Matrices

Matrix and its Various Types

Matrices and their Related Terminology

- A **matrix** is an ordered rectangular array of numbers and functions. The numbers or functions in a matrix are called the elements of the matrix.

  For example, if the marks obtained by Smita in English, Maths and Science are 84, 93, and 89 respectively and the marks scored by Gopal are 81, 90, and 92 respectively, then this can be represented in the form of matrix as $\begin{bmatrix} 84 & 81 \\ 93 & 90 \\ 89 & 92 \end{bmatrix}$ or as $\begin{bmatrix} 84 & 93 & 89 \\ 81 & 90 & 92 \end{bmatrix}$.

  Here, in the first matrix, the vertical lines of elements represent the marks obtained by Smita and Gopal respectively. In the second matrix, the horizontal lines of elements represent the marks obtained by Smita and Gopal.

- The matrices are usually denoted by capital letters. The horizontal line of elements is known as the row of matrix and the vertical line of elements is known as the column of matrix.

- A matrix with m rows and n columns is known as the matrix of order $m \times n$ or an $m \times n$ matrix.

  For example: $A = \begin{bmatrix} 4 & 1 \\ 3 & 9 \\ 8 & 2 \end{bmatrix}$ is a 3 × 2 matrix and $B = \begin{bmatrix} 8 & 3 & 9 \\ 1 & 6 & 7 \end{bmatrix}$ is a 2 × 3 matrix.

- In general, an $m \times n$ matrix represents the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{21} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

It can also be represented as $A = [a_{ij}]_{m \times n},\ 1 \le i \le m,\ 1 \le j \le n;\ i, j \in \mathbf{N}$.

This means that the i^{th} row has elements $a_{i1}, a_{i2} \ldots a_{ij} \ldots a_{in}$ and the j^{th} column has elements $a_{1j}, a_{2j} \ldots a_{ij} \ldots a_{mj}$.

- An $m \times n$ matrix has mn number of elements.

Types of Matrices

- A matrix is said to be a **column matrix** if it has only one column.

  For example: $\begin{bmatrix} 2 \\ 7 \\ 8 \end{bmatrix}$ is a column matrix.

- A matrix is said to be a **row matrix** if it has only one row.

  For example: $\begin{bmatrix} 7 & 5 & 1 \end{bmatrix}$ is a row matrix.

- A matrix in which the number of rows is equal to the number of columns is called a **square matrix**. In a square matrix, $m = n$. Therefore, it is known as a square matrix of order n. It is represented by $A = [a_{ij}]_{n \times n}$

  For example: $\begin{bmatrix} 3 & 11 & 2 \\ 5 & 4 & 1 \\ 7 & 9 & 8 \end{bmatrix}$ is a square matrix of order 3.

- A matrix is said to be a **rectangular matrix** if the number of rows is not equal to the number of columns.

  For example: $\begin{bmatrix} 8 & 3 & 9 \\ 1 & 6 & 7 \end{bmatrix}$ is a rectangular matrix.

- In a square matrix $A = [a_{ij}]_{n \times n}$, the elements $a_{11}, a_{22} \ldots a_{nn}$ are called the **diagonal elements** of $A = [a_{ij}]_{n \times n}$. For instance, for the square matrix $\begin{bmatrix} 3 & 11 & 2 \\ 5 & 4 & 1 \\ 7 & 9 & 8 \end{bmatrix}$, the diagonal elements are 3, 4, and 8. If all the non-diagonal elements of a square matrix are zero, then the square matrix is known as a **diagonal matrix** i.e., a square matrix $A = \left[a_{ij}\right]_{n\times n}$ is a diagonal matrix if $a_{ij} = 0$, whenever $i \neq j$.

For example: $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ are diagonal matrices.

- A diagonal matrix in which all the diagonal elements are equal is known as a scalar matrix i.e., square matrix $A = \left[a_{ij}\right]_{n\times n}$ is a **scalar matrix** if

  $a_{ij} = 0$, when $i \neq j$

  $a_{ij} = k$, when $i = j$, where k is a constant.

  For example: $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$, $\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ are scalar matrices.

- A square matrix in which all the diagonal elements are 1 and all other elements are zero is known as an **identity matrix** i.e., $A = \left[a_{ij}\right]_{n\times n}$ is an identity matrix if

  $a_{ij} = 1$, when $i = j$

  $a_{ij} = 0$, when $i \neq j$

  For example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 2 and 3 respectively.

- A matrix is known as a **zero** or **null matrix** if all its elements are zero.

  For example: [0], $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are all zero matrices.

Let us solve some examples on matrices.

Example 1:

The lengths of the sides of two triangles are 2 cm, 5 cm, 8 cm and 3 cm, 4 cm, 5 cm. Represent this in the form of a 2 × 3 matrix.

