Chapter 4

Operations Research - I

CHAPTER HIGHLIGHTS

- *Linear Programming*
- *Contents of a Linear Programming Problem*
- *Formulation of Linear Programming Problem*
- *Simplex Method*
- *Duality in Linear Programming*
- *Primal of Dual Problem*
- *Transportation Problem*
- *Feasible Solution*
- *Optimal Solution*
- *Check of Optimality(Modi Method)*
- *Assignment Problem*

Linear Programming

Linear programming is a very important tool of operations research which is being widely used in the field of production management. Every business has limited resources (men, material, money, etc). Programming problems generally deal with determining optimum allocation of limited resources to meet the set of given objectives. The objective is to optimise the total profit minimising the total cost using the limited resources subjected to various restrictions.

Linear programming deals with a class of problems in which all relations among the variables are linear. Therefore, linear programming problems include a set of simultaneous linear equations or inequalities which represent the conditions of the problem. It also contains a linear expression which represents the objective function.

Linear programming may be defined as a method for determining the optimum program of inter-dependent activities with the use of available resources. The ultimate objective of linear programming is to maximise the profit or minimise the cost subjected to the constraints specified in the problem.

Contents of a Linear Programming Problem

The contents of a linear programming problem are the following:

Objective Function

It is in the form of a linear equation $Z = a_1x_1 + a_2x_2 + a_3x_3 + \ldots\ldots\ldots\ldots$ where $a_1, a_2, a_3\ldots\ldots\ldots\ldots\ldots\ldots$ are constants and $x_1, x_2, x_3\ldots\ldots\ldots$ are decision variables.

Linear Constraints

These are the restrictions imposed on the decision variables of the objective function. These are also in the form of linear equations

$$c_1x_1 + c_2x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \begin{matrix} < \\ = \\ > \end{matrix} C$$

$$d_1x_1 + d_2x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \begin{matrix} < \\ = \\ > \end{matrix} d, \text{ etc.}$$

Feasible Solution

These are the set of values of decision variables which satisfy the set of constraints.

Optimal Solution

Among the feasible solutions, the one which optimises the objective function is known as optimal solution.

Formulation of LPP

The first step to seek the solution of LPP is to formulate a mathematical model for the defined problem.

Step 1: Understand the objective of the problem, whether it is a maximising problem or a minimising problem.

Step 2: Express quantitatively, the objective of the problem as a mathematical function in terms of the assumed decision variables.

Step 3: Express the constraints also in terms of decision variables. It indicates the availability of resources. These are also linear equations or inequalities.

Step 4: Express the non-negative conditions of the decision variables $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$, etc……..

A standard linear programming problem will be of the type,

x_1, x_2, x_3..........................to optimise

$Z = a_1x_1 + a_2x_2 + a_3x_3$..................

Subjected to the constraints

$b_1x_1 + b_2x_2 + b_3x_3 + \ldots\ldots\ldots\ldots \overset{<}{\underset{>}{=}} b$

$c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots\ldots\ldots \overset{<}{\underset{>}{=}} c$

$d_1x_1 + d_2x_2 + d_3x_3 + \ldots\ldots\ldots\ldots \overset{<}{\underset{>}{=}} d$

....

....

$x_1, x_2, x_3, x_4, \ldots\ldots\ldots\ldots \geq 0.$

For example:

1. Consider a firm manufacturing two products A and B, which have to pass through two departments: cutting and finishing to get its final shape. One unit of A requires 1 hr in the cutting department, 2 hrs in the finishing department. One unit of B requires 3 hrs in the cutting department and 1 hr in the finishing department. The total number of hrs available in the cutting department is 180 hrs/month and in the finishing department, it is 140 hrs/month. If we sell one unit of 'A' in the market, we get a profit of ₹15, and by selling one unit of B, we get a profit of ₹18 utilising all the resources to maximise the profit, how many numbers of A are to be produced? How many numbers of B are to be produced?

The problem mentioned above is a linear programming problem. It is to be formulated to a mathematical model.

Let us that assume 'x' units of 'A' are to be produced and 'y' units of B are to be produced.

Then the objective function which is to be maximised is $Z = 15x + 18y$; ₹15, ₹18, being the profits of A and B respectively.

