ddot{x}$$

or, $$\ddot{x} = -\left(\frac{\kappa}{m} - \omega^2\right)x \qquad (1)$$



Thus the sought time period

$$T = \frac{2\pi}{\sqrt{\frac{\kappa}{m} - \omega^2}} = 0\cdot 7\,s$$

It is obvious from Eqn (1) that the sleeve will not perform small oscillations if

$\omega \geq \sqrt{\frac{\kappa}{m}}$ 10 rod/s.

Q.32. A plank with a bar placed on it performs horizontal harmonic oscillations with amplitude a = 10 cm. Find the coefficient of friction between the bar and the plank if the former starts sliding along the plank when the amplitude of oscillation of the plank becomes less than T = 1.0 s.

Ans. When the bar is about to start sliding along the plank, it experiences the maximum restoring force which is being provided by the limiting friction, Thus

$$kN = m\omega_0^2 a \qquad \text{or,} \quad kmg = m\omega_0^2 a$$

or, $$k = \frac{\omega_0^2 a}{g} = \frac{a}{g}\left(\frac{2\pi}{T}\right)^2 = 4\,s.$$

Q.33. Find the time dependence of the angle of deviation of a mathematical pendulum 80 cm in length if at the initial moment the pendulum

(a) was deviated through the angle 3.0° and then set free without push;

(b) was in the equilibrium position and its lower end was imparted the horizontal velocity 0.22 m/s;

(c) was deviated through the angle 3.0° and its lower end was imparted the velocity 0.22 m/s directed toward the equilibrium position.

Ans. The natural angular frequency of a mathematical pendulum equals

$$\omega_0 = \sqrt{g/l}$$

(a) We have the solution of S.H.M. equation in angular form :

$$\theta = \theta_m \cos(\omega_0 t + \alpha)$$

If at the initial moment i.e. at t = 0 , $\theta = \theta_m$ than $\alpha = 0$.
Thus the above equation takes the form

$$\theta = \theta_m \cos \omega_0 t$$

$$= \theta_m \cos\sqrt{\frac{g}{l}}\, t = 3^\circ \cos\sqrt{\frac{9{\cdot}8}{0{\cdot}8}}\, t$$

Thus $\quad 0 = 3^\circ \cos 3.5\, t$

(b) The S.H.M. equation in angular form :

$$\theta = \theta_m \sin(\omega_0 t + \alpha)$$

If at the initial moment t = 0 , $\theta = 0$, then $\alpha = 0$ .Then the above equation takes the form

$$\theta = \theta_m \sin \omega_0 t$$

Let v_0 be the velocity of the lower end of pendulum at $\theta = 0$, then from conserved of mechanical energy of oscillaton

$$E_{mean} = E_{extreme} \quad \text{or} \quad T_{mean} = U_{extrem}$$

or, $$\frac{1}{2} m v_0^2 = m g l (1 - \cos\theta_m)$$

Thus

$$\theta_m = \cos^{-1}\left(1 - \frac{v_0^2}{2gl}\right) = \cos^{-1}\left[1 - \frac{(0\cdot 22)^2}{2 \times 9\cdot 8 \times 0\cdot 8}\right] = 4\cdot 5^c$$

Thus the sought equation becomes

$$\theta = \theta_m \sin \omega_0 t = 4{\cdot}5^\circ \sin 3{\cdot}5\, t$$

(c) Let θ_O and v_0 be the angular deviation and linear velocity at t = 0.
As the mechanical energy of oscillation of the mathematical pendulum is conservation

$$\frac{1}{2} m v_0^2 + m g l (1 - \cos\theta_0) = m g l (1 - \cos\theta_m)$$

or, $$\frac{v_0^2}{2} = g l (\cos\theta_0 - \cos\theta_m)$$

Thus $$\theta_m = \cos^{-1}\left\{\cos\theta_0 - \frac{v_0^2}{2 g l}\right\} = \cos^{-1}\left\{\cos 3^\circ - \frac{(0{\cdot}22)^2}{2 \times 9{\cdot}8 \times 0{\cdot}8}\right\} = 5{\cdot}4^\circ$$

Then from θ = 5.4° sin (3.5 t + α), we see that $\sin\alpha = \frac{3}{5.4}$ and $\cos\alpha < 0$ because the velocity is directed towards the centre. Thus

$\alpha = \frac{\pi}{2} + 1.0$ radians and we get the answer.

Q.34. A body A of mass m_1 = 1.00 kg and a body B of mass m_2 = 4.10 kg are interconnected by a spring as shown in Fig. 4.9. The body A performs free vertical harmonic oscillations with amplitude a = 1.6 cm and frequency = 25 s^{-1}. Neglecting the mass of the spring, find the maximum and minimum values of force that this system exerts on the bearing surface.

Ans. While the body A is at its upper extreme position, the spring is obviously elongated by the amount

$$\left(a - \frac{m_1 g}{\kappa}\right).$$

If we indicate y-axis in vertically downward direction, Newton's second law of motion in projection form i.e. $F_y = m w_y$ for body A gives :

$$m_1 g + \kappa\left(a - \frac{m_1 g}{\kappa}\right) = m_1 \omega^2 a \quad \text{or,} \quad \kappa\left(a - \frac{m_1 g}{\kappa}\right) = m_1(\omega^2 a - g) \qquad (1)$$

(Because at any extreme position the magnitude of acceleration of an oscillating body equals $\omega^2 a$ and is restoring in nature.)
If N be the normal force exerted by the floor on the body B, while the body A is at its upper extreme position, from Newton's second law for body B

$$N + \kappa\left(a - \frac{m_1 g}{\kappa}\right) = m_2 g$$

or, $$N = m_2 g - \kappa\left(a - \frac{m_1 g}{\kappa}\right) = m_2 g - m_1(\omega^2 a - g) \text{ (using Eqn. 1)}$$

Hence $$N = (m_1 + m_2) g - m_1 \omega^2 a$$

When the body A is at its lower extreme position, the spring is compressed by the distance

$$\left(a + \frac{m_1 g}{\kappa}\right).$$

From Newton's second law in projection fonn i.e. $F_y = mw_y$ for body A at this state:

$$m_1 g - \kappa\left(a + \frac{m_1 g}{\kappa}\right) = m_1(-\omega^2 a) \quad \text{or,} \quad \kappa\left(a + \frac{m_1 g}{\kappa}\right) = m_1(g + \omega^2 a) \qquad (3)$$

In this case if N' be the normal force exerted by the floor on the body B, From Newton's second law

for body B we get: $N' = \kappa\left(a + \frac{m_1 g}{\kappa}\right) + m_2 g = m_1(g + \omega^2 a) + m_2 g$ (using Eqn. 3)

Hence $N' = (m_1 + m_2) g + m_1 \omega^2 a$

From Newton's third law the magnitude of sought forces are N' and N, respectively.

Q.35. A plank with a body of mass m placed on it starts moving straight up according to the law y = a (1 — cos ωt), where y is the displacement from the initial position, $\omega = 11\ s^{-1}$.

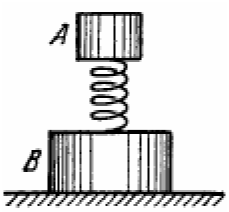


Fig. 4.9.

Find: (a) the time dependence of the force that the body exerts on the plank if a = 4.0 cm; plot this dependence;
(b) the minimum amplitude of oscillation of the plank at which the body starts falling behind the plank;
(c) the amplitude of oscillation of the plank at which the body springs up to a height h = 50 cm relative to the initial position (at the moment t = 0).

Ans. (a) For the block from Newton's second law in projection form $F_y = mw_y$

$N - mg = m\ddot{y}$ (1)

But from $y = a(1 - \cos\omega t)$

We get $\ddot{y} = \omega^2 a \cos\omega t$ (2)

From Eqns (1) and (2)

$$N = mg\left(1 + \frac{\omega^2 a}{g}\cos\omega t\right) \qquad (3)$$

From Newtons's third law the force by which the body m exerts on the block is directed vertically downward and equls

$$N = mg\left(1 + \frac{\omega^2 a}{g}\cos\omega t\right)$$

(b) When the body m starts, falling behind the plank or looking contact, N = 0, (because the normal reaction is the contact force). Thus from Eqn. (3)

$$mg\left(1 + \frac{\omega^2 a}{g}\cos\omega t\right) = 0 \quad \text{for some } t.$$

Hence

$$a_{min} = g/\omega^2 = 8 \text{ cm.}$$

(c) We observe that the motion takes place about the mean position y = a. A t the initial instant y = 0. As shown in (b) the normal reaction vanishes at a height (g/ω^2) above the position of equilibrium and the body flies off as a free body. The speed of the body at a distance (g/ω^2) from the equilibrium position is $\omega\sqrt{a^2 - (g/\omega^2)^2}$, so that the condition of the problem gives

$$\frac{[\omega\sqrt{a^2 - (g/\omega^2)^2}\,]^2}{2g} + \frac{g}{\omega^2} + a = h$$

Hence solving the resulting quadratic equation and taking the positive roof,

$$a = -\frac{g}{\omega^2} + \sqrt{\frac{2hg}{\omega^2}} = 20 \text{ cm.}$$

Mechanical Oscillations (Part -4)

Q.36. A body of mass in was suspended by a non-stretched spring, and then set free without push. The stiffness of the spring is x. Neglecting the mass of the spring, find:
(a) the law of motion y (t) , where y is the displacement of the body from the equilibrium position;
(b) the maximum and minimum tensions of the spring in the process of motion.

Ans. (a) Let y (t) = displacement of the body from the end of the unstarched position of the spring (not the equilibrium position). Then

$$m\ddot{y} = -\kappa y + mg$$

This equation has the solution of the form

$$y = A + B\cos(\omega t + \alpha)$$

if $-m\omega^2 B\cos(\omega t + \alpha) = -\kappa[A + B\cos(\omega t + \alpha)] + mg$

Then $\omega^2 = \frac{\kappa}{m}$ and $A = \frac{mg}{\kappa}$

we have $y = 0$ and $\dot{y} = 0$ at $t = 0$. So

$$-\omega B\sin\alpha = 0$$

$$A + B\cos\alpha = 0$$

Since $B > 0$ and $A > 0$ we must have $\alpha = \pi$

$$B = A = \frac{mg}{\kappa}$$

And $y = \frac{mg}{\kappa}(1 - \cos\omega t)$

(b) Tension in the spring is

$$T = \kappa y = mg(1 - \cos\omega t)$$

so $T_{max} = 2mg$, $T_{min} = 0$

Q.37. A particle of mass in moves due to the force F = — αmr, where α is a positive constant, r is the radius vector of the particle relative to the origin of coordinates. Find the trajectory of its motion if at the initial moment r = r_oi and the velocity v = v_oj, where i and j are the unit vectors of the x and y axes.

Ans. In accordance with the problem

$$\vec{F} = -\alpha m \vec{r}$$

So, $$m(\ddot{x}\,\vec{i} + \ddot{y}\,\vec{j}) = -\alpha m(x\vec{i} + y\vec{j})$$

Thus $$\ddot{x} = -\alpha x \text{ and } \ddot{y} = -\alpha y$$

Hence the solution of the differential equation

$$\ddot{x} = -\alpha x \quad \text{becomes } x = a\cos(\omega_0 t + \delta), \text{ where } \omega_0^2 = \alpha \qquad (1)$$

So, $$\dot{x} = -a\omega_0 \sin(\omega_0 t + \alpha) \qquad (2)$$

From the initial conditions of the problem, $v_x = 0$ and $x = r_0$ at $t = 0$

So from Eqn. (2) a - 0, and Eqn takes the form

$x = r_0 \cos \omega_0 t \quad$ so, $\cos \omega_0 t = x/r_0$ (3)

One of the solution of the other differential Eqn $\ddot{y} = -\alpha y$, becomes

$y = a' \sin(\omega_0 t + \delta')$, where $\omega_0^2 = \alpha$ (4)

From the initial condition, y = 0 at t = 0, so δ' = 0 and Eqn (4) becomes

y = a' sin ω_0t(S)

Differentiating w.r.t time we get

$\dot{y} = a' \omega_0 \cos \omega_0 t$ (6)

But from the initial condition of the problem, $\dot{y} = v_0$ at $t = 0$,

$v_0 = a'\omega_0 \quad$ or, $\quad a' = v_0/\omega_0$

Using it in Eqn (5), we get

$y = \frac{v_0}{\omega_0}\sin \omega_0 t \quad$ or $\quad \sin \omega_0 t = \frac{\omega_0 y}{v_0}$ (7)

Squaring and adding Eqns (3) and (7) we get:

$$\sin^2 \omega_0 t + \cos^2 \omega_0 t = \frac{\omega_0^2 y^2}{v_0^2} + \frac{x^2}{r_0^2}$$

Or, $$\left(\frac{x}{r_0}\right)^2 + \alpha\left(\frac{y}{v_0}\right)^2 = 1 \left(\text{as} \quad \alpha, = \omega_0^2 \right)$$

Q.38. A body of mass m is suspended from a spring fixed to the ceiling of an elevator car. The stiffness of the spring is x. At the moment t = 0 the car starts going up with an acceleration ω. Neglecting the mass of the spring, find the law of motion y (t) of the body rela- tive to the elevator car if y (0) = 0 and $\dot{y}$ (0) = 0. Consider the following two cases:
(a) ω = const;
(b) ω = αt, where α is a constant.

Ans. (a) As the elevator car is a translating non-inertial frame, therefore the body m will experience an inertial force m w directed downward in addition to the real forces in the elevator's frame. From the Newton's second law in projection form $F_y = mw_y$ for the body in the frame of elevator car:

$$-\kappa\left(\frac{mg}{\kappa}+y\right)+mg+mw = m\ddot{y} \qquad \text{(A)}$$

(Because the initial elongation in the spring is m g/K)

so, $$m\ddot{y} = -\kappa y + mw = -\kappa\left(y-\frac{mw}{\kappa}\right)$$

or, $$\frac{d^2}{dt^2}\left(y-\frac{mw}{\kappa}\right) = -\frac{\kappa}{m}\left(y-\frac{mw}{\kappa}\right) \qquad (1)$$

Eqn. (1) shows that the motion of the body m is S.H.M. and its solution becomes

$$y-\frac{mw}{\kappa} = a\sin\left(\sqrt{\frac{\kappa}{m}}\,t+\alpha\right) \qquad (2)$$

Differentiating Eqn (2) w.r.t time

$$\dot{y} = a\sqrt{\frac{\kappa}{m}}\cos\left(\sqrt{\frac{\kappa}{m}}\,t+\alpha\right) \qquad (3)$$

Using the initial condition y (0) = 0 in Eqn (2), we get:

$$a\sin\alpha = -\frac{mw}{\kappa}$$

And using the other initial condition $\dot{y}(0) = 0$ in Eqn (3) we get $$a\sqrt{\frac{\kappa}{m}}\cos\alpha = 0$$

Thus $$\alpha = -\alpha/2 \text{ and } a = \frac{m\,w}{\kappa}$$

Hence using these values in Eqn (2), we get

$$y = \frac{m\,w}{\kappa}\left(1-\cos\sqrt{\frac{\kappa}{m}}\,t\right)$$

(b) Proceed up to Eqn.(l). The solution of this differential Eqn be of the form

$$y-\frac{m\,w}{\kappa} = a\sin\left(\sqrt{\frac{\kappa}{m}}\,t+\delta\right)$$

or, $$y-\frac{\alpha\,t}{\kappa/m} = a\sin\left(\sqrt{\frac{\kappa}{m}}\,t+\delta\right)$$

or, $$y-\frac{\alpha\,t}{\omega_0^2} = a\sin(\omega_0\,t+\delta)\ \left(\text{wher}\ \ \omega_0 = \sqrt{\frac{\kappa}{m}}\right) \tag{4}$$

From the initial condition that at $t = 0$, $y(0) = 0$, so $0 = a\sin\delta$ or $\delta = 0$

Thus Eqn.(4) takes the from : $y-\frac{\alpha\,t}{\omega_0^2} = a\sin\omega_0\,t$ (5)

Differentiating Eqn. (5) we get : $\dot{y}-\frac{\alpha}{\omega_0^2} = a\,\omega_0\cos\omega\,t$ (6)

But from the other initial condition $\dot{y}(0) = 0$ at $t = 0$.