Solution:

A 2 × 3 matrix is represented as $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$.

The lengths of sides can be represented in the form of matrix as

$$\begin{bmatrix} 2 & 5 & 8 \\ 3 & 4 & 5 \end{bmatrix}$$

Here, the first row represents the sides of the first triangle and the second row represents the sides of the second triangle.

Example 2:

Construct a square matrix of order 3 whose elements are given by $a_{ij} = \frac{1}{2}|2i - j|$.

Solution:

A 3 × 3 matrix is represented as $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

$$a_{11} = \frac{1}{2}|2\cdot 1 - 1| = \frac{1}{2},\ a_{12} = \frac{1}{2}|2\cdot 1 - 2| = 0,\ a_{13} = \frac{1}{2}|2\cdot 1 - 3| = \frac{1}{2}|-1| = \frac{1}{2}$$
$$a_{21} = \frac{1}{2}|2\cdot 2 - 1| = \frac{3}{2},\ a_{22} = \frac{1}{2}|2\cdot 2 - 2| = 1,\ a_{23} = \frac{1}{2}|2\cdot 2 - 3| = \frac{1}{2}$$
$$a_{31} = \frac{1}{2}|2\cdot 3 - 1| = \frac{5}{2},\ a_{32} = \frac{1}{2}|2\cdot 3 - 2| = 2,\ a_{31} = \frac{1}{2}|2\cdot 3 - 3| = \frac{3}{2}$$

Thus, the required matrix is $\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} \end{bmatrix}$.

Equality of Matrices

- Two matrices $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are said to be equal (denoted as $A = B$) if they are of the same order and each element of A is equal to the corresponding element of B i.e., $a_{ij} = b_{ij}$ for all i and j.

For example: $\begin{bmatrix} 15 & 11 \\ 7 & 2 \end{bmatrix}$ and $\begin{bmatrix} 15 & 11 \\ 7 & 2 \end{bmatrix}$ are equal but $\begin{bmatrix} 15 & 11 \\ 7 & 2 \end{bmatrix}$ and $\begin{bmatrix} 7 & 2 \\ 15 & 11 \end{bmatrix}$ are not equal.

Solved Examples

Example 1

If $\begin{bmatrix} 7 & x-y \\ 13 & 3y+z \end{bmatrix} = \begin{bmatrix} 2x+y & 5 \\ 2x+y+z & 3 \end{bmatrix}$, then find the values of x, y and z.

Solution:

Since the corresponding elements of equal matrices are equal,

$2x + y = 7$… (1)

$x - y = 5$… (2)

$2x + y + z = 13$… (3)

$3y + z = 3$… (4)

On solving equations (1) and (2), we obtain $x = 4$ and $y = -1$.

On substituting the value of y in equation (4), we obtain $z = 6$.

Thus, the values of x, y and z are 4, −1 and 6 respectively.

Example 2

If $A = \begin{bmatrix} a+b & 2b-c \\ 4 & b \end{bmatrix}$ and $B = \begin{bmatrix} -a & 3c \\ 4 & -b+a+5 \end{bmatrix}$ are two equal matrices, then find matrix A and matrix B.

Solution:

The given matrices are equal. Therefore, their corresponding elements are equal. On comparing the corresponding elements, we obtain

$a + b = -a$

$\Rightarrow 2a + b = 0 \ldots (1)$

$2b - c = 3c$

$\Rightarrow 4c - 2b = 0 \ldots (2)$

$b = -b + a + 5$

$\Rightarrow -a + 2b = 5 \ldots (3)$

On solving equations (1) and (3), we obtain $a = -1$ and $b = 2$.

On substituting the value of b in equation (2), we obtain $c = 1$.

In matrix A,

$a_{11} = a + b = -1 + 2 = 1$

$a_{12} = 2b - c = 2(2) - 1 = 3$

$a_{22} = b = 2$

$$\therefore A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

Since A and B are equal matrices, B is the same as matrix A.

Thus, matrix A and matrix B are $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$.

Addition and Subtraction of Matrices

Addition of Matrices and its Properties

- The sum of two matrices is obtained by adding their corresponding elements. Two matrices can be added only if their orders are same.

- In general, if $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are two matrices of order $m \times n$, then the sum of these matrices is given by $C=\left[c_{ij}\right]_{m\times n}$, where $c_{ij} = a_{ij} + b_{ij}$.

  For example: $\begin{bmatrix} 15 & 11 \\ 7 & 2 \end{bmatrix}+\begin{bmatrix} 7 & 2 \\ 5 & 1 \end{bmatrix}=\begin{bmatrix} 15+7=22 & 11+2=13 \\ 7+5=12 & 2+1=3 \end{bmatrix}=\begin{bmatrix} 22 & 13 \\ 12 & 3 \end{bmatrix}$

- The sum of matrices whose order is not the same is not defined.
- If $A=\left[a_{ij}\right]$, $B=\left[b_{ij}\right]$, $C=\left[c_{ij}\right]$ are three matrices of the same order, then their addition satisfies the following properties:
- The addition of matrices is commutative i.e., $A + B = B + A$
- The addition of matrices is associative i.e., $(A + B) + C = A + (B + C)$
- For every matrix A, there exists a zero matrix O of the same order such that $A + O = O + A = A$. O is the additive identity of the matrix addition.
- For every matrix A, there exists another matrix $-A$ of same order such that $A + (-A) = (-A) + A = O$. $(-A)$ is the additive inverse of A.

Subtraction of Matrices

- The difference of two matrices $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ of the same order $m \times n$, (say B from A) is given by $C=\left[c_{ij}\right]_{m\times n}$, where $c_{ij} = a_{ij} - b_{ij}$.

  For example: $\begin{bmatrix} 5 & 9 \\ 7 & 2 \end{bmatrix}-\begin{bmatrix} 7 & 0 \\ 3 & 1 \end{bmatrix}=\begin{bmatrix} 5-7=-2 & 9-0=9 \\ 7-3=4 & 2-1=1 \end{bmatrix}=\begin{bmatrix} -2 & 9 \\ 4 & 1 \end{bmatrix}$

- The difference of matrices whose orders are not the same is not defined.
- $A - B \neq\neq B - A$

Solved Examples

Example 1

If $\begin{bmatrix} x & 3y \\ 2 & -6 \end{bmatrix}+\begin{bmatrix} -4y & 2x \\ -3 & 7 \end{bmatrix}=\begin{bmatrix} -3 & 5 \\ -1 & 1 \end{bmatrix}$, then find the values of x and y.

Solution:

According to the given information,

$$\begin{bmatrix} x & 3y \\ 2 & -6 \end{bmatrix} + \begin{bmatrix} -4y & 2x \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x-4y & 2x+3y \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ -1 & 1 \end{bmatrix}$$

On equating the corresponding elements, we obtain

$x - 4y = -3$… (1)

$2x + 3y = 5$… (2)

On solving equations (1) and (2), we obtain $x = 1$ and $y = 1$.

Thus, the values of x and y are 1 and 1 respectively.

Example 2

If $A = \begin{bmatrix} \frac{3}{2} & 5 \\ \frac{7}{3} & 3 \end{bmatrix}, B = \begin{bmatrix} -\frac{5}{2} & -7 \\ \frac{2}{3} & 1 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & -2 \\ 3 & 3 \end{bmatrix}$ are three matrices, then prove that $A + B - C$ is an identity matrix.

Solution:

$$A+B-C = \begin{bmatrix} \frac{3}{2} & 5 \\ \frac{7}{3} & 3 \end{bmatrix} + \begin{bmatrix} -\frac{5}{2} & -7 \\ \frac{2}{3} & 1 \end{bmatrix} - \begin{bmatrix} -2 & -2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - \frac{5}{2} - (-2) & 5-7-(-2) \\ \frac{7}{3} + \frac{2}{3} - 3 & 3+1-3 \end{bmatrix}$$

$$= \begin{bmatrix} -1+2 & -2-(-2) \\ 3-3 & 4-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, $A + B - C$ is an identity matrix.

Multiplication of A Matrix by a Scalar and Related Properties

Multiplication of a Matrix by a Scalar

- A matrix A, when multiplied by a scalar k, gives another matrix kA, which is obtained by multiplying each element of matrix A by the scalar k. In general, if $A=\left[a_{ij}\right]_{m\times n}$, then $kA=k\left[a_{ij}\right]_{m\times n}=\left[k\left(a_{ij}\right)\right]_{m\times n}$ i.e., $(i, j)^{th}$ element of kA is ka_{ij}, for all values of i and j.

  For example: If $A=\begin{bmatrix}4 & -8\\ -12 & 20\end{bmatrix}$, then $3A$ is given by $3A=\begin{bmatrix}4\times 3 & -8\times 3\\ -12\times 3 & 20\times 3\end{bmatrix}=\begin{bmatrix}12 & -24\\ -36 & 60\end{bmatrix}$

- The negative of a matrix is obtained by multiplying the matrix with −1 i.e., if A is a matrix, then its negative is $(-1)A = -A$.