So, from Eqn.(6) $$-\frac{\alpha}{\omega_0^2} = a\,\omega_0 \quad \text{or} \quad a = -\alpha/\omega_0^3$$

Putting the value of a in Eqn. (5), we get the sought $y(t)$. i.e.

$$y-\frac{\alpha\,t}{\omega_0^2} = -\frac{\alpha}{\omega_0^3}\sin\omega_0\,t \quad \text{or} \quad y = \frac{\alpha}{\omega_0^3}(\omega_0\,t-\sin\omega_0\,t)$$

Q.39. A body of mass m = 0.50 kg is suspended from a rubber cord with elasticity coefficient k = 50 Is17m. Find the maximum distance over which the body can be pulled down for the body's oscillations to remain harmonic. What is the energy of oscillation in this case?

Ans. There is an important difference between a rubber cord or steel coire and a spring. A spring can be pulled or compressed and in both cases, obey’s Hooke’s law. But a

rubber cord becomes loose when one tries to compress it and does not then obey Hooke's law. Thus if we suspend a 'body by a rubber cord it stretches by a distance m g/k in reaching the equilibrium configuration. If we further stretch it by a distance A hit will execute harmonic oscillations when released if A h ≤ m g / k because only in this case will the cord remain taut and obey Hooke's law.
Thus A $h_{max} \leq$ m g / k
The energy of oscillation in this case is

$$\frac{1}{2}\kappa(\Delta h_{max})^2 = \frac{1}{2}\frac{m^2g^2}{\kappa}$$

Q.40. A body of mass m fell from a height h onto the pan of a spring balance (Fig. 4.10). The masses of the pan and the spring are negligible, the stiffness of the latter is x. Having stuck to the pan, the body starts performing harmonic oscillations in the vertical direction. Find the amplitude and the energy of these oscillations.

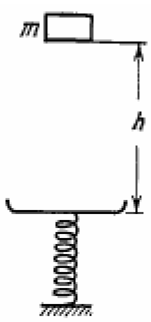


Fig. 4.10.

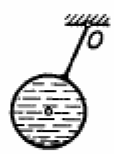


Fig. 4.11.

Ans. As the pan is of negligible mass, there is no loss of kinetic energy even though the collision is inelastic. The mechanical energy of the body m in the field generated by the joint action of both the gravity force and the elastic force is conserved i.e. ΔE = 0. During the motion of the body m from the initial to the final (position of maximum compression of the spring) position ΔT = 0, and therefore

$$\Delta U = \Delta U_{gr} + \Delta U_{sp} = 0$$

Or $$-mg(h+x) + \frac{1}{2}\kappa x^2 = 0$$

On solving the quadratic equation:

$$x = \frac{mg}{\kappa} \pm \sqrt{\frac{m^2g^2}{\kappa^2} + \frac{2mgh}{\kappa}}$$

As minus sign is not acceptable

$$x = \frac{mg}{\kappa} + \sqrt{\frac{m^2g^2}{\kappa^2} + \frac{2mgh}{\kappa}}$$

If the body m were at rest on the spring, the corresponding position of m will be its equilibrium position and at this position the resultant force on the body m will be zero. Therefore the equilibrium compression Δx (say) due to the body m will be given by

$$\kappa \Delta x = mg \quad \text{or} \quad \Delta x = mg/\kappa$$

Therefore seperation between the equilibrium position and one of the extreme position i.e. the sought amplitude

$$a = x - \Delta x = \sqrt{\frac{m^2g^2}{\kappa^2} + \frac{2mgh}{\kappa}}$$

The mechanical eneigy of oscillation which is con served equals E = $U_{extreme}$, because at the extreme position kinetic eneigy becomes zero.
Although the weight of body mis a conservative force, it is not restoring in this problem , hence U extreme is only concerned with the spring force . Therefore

$$E = U_{extreme} = \frac{1}{2}\kappa a^2 = mgh + \frac{m^2g^2}{2\kappa}$$

Q.41. Solve the foregoing problem for the case of the pan having a mass M. Find the oscillation amplitude in this case.

Ans. Unlike the previous (4.40) problem the kinetic energy of body m decreases due to the perfectly inelastic collision with the pan. Obviously the body m comes to strike the pan with velocity $v_0 = \sqrt{2gh}$. If v be the common velocity of the "body m + pan " system due to the collision then from the conservation of linear momentum

$$m v_0 = (M + m) v$$

$$v = \frac{m v_0}{(M+m)} = \frac{m\sqrt{2gh}}{(M+m)}$$

At the moment the body m strikes the pan, the spring is com pressed due to the weight of "he pan by the amount M g/ k . If l be the further com pression of the spring due to the velocity acquired by the "pan - body m " system , then from the conservation of m mechanical energy of the said system in the Geld generated by the joint action of both the gravity and spring forces

$$\frac{1}{2}(M+m)v^2+(M+m)gl=\frac{1}{2}\kappa\left(\frac{Mg}{\kappa}+l\right)^2-\frac{1}{2}\kappa\left(\frac{Mg}{\kappa}\right)^2$$

or, $$\frac{1}{2}(M+m)\frac{m^2 2gh}{(M+m)}+(M+m)gl=\frac{1}{2}\kappa\left(\frac{Mg}{\kappa}\right)^2+\frac{1}{2}\kappa l^2+Mgl-\frac{1}{2}\kappa\left(\frac{Mg}{\kappa}\right)^2 \qquad \text{(Using 1)}$$

or, $$\frac{1}{2}\kappa l^2-mgl-\frac{m^2 gh}{(m+M)}=0$$

Thus $$l=\frac{mg\pm\sqrt{m^2g^2+\frac{2\kappa ghm^2}{M+m}}}{\kappa}$$

As minus sign is not acceptable

$$l=\frac{mg}{\kappa}+\frac{1}{\kappa}\sqrt{m^2g^2+\frac{2\kappa m^2 gh}{(M+m)}}$$

If the oscillating "pan + body m" system were at rest it correspond to their equilbijium position i.e. the spring were com pressed by $\frac{(M+m)g}{\kappa}$ therefore the am plitude of oscillation

$$a=l-\frac{mg}{\kappa}=\frac{mg}{\kappa}\sqrt{1+\frac{2h\kappa}{mg}}$$

The mechanical energy of oscillation which is only conserved with the restoring forces

becomes $E=U_{extreme}=\frac{1}{2}\kappa a^2$ (Because spring force is the only restoring force not the weight of the body)

Alternately $E=T_{mean}=\frac{1}{2}(M+m)a^2\omega^2$

thus $E=\frac{1}{2}(M+m)a^2\left(\frac{\kappa}{M+m}\right)=\frac{1}{2}\kappa a^2$

Q.42. A particle of mass m moves in the plane xy due to the force varying with velocity as $\mathbf{F}=a(\dot{y}\mathbf{i}-\dot{x}\mathbf{j})$, where a is a positive constant, i and j are the unit vectors of the x and y axes. At the initial moment t = 0 the particle was located at the point x = y = 0 and possessed a velocity v_0 directed along the unit vector j. Find the law of motion x (t) , y (t) of the particle, and also the equation of its trajectory.

Ans.

We have $\vec{F} = a(\dot{y}\vec{i} - \dot{x}\vec{j})$

or, $$m(\ddot{x}\vec{i} + \ddot{y}\vec{j}) = a(\dot{y}\vec{i} - \dot{x}\vec{j})$$

So, $$m\ddot{x} = a\dot{y} \text{ and } m\ddot{y} = -a\dot{x} \quad (1)$$

From the initial condition, at $t = 0$, $\dot{x} = 0$ and $y = 0$

So, integrating Eqn, $m\ddot{x} = a\dot{y}$

we get $$m\dot{x} = ay \text{ or } \dot{x} = \frac{a}{m}y \quad (2)$$

Using Eqn (2) in the Eqn $m\ddot{y} = -a\dot{x}$, we get

$$m\ddot{y} = -\frac{a^2}{m}y \text{ or } \ddot{y} = -\left(\frac{a}{m}\right)^2 y \quad (3)$$

one of the solution of differential Eqn (3) is y = A sin (ω_0t + α) , where ω_0 = a/ m .
As at r = 0 ,y » 0 , s o the solution takes the form y = A sin ω_0t

On differentiating w.r.t time $\dot{y} = A\,\omega_0 \cos \omega_0 t$

From the initial condition of the problem, at $t = 0$, $\dot{y} = v_0$

So $v_0 = A\omega_0$ or $A = v_0/\omega_0$

Thus $y = (v_0/\omega_0)\sin \omega_0 t$ (4)

Thus from (2) $\dot{x} = v_0 \sin \omega_0 t$ so integrating

$$x = B - \frac{v_0}{\omega_0}\cos \omega_0 t \quad (5)$$

On using $x = 0$ at $t = 0$, $B = \frac{v_0}{\omega_0}$

Hence finally $$x = \frac{v_0}{\omega_0}(1 - \cos \omega_0 t) \quad (6)$$

Hence from Eqns (4) and (6) we get

$$[x - (v_0/\omega_0)]^2 + y^2 = (v_0/\omega_0)^2$$

Which is the equation of a circle of radius (V_o/ω_0) with the centre at the point
$x_0 = v_0/\omega_0$, $y_0 = 0$

Q.43. A pendulum is constructed as a light thin-walled sphere of radius R filled up with water and suspended at the point 0 from a light rigid rod (Fig. 4.1.1). The distance between the point 0 and the centre of the sphere is equal to l. How many

times will the small oscillations of such a pendulum change after the water freezes? The viscosity of water and the change of its volume on freezing are to be neglected.

Ans. If water has frozen, the system consisting of the light rod and the frozen water in the hollow sphere constitute a compound (physical) pendulum to a very good approximation because we can take the whole system to be rigid. For such systems the time period is given by

$$T_1 = 2\pi\sqrt{\frac{l}{g}}\sqrt{1+\frac{k^2}{l^2}} \qquad \text{where} \quad k^2 = \frac{2}{5}R^2 \text{ is the radius of gyration of the sphere.}$$

The situation is different when water is unfrozen. When dissipative forces (viscosity) are neglected, we are dealing with ideal fluids. Such fluids instantaneously respond to (unbalanced) internal stresses. Suppose the sphere with liquid water actually executes small rigid oscillations.

Then the portion of the fluid above the centre of the sphere will have a greater acceleration than the portion below the centre because the linear acceleration of any element is in this case, equal to angular acceleration of the element multiplied by the distance of the element from the centre of suspension (Recall that we are considering small oscillations).

Then, as is obvious in a frame moving with the centre of mass, there will appear an unbalanced couple (not negated by any pseudo forces) which will cause the fluid to move rotationally so as to destroy differences in acceleration. Thus for this case of ideal fluids the pendulum must move in. such a way that the elements of the fluid all undergo the same acceleration. This implies that we have a simple (mathematical) pendulum with the time period:

$$T_0 = 2\pi\sqrt{\frac{l}{g}}$$

Thus $$T_1 = T_0\sqrt{1+\frac{2}{5}\left(\frac{R}{l}\right)^2}$$

(One expects that a liquid with very small viscosity will have a time period close T_0 while one with high viscosity will have a time period closer to 7l.)

Q.44. Find the frequency of small oscillations of a thin uniform vertical rod of mass m and length l hinged at the point 0 (Fig. 4.12). The combined stiffness of the springs is equal to x. The mass of the springs is negligible.

Ans. Let us locate the rod at the position when it makes an angle 0 from the vertical. In this problem both, the gravity and spring forces are restoring conservative forces, thus from the conservation of mechanical energy of oscillation of the oscillating system:

$$\frac{1}{2}\frac{m\,l^2}{3}2\,\dot{\theta}\,\ddot{\theta}+\frac{m\,g\,l}{2}\sin\theta\,\dot{\theta}+\frac{1}{2}\kappa\,l^2\,2\,\theta\,\dot{\theta}=0 \quad = \text{constant}$$

Differentiating w.r.t. time, we get :

$$\frac{1}{2}\frac{m\,l^2}{3}2\,\dot{\theta}\,\ddot{\theta}+\frac{m\,g\,l}{2}\sin\theta\,\dot{\theta}+\frac{1}{2}\kappa\,l^2\,2\,\theta\,\dot{\theta}=0$$

$$\ddot{\theta}=-\frac{3\,g}{2\,l}\left(1+\frac{\kappa\,l}{m\,g}\right)\theta$$

Hence,
$$\omega_0=\sqrt{\frac{3\,g}{2\,l}\left(1+\frac{\kappa\,l}{mg}\right)}.$$

Q.45. A uniform rod of mass m = 1.5 kg suspended by two identical threads l = 90 cm in length (Fig. 4.13) was turned through a small angle about the vertical axis passing through its middle point C. The threads deviated in the process through an angle α = 5.0°. Then the rod was released to start performing small oscillations. Find:
(a) the oscillation period;
(b) the rod's oscillation energy.

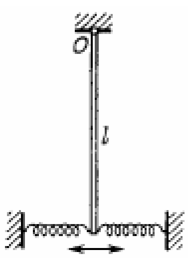


Fig. 4.12.

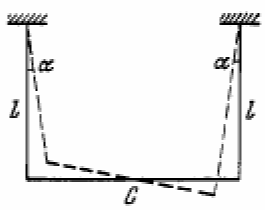


Fig. 4.13.

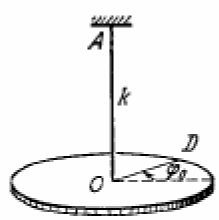
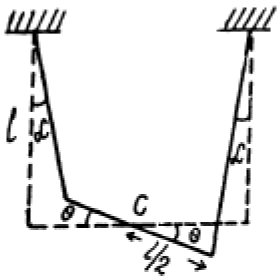


Fig. 4.14.

Ans. (a) Let us locate the system when the threads are deviated through an angle α' < α , during the oscillations of the system (Fig.). From the conservation of mechanical energy of the system:

$$\frac{1}{2}\frac{mL^2}{12}\dot{\theta}^2+m\,g\,l\,(1-\cos\alpha')=\text{constant} \qquad (1)$$

Where L is the length of the rod, 0 is the angular deviation of the rod from its equilibrium position i.e. θ = 0.