- If $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are two matrices of the same order i.e., $m \times n$, and k and l are scalars, then A and B satisfies the following properties:

- $k(A + B) = kA + kB$
- $(k + l) A = kA + lA$
  - In order to understand the concept of multiplication of a matrix by a scalar in a better way,

Solved Examples

Example 1:

Find the value of $2A - 3B$, where $A=\begin{bmatrix}4 & -3\\ 7 & 3\end{bmatrix}$ and $B=\begin{bmatrix}7 & -1\\ 5 & -4\end{bmatrix}$.

Solution:

$$A=\begin{bmatrix}4 & -3\\ 7 & 3\end{bmatrix}$$

$$\Rightarrow 2A=\begin{bmatrix}4\times 2 & -3\times 2\\ 7\times 2 & 3\times 2\end{bmatrix}=\begin{bmatrix}8 & -6\\ 14 & 6\end{bmatrix}$$

$$B=\begin{bmatrix}7 & -1\\ 5 & -4\end{bmatrix}$$

$$\Rightarrow 3B=\begin{bmatrix}3\times 7 & 3\times -1\\ 3\times 5 & 3\times -4\end{bmatrix}=\begin{bmatrix}21 & -3\\ 15 & -12\end{bmatrix}$$

$$\therefore 2A-3B=\begin{bmatrix}8 & -6\\ 14 & 6\end{bmatrix}-\begin{bmatrix}21 & -3\\ 15 & -12\end{bmatrix}=\begin{bmatrix}-13 & -3\\ -1 & 18\end{bmatrix}$$

Example 2:

If $3\begin{bmatrix}-2 & 4\\ 1 & 3\end{bmatrix}-5\begin{bmatrix}a & 1\\ 2 & b\end{bmatrix}=\begin{bmatrix}-1 & 7\\ -7 & -1\end{bmatrix}$, then find the values of *a* and *b*.

Solution:

$$3\begin{bmatrix}-2 & 4\\ 1 & 3\end{bmatrix}-5\begin{bmatrix}a & 1\\ 2 & b\end{bmatrix}=\begin{bmatrix}-1 & 7\\ -7 & -1\end{bmatrix}$$
$$\Rightarrow\begin{bmatrix}-6 & 12\\ 3 & 9\end{bmatrix}+\begin{bmatrix}-5a & -5\\ -10 & -5b\end{bmatrix}=\begin{bmatrix}-1 & 7\\ -7 & -1\end{bmatrix}$$
$$\Rightarrow\begin{bmatrix}-6-5a & 12-5\\ 3-10 & 9-5b\end{bmatrix}=\begin{bmatrix}-1 & 7\\ -7 & -1\end{bmatrix}$$

On equating the corresponding elements, we obtain

$-6 - 5a = -1$

$\Rightarrow a = -1$

$9 - 5b = -1$

$\Rightarrow b = 2$

Thus, the values of *a* and *b* are −1 and 2 respectively.

Multiplication of Matrices and Related Properties

Multiplication of Matrices

- The product of two matrices *A* and *B* is defined only if the number of columns of *A* is equal to the number of rows of *B* i.e., if $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$, only then *AB* is defined.
- If *A* is a matrix of order $m\times n$ and *B* is a matrix of order $n\times p$, then their product, say, $C = AB$, is of the order $m\times p$. To find the $(i, k)^{th}$ element, (c_{ik}) of matrix C (= *AB*), multiply the elements of the i^{th} row of *A* and the elements of the k^{th} column of *B* and take the sum of all their products.
- If the product of two matrices is a zero matrix, then it is not necessary that one of the matrices is a zero matrix.

Properties of Multiplication of Matrices

The multiplication of three matrices A, B and C satisfies the following properties:

- The multiplication of matrices is non-commutative i.e., the product AB and BA may or may not be equal.

  For example: If $A=\begin{bmatrix}3 & 1\\ 5 & 4\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 0\\ 4 & 5\end{bmatrix}$, then $AB=\begin{bmatrix}1 & 5\\ 11 & 20\end{bmatrix}$

  However, $BA=\begin{bmatrix}-3 & -1\\ 37 & 24\end{bmatrix}$

  Here, $AB \neq BA$.

  Consider,

  $A=\begin{bmatrix}2 & 0\\ 0 & 6\end{bmatrix}$ and $B=\begin{bmatrix}-5 & 0\\ 0 & 1\end{bmatrix}$

  $$AB=\begin{bmatrix}2 & 0\\ 0 & 6\end{bmatrix}\begin{bmatrix}-5 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-10 & 0\\ 0 & 6\end{bmatrix}$$
  $$BA=\begin{bmatrix}-5 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 6\end{bmatrix}=\begin{bmatrix}-10 & 0\\ 0 & 6\end{bmatrix}$$

  Here, $AB = BA$

  - The multiplication of diagonal matrices of the same order is commutative.

- The multiplication of matrices is associative i.e., $(AB)\,C = A\,(BC)$.
- Distributive law is satisfied for any three matrices.

  $A(B + C) = AB + AC$

  $(A + B)C = AC + BC$

- For every square matrix A, there exists an identity matrix I of the same order such that $AI = IA = A$.

Solved Examples

Example 1:

If $\begin{bmatrix} 4 & 3 \\ -2 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 7 \\ 9 \\ 20 \end{bmatrix}$, then what are the values of *x* and *y*?

Solution:

$$\begin{bmatrix} 4 & 3 \\ -2 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 7 \\ 9 \\ 20 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4x+3y \\ -2x+y \\ 4y \end{bmatrix}=\begin{bmatrix} 7 \\ 9 \\ 20 \end{bmatrix}$$

$\therefore 4x + 3y = 7$ . . . (1)

$-2x + y = 9$ . . . (2)

$4y = 20$ . . . (3)

From equation (3), we obtain $y = 5$.
On substituting this value in equation (2), we obtain $x = -2$.
Thus, the values of *x* and *y* are −2 and 5 respectively.

Example 2:

If $A=\begin{bmatrix} 0 & 3 \\ 1 & 4 \end{bmatrix}, B=\begin{bmatrix} -2 & 1 \\ 3 & 5 \end{bmatrix}$ and $C=\begin{bmatrix} 6 & 0 \\ 3 & 1 \end{bmatrix}$, then prove that

$A(B + C) = AB + AC$.

Solution:

$$B+C=\begin{bmatrix}-2 & 1\\ 3 & 5\end{bmatrix}+\begin{bmatrix}6 & 0\\ 3 & 1\end{bmatrix}=\begin{bmatrix}4 & 1\\ 6 & 6\end{bmatrix}$$

$$\therefore A(B+C)=\begin{bmatrix}0 & 3\\ 1 & 4\end{bmatrix}\begin{bmatrix}4 & 1\\ 6 & 6\end{bmatrix}=\begin{bmatrix}0+18 & 0+18\\ 4+24 & 1+24\end{bmatrix}=\begin{bmatrix}18 & 18\\ 28 & 25\end{bmatrix} \qquad ...(1)$$

$$AB=\begin{bmatrix}0 & 3\\ 1 & 4\end{bmatrix}\begin{bmatrix}-2 & 1\\ 3 & 5\end{bmatrix}=\begin{bmatrix}0+9 & 0+15\\ -2+12 & 1+20\end{bmatrix}=\begin{bmatrix}9 & 15\\ 10 & 21\end{bmatrix}$$

$$AC=\begin{bmatrix}0 & 3\\ 1 & 4\end{bmatrix}\begin{bmatrix}6 & 0\\ 3 & 1\end{bmatrix}=\begin{bmatrix}0+9 & 0+3\\ 6+12 & 0+4\end{bmatrix}=\begin{bmatrix}9 & 3\\ 18 & 4\end{bmatrix}$$

$$\therefore AB+AC=\begin{bmatrix}9 & 15\\ 10 & 21\end{bmatrix}+\begin{bmatrix}9 & 3\\ 18 & 4\end{bmatrix}=\begin{bmatrix}18 & 18\\ 28 & 25\end{bmatrix} \qquad ...(2)$$

From (1) and (2), we obtain

A (B + C) = AB + AC

Thus, the result is proved.

Transpose of a Matrix and Related Properties

Transpose of a Matrix

- If $A=\left[a_{ij}\right]_{m\times n}$ is a matrix of order $m \times n$, then the matrix obtained by interchanging its rows and columns is known as the transpose of A.
- It is denoted by A'. In general, if $A=\left[a_{ij}\right]_{m\times n}$, then $A'=\left[a_{ji}\right]_{n\times m}$

  For example: If $A=\begin{bmatrix}-3 & 2\\ 4 & 4\\ 5 & 6\end{bmatrix}$, then

$$A'=\begin{bmatrix}-3 & 2\\ 4 & 4\\ 5 & 6\end{bmatrix}'=\begin{bmatrix}-3 & 4 & 5\\ 2 & 4 & 6\end{bmatrix}$$

Properties of Transpose of Matrix

- If A and B are two matrices, then

- $(A')' = A$
- $(KA')' = KA$, where K is a constant
- $(A+B)' = A' + B'$
- $(AB)' = B'A'$

Note: Transpose of a row matrix is a column matrix and vice versa.

$$\text{For example, } \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix},\ \begin{bmatrix} a & b \end{bmatrix}^T = \begin{bmatrix} a \\ b \end{bmatrix}$$

Solved Examples

Example 1:

If $A = \begin{bmatrix} 4 & -2 \\ -1 & 6 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 4 \\ -2 & -3 \\ -1 & 2 \end{bmatrix}$, then prove that $(A+B)' = A' + B'$.