Differentiating Eqn. (1) w.r.t. time

$$\frac{1}{2}\frac{mL^2}{12}2\dot{\theta}\ddot{\theta} + m g l \sin\alpha' \dot{\alpha}' = 0$$

So, $$\frac{L^2}{12}\dot{\theta}\ddot{\theta} + g l \alpha' \dot{\alpha}' = 0 \text{ (for small } \alpha', \ \sin\alpha' \approx \alpha' \text{)} \qquad (2)$$

But from the Fig.

$$\frac{L}{2}\theta = l\alpha' \text{ or } \alpha' = \frac{L}{2l}\theta$$

So, $$\dot{\alpha}' = \frac{L}{2l}\dot{\theta}$$

Putting these values of α' and $\frac{d\alpha'}{dt}$ in Eqn. (2) we get

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{l}\theta$$

Thus the sought time period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{3g}}$$

(b) The sought oscillation energy

$$E = U_{extreme} = m g l (1 - \cos\alpha) = m g l 2 \sin^2\frac{\alpha}{2}$$

$$= m g l 2 \frac{\alpha^2}{4} = \frac{m g l \alpha^2}{2} \text{ (because for small angle } \sin\theta \approx \theta \text{)}$$

Q.46. An arrangement illustrated in Fig. 4.14 consists of a horizontal uniform disc D of mass m and radius R and a thin rod AO whose torsional coefficient is equal to k. Find the amplitude and the energy of small torsional oscillations if at the initial

moment the disc was deviated through an angle φ_0 from the equilibrium position and then imparted an angular velocity $\dot{\varphi}_0$.

Ans.

The K.E. of the disc is $\frac{1}{2} I \dot{\varphi}^2 = \frac{1}{2}\left(\frac{mR^2}{2}\right)\dot{\varphi}^2 = \frac{1}{4} m R^2 \dot{\varphi}^2$

The torsional potential energy is $\frac{1}{2} k \varphi^2$. Thus the total energy is :

$$\frac{1}{4} m R^2 \dot{\varphi}^2 + \frac{1}{2} k \varphi^2 = \frac{1}{4} m R^2 \dot{\varphi}_0^2 + \frac{1}{2} k \varphi_0^2$$

By definition of the amplitude φ_m, $\dot{\varphi} = 0$ when $\varphi = \varphi_m$. Thus total energy is

$$\frac{1}{2} k \varphi_m^2 = \frac{1}{4} m R^2 \dot{\varphi}_0^2 + \frac{1}{2} k \varphi_0^2$$

Or, $$\varphi_m = \varphi_0 \sqrt{1 + \frac{mR^2}{2k} \frac{\dot{\varphi}_0^2}{\varphi_0^2}}$$

Q.47. A uniform rod of mass m and length l performs, small oscillations about the horizontal axis passing through its upper end. Find the mean kinetic energy of the rod averaged over one oscillation period if at the initial moment it was deflected from the vertical by an angle θ_0 and then imparted an angular velocity $\dot{\theta}_0$.

Ans. Moment of inertia of the rod equals $\frac{ml^2}{3}$ about its one end and perpendicular to its length

Thus rotational kinetic energy of the rod = $\frac{1}{2}\left(\frac{ml^2}{3}\right)\dot{\theta}^2 = \frac{ml^2}{6}\dot{\theta}^2$

when the rod is displaced by an angle 6 its C.G. goes up by a distance

$$\frac{l}{2}(1 - \cos\theta) = \frac{l\theta^2}{4} \text{ for small } \theta.$$

Thus the P.E. becomes: $mg\frac{l\theta^2}{4}$

As the mechanical energy of oscillation of the rod is conserved.

$$\frac{1}{2}\left(\frac{ml^2}{3}\right)\dot{\theta}^2 + \frac{1}{2}\left(\frac{mgl}{2}\right)\theta^2 = \text{Constant}$$

on differentiating w x t time and for the simplifies we get: $\ddot{\theta} = -\frac{3g}{2l}\theta$ for small θ.
we see that the angular frequency ω is

$$= \sqrt{3g/2l}$$

we write the general solution of the angular oscillation as:

$$\theta = A\cos\omega t + B\sin\omega t$$

But $\theta = \theta_0$ at $t = 0$, so $A = \theta_0$

and $\dot{\theta} = \dot{\theta}_0$ at $t = 0$, so

$$B = \dot{\theta}_0/\omega$$

Thus $\theta = \theta_0\cos\omega t + \frac{\dot{\theta}_0}{\omega}\sin\omega t$

Thus the ICE. of the rod

$$T = \frac{ml^2}{6}\dot{\theta}^2 = [-\omega\theta_0\sin\omega t + \dot{\theta}_0\cos\omega t]^2$$

$$= \frac{ml^2}{6}[\dot{\theta}_0^2\cos^2\omega t + \omega^2\theta_0^2\sin^2\omega t - 2\omega\theta_0\dot{\theta}_0\sin\omega t\cos\omega t]$$

On averaging over one time period the last term vanishes and

$< \sin^2\omega t > = < \cos^2\omega t > = 1/2$. Thus

$$< T > = \frac{1}{12}ml^2\dot{\theta}_0^2 + \frac{1}{8}mgl^2\theta_0^2 \quad (\text{where } \omega^2 = 3g/2l)$$

Q.48. A physical pendulum is positioned so that its centre of gravity is above the suspension point. From that position the pendulum started moving toward the stable equilibrium and passed it with an angular velocity ω. Neglecting the friction find the period of small oscillations of the pendulum.

Ans. Let l = distance between the C.G. (C) of the pendulum and, its point of suspension 0 Originally the pendulum is in inverted position and its C.G. is above O. When it falls to the normal (stable) position of equilibrium its C.G. has fallen by a distance 2l. In the equilibrium

position the total energy is equal to K.E. = $\frac{1}{2}I\omega^2$ and we have from energy conservation :

$$\frac{1}{2}I\omega^2 = mg2l \quad \text{or} \quad I = \frac{4mgl}{\omega^2}$$

Angular frequency of oscillation for a physical pendulum is given by $\omega_0^2 = m g l/I$

$$T = 2\pi \sqrt{\frac{I}{m g l}} = 2\pi \sqrt{\frac{4 m g l/\omega^2}{m g l}} = \frac{4\pi}{3}$$

Q.49. A physical pendulum performs small oscillations about the horizontal axis with frequency $\omega_1 = 15.0\ s^{-1}$. When a small body of mass m = 50 g is fixed to the pendulum at a distance l = 20 cm below the axis, the oscillation frequency becomes equal to $\omega_2 = 10.0\ s^{-1}$. Find the moment• of inertia of the pendulum relative to the oscillation axis.

Ans. Let, moment of inertia of the pendulum, about the axis, concerned is 7, then writing N_z - $I\beta_D$ for the pendulum,

$$- m g x \sin a\, \theta = I \dot{\theta} \quad \text{or,} \quad \dot{\theta} = - \frac{m g x}{I} \theta \qquad \text{(For small } \theta \text{)}$$

Which is the required equation for S.H.M. So, the frequency of oscillation,

$$\omega_1 = \sqrt{\frac{M g x}{I}} \quad \text{or,} \quad x = \frac{I}{M g} \sqrt{\omega_1^2} \qquad (1)$$

Now, when the mass m is attached to the pendulum, at a distance / below the oscillating axis,

$$- M g x \sin \theta' - m g l \sin \theta' = (I + m l^2) \frac{d^2 \theta'}{d t^2}$$

$$\text{or,} \qquad - \frac{g (M x + m l)}{(I + m l^2)} \theta' = \frac{d^2 \theta}{d t^2}, \text{ (For small } \theta \text{)}$$

which is again the equation of S.H.M., So, the new frequency,

$$\omega_2 = \sqrt{\frac{g (M x + m l)}{(I + m l^2)}} \qquad (2)$$

Solving Eqns. (1) and (2),

$$\omega_2 = \sqrt{\frac{g\,((I/g)\,\omega_1^2 + m\,l)}{(I + m\,l^2)}}$$

or, $$\omega_2^2 = \frac{I\,\omega_1^2 + m\,g\,l}{I + m\,l^2}$$

or, $$I\,(\omega_2^2 - \omega_1^2) = m\,g\,l - m\,\omega_2^2\,l^2$$

and hence, $$I = m\,l^2\,(\omega_2^2 - g/l)\Big/(\omega_1^2 - \omega_2^2) = 0{\cdot}8\ \mathrm{g}\cdot\mathrm{m}^2$$

Q.50. Two physical pendulums perform small oscillations about the same horizontal axis with frequencies ω_1 and ω_2. Their moments of inertia relative to the given axis are equal to I_1 and I_2 respectively. In a state of stable equilibrium the pendulums were fastened rigidly together. What will be the frequency of small oscillations of the compound pendulum?

Ans.

$$I_1\,\ddot{\theta} = -\omega_1^2 I_1\,\theta + G$$
$$I_2\,\ddot{\theta} = -\omega_2^2 I_2\,\theta - G$$

where $\pm$ G is the to rque of mutual in teractions. We have written the restoring forces on each pendulum in the absence of the other as $-\omega_1^2 I_1\,\theta$ and $-\omega_2^2 I_2\,\theta$ respectively . Then

$$\ddot{\theta} = -\frac{I_1\,\omega_1^2 + I_2\,\omega_2^2}{I_1 + I_2}\,\theta = -\omega^2\,\theta$$

Hence $$\omega = \sqrt{\frac{I_1\,\omega_1^2 + I_2\,\omega_2^2}{I_1 + I_2}}$$

Mechanical Oscillations (Part -5)

Q.51. A uniform rod of length 1 performs small oscillations about the horizontal axis 00' perpendicular to the rod and passing through one of its points. Find the distance between the centre of inertia of the rod and the axis 00' at which the oscillation period is the shortest. What is it equal to?

Ans. Let us locate the rod when it is at small angular position 6 relative to its equilibrium position. If a be the sought distance, the n from the conservation of mechanical energy of oscillation

$$m g a (1 - \cos\theta) + \frac{1}{2} I_{OO'} (\dot{\theta})^2 = \text{constant}$$

Differentiating w.r.t= time we get:

$$m g a \sin\theta\, \dot{\theta} + \frac{1}{2} I_{OO'} 2 \dot{\theta} \ddot{\theta} = 0$$

But $I_{OO'} = \frac{m l^2}{12} + m a^2$ and for small θ, $\sin\theta = \theta$, we get

$$\ddot{\theta} = -\left(\frac{g a}{\frac{l^2}{12} + a^2}\right)\theta$$

Hence the time period of one full osscillation becomes

$$T = 2\pi \sqrt{\frac{\frac{l^2}{12} + a^2}{a g}} \quad \text{or} \quad T^2 = \frac{4\pi^2}{g}\left(\frac{l^2}{12 a} + a\right)$$

For $T_{\min}$, obviously $\frac{d}{da}\left(\frac{l^2}{12 a} + a\right) = 0$

So, $-\frac{l^2}{12 a^2} + 1 = 0$ or $a = \frac{l}{2\sqrt{3}}$

Hence $T_{\min} = 2\pi \sqrt{\frac{l}{g\sqrt{3}}}$

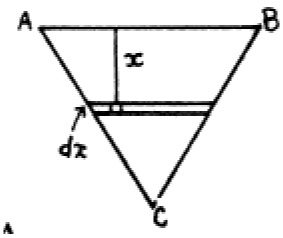
Q.52. A thin uniform plate shaped as an equilateral triangle with a height h performs small oscillations about the horizontal axis coinciding with one of its sides. Find the oscillation period and the reduced length of the given pendulum.

Ans. Consider the moment of inertia of the triangular plate about AB.

$$I = \iint x^2\, dm = \iint x^2 \rho\, dx\, dy$$

$$= \int_0^h x^2 \rho\, dx \frac{h-x}{h} \cdot \frac{2h}{\sqrt{3}} = \int_0^h x^2 \frac{2\rho}{\sqrt{3}} (h-x)\, dx$$

$$= \frac{2\rho}{\sqrt{3}} \left(\frac{h^4}{3} - \frac{h^4}{4} \right) = \frac{\rho h^4}{6\sqrt{3}} = \frac{m h^2}{6}$$



On using the area of the triangle $\Delta ABC = \frac{h^2}{\sqrt{3}}$ and $m = \rho \Delta$.

Thus K.E. $= \frac{1}{2} \frac{m h^2}{6} \dot{\theta}^2$

P.E. $= m g \frac{h}{3} (1 - \cos\theta) = \frac{1}{2} m g h \frac{\theta^2}{3}$

Here 6 is the angle that the instantaneous plane of the plate makes with the equilibrium position which is vertical. (The plate rotates as a rigid body)

Thus $$E = \frac{1}{2} \frac{m h^2}{6} \dot{\theta}^2 + \frac{1}{2} \frac{m g h}{3} \theta^2$$

Hence $$\omega^2 = \frac{2g}{h} = \frac{m g h}{3} \Big/ \frac{m h^2}{6}$$

$$T = 2\pi \sqrt{\frac{h}{2g}} = \pi \sqrt{\frac{2h}{g}}. \quad \text{and} \quad l_{reduced} = h/2.$$

Q.53. A smooth horizontal disc rotates about the vertical axis 0 (Fig. 4.15) with a constant angular velocity ω. A thin uniform rod AB of length 1 performs small oscillations about the vertical axis A fixed to the disc at a distance a from the axis of the disc. Find the frequency ω_0 of these oscillations.

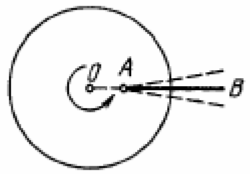


Fig. 4.15.

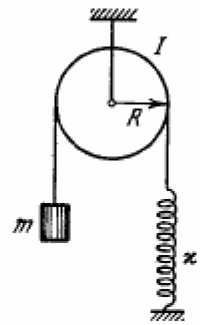
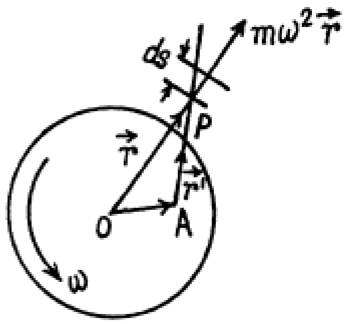


Fig. 4.16.

Ans. Let us go to the rotating frame, in which the disc is stationary. In this frame die rod is subjected to Coriolis and centrifugal forces, $\mathbf{F}_{cor}$ and $\mathbf{F}_{cf}$, where



(as OA is constant)

Where r is the position of an elemental mass of the rod (Fig.) with respect to point O (disc's centre) and

$$\mathbf{F}_{cor} = \int 2\,dm\,(\mathbf{v}' \times \vec{\omega}_0) \text{ and } \mathbf{F}_{cf} = \int dm\,\omega_0^2\,\mathbf{r},$$

$$\mathbf{v}' = \frac{d\mathbf{r}'}{dt}$$

As $$\mathbf{r} = \mathbf{OP} = \mathbf{OA} + \mathbf{AP}$$

So, $$\frac{d\mathbf{r}}{dt} = \frac{d(\mathbf{AP})}{dt} = \mathbf{v}'$$

As the rod is vibrating transversely, so v' is directed perpendicular to the length of the rod.

Hence $2\,dm\,(\mathbf{v}' \times \vec{\omega})$ for each elemental mass o f the rod is directed along PA. Therefore

the net torque of Coriolis about A becomes zero. The not torque of centrifugal force about point A:

Now, $\vec{\tau}_{cf(A)} = \int \mathbf{AP} \times \, dm\,\omega_0^2 \mathbf{r} = \int \mathbf{AP} \times \left(\frac{m}{l}\right) ds\,\omega_0^2 (\mathbf{OA} + \mathbf{AP})$

$$= \int \mathbf{AP} \times \left(\frac{m}{l} ds\right) \omega_0^2 \mathbf{OA} = \int \frac{m}{l} ds\,\omega_0^2 s\,a \sin\theta\,(-\mathbf{k})$$

$$= \frac{m}{l} \omega_0^2 a \sin\theta\,(-\mathbf{k}) \int_0^l s\,ds = m\,\omega_0^2 a \frac{l}{2} \sin\theta\,(-\mathbf{k})$$

So, $\tau_{cf(Z)} = \vec{\tau}_{cf(A)} \cdot \mathbf{k} = -m\,\omega_0^2 a \frac{l}{2} \sin\theta$

According to the equation of rotational dynamics : $\tau_{A(Z)} = I_A \alpha_Z$

or, $-m\,\omega_0^2 a \frac{l}{2} \sin\theta = \frac{m l^2}{3} \ddot{\theta}$

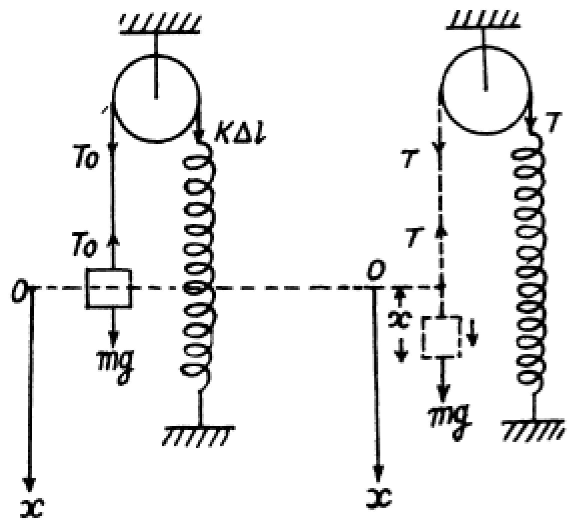
or, $\ddot{\theta} = -\frac{3}{2} \frac{\omega_0^2 a}{l} \sin\theta$

Thus, for small θ, $\ddot{\theta} = -\frac{3}{2} \frac{\omega^2 a}{2l} \theta$

This implies that the frequency ω_0 of oscillation is $\omega_0 = \sqrt{\frac{3\,\omega^2 a}{2l}}$

Q.54. Find the frequency of small oscillations of the arrangement illustrated in Fig. 4.16. The radius of the pulley is R, its moment of inertia relative to the rotation axis is I, the mass of the body is m, and the spring stiffness is x. The mass of the thread and the spring is negligible, the thread does not slide over the pulley, there is no friction in the axis of the pulley.

Ans. The physical system consists with a pulley and the block. Choosing an intertial frame, let us direct the x-axis as shown in the figure.



Initially the system is in equilibrium position. Now from the condition of translation equilibrium for the block

T_o = ms (1)
Similarly for the rotational equilibrium of the pulley

k Δ/R = T_0R
or. T_0 = k Δ l (2)

from Eqns. (1) and (2) $\Delta l = \frac{mg}{\kappa}$ (3)

Now let us disturb the equilibrium of the system no matter in which way to analyse its motion. At an arbitrary position shown in the figure, from Newton's second law of motion for the block

$$F_x = m w_x$$
$$mg - T = mw = m\ddot{x} \qquad (4)$$

Similarly for the pulley

$$N_z = I \beta_z$$
$$TR - \kappa(\Delta l + x)R = I\ddot{\theta} \qquad (5)$$

But $$w = \beta R \quad \text{or,} \quad \ddot{x} = R\ddot{\theta} \qquad (6)$$

from (5) and (6) $$TR - \kappa(\Delta l + x)R = \frac{I}{R}\ddot{x} \qquad (7)$$

Solving (4) and (7) using the initial condition of the problem

$$-\kappa R x = \left(m R + \frac{I}{R}\right) \ddot{x}$$

or, $$\ddot{x} = -\left(\frac{\kappa}{m + \frac{I}{R^2}}\right) x$$

Hence the sought time period, $T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{m + I/R^2}{\kappa}}$

Note: we may solve this problem by using die conservation of mechanical energy also

Q.55. A uniform cylindrical pulley of mass M and radius R can freely rotate about the horizontal axis 0 (Fig. 4.17). The free end of

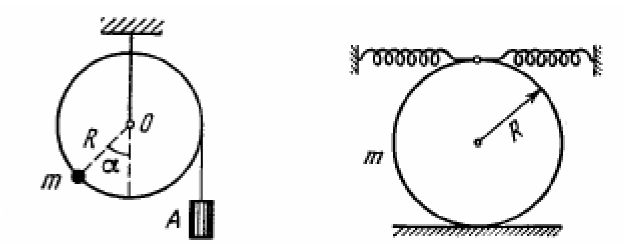


Fig. 4.17. Fig. 4.18.

a thread tightly wound on the pulley carries a deadweight A. At a certain angle α it counterbalances a point mass in fixed at the rim of the pulley. Find the frequency of small oscillations of the arrangement.

Ans. At the equlibrium position, $N_{oz} = 0$ (Net torque about 0)

So, $m_A g R - m g R \sin\alpha = 0$ or $m_A = m \sin\alpha$ (1)

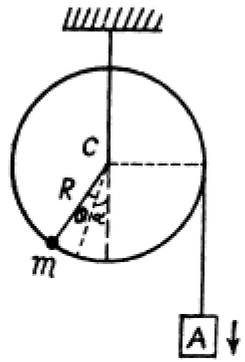
From the equation of rotational dynamics of a solid body about the stationary axis (say 2 -axis) of rotation i.e. from $N_z = I\beta_z$

when the pulley is rotated by the small angular displacement 0 in clockwise sense relative to the equilibrium position (Fig.), we get :

$$m_A g R - m g R \sin(\alpha + \theta) = \left[\frac{MR^2}{2} + mR^2 + m_A R^2\right]\ddot{\theta}$$

Using Eqn. (1)

$$m g \sin\alpha - m g(\sin\alpha\cos\theta + \cos\alpha\sin\theta) = \left\{\frac{MR + 2m(1+\sin\alpha)R}{2}\right\}\ddot{\theta}$$



But for small θ, we may write cos θ = 1 and sin θ = θ

Thus we have

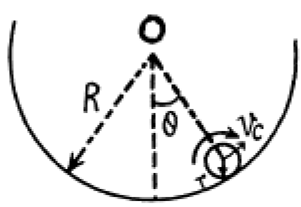
$$m g \sin\alpha - m g(\sin\alpha + \cos\alpha\,\theta) = \frac{\{MR + 2m(1+\sin\alpha)R\}}{2}\ddot{\theta}$$

Hence, $$\ddot{\theta} = -\frac{2 m g \cos\alpha}{[MR + 2m(1+\sin\alpha)R]}\theta$$

Hence the sought angular frequency $\omega_0 = \sqrt{\dfrac{2 m g \cos\alpha}{MR + 2mR(1+\sin\alpha)}}$

Q.56. A solid uniform cylinder of radius r rolls without sliding along the inside surface of a cylinder of radius R, performing small oscillations. Find their period.

Ans. Let us locate solid cylinder when it is displaced from its stable equilibrium position by the small angle θ during its oscillations (Fig.). If V_c be the instantaneous speed of the C.M. (C) of the solid cylinder which is in pure rolling, then its angular velocity about its own centre C is



$\omega = v_c/r$ (1)

Since C moves in a circle of radius (R - r), the speed of C at the same m oment can be written as

$$v_c = \dot{\theta}(R-r) \qquad (2)$$

Thus from Eqns (1) and (2)

$$\omega = \dot{\theta}\frac{(R-r)}{r} \qquad (3)$$

As the mechanical energy of oscillation of the solid cylinder is conserved, i.e. E = T + U = constant

$$\text{So,} \quad \frac{1}{2}m v_c^2 + \frac{1}{2}I_c\omega^2 + m g(R-r)(1-\cos\theta) = \text{constant}$$

(Where m is the mass of solid cylinder and I_c is the moment of inertia of the solid cylinder about an axis passing through its C .M. (C) and perpendicular to the plane of Fig. of solid cylinder)

$$\text{or,} \quad \frac{1}{2}m\omega^2 r^2 + \frac{1}{2}\frac{m r^2}{2}\omega^2 + m g(R-r)(1-\cos\theta) = \text{constant} \quad (\text{using Eqn (1) and } I_c = m r^2/2)$$

$$\frac{3}{4}r^2(\dot{\theta})^2\frac{(R-r)^2}{r^2} + g(R-r)(1-\cos\theta) = \text{constant, (using Eqn. 3)}$$

Differentiating w.r.t time

$$\frac{3}{4}(R-r)2\dot{\theta}\ddot{\theta} + g\sin\theta\,\dot{\theta} = 0$$

$$\text{So,} \quad \ddot{\theta} = -\frac{2g}{3(R-r)}\theta, \text{ (because for small } \theta, \sin\theta \approx \theta\text{)}$$

$$\text{Thus} \quad \omega_0 = \sqrt{\frac{2g}{3(R-r)}}$$

Hence the sought time period $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{3(R-r)}{2g}}$

Q.57. A solid uniform cylinder of plass m performs small oscillations due to the action of two springs whose combined stiffness is equal to x (Fig. 4.18). Find the period of these oscillations in the absence of sliding.

Ans. Let k_1 and k_2 be the spring constant o f left and right sides springs. As the rolling o f th< solid cylinder is pure its lowest point becomes the instantaneous centre of rotation. If θ b(the small angular displacement of its upper most point relative to its equilibrium position, the deformation o f each spring becomes (IR Q) . Since the mechanical energy o f oscillation of the solid cylinder is conserved, E = T+ U = constant

i.e. $\frac{1}{2}I_p(\dot{\theta})^2+\frac{1}{2}\kappa_1(2R\theta)^2+\frac{1}{2}\kappa_2(2R\theta)^2 = \text{constant}$

Differentiating w.r.t time

$$\frac{1}{2}I_p 2\dot{\theta}\ddot{\theta}+\frac{1}{2}(\kappa_1+\kappa_2)4R^2 2\theta\dot{\theta} = 0$$

or,

$$\left(\frac{mR^2}{2}+mR^2\right)\ddot{\theta}+4R^2\kappa\theta = 0$$

$$\left(\text{Because } I_P = I_C+mR^2 = \frac{mR^2}{2}+mR^2\right)$$

Hence

$$\ddot{\theta} = -\frac{8}{3}\frac{\kappa}{m}\theta$$

Thus $\omega_0 = \frac{8\kappa}{3m}$ and sought time period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{3m}{8\kappa}} = \pi\sqrt{\frac{3m}{2\kappa}}$$

Q.58. Two cubes with masses m1 a nd m2 were interconnected by a weightless spring of stiffness x and placed on a smooth horizontal surface. Then the cubes were drawn closer to each other and released simultaneously. Find the natural oscillation frequency of the system.

Ans. In the C.M. frame (which is rigidly attached with the centre of mass of the two cubes) the cubes oscillates. We know that the kinetic energy of two body system

equals $\frac{1}{2}\mu v_{rel}^2$, where p. is the reduced mass and is the modulus of velocity of any one

body particle relative to other. From the conservation of mechanical energy of

oscillation :

$$\frac{1}{2}\kappa x^2+\frac{1}{2}\mu\left\{\frac{d}{dt}(l_0+x)\right\}^2 = \text{constant}$$

Here l_0 is the natural length of the spring.

Differenting the above equation w.r.t time, we get:

$$\frac{1}{2}\kappa\, 2x\dot{x}+\frac{1}{2}\mu\, 2\dot{x}\ddot{x} = 0 \left[\text{becomes } \frac{d(l_0+x)}{dt} = \dot{x}\right]$$

$$\text{Thus } \ddot{x} = -\frac{\kappa}{\mu}x \quad \left(\text{where } \mu = \frac{m_1 m_2}{m_1+m_2}\right)$$

Hence the natural frequency of oscillation: $\omega_0 = \sqrt{\frac{\kappa}{\mu}}$ where $\mu = \frac{m_1 m_2}{m_1+m_2}$

Q.59. Two balls with masses m_1 = 1.0 kg and m_2 = 2.0 kg are slipped on a thin smooth horizontal rod (Fig. 4.19). The balls are

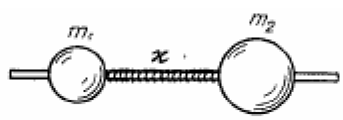


Fig. 4.19.

interconnected by a light spring of stiffness x = 24 N/m. The left-hand ball is imparted the initial velocity v_1 = 12 cm/s. Find:

(a) the oscillation frequency of the system in the process of motion;
(b) the energy and the amplitude of oscillations.

Ans. Suppose the balls 1 & 2 are displaced by x_1, x_2 from their initial position. Then the eneigy is

$$E = \frac{1}{2}m_1\dot{x}_1^2 + m_2\dot{x}^2 + \frac{1}{2}k(x_1-x_2)^2 = \frac{1}{2}m_1 v_1^2$$

Also total momentum is : $m_1\dot{x}_1 + m_2\dot{x}_2 = m_1 v_1$

Define $$X = \frac{m_1x_1 + m_1x_2}{m_1+m_2},\ x = x_1 - x_2$$

Then $$x_1 = X + \frac{m_2}{m_1+m_2}x,\quad x_2 = X - \frac{m_1}{m_1+m_2}x$$

$$E = \frac{1}{2}(m_1 + m_2)\dot{X}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{x}^2 + \frac{1}{2}kx^2$$

Hence $$\dot{X} = \frac{m_1 v_1}{m_1 + m_2}$$

So $$\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m_1 v_1^2 - \frac{1}{2}\frac{m_1^2 v_1^2}{m_1 + m_2} = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}v_1^2$$

We see $\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{3 \times 24}{2}} = 6\,s^{-1}$, when $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{2}{3}$ kg.

$$\frac{1}{2}\frac{m_1 m_2}{m_2 + m_2}v_1^2 = \frac{1}{2}\frac{2}{3} \times (0.12)^2 = 48 \times 10^{-4} = 4.8 \text{ mJ}$$

We have $x = a \sin(\omega t + \alpha)$

Initially $x = 0$ at $t = 0$ so $\alpha = 0$

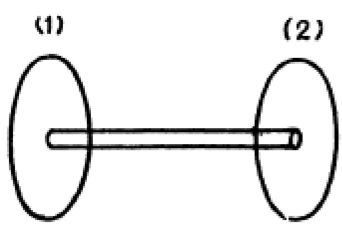
Then $x = a \sin \omega t$. Also $\dot{x} = v_1$ at $t = 0$.