Solution:

$$A + B = \begin{bmatrix} 4 & -2 \\ -1 & 6 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 7 & 4 \\ -2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 4+7 & -2+4 \\ -1-2 & 6-3 \\ 3-1 & 1+2 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 11 & 2 \\ -3 & 3 \\ 2 & 3 \end{bmatrix}$$

$$\Rightarrow (A+B)' = \begin{bmatrix} 11 & 2 \\ -3 & 3 \\ 2 & 3 \end{bmatrix}' = \begin{bmatrix} 11 & -3 & 2 \\ 2 & 3 & 3 \end{bmatrix} \ldots (1)$$

$$A=\begin{bmatrix} 4 & -2 \\ -1 & 6 \\ 3 & 1 \end{bmatrix} \Rightarrow A'=\begin{bmatrix} 4 & -1 & 3 \\ -2 & 6 & 1 \end{bmatrix}$$

$$B=\begin{bmatrix} 7 & 4 \\ -2 & -3 \\ -1 & 2 \end{bmatrix} \Rightarrow B'=\begin{bmatrix} 7 & -2 & -1 \\ 4 & -3 & 2 \end{bmatrix}$$

$$A'+B'=\begin{bmatrix} 4 & -1 & 3 \\ -2 & 6 & 1 \end{bmatrix}+\begin{bmatrix} 7 & -2 & -1 \\ 4 & -3 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 11 & -3 & 2 \\ 2 & 3 & 3 \end{bmatrix} \ldots (2)$$

From (1) and (2), we obtain

$$(A+B)' = A'+B'$$

Thus, given result is proved.

Example 2:

If $A=\begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}$ and $B=\begin{bmatrix} 7 & 3 \\ -4 & 2 \end{bmatrix}$, then prove that $(AB)' = B'A'$.

Solution:

$$AB=\begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ -4 & 2 \end{bmatrix}=\begin{bmatrix} 14-4 & 6+2 \\ 35-24 & 15+12 \end{bmatrix}$$

$$=\begin{bmatrix} 10 & 8 \\ 11 & 27 \end{bmatrix}$$

$$(AB)' = \begin{bmatrix} 10 & 8 \\ 11 & 27 \end{bmatrix}' = \begin{bmatrix} 10 & 11 \\ 8 & 27 \end{bmatrix} \quad \dots (1)$$

$$B = \begin{bmatrix} 7 & 3 \\ -4 & 2 \end{bmatrix}$$

$$\Rightarrow B' = \begin{bmatrix} 7 & -4 \\ 3 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}$$

$$\Rightarrow A' = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 7 & -4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 14-4 & 35-24 \\ 6+2 & 15+12 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 11 \\ 8 & 27 \end{bmatrix} \quad \dots (2)$$

From (1) and (2), we obtain $(AB)' = B'A'$.

Thus, the given result is proved.

Example 3:

If $A = \begin{bmatrix} -3 & 2 \\ 4 & 4 \\ 5 & 6 \end{bmatrix}$, then determine the transpose of *A*.

Solution:

The transpose of *A* is given by

$$A' = \begin{bmatrix} -3 & 2 \\ 4 & 4 \\ 5 & 6 \end{bmatrix}' = \begin{bmatrix} -3 & 4 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Example 4:
Let $m \times n$ be the order a matrix *A*. If the order of the transpose of the matrix *A* is 4×3 then find the value of *m* and *n*.

Solution:
Order of the matrix A = $m \times n$
Order of transpose of the matrix A = $n \times m = 4 \times 3$
Therefore, $n = 4$ and $m = 3$.

Example 5:

$$\text{Let A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, \text{ then show that } (A')' + A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \\ 10 & 12 \end{pmatrix}$$

Answer:
Here,

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \Rightarrow A' = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

$$\Rightarrow (A')' = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

Now,

$$(A')' + A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$(A')' + A = 2\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$(A')' + A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \\ 10 & 12 \end{pmatrix}$$

Symmetric and Skew Symmetric Matrices

- A square matrix A = $[a_{ij}]$ of order n is said to be symmetric, if $A' = A$. That is, $[a_{ij}] = [a_{ji}]$ for all values of i and j.

    For example, consider the matrix:

$$A=\begin{bmatrix}3 & -4 & 1\\ -4 & 3 & 0\\ 1 & 0 & 3\end{bmatrix}$$

$$A'=\begin{bmatrix}3 & -4 & 1\\ -4 & 3 & 0\\ 1 & 0 & 3\end{bmatrix}$$

$\therefore A = A'$

Thus, A is a symmetric matrix.