So $\omega a = v_1$ and hence $a = \frac{v_1}{\omega} = \frac{12}{6} = 2$ cm.

Q.60. Find the period of small torsional oscillations of a system consisting of two discs slipped on a thin rod with torsional coefficient k. The moments of inertia of the discs relative to the rod's axis are equal to I_1 and l_2.

Ans. Suppose the disc 1 rotates by angle θ_1 and the disc 2 by angle θ_2 in the opposite sense. Then total torsion of the rod = $\theta_1 + \theta_2$

and torsional P.E. $= \frac{1}{2}\kappa(\theta_1 + \theta_2)^2$



The KJE. of the system (neglecting the moment of inertia of the rod) is

$$\frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2$$

So total eneigy of the rod

$$E = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}\kappa(\theta_1 + \theta_2)^2$$

We can put the total angular momentum of the rod equal to zero since the frequency associated with the rigid rotation of the whole system must be zero (and is known).

Thus $$I_1 \theta_1 = I_2 \dot{\theta}_2 \quad \text{or} \quad \frac{\dot{\theta}_1}{1/I_1} = \frac{\dot{\theta}_2}{1/I_2} = \frac{\dot{\theta}_1 + \dot{\theta}_2}{1/I_1 + 1/I_2}$$

So $$\dot{\theta}_1 = \frac{I_2}{I_1 + I_2}(\dot{\theta}_1 + \dot{\theta}_2) \quad \text{and} \quad \dot{\theta}_2 = \frac{I_1}{I_1 + I_2}(\dot{\theta}_1 + \dot{\theta}_2)$$

and $$E = \frac{1}{2}\frac{I_1 I_2}{I_1 + I_2}(\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2}\kappa(\theta_1 + \theta_2)^2$$

The angular oscillation, frequency corresponding to this is

$$\omega^2 = \kappa \Big/ \frac{I_1 I_2}{I_1 + I_2} = \kappa / I' \quad \text{and} \quad T = 2\pi\sqrt{\frac{I'}{\kappa}}, \text{ where } I' = \frac{I_1 I_2}{I_1 + I_2}$$

Q.61. A mock-up of a CO_2 molecule consists of three balls interconnected by identical light springs and placed along a straight line in the state of equilibrium. Such a system can freely perform oscillations of two types, as shown by the arrows in Fig. 4.20. Knowing the masses of the atoms, find the ratio of frequencies of these oscillations.

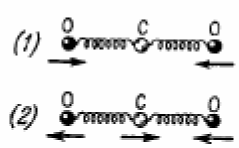


Fig. 4.20.

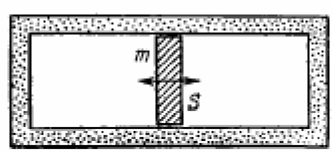
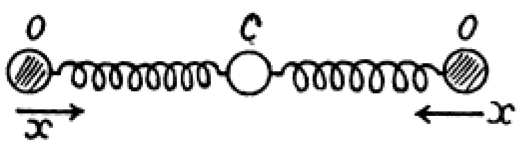


Fig. 4.21.

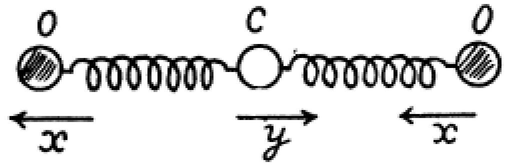
Ans. In the first mode the carbon atom remains fixed and the oxygen atoms move in equal & opposite steps. Then total energy is



$$\frac{1}{2}2\,m_0\dot{x}^2 + \frac{1}{2}2\,\kappa x^2$$

Where x is the displacement of one of the 0 atom (say left one). Thus

$$\omega_1^2 = \kappa / m_0 .$$



In this mode the oxygen atoms move in equal steps in the same direction but the carbon atom moves in such a way as to keep the centre of mass fixed.

Thus $$2\, m_0 x + m_c y = 0 \quad \text{or, } y = -\frac{2\, m_0}{m_c} x$$

$$\text{KE.} = \frac{1}{2} 2\, m_0 \dot{x}^2 + \frac{1}{2} m_c \left(\frac{2\, m_0}{m_c} \dot{x} \right)^2 = \frac{1}{2} 2\, m_0 \dot{x}^2 + \frac{1}{2} 2\, m_0 \frac{2\, m_0}{m_c} \dot{x}^2 \qquad = \frac{1}{2} 2\, m_0 \left(1 + \frac{2\, m_0}{m_c} \right) \dot{x}^2$$

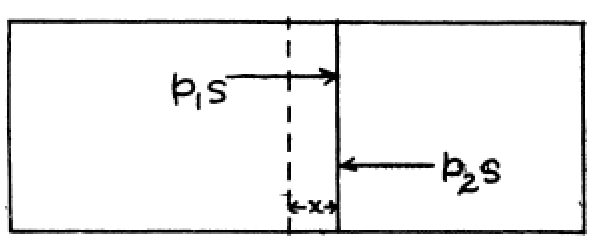
$$\text{P.E.} = \frac{1}{2} k \left(1 + \frac{2\, m_0}{m_c} \right)^2 x^2 \qquad + \frac{1}{2} \kappa \left(1 + \frac{2\, m_0}{m_c} \right)^2 x^2 = \frac{1}{2} 2\, \kappa \left(1 + \frac{2\, m_0}{m_c} \right)^2 x^2$$

Thus $$\omega_2^2 = \frac{\kappa}{m_0} \left(1 + \frac{2\, m_0}{m_c} \right) \text{ and } \omega_2 = \omega_1 \sqrt{1 + \frac{2\, m_0}{m_c}}$$

Hence, $$\omega_2 = \omega_1 \sqrt{1 + \frac{32}{12}} = \omega_1 \sqrt{\frac{11}{3}} = 1{\cdot}91\, \omega_1$$

Q.62. In a cylinder filled up with ideal gas and closed from both ends there is a piston of mass m, and cross-sectional area S (Fig. 4.21).
In equilibrium the piston divides the cylinder into two equal parts, each with volume V_o. The gas pressure is P_o. The piston was slighlty displaced from the equilibrium position and released. Find its oscillation frequency, assuming the processes in the gas to be adiabatic and the friction negligible.

Ans. Let, us displace the piston through small distance x, towards right, then from $F_x = mw_x$



or, $(p_1 - p_1) S = -m \ddot{x}$ (1)

But, the process is adiabatic, so from $P V^{\gamma} =$ const.

$$p_2 = \frac{p_0 V_0^\gamma}{(V_0 - Sx)^\gamma} \quad \text{and} \quad p_1 = \frac{p_0 V_0^\gamma}{(V_0 + Sx)^\gamma},$$

As the new volumes of the left and the right parts are now (V_o + S_x) and (V_o - S_x) respectively.

So, the Eqn (1) becomes.

$$\frac{p_0 V_0^\gamma S}{m}\left\{\frac{1}{(V_0 - Sx)^\gamma} - \frac{1}{(V_0 + Sx)^\gamma}\right\} = -\ddot{x}$$

or,
$$\frac{p_0 V_0^\gamma S}{m}\left\{\frac{(V_0 + Sx)^\gamma - (V_0 - Sx)^\gamma}{(V_0^2 - S^2 x^2)^\gamma}\right\} = -\ddot{x}$$

or,
$$\frac{p_0 V_0^\gamma S}{m}\left\{\frac{\left(1 + \frac{\gamma S x}{V_0}\right) - \left(1 - \frac{\gamma S x}{V_0}\right)}{V_0^\gamma\left(1 - \frac{\gamma S^2 x^2}{V_0^2}\right)}\right\} = -\ddot{x}$$

Neglecting the term $\frac{\gamma S^2 x^2}{V_0^2}$ in the denominator, as it is very sm all, we get

$$\ddot{x} = -\frac{2 p_0 S^2 \gamma x}{m V_0},$$

Which is the equation for S.H.M. and hence the oscillating frequency.

$$\omega_0 = S\sqrt{\frac{2 p_0 \gamma}{m V_0}}$$

Q.63. A small ball of mass m = 21 g suspended by an insulating thread at a height h = 12 cm from a large horizontal conducting plane performs small oscillations (Fig. 4.22). After a charge q had been imparted to the ball, the oscillation period changed η = 2.0 times. Find q.

Ans. In the absence of the charge, the oscillation period of the ball
$T = 2\pi\sqrt{l/g}$

When we impart the charge q to the ball, it will be influenced by the induced charges on the conducting plane. From the electric image method the electric force on the ball by the plane $\frac{q^2}{4\pi\varepsilon_0(2h)^2}$ and is directed downward . Thus in this case the effective acceleration of the ball

$$g' = g + \frac{q^2}{16\pi\varepsilon_0 m h^2}$$

And the corresponding time period

$$T' = 2\pi\sqrt{\frac{l}{g'}} = 2\pi\sqrt{\frac{l}{g + \frac{q^2}{16\pi\varepsilon_0 m h^2}}}$$

From the condition of the problem

$$T = \eta T'$$

So, $$T^2 = \eta^2 T'^2 \quad \text{or} \quad \frac{1}{g} = \eta^2 \left(\frac{1}{g + \frac{q^2}{16\pi\varepsilon_0 m h^2}}\right)$$

Thus on solving

$$q = 4h\sqrt{\pi\varepsilon_0 m g(\eta^2 - 1)} = 2\,\mu C$$

Q.64. A small magnetic needle performs small oscillations about an axis perpendicular to the magnetic induction vector. On changing the magnetic induction the needle's oscillation period decreased η = 5.0 times. How much and in what way was the magnetic induction changed? The oscillation damping is assumed to be negligible.

Ans. In a magnetic field of induction B the couple on the magnet is - MB sinθ = - MBθ equating this to $I\ddot{\theta}$ we get

$$I\ddot{\theta} + MB\theta = 0$$

or $$\omega^2 = \frac{MB}{I} \quad \text{or} \quad T = 2\pi\sqrt{\frac{I}{MB}}.$$

Given $$T_2 = T_1/\eta$$

o. $$\sqrt{\frac{1}{B_2}} = \sqrt{\frac{1}{B_1}}\cdot\frac{1}{\eta} \quad \text{or} \quad \frac{1}{B_2} = \frac{1}{B_1}\cdot\frac{1}{\eta^2}$$

or $$B_2 = \eta^2 B_1$$

The induction of the field increased η^2 times.

Q.65. A loop (Fig. 4.23) is formed by two parallel conductors connected by a solenoid with inductance L and a conducting rod of mass m which can freely (without friction) slide over the conductors. The conductors are located in a horizontal plane in a uniform vertical magnetic field with induction B. The distance between the conductors is equal to 1. At the moment t = 0 the rod is imparted an initial velocity v_o directed to the right. Find the law of its motion x (t) if the electric resistance of the loop is negligible.

Ans. We have in the circuit at a certain instant of time (t), from Faraday's law of electromagnetic induction :

$$L\frac{di}{dt} = Bl\frac{dx}{dt} \quad \text{or} \quad L\,di = Bl\,dx$$

As at $t = 0,\ x = 0,$ so $L\,i = Blx$ or $i = \frac{Bl}{L}x(1)$

For the rod from the second law of motion $F_x = mw_x$

$-ilB = m\ddot{x}$

Using Eqn. (I), we get : $\ddot{x} = -\left(\frac{l^2B^2}{mL}\right)x = -\omega_0^2 x$ (2)

where $\omega_0 = lB/\sqrt{mL}$

The solution of the above differential equation is of the form
$x = a\sin(\omega_0 t + \alpha)$

From the initial condition, at = 0, x = 0 , so α = 0

Hence, $x = a\sin\omega_0 t$ (3)

Differentiating w.r.t time, $\dot{x} = a\,\omega_0\cos\omega_0 t$

But from the initial condition of the problem at $t = 0,\ \dot{x} = v_0$

Thus $v_0 = a\,\omega_0$ or $a = v_0/\omega_0$ (4)

Putting the value of a from Eqn. (4) into Eqn. (3), we obtained

$$x = \frac{v_0}{\omega_0}\sin\omega_0 t\left(\text{where } \omega_0 = \frac{lB}{\sqrt{mL}}\right)$$

Mechanical Oscillations (Part - 6)

Q.66. A coil of inductance L connects the upper ends of two vertical copper bars separated by a distance l. A horizontal conducting connector of mass m starts falling with zero initial velocity along the bars without losing contact with them. The whole system is located in a uniform magnetic field with induction B perpendicular to the plane of the bars. Find the law of motion x (t) of the connector.

Ans. As the connector moves, an emf is set up in the circuit and a current flows, since the emf is

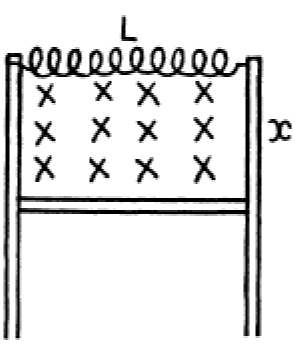
$\xi = -B\,l\,\dot{x}$, we must have : $-B\,l\,\dot{x} + L\dfrac{dI}{dt} = 0$

so, $I = B\,l\,x/L$

Provided x is measured from the initial position.

We then have

$$m\ddot{x} = -\frac{B\,l\,x}{L}.B.l + mg$$



for by Lenz's law the induced current will oppose downward sliding. Finally

$$\ddot{x} + \frac{(B\,l)^2}{m\,L}x = g$$

On putting

$$\omega_0 = \frac{Bl}{\sqrt{mL}}$$

$$\ddot{x} + \omega_0^2 x = g$$

A solution of this equation is $x = \frac{g}{\omega_0^2} + A\cos(\omega_0 t + \alpha)$

But $x = 0$ and $\dot{x} = 0$ at $t = 0$. This gives

$$x = \frac{g}{\omega_0^2}(1 - \cos\omega_0 t).$$

Q.67. A point performs damped oscillations according to the law $x = a_0 e^{-\beta t}\sin\omega t$. Find: (a) the oscillation amplitude and the velocity of the point at the moment t = 0; (b) the moments of time at which the point reaches the extreme positions.

Ans. We are given $x = a_0 e^{-\beta t}\sin\omega t$

(a) The velocity of the point at t = 0 is obtained from

$v_0 = (\dot{x})_{t=0} = \omega a_0$

The term "oscillation amplitude at the moment t = 0" is meaningless. Probably the implication is the amplitude for $t << \frac{1}{\beta}$. Then $x = a_0$ sin ωt and amplitude is a_o.

(b) $\dot{x} = (-\beta a_0 \sin\omega t + \omega a_0 \cos\omega t)e^{-\beta t} = 0$

when the displacement is an extremum. Then

$$\tan\omega t = \frac{\omega}{\beta}$$

or $$\omega t = \tan^{-1}\frac{\omega}{\beta} + n\pi,\ n = 0, 1, 2, \ldots$$

Q.68. A body performs torsional oscillations according to the law $\varphi = \varphi_0 e^{-\beta t}\cos\omega t$. Find:

(a) the angular velocity $\dot{\varphi}$ and the angular acceleration $\ddot{\varphi}$ of the body at the moment t = 0; (b) the moments of time at which the angular velocity becomes maximum.