Note: If $\begin{bmatrix}p & q\\ r & s\end{bmatrix}$ is a symmetric matrix of order 2×22×2, then $q = r$.

- A square matrix $A = [a_{ij}]$ is said to be skew symmetric if $A' = -A$. That is, $[a_{ji}] = -[a_{ij}]$ for all values of i and j

For example, consider $A=\begin{bmatrix}0 & -3 & -2\\ 3 & 0 & -1\\ 2 & 1 & 0\end{bmatrix}$

$$A'=\begin{bmatrix}0 & 3 & 2\\ -3 & 0 & 1\\ -2 & -1 & 0\end{bmatrix}=-\begin{bmatrix}0 & -3 & -2\\ 3 & 0 & -1\\ 2 & 1 & 0\end{bmatrix}$$

$\therefore A' = -A$

Thus, A is a skew symmetric matrix.

Note: If $\begin{bmatrix}0 & q\\ r & 0\end{bmatrix}$ is a skew symmetric matrix of order 2×22×2, then $q = -r$ or $r = -q$.

- If A is square matrix with real elements, then $A + A'$ is a symmetric matrix and $A - A'$ is a skew symmetric matrix.
- Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

Solved Examples

Example 1:

If $A=\begin{bmatrix}-7 & 3\\ 4 & -2\end{bmatrix}$, then prove that $A + A'$ is symmetric and $A - A'$ is skew symmetric.

Solution:

$$A=\begin{bmatrix}-7 & 3\\ 4 & -2\end{bmatrix}$$
$$A'=\begin{bmatrix}-7 & 4\\ 3 & -2\end{bmatrix}$$

$$A+A'=\begin{bmatrix}-7 & 3\\ 4 & -2\end{bmatrix}+\begin{bmatrix}-7 & 4\\ 3 & -2\end{bmatrix}=\begin{bmatrix}-14 & 7\\ 7 & -4\end{bmatrix}$$
$$\text{Now,}(A+A')'=\begin{bmatrix}-14 & 7\\ 7 & -4\end{bmatrix}$$
$$\therefore (A+A')'=A+A'$$

Thus, $A + A'$ is a symmetric matrix.

$$A-A'=\begin{bmatrix}-7 & 3\\ 4 & -2\end{bmatrix}-\begin{bmatrix}-7 & 4\\ 3 & -2\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$
$$\Rightarrow (A-A')'=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=-\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$
$$\therefore (A-A')'=-(A-A')$$

Thus, $A - A'$ is a skew symmetric matrix.

Example 2:

If $A=\begin{bmatrix}-\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$, then express A as the sum of a symmetric and a skew symmetric matrix.

Solution:

We can write A as:

$$A=\frac{1}{2}(A+A')+\frac{1}{2}(A-A')$$
$$A=P+Q$$
$$A=\begin{bmatrix}-\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$
$$A'=\begin{bmatrix}-\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$
$$P=\frac{1}{2}(A+A')$$
$$P=\frac{1}{2}\left\{\begin{bmatrix}-\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}+\begin{bmatrix}-\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\right\}$$
$$P=\frac{1}{2}\begin{bmatrix}-2\cos\theta & 0\\ 0 & 2\cos\theta\end{bmatrix}$$
$$P=\begin{bmatrix}-\cos\theta & 0\\ 0 & \cos\theta\end{bmatrix}$$
$$P'=\begin{bmatrix}-\cos\theta & 0\\ 0 & \cos\theta\end{bmatrix}$$
$$\therefore P=P'$$

$\therefore$ *P* is a symmetric matrix.

$$Q=\frac{1}{2}(A-A')$$
$$Q=\frac{1}{2}\left\{\begin{bmatrix}-\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}-\begin{bmatrix}-\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\right\}$$
$$Q=\frac{1}{2}\begin{bmatrix}0 & 2\sin\theta\\ -2\sin\theta & 0\end{bmatrix}$$
$$Q=\begin{bmatrix}0 & \sin\theta\\ -\sin\theta & 0\end{bmatrix}$$
$$Q'=\begin{bmatrix}0 & -\sin\theta\\ \sin\theta & 0\end{bmatrix}=-\begin{bmatrix}0 & \sin\theta\\ -\sin\theta & 0\end{bmatrix}$$
$$\therefore Q'=-Q$$

Therefore, Q is a skew symmetric matrix.

$$\Rightarrow \begin{bmatrix} -\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = P + Q = \begin{bmatrix} -\cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix} + \begin{bmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{bmatrix}$$

Where, P is a symmetric matrix and Q is a skew symmetric matrix

Elementary Transformation and Inverse of a Matrix

Elementary Transformation of a Matrix

- Interchange of any two rows or columns of a matrix.
- It is denoted as $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$.