Ans.

Given $\varphi = \varphi_0 e^{-\beta t} \cos \omega t$

we have $\dot{\varphi} = -\beta \varphi - \omega \varphi_0 e^{-\beta t} \sin \omega t$

$$\ddot{\varphi} = -\beta \dot{\varphi} + \beta \omega \varphi_0 e^{-\beta t} \sin \omega t - \omega^2 \varphi_0 e^{-\beta t} \cos \omega t$$

$$= \beta^2 \varphi + 2 \beta \omega \varphi_0 e^{-\beta t} \sin \omega t - \omega^2 \varphi$$

so

$(\dot{\varphi})_0 = -\beta \varphi_0$, $(\ddot{\varphi})_0 = (\beta^2 - \omega^2) \varphi_0$

$\dot{\varphi} = -\varphi_0 e^{-\beta t} (\beta \cos \omega t + \omega \sin \omega t)$ becomes maximum (or minimum) when

$$\ddot{\varphi} = \varphi_0 (\beta^2 - \omega^2) e^{-\beta t} \cos \omega t + 2 \beta \omega \varphi_0 e^{-\beta t} \sin \omega t = 0$$

or $$\tan \omega t = \frac{\omega^2 - \beta^2}{2 \beta \omega}$$

and $$t_n = \frac{1}{\omega} \left[\tan^{-1} \frac{\omega^2 - \beta^2}{2 \beta \omega} + n \pi \right], \quad n = 0, 1, 2, \dots$$

Q.69. A point performs damped oscillations with frequency ω and damping coefficient according to the law (4.1b). Find the initial amplitude a, and the initial phase a if at the moment t = 0 the displacement of the point and its velocity projection are equal to

(a) $x(0) = 0$ and $v_x(0) = \dot{x}_0$;

(b) $x(0) = x_0$ and $v_x(0) = 0$.

Ans.

We write $x = a_0 e^{-\beta t} \cos(\omega t + \alpha)$.

(a) $x(0) = 0 \Rightarrow \alpha = \pm \frac{\pi}{2} \Rightarrow x = \mp a_0 e^{-\beta t} \sin \omega t$

$$\dot{x}(0) = (\dot{x})_{t=0} = \mp \omega a_0$$

Since a0 is + ve, we must choose the upper sign if $\dot{x}(0) < 0$ and the lower sign if

$\dot{x}(0) > 0$. Thus

$$a_0 = \frac{|\dot{x}(0)|}{\omega} \quad \text{and} \quad \alpha = \begin{cases} +\frac{\pi}{2} & \text{if } \dot{x}(0) < 0 \\ -\frac{\pi}{2} & \text{if } \dot{x}(0) > 0 \end{cases}$$

(b)

we write $x = \text{Re}\, A\, e^{-\beta t + i\omega t}$, $A = a_0 e^{i\alpha}$

Then $\dot{x} = v_x = \text{Re}\,(-\beta + i\omega) A e^{-\beta t + i\omega t}$

From $v_x(0) = 0$ we get $Re\,(-\beta + i\omega) A = 0$

This implies $A = \pm\, i(\beta + i\omega) B$ where B is real and positive. Also

$$x_0 = \text{Re}\, A = \mp\, \omega B$$

Thus $B = \frac{|x_0|}{\omega}$ with $+$ sign in A if $x_0 < 0$

$-$ sign in A if $x_0 > 0$

So $A = \pm\, i \frac{\beta + i\omega}{\omega} |x_0| = \left(\mp 1 + \pm \frac{i\beta}{\omega}\right) |x_0|$

Finally $a_0 = \sqrt{1 + \left(\frac{\beta}{\omega}\right)^2}\, |x_0|$

$$\tan\alpha = \frac{-\beta}{\omega},\ \alpha = \tan^{-1}\left(\frac{-\beta}{\omega}\right)$$

α is in the 4th quadrant $\left(-\frac{\pi}{2} < \alpha < 0\right)$ if $x_0 > 0$ and α is in the 2nd quadrant $\left(\frac{\pi}{2} < \alpha < \pi\right)$ if $x_0 < 0$.

Q.70. A point performs damped oscillations with frequency ω = = 25 s^{-1}. Find the damping coefficient l if at the initial moment the velocity of the point is equal to zero and its displacement from the equilibrium position is η = 1.020 times less than the amplitude at that moment.

Ans.

$x = a_0 e^{-\beta t} \cos(\omega t + \alpha)$

Then $(\dot{x})_{t=0} = -\beta a_0 \cos\alpha - \omega a_0 \sin\alpha = 0$

or $\tan\alpha = -\frac{\beta}{\omega}.$

Also $(x)_{t=0} = a_0 \cos\alpha = \frac{a_0}{\eta}$

$$\sec^2\alpha = \eta^2,\ \tan\alpha = -\sqrt{\eta^2 - 1}$$

Thus $\beta = \omega\sqrt{\eta^2 - 1}$

(We have taken the amplitude at $t = 0$ to be a_0).

Q.71. A point performs damped oscillations with frequency ω and damping coefficient β. Find the velocity amplitude of the point as a function of time t if at the moment t = 0

(a) its displacement amplitude is equal to a_0;
(b) the displacement of the point x (0) = 0 and its velocity pro- jection $v_x(0) = \dot{x}_0$.

Ans.

We write $x = a_0 e^{-\beta t} \cos(\omega t + \alpha)$

$$= Re\, A e^{-\beta t + i\omega t},\ A = a_0 e^{i\alpha}$$

$$\dot{x} = Re\, A(-\beta + i\omega) e^{-\beta t + i\omega t}$$

Velocity amplitude as a function of time is defined in the following manner.

Put $t = t_0 + \tau,$

then

$$x = Re\, A e^{-\beta(t_0+\tau)}\, e^{i\omega(t_0+\tau)}$$

$$= Re\, A e^{-\beta t_0}\, e^{i\omega t_0 + i\omega\tau} = Re A\, e^{-\beta t_0} e^{i\omega t}$$

For $\tau << \frac{1}{\beta}$ This means that the displacement amplitude around the

time t_0 is $a_0 e^{-\beta t_0}$ and we can say that the displacement amplitude at

time t is $a_0 e^{-\beta t}$. Similarly for the velocity amplitude.

Clearly
(a) Velocity amplitude at time $t = a_0 \sqrt{\beta^2 + \omega^2}\, e^{-\beta t}$

Since $A(-\beta + i\omega) = a_0 e^{i\alpha}(-\beta + i\omega)$

$$= a_0 \sqrt{\beta^2 + \omega^2}\, e^{i\gamma}$$

Where y is another constant

(b) $x(0) = 0 \Rightarrow Re\, A = 0$ or $A = \pm i a_0$

where a_0 is real and positive.

Also

$$v_x(0) = \dot{x}_0 = Re \pm i a_0(-\beta + i\omega)$$
$$= \mp \omega a_0$$

Thus $a_0 = \frac{|\dot{x}_0|}{\omega}$ and we take - (+) sign if x_0 is negative (positive). Finally the velocity amplitude is obtained as

$$\frac{|\dot{x}_0|}{\omega}\sqrt{\beta^2 + \omega^2}\, e^{-\beta t}.$$

Q.72. There are two damped oscillations with the following periods T and damping coefficients β: T_1 = 0.10 ms, β_1 = 100s^{-1} and T_2 = 10ms, β_2 = 10s^{-1}. Which of them decays faster?

Ans. The first oscillation decays faster in time. But if one takes the natural time scale, the period T for each oscillation, the second oscillation attenuates faster during that period.

Q.73. A mathematical pendulum oscillates in a medium for which the logarithmic damping decrement is equal to 20 = 1.50. What will be the logarithmic damping decrement if the resistance of the medium increases n = 2.00 times? How many times has the resistance of the medium to be increased for the oscillations to become impossible?

Ans. By definition of the logarithmic decrement $\left(\lambda = \beta\frac{2\pi}{\omega}\right)$ we get for the original

decrement

λ_0

$$\lambda_0 = \beta\frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}} \quad \text{and finally } \lambda = \frac{2\pi n\beta}{\sqrt{\omega_0^2 - n^2\beta^2}}$$

Now $$\frac{\beta}{\sqrt{\omega_0^2 - \beta^2}} = \frac{\lambda_0}{2\pi} \quad \text{or} \quad \frac{\beta}{\omega_0} = \frac{\lambda_0/2\pi}{\sqrt{1 + \left(\frac{\lambda_0}{2\pi}\right)^2}}$$

so

$$\frac{\lambda/2\pi}{\sqrt{1+\left(\frac{\lambda}{2\pi}\right)^2}} = \frac{n\frac{\lambda_0}{2\pi}}{\sqrt{1+\left(\frac{\lambda_0}{2\pi}\right)^2}}$$

Hence

$$\frac{\lambda}{2\pi} = \frac{n\lambda_0/2\pi}{\sqrt{1-(n^2-1)\left(\frac{\lambda_0}{2\pi}\right)^2}}$$

For critical damping $\omega_0 = n_c\beta$

$$\frac{1}{n_c} = \frac{\beta}{\omega_0} = \frac{\lambda_0/2\pi}{\sqrt{1+\left(\frac{\lambda_0}{2\pi}\right)^2}} \quad \text{or} \quad n_c = \sqrt{1+\left(\frac{2\pi}{\lambda_0}\right)^2}$$

Q.74. A deadweight suspended from a weightless spring extends it by Δx = 9.8 cm. What will be the oscillation period of the dead Weight when it is pushed slightly in the vertical direction? The logarithmic damping decrement is equal to λ = 3.1.

Ans. The Eqn of the dead weight is

$$m\ddot{x} + 2\beta m\dot{x} + m\omega_0^2 x = mg$$

so

$$\Delta x = \frac{g}{\omega_0^2} \quad \text{or} \quad \omega_0^2 = \frac{g}{\Delta x}.$$

Now

$$\lambda = \frac{2\pi\beta}{\omega} = \frac{2\pi\beta}{\sqrt{\omega_0^2-\beta^2}} \quad \text{or} \quad \frac{\omega_0}{\sqrt{\omega_0^2-\beta^2}} = \sqrt{1+\left(\frac{\lambda}{2\pi}\right)^2}$$

Thus

$$T = \frac{2\pi}{\sqrt{\omega_0^2-\beta^2}} = \frac{2\pi}{\omega_0}\sqrt{1+\left(\frac{\lambda}{2\pi}\right)^2}$$

$$= 2\pi\sqrt{\frac{\Delta x}{g}}\sqrt{1+\left(\frac{\lambda}{2\pi}\right)^2} = \sqrt{\frac{\Delta x}{g}(4\pi^2+\lambda^2)} = 0.70 \text{ sec.}$$

Q.75. Find the quality factor of the oscillator whose displacement amplitude decreases η = 2.0 times every n = 110 oscillations.

Ans. The displacement amplitude decrease η times every n oscillations. Thus

$$\frac{1}{\eta} = e^{-\beta \cdot \frac{2\pi}{\omega} \cdot n}$$

or $$\frac{2\pi n \beta}{\omega} = \ln \eta \quad \text{or} \quad \frac{\beta}{\omega} = \frac{\ln \eta}{2\pi n}.$$

So $$Q = \frac{\omega}{2\beta} = \frac{\pi n}{\ln \eta} \approx 499.$$

Q.76. A particle was displaced from the equilibrium position by a distance l = 1.0 cm and then left alone. What is the distance that the particle covers in the process of oscillations till the complete stop, if the logarithmic damping decrement is equal to λ = 0.020?

Ans.

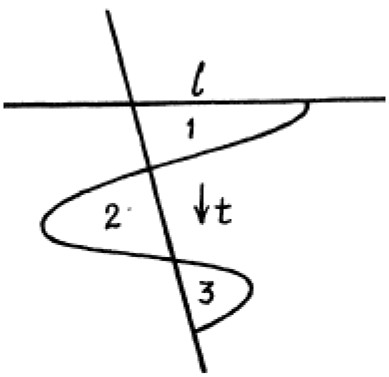
From $x = a_0 e^{-\beta t} \cos(\omega t + \alpha)$, we get using

$$(x)_{t=0} = l = a_0 \cos \alpha$$

$$0 = (\dot{x})_{t=0} = -\beta a_0 \cos \alpha - \omega a_0 \sin \alpha$$

Then $\tan \alpha = -\frac{\beta}{\omega}$ or $\cos \alpha = \frac{\omega}{\sqrt{\omega^2 + \beta^2}}$

and $$x = \frac{l\sqrt{\omega^2 + \beta^2}}{\omega} e^{-\beta t} \cos\left(\omega t - \tan^{-1} \frac{\beta}{\omega}\right)$$



$x = 0$ at $t = \frac{1}{\omega}\left(n\pi + \frac{\pi}{2} + \tan^{-1} \frac{\beta}{\omega}\right)$

Total distance travelled in the first lap $= l$

To get the maximum displacement in the second lap we note that

$$\dot{x} = \left[-\beta \cos\left(\omega t - \tan^{-1} \frac{\beta}{\omega}\right) - \omega \sin\left(\omega t - \tan^{-1} \frac{\beta}{\omega}\right)\right] \times \frac{l\sqrt{\omega^2 + \beta^2}}{\omega} e^{-\beta t} = 0$$

$\omega t = \pi, 2\pi, 3\pi, \ldots$ etc.

Thus $\qquad \ddot{x}_{max} = -a_0 e^{-\pi\beta/\omega}\cos\alpha = -l e^{-\pi\beta/\omega}$ for $t = \pi/\omega$

so, distance traversed in the 2nd lap $= 2 l e^{-\pi\beta/\omega}$

Continuing total distance traversed $= l + 2 l e^{-\pi\beta/\omega} + 2 l e^{-2\pi\beta/\omega} + \ldots\ldots$

$$= l + \frac{2 l e^{-\pi\beta/\omega}}{1 - e^{-\beta\pi/\omega}} = l + \frac{2 l}{e^{\beta\pi/\omega} - 1}$$

$$= l \frac{e^{\beta\pi/\omega} + 1}{e^{\beta\pi/\omega} - 1} = l \frac{1 + e^{\lambda/2}}{e^{\lambda/2} - 1}$$

Where $\lambda = \frac{2\pi\beta}{\omega}$, is the logarithmic decrement Substitution gives 2 metres.

Q.77. Find the quality factor of a mathematical pendulum l = = 50 cm long if during the time interval δ = 5.2 min its total mechanical energy decreases η = 4.0.10^4 times.

Ans. For an undamped oscillator the mechanical energy $E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2$ is conserved. For a damped oscillator.

$$x = a_0 e^{-\beta t}\cos(\omega t + \alpha), \; \omega = \sqrt{\omega_0^2 - \beta^2}$$

and $$E(t) = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2$$

$$= \frac{1}{2} m a_0^2 e^{-2\beta t}\left[\beta^2\cos^2(\omega t + \alpha) + 2\beta\omega\cos(\omega t + \alpha) \times \sin(\omega t + \alpha) + \omega^2\sin^2(\omega t + \alpha)\right]$$
$$+ \frac{1}{2} m a_0^2 \omega_0^2 e^{-2\beta t}\cos^2(\omega t + \alpha)$$

$$= \frac{1}{2} m a_0^2 \omega_0^2 e^{-2\beta t} + \frac{1}{2} m a_0^2 \beta^2 e^{-2\beta t}\cos(2\omega t + 2\alpha) + \frac{1}{2} m a_0^2 \beta\omega e^{-2\beta t}\sin(2\omega t + 2\alpha)$$

If $\beta << \omega$, then the average of the last two terms over many oscillations about the time t will vanish and

$$< E(t) > = \frac{1}{2} m a_0^2 \omega_0^2 e^{-2\beta t}$$

And this is the relevant mechanical energy.