For example, consider $A = \begin{bmatrix} 2 & 4 & -5 \\ -3 & 6 & 1 \end{bmatrix}$

Applying $C_2 \leftrightarrow\leftrightarrow C_3$, we obtain $A = \begin{bmatrix} 2 & -5 & 4 \\ -3 & 1 & 6 \end{bmatrix}$

- Elements of any row or column multiplied by a non-zero number.
- It can be denoted as $R_i \leftrightarrow kR_i$ of $C_i \leftrightarrow kC_i$, where k is a non-zero constant.

For example, applying $R_1 \leftrightarrow \frac{1}{5} R_1$ to $A = \begin{bmatrix} 5 & 10 \\ -3 & 0 \end{bmatrix}$, we obtain $\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}$.

- Addition to the elements of any row or column; the corresponding elements of any other row or column multiplied by any non-zero number.

It is denoted as $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$.

For example, applying $R_1 \to (R_1 - 2R_2)$ to the matrix $\begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}$, we obtain $\begin{bmatrix} 0 & -7 \\ 2 & 3 \end{bmatrix}$.

Inverse of a Matrix

- If A is an invertible square matrix of order m and if there exists another square matrix of order m, such that $AB = BA = I$, then B is known as the inverse of A and is denoted as A^{-1}.

- Inverse of rectangular matrix is not defined.
- The inverse of a square matrix, if it exists, is unique.
- If A is the inverse of B, then B is the inverse of A.
- If A and B are invertible matrices of same order, then $(AB)^{-1} = B^{-1}A^{-1}$

- To find the inverse of an invertible matrix A, elementary operations can be applied. The matrix A can be written as $A = IA$ ($A = AI$), then elementary row (column) operations are applied to obtain $I = BA$ ($I = AB$). Hence, B is the inverse of A.

Solved Examples

Example 1:

Find the inverse of the matrix $\begin{bmatrix} 2 & 6 & 0 \\ -1 & 8 & 1 \\ 4 & 2 & 0 \end{bmatrix}$.

Solution:

Let $A = \begin{bmatrix} 2 & 6 & 0 \\ -1 & 8 & 1 \\ 4 & 2 & 0 \end{bmatrix}$

Now, $A = IA$

$\therefore \begin{bmatrix} 2 & 6 & 0 \\ -1 & 8 & 1 \\ 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Applying $R_3 \to R_3 + 4R_2,\ R_2 \to R_2 + \frac{1}{2}R_1,$

$$\begin{bmatrix} 2 & 6 & 0 \\ 0 & 11 & 1 \\ 0 & 34 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 - 3R_2,$

$$\begin{bmatrix} 2 & 6 & 0 \\ 0 & 11 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{3}{2} & 1 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - R_3,$

$$\begin{bmatrix} 2 & 6 & 0 \\ 0 & 10 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ -\frac{3}{2} & 1 & 1 \end{bmatrix} A$$

Applying $R_2 \to \frac{1}{10}R_2,$

$$\begin{bmatrix} 2 & 6 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{5} & 0 & -\frac{1}{10} \\ -\frac{3}{2} & 1 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 6R_2,\ R_3 \to R_3 - R_2,$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & 0 & +\frac{3}{5} \\ \frac{1}{5} & 0 & -\frac{1}{10} \\ -\frac{17}{10} & 1 & \frac{11}{10} \end{bmatrix} A$$

Applying $R_1 \to \frac{1}{2} R_1$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} & 0 & +\frac{3}{10} \\ \frac{1}{5} & 0 & -\frac{1}{10} \\ -\frac{17}{10} & 1 & \frac{11}{10} \end{bmatrix} A$$

$$\therefore A^{-1} = \begin{bmatrix} -\frac{1}{10} & 0 & \frac{3}{10} \\ \frac{1}{5} & 0 & -\frac{1}{10} \\ -\frac{17}{10} & 1 & \frac{11}{10} \end{bmatrix}$$

Example 2:

Find the inverse of the matrix $\begin{bmatrix} -3 & -9 \\ 6 & 12 \end{bmatrix}$.

Solution:

Let $A = \begin{bmatrix} -3 & -9 \\ 6 & 12 \end{bmatrix}$

Now, $A = IA$

$$\therefore \begin{bmatrix} -3 & -9 \\ 6 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 + 2R_1$,

$$\begin{bmatrix} -3 & -9 \\ 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} A$$

Applying $R_2 \to -\frac{1}{6}R_2,\ R_1 \to -\frac{1}{3}R_1$,

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{-1}{6} \end{bmatrix} A$$

Applying $R_1 \to R_1 - 3R_2$,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{6} \end{bmatrix} A$$

$$\therefore\ A^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{6} \end{bmatrix}$$