In time δ this decreases by a factor $\frac{1}{\eta}$ so

$$e^{-2\beta\tau} = \frac{1}{\eta} \quad \text{or} \quad \tau = \frac{\ln\eta}{2\beta}.$$

$$\beta = \frac{\ln\eta}{2\tau}$$

and $$\lambda = \frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}} = \frac{2\pi}{\sqrt{\left(\frac{\omega_0}{\beta}\right)^2 - 1}} = \frac{2\pi}{\sqrt{\frac{4g\tau^2}{l\ln^2\eta} - 1}} \quad \text{since } \omega_0^2 = \frac{g}{l}$$

and $$Q = \frac{\pi}{\lambda} = \frac{1}{2}\sqrt{\frac{4g\tau^2}{l\ln^2\eta} - 1} \approx 130.$$

Q.78. A uniform disc of radius R = 13 cm can rotate about a horizontal axis perpendicular to its plane and passing through the edge of the disc. Find the period of small oscillations of that disc if the logarithmic damping decrement is equal to λ = 1.00.

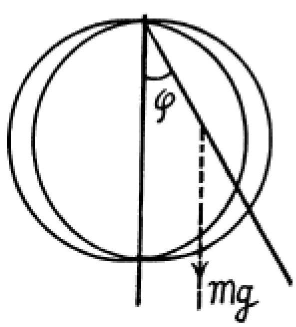
Ans. The restoring couple is

$$\Gamma = -mgR\sin\varphi \approx -mgR\varphi$$

The moment of inertia is

$$I = \frac{3mR^2}{2}$$

Thus for undamped oscillations



$$\frac{3mR^2}{2}\ddot{\varphi} + mgR\varphi = 0$$

so, $$\omega_0^2 = \frac{2g}{3R}$$

Also $$\lambda = \frac{2\pi\beta}{\omega} = \frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}}$$

Hence $$\frac{\beta}{\sqrt{\omega_0^2-\beta^2}} = \frac{\lambda}{2\pi} \quad \text{or} \quad \frac{\omega_0}{\sqrt{\omega_0-\beta^2}} = \sqrt{1+\left(\frac{\lambda}{2\pi}\right)^2}$$

Hence finally the period T of small oscillation comes to

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_0}\times\frac{\omega_0}{\sqrt{\omega_0^2-\beta^2}}\ 2\pi\sqrt{\frac{3R}{2g}\left(1+\left(\frac{\lambda}{2\pi}\right)^2\right)}$$

$$= \sqrt{\frac{3R}{2g}(4\pi^2+\lambda^2)} = 0.90 \text{ sec.}$$

Q.79. A thin uniform disc of mass m and radius R suspended by an elastic thread in the horizontal plane performs torsional oscillations in a liquid. The moment of elastic forces emerging in the thread is equal to N = αφ, where a is a constant and IT is the angle of rotation from the equilibrium position. The resistance force acting on a unit area of the disc is equal to F_1 = ηv, where is a constant and η is the velocity of the given element of the disc relative to the liquid. Find the frequency of small oscillation.

Ans. Let us calculate the moment G_1 of all the resistive forces on the disc. When the disc rotates an element (rdrdθ) with coordinates (r,θ) has a velocity $r\dot{\varphi}$, where φ is the instantaneous angle of rotation from the equilibrium position and r is measured from the centre. Then

$$G_1 = \int_0^{2\pi} d\theta \int_0^R dr\cdot r\cdot(F_1\times r)$$

$$= \int_0^R \eta\, r\dot{\varphi} r^2 d\gamma\times 2\pi = \frac{\eta\pi R^4}{2}\dot{\varphi}$$

Also moment of inertia = $\frac{mR^2}{2}$

Thus $$\frac{mR^2}{2}\ddot{\varphi}+\frac{\pi\eta R^4}{2}\dot{\varphi}+\alpha\varphi = 0$$

or $$\ddot{\varphi}+2\frac{\pi\eta R^2}{2m}\dot{\varphi}+\frac{2\alpha}{mR^2}\varphi = 0$$

Hence $$\omega_0^2 = \frac{2\alpha}{mR^2} \quad \text{and} \quad \beta = \frac{\pi\eta R^2}{2m}$$

and angular frequency $$\omega = \sqrt{\left(\frac{2\alpha}{mR^2}\right)-\left(\frac{\pi\eta R^2}{2m}\right)^2}$$

Note:- normally by frequency we mean $\frac{\omega}{2\pi}$

Q.80. A disc A of radius R suspended by an elastic thread between two stationary planes (Fig. 4.24) performs torsional oscillations about its axis 00'. The moment of inertia of the disc relative to that axis is equal to I, the clearance between the disc and each of the planes is equal to h, with h << R. Find the viscosity of the gas surrounding the disc A if the oscillation period of the disc equals T and the logarithmic damping decrement, λ.

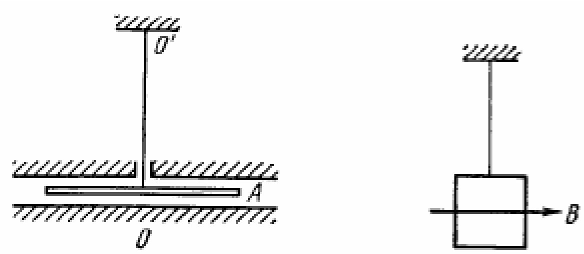


Fig. 4.24. Fig. 4.25.

Ans. From the law of viscosity, force per unit area = $\eta \frac{dv}{dx}$

So when the disc executes torsional oscillations the resistive couple on it is

$$= \int_0^R \eta \cdot 2\pi r. \frac{r\dot{\varphi}}{h} \cdot r \cdot dr \times 2 = \frac{\eta \pi R^4}{h} \dot{\varphi}$$

(factor 2 for the two sides of the disc; see the figure in the book) where φ is torsion. The equation of motion is

$$I\ddot{\varphi} + \frac{\eta \pi R^4}{h} \dot{\varphi} + c\varphi = 0$$

Comparing with $\ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2 \varphi = 0$ we get

$$\beta = \eta \pi R^4 / 2hI$$

Now the logarithmic decrement λ is given by $\lambda = \beta T$, $T =$ time period
Thus $\eta = 2\lambda h I / \pi R^4 T$

Mechanical Oscillations (Part - 7)

Q.81. A conductor in the shape of a square frame with side a suspended by an elastic thread is located in a uniform horizontal magnetic field with induction B. In equilibrium the plane of the frame is parallel to the vector B (Fig. 4.25). Having been displaced from the equilibrium position, the frame performs small oscillations about a vertical axis passing through its centre. The moment of inertia of the frame relative to that axis is equal to I, its electric resistance is R. Neglecting the inductance of the frame, find the time interval after which the amplitude of the frame's deviation angle decreases e-fold.

Ans. If φ = angle of deviation of the frame from its normal position, then an e.m.f. $\varepsilon = B a^2 \dot{\varphi}$

Is induced in the frame in the displaced position and a current $\frac{\varepsilon}{R} = \frac{B a^2 \dot{\varphi}}{R}$ flows in it. A couple

$$\frac{B a^2 \dot{\varphi}}{R} \cdot B \cdot a \cdot a = \frac{B^2 a^4}{R} \dot{\varphi}$$

Then acts on the frame in addition to any elastic restoring couple c φ. We write the equation of the frame as

$$I \ddot{\varphi} + \frac{B^2 a^4}{R} \dot{\varphi} + c \varphi = 0$$

Thus $\beta = \frac{B^2 a^4}{2 I R}$ where β is defined in the book.

Amplitude of oscillation die out according to $e^{-\beta t}$ so time required for the oscillations to decrease to $\frac{1}{e}$ of its value is

$$\frac{1}{\beta} = \frac{2 I R}{B^2 a^4}$$

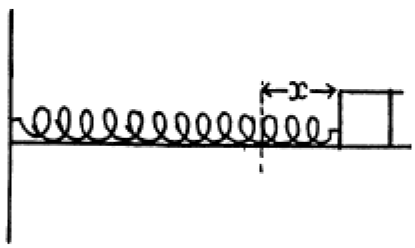
Q.82. A bar of mass m = 0.50 kg lying on a horizontal plane with a friction coefficient k = 0.10 is attached to the wall by means of a horizontal non-deformed spring. The stiffness of the spring is equal to x = 2.45 N/cm, its mass is negligible. The bar was displaced so that the spring was stretched by x_0 = 3.0 cm, and then released. Find:

(a) the period of oscillation of the bar;
(b) the total number of oscillations that the bar performs until it stops completely.

Ans. We shall denote the stiffness constant by k . Suppose the spring is stretched by x_0 . The bar in then subject to two horizontal forces (1) restoring force - k x and (2) friction kmg opposing motion. If

$$x_0 > \frac{kmg}{\kappa} = \Delta$$

The bar will come back.
(If $x_0 \leq \Delta$, the bar will stay put) The equation of the bar when it is moving to the left is



$$m\ddot{x} = -\kappa x + kmg$$

This equation has the solution

$$x = \Delta + (x_0 - \Delta) \cos\sqrt{\frac{k}{m}}\, t$$

Where we have used $x = x_0, \dot{x} = 0$ at $t = 0$ This solution is only valid till the bar comes to rest. This happens at

$$t_1 = \pi/\sqrt{\frac{k}{m}}$$

And at that time $x = x_1 = 2\Delta - x_0.$ if $x_0 > 2\Delta$ the tendency of the rod will now be to move to the right . (if $\Delta < x_0 < 2\Delta$ the rod will stay put now) Now the equation for rightward motion becomes'*

$$m\ddot{x} = -\kappa x - kmg$$

(The friction force has reversed).

We notice that the rod will move to the right only if

$\kappa(x_0 - 2\Delta) > k\,mg$ i.e. $x_0 > 3\Delta$

In this case the solution is

$$x = -\Delta + (x_0 - 3\Delta)\cos\sqrt{\frac{k}{m}}\,t$$

Since $x = 2\Delta - x_0$ and $\dot{x} = 0$ at $t = t_1 = \pi/\sqrt{\frac{k}{m}}$

The rod will next come to rest at

$$t = t_2 = 2\pi/\sqrt{\frac{k}{m}}$$

and at that instant $x = x_2 = x_0 - 4\Delta$. How ever the rod will stay put unless $x_0 > 5\Delta$. Thus

(a) time period of one full oscillation $= 2\pi/\sqrt{\frac{k}{m}}$.

(b) There is no oscillation if $0 < x_0 < \Delta$

One half oscillation if $\Delta < x_0 < 3\Delta$

2 half oscillation if $3\Delta < x_0 < 5\Delta$ etc.

We can say that the number of full oscillations is one half of the integer

where $n = \left[\frac{x_0 - \Delta}{2\Delta}\right]$

Where [x] = smallest non-negative integer greater than x.

Q.83. A ball of mass m can perform undamped harmonic oscillations about the point x = 0 with natural frequency ω_0. At the moment t = 0, when the ball was in equilibrium, a force $F_x = F_0 \cos \omega t$ coinciding with the x axis was applied to it. Find the law of forced oscillation x (t) for that ball.

Ans. The equation of motion of the ball is

$$m(\ddot{x} + \omega_0^2 x) = F_0 \cos\omega t \,.$$

This equation has the solution

$$x = A\cos(\omega_0 t + \alpha) + B\cos\omega t$$

Where A and a are arbitrary and B is obtained by substitution in the above equation

$$B = \frac{F_0/m}{\omega_0^2 - \omega^2}$$

The conditions $x = 0$, $\dot{x} = 0$ at $t = 0$ give

$$A\cos\alpha + \frac{F_0/m}{\omega_0^2 - \omega^2} = 0 \quad \text{and} \quad -\omega_0 A \sin\alpha = 0$$

This gives $\alpha = 0$, $$A = -\frac{F_0/m}{\omega_0^2 - \omega^2} = \frac{F_0/m}{\omega^2 - \omega_0^2}$$

Finally, $$x = \frac{F_0/m}{\omega^2 - \omega_0^2}(\cos\omega_0 t - \cos\omega t)$$

Q.84. A particle of mass m can perform undamped harmonic oscillations due to an electric force with coefficient k. When the particle was in equilibrium, a permanent force F was applied to it for δ seconds. Find the oscillation amplitude that the particle acquired after the action of the force ceased. Draw the approximate plot x (t) of oscillations. Investigate possible cases.

Ans. We have to look for solutions of the equation

$$m\ddot{x} + kx = F,\ 0 < t_1 < \tau,$$
$$m\ddot{x} + kx = 0,\ t > \tau$$

subject to $x(0) = \dot{x}(0) = 0$ where F is constant.

The solution of this equation will be sought in the form

$$x = \frac{F}{k} + A\cos(\omega_0 t + \alpha),\ 0 \le t \le \tau$$
$$x = B\cos(\omega_0(t - \tau) + \beta),\ t > \tau$$

A and α will be determined from the boundary condition at t = 0.

$$0 = \frac{F}{k} + A\cos\alpha$$
$$0 = -\omega_0 A\sin\alpha$$

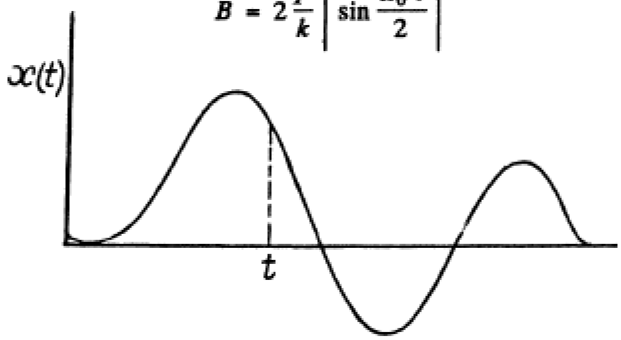
Thus $\alpha = 0$ and $A = -\frac{F}{k}$ and $x = \frac{F}{k}(1 - \cos\omega_0 t)$ $0 \le t < \tau.$

B and β will be determined by the continuity of x and $\dot{x}$ at $t = \tau$. Thus

$$\frac{F}{k}(1 - \cos\omega_0\tau) = B\cos\beta \quad \text{and} \quad \phi_0 \frac{F}{k}\sin\omega_0\tau = -\phi_0 B\sin\beta$$

Thus
$$B^2 = \left(\frac{F}{k}\right)^2 (2 - 2\cos\omega_0\tau)$$

or
$$B = 2\frac{F}{k}\left|\sin\frac{\omega_0\tau}{2}\right|$$



Q.85. A ball of mass m, when suspended by a spring stretches the latter by Δl. Due to external vertical force varying according to a harmonic law with amplitude F_0 the ball performs forced oscillations. The logarithmic damping decrement is equal to λ. neglecting the mass of the spring, find the angular frequency of the external force at which the displacement amplitude of the ball is maximum. What is the magnitude of that amplitude?

Ans. For the spring $mg = \kappa\Delta l$

Where k is its stuffiness coefficient. Thus

$$\omega_0^2 = \frac{\kappa}{m} = \frac{g}{\Delta l},$$

The equation of motion of the ball is

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

Here
$$\lambda = \frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}} \quad \text{or} \quad \frac{\beta}{\omega} = \frac{\lambda/2\pi}{\sqrt{1 + (\lambda/2\pi)^2}}$$

To find the solution of the above equation we look for the solution of the auxiliary equation

$$\ddot{z} + 2\beta \dot{z} + \omega_0^2 z = \frac{F_0}{m} e^{i\omega t}$$

Clearly we can take $\text{Re}\, z = x$. Now we look for a particular integral for z of the form $z = A e^{i\omega t}$

Thus, substitution gives A and we get

$$z = \frac{(F_0/m)\, e^{i\omega t}}{\omega_0^2 - \omega^2 + 2 i \beta \omega}$$

So taking the real part

$$x = \frac{(F_0/m)\left[(\omega_0^2 - \omega^2)\cos\omega t + 2\beta\omega\sin\omega t\right]}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

$$= \frac{F_0}{m}\frac{\cos(\omega t - \varphi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}, \quad \varphi = \tan^{-1}\frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

The amplitude of this oscillation is maximum when the denominator is minimum. Tthis happens when

$\omega^4 - 2\omega_0^2\omega^2 + 4\beta^2\omega^2 + \omega_0^4 = (\omega^2 - \omega_0^2 + 2\beta^2) + 4\beta^2\omega_0^2 - 4\beta^4$ is minimum. i.e for $\omega^2 = \omega_0^2 - 2\beta^2$

Thus $$\omega_{res}^2 = \omega_0^2\left(1 - \frac{2\beta^2}{\omega_0^2}\right)$$

$$= \frac{g}{\Delta l}\left[1 - \frac{2\left(\frac{\lambda}{2\pi}\right)^2}{1 + \left(\frac{\lambda}{2\pi}\right)^2}\right] = \frac{g}{\Delta l}\,\frac{1 - \left(\frac{\lambda}{2\pi}\right)^2}{1 + \left(\frac{\lambda}{2\pi}\right)^2}$$

and $$a_{res} = \frac{F_0/m}{\sqrt{4\beta^2\omega_0^2 - 4\beta^4}} = \frac{F_0/m}{2\beta\sqrt{\omega_0^2 - \beta^2}} = \frac{F_0/m}{2\beta^2}\cdot\frac{\lambda}{2\pi}$$

$$= \frac{F_0}{2m\omega_0^2}\cdot\frac{1 + \left(\frac{\lambda}{2\pi}\right)^2}{\lambda/2\pi} = \frac{F_0\,\Delta l\,\lambda}{4\pi m g}\left(1 + \frac{4\pi^2}{\lambda^2}\right)$$

Q.86. The forced harmonic oscillations have equal displacement amplitudes at frequencies $\omega_1 = 400\ s^{-1}$ and $\omega_2 = 600 s^{-1}$. Find the resonance frequency at which the displacement amplitude is maximum.

Ans.

Since $a = \dfrac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2 + 2\beta^2)^2 + 4\beta^2(\omega_0^2 - \beta^2)}}$

we must have $\omega_1^2 - \omega_0^2 + 2\beta^2 = -(\omega_2^2 - \omega_0^2 + 2\beta^2)$

or $\omega_0^2 - 2\beta^2 = \dfrac{\omega_1^2 + \omega_2^2}{2} = \omega_{res}^2$

Q.87. The velocity amplitude of a particle is equal to half the maximum value at the frequencies ω_1 and ω_2 of external harmonic force. Find:
(a) the frequency corresponding to the velocity resonance;
(b) the damping coefficient β and the damped oscillation frequency co of the particle.

Ans.

$$x = \frac{F_0}{m}\frac{(\omega_0^2 - \omega^2)\cos\omega t + 2\beta\omega\sin\omega t}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}}$$

Then $\dot{x} = \dfrac{F_0\omega}{m}\dfrac{2\beta\omega\cos\omega t + (\omega^2 - \omega_0^2)\sin\omega t}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$

Thus the velocity amplitude is

$$V_0 = \frac{F_0\omega}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$
$$= \frac{F_0}{m\sqrt{\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 + 4\beta^2}}$$

This is maximum when $\omega^2 = \omega_0^2 = \omega_{res}^2$

And then $V_{0\,res} = \dfrac{F_0}{2m\beta}$.

Now at half maximum $\left(\dfrac{\omega_0^2}{\omega} - \omega\right)^2 = 12\beta^2$

or $\omega^2 \pm 2\sqrt{3}\,\beta\omega - \omega_0^2 = 0$

$$\omega = \mp\beta\sqrt{3} + \sqrt{\omega_0^2 + 3\beta^2}$$

Where we have rejected a solution with - ve sign before there decal. Writing

$$\omega_1 = \sqrt{\omega_0^2 + 3\beta^2} + \beta\sqrt{3}\ , \quad \omega_2 = \sqrt{\omega_0^2 + 3\beta^2} - \beta\sqrt{3}$$

we get (a) $\omega_{res} = \omega_0 = \sqrt{\omega_1 \omega_2}$ (Velocity resonance frequency)

$\beta = \dfrac{|\omega_1 - \omega_2|}{2\sqrt{3}}$ and damped **oscillati**on frequency

$$\sqrt{\omega_0^2 - \beta^2} = \sqrt{\omega_1 \omega_2 - \frac{(\omega_1 - \omega_2)^2}{12}}$$

Q.88. A certain resonance curve describes a mechanical oscillating system with logarithmic damping decrement λ = 1.60. For this curve find the ratio of the maximum displacement amplitude to the displacement amplitude at a very low frequency.

Ans. In general for displacement amplitude

$$a = \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$= \frac{F_0}{m} \frac{1}{\sqrt{(\omega^2 - \omega_0^2 + 2\beta^2)^2 + 4\beta^2(\omega_0^2 - \beta^2)}}$$

Thus $\qquad \eta = \dfrac{a_{res}}{a_{low}} = \dfrac{\omega_0^2}{\sqrt{4\beta^2(\omega_0^2 - \beta^2)}} = \dfrac{\omega_0^2}{2\beta\sqrt{\omega_0^2 - \beta^2}}$

But $\qquad \dfrac{\beta}{\omega_0} = \dfrac{\lambda/2\pi}{\sqrt{1 + (\lambda/2\pi)^2}}\ , \quad \dfrac{\lambda}{2\pi} = \dfrac{\beta}{\sqrt{\omega_0^2 - \beta^2}}$

Hence $\qquad \eta = \dfrac{\omega_0^2}{2\beta^2} \cdot \dfrac{\lambda}{2\pi} = \dfrac{1}{2} \dfrac{1 + \left(\dfrac{\lambda}{2\pi}\right)^2}{\dfrac{\lambda}{2\pi}} = 2.90$

Q.89. Due to the external vertical force $F_x = F_0 \cos \omega t$ a body suspended by a spring performs forced steady-state oscillations according to the law $x = a \cos(\omega t — \varphi)$. Find the work performed by the force F during one oscillation period.

Ans. The work done in one cycle is

$$A = \int F\,dx = \int_0^T F v\,dt = \int_0^T F_0 \cos \omega t\,(-\omega a \sin(\omega t - \varphi))\,dt$$

$$= \int_0^T F_0 \omega a\,(-\cos \omega t \sin \omega t \cos \varphi + \cos^2 \omega t \sin \varphi)\,dt$$

$$= \frac{1}{2} F_0 \omega a \frac{T}{2} \sin \varphi = \pi a F_0 \sin \varphi$$

Q.90. A ball of mass m = 50 g is suspended by a weightless spring with stiffness x = 20.0 N/m. Due to external vertical harmonic force with frequency ω = 25.0 s^{-1} the ball performs steady-state oscillations with amplitude a = 1.3 cm. In this case the displace- ment of the ball lags in phase behind the external force by

$\varphi = \frac{3}{4}\pi.$

Find: (a) the quality factor of the given oscillator; (b) the work performed by the external force during one oscillation period.

Ans. In the formula $x = a \cos(\omega t - \varphi)$

we have

$$a = \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$\tan \varphi = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

Thus $\beta = \frac{(\omega_0^2 - \omega^2)\tan \varphi}{2\omega}.$

Hence $\omega_0 = \sqrt{K/m} = 20\ s^{-1}.$

And (a) the quality factor

$$Q = \frac{\pi}{\beta T} = \frac{\sqrt{\omega_0^2 - \beta^2}}{2\beta} = \frac{1}{2}\sqrt{\frac{4\omega^2\omega_0^2}{(\omega_0^2 - \omega^2)^2 \tan^2 \varphi} - 1} = 2.17$$

(b) work done is

$$A = \pi a F_0 \sin \varphi$$

$$= \pi m a^2 \sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}\,\sin \varphi = \pi m a^2 \times 2\beta\omega$$

$$= \pi m a^2 (\omega_0^2 - \omega^2)\tan \varphi = 6\ \text{mJ}.$$

Q.91. A ball of mass m suspended by a weightless spring can perform vertical oscillations with damping coefficient β. The natural oscillation frequency is equal to ω_0. Due to the external vertical force varying as $F_x = F_o$, cos ωt the ball performs steady-state harmonic oscillations. Find:
(a) the mean power (P), developed by the force F, averaged over one oscillation period;
(b) the frequency ω of the force F at which (P) is maximum; what is $(P)_{max}$ equal to?

Ans. Here as usual $\tan\varphi = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$ where φ is the phase lag of the displacement

$$x = a\cos(\omega t - \varphi),\ a = \frac{F_0}{m}\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}}$$

(a) Mean power developed by the force over one oscillation period

$$= \frac{\pi F_0 a \sin\varphi}{T} = \frac{1}{2}F_0 a\omega\sin\varphi$$

$$= \frac{F_0^2}{m}\frac{\beta\omega^2}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2} = \frac{F_0^2\beta}{m}\frac{1}{\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 + 4\beta^2}$$

(b) Mean power < P > is maximum when $\omega = \omega_0$ (for the denominator is then minimuir Also

$$<P>_{max} = \frac{F_0^2}{4m\beta}$$

Q.92. An external harmonic force F whose frequency can be varied, with amplitude maintained constant, acts in a vertical direction on a ball suspended by a weightless spring. The damping coefficient is times less than the natural oscillation frequency ω_o of the ball. How much, in per cent, does the mean power (P) developed by the force F at the frequency of displacement resonance differ from the maximum mean power $(P)_{max}$? Averaging is performed over one oscillation period.

Ans. Given $\beta = \omega_0/\eta$. Then from the previous problem

$$<P> = \frac{F_0^2\omega_0}{\eta m}\cdot\frac{1}{\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 + 4\frac{\omega_0^2}{\eta^2}}$$

At displacement resonance $\omega = \sqrt{\omega_0^2 - 2\beta^2}$

$$<P>_{res} = \frac{F_0^2 \omega_0}{\eta m} \frac{1}{\frac{4\beta^4}{\omega_0^2 - 2\beta^2} + \frac{4\omega_0^2}{\eta^2}} = \frac{F_0^2 \omega_0}{\eta m} \frac{1}{\frac{4\omega_0^4/\eta^4}{\omega_0^2\left(1 - \frac{2}{\eta^2}\right)} + 4\frac{\omega_0^2}{\eta^2}}$$

$$= \frac{F_0^2}{4\eta m \omega_0} \frac{\eta^2}{\frac{1}{\eta^2 - 2} + 1} = \frac{F_0^2 \eta}{4 m \omega_0} \frac{\eta^2 - 2}{\eta^2 - 1}$$

while $$<P>_{\max} = \frac{F_0^2 \eta}{4 m \omega_0}.$$

Thus $$\frac{<P>_{\max} - <P>_{res}}{<P>_{\max}} = \frac{100}{\eta^2 - 1}\ \%$$

Q.93. A uniform horizontal disc fixed at its centre to an elastic vertical rod performs forced torsional oscillations due to the moment of forces $N_z = N_m \cos \omega t$. The oscillations obey the law $\varphi = \varphi_m \cos(\omega t - \alpha)$. Find: (a) the work performed by friction forces acting on the disc during one oscillation period; (b) the quality factor of the given oscillator if the moment of inertia of the disc relative to the axis is equal to I.

Ans. The equation of the disc is

$$\ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2 \varphi = \frac{N_m \cos \omega t}{I}$$

Then as before

$$\varphi = \varphi_m \cos(\omega t - \alpha)$$

where

$$\varphi_m = \frac{N_m}{I[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]^{1/2}}, \quad \tan\alpha = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

(a) Work performed by frictional forces

$$= -\int N_r\, d\varphi \quad \text{where } N_r = -2I\beta\dot{\varphi} = -\int_0^T 2\beta I \dot{\varphi}^2 dt = -2\pi\beta\omega I \varphi_m^2$$

$$= -\pi I \varphi_m^2 [(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]^{1/2} \sin\alpha = -\pi N_m \varphi_m \sin\alpha$$

(b) The quality factor

$$Q = \frac{\pi}{\lambda} = \frac{\pi}{\beta T} = \frac{\sqrt{\omega_0^2 - \beta^2}}{2\beta} = \frac{\omega\sqrt{\omega_0^2 - \beta^2}}{(\omega_0^2 - \omega^2)\tan\alpha} = \frac{1}{2\tan\alpha}\left\{\frac{4\,\omega^2\,\omega_0^2}{(\omega_0^2 - \omega^2)^2} - \frac{4\,\beta^2\,\omega^2}{(\omega_0^2 - \omega^2)^2}\right\}^{1/2}$$

$$= \frac{1}{2\tan\alpha}\left\{\frac{4\,\omega^2\,\omega_0^2 I^2\,\varphi_m^2}{N_m^2\cos^2\alpha} - \tan^2\alpha\right\}^{1/2} \quad \text{since} \quad \omega_0^2 = \omega^2 + \frac{N_m}{I\,\varphi_m}\cos\alpha$$

$$= \frac{1}{2\sin\alpha}\left\{\frac{4\,\omega^2\,\omega_0^2 I^2\,\varphi_m^2}{N_m^2} - \sin^2\alpha\right\}^{1/2}$$

$$= \frac{1}{2\sin\alpha}\left\{\frac{4\,\omega^2 I^2\,\varphi_m^2}{N_m^2}\left(\omega^2 + \frac{N_m\cos\alpha}{I\,\varphi_m}\right) + 1 - \cos^2\alpha\right\}^{1/2}$$

$$= \frac{1}{2\sin\alpha}\left\{\frac{4 I^2\,\varphi_m^2}{N_m^2}\,\omega^4 + \frac{4\,I\,\varphi_m}{N_m}\,\omega^2\cos\alpha + \cos^2\alpha - 1\right\}^{1/2} = \frac{1}{2\sin\alpha}\left\{\left(\frac{2\,I\,\varphi_m\,\omega^2}{N_m} + \cos\alpha\right)^2 - 1\right\}^{1/2}$$