The constraints are the availability of time in the cutting and finishing departments. If 'x' units of A and 'y' units of B are produced, the time consumed at the cutting departments

$$1x + 3y \leq 180.$$

The time consumed at the finishing department is

$2x + y \leq 140.$

The problem can be stated as follows:

Maximise $Z = 15x + 18y$

Subjected to $x + 3y \leq 180$

$2x + y \leq 140$

$x, y \geq 0.$

The problem can be tabulated as shown below:

	₹15 Product A (x)	₹18 Product B (x)	
Cutting	1	3	180
Finishing	2	1	140

$Z = 15x + 18y$, subjected to

$x + 3y \leq 180$

$2x + y \leq 140.$

2. Consider the case of a firm producing two gadgets X and Y. It has to pass through the forging and then through the treatment departments. The man hours available, and the profits gain through unit selling of each are given in the following chart.

	Gadget X Profit/unit = 30	Gadget Y Profit/unit = 20	Hours (available)
Forging	10 hrs	6 hrs	1000 hrs
Treatment	5 hrs	4 hrs	600 hrs

To formulate the mathematical model of LPP, objective function

$$Z = 30x + 20\,y$$

where, 'x' is the quantity of X produced, 'y' is the quantity of Y produced.

Constraints are

$10x + 6y \leq 1000$

$5x + 4y \leq 600$

$x, y \geq 0$

∴ The model is

Maximise $Z = 30x + 20y$

Subjected to $10x + 6y \leq 1000$

$5x + 4y \leq 600$

$x, y \geq 0.$

Once the LPP is formulated, it can be solved either graphically or by simplex method. Graphical method is usually done when the variables involved are only 2 (maximum). In the case of 3 or more variables, graphical methods may result in errors.

Another method is known as the simplex method. It is a powerful method used to solve any linear programming problem. It is an iterative method.

Solved Example

Example 1: A furniture manufacturer makes two products p_1 and p_2. Two machines A and B are performing the operations, p_1 requires 2 hrs on A and 6 hrs on B. p_2 requires 5 hrs on A only. 16 hrs are available on A, 30 hrs are available on B. p_1 fetches a profit of ₹1 per unit and p_2 a profit of ₹5 per unit. The maximum profit shall be

(A) 11 units (B) 16 units (C) 18 units (D) 20 units.

Solution:

	Profit/unit = 1 Product p_1	Profit/unit = 5 Product p_2	Total hrs available
Machine A	2	5	16
Machine B	6	0	30

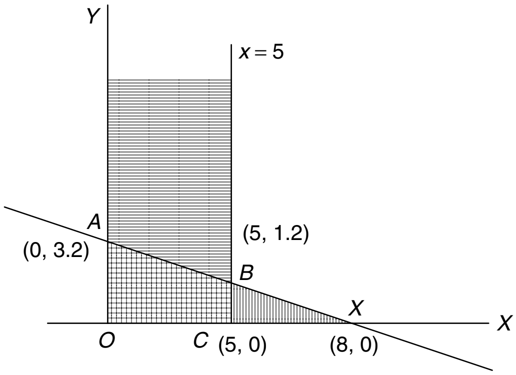
Let 'x' units of p_1 and 'y' units of p_2 be produced. The problem becomes

Maximise $Z = x + 5y$

Subjected to $2x + 5y \leq 16$

$6x \leq 30, x, y \geq 0$

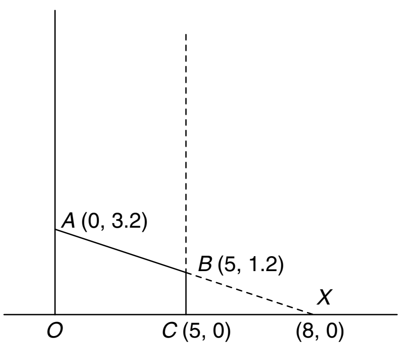
Prepare a graph taking 'x' along the x-axis and 'y' along the y axis.



Represent the constraints on the graph $2x + 5y \leq 16$.
Consider it as $2x + 5y = 16$
When $y = 0, x = 8$, when $x = 0, y = 3.2$
(8, 0) and (0, 3.2) are the points on the line. The cross hatched region gives the inequality $2x + 5y \leq 0$
Next constraint is $6x \leq 30$

$x \leq 5.$

Consider $x = 5$
It is the horizontally hatched region. The region satisfying both the constraints is $OABC$. The objective function is $Z = x + 5y$.



Point B can be found by solving
$x = 5, 2x + 5y = 16$

$10 + 5y = 16$

$5y = 16$

$y = 1.2$

$B = (5, 1.2).$

Substitute the points O, A, B and C in the objective functions.
At A (0, 3.2)

$$Z = x + 5y = 0 + 5 \times 3.2 = 16$$

At B

$$Z = 5 + 5 \times 1.2 = 5 + 6 = 11$$

At C (5, 0)

$$Z = x + 5y = 5$$

Among the three, Z is maximum at $x = 0$,

$y = 3.2$

$Z_{max} = 16.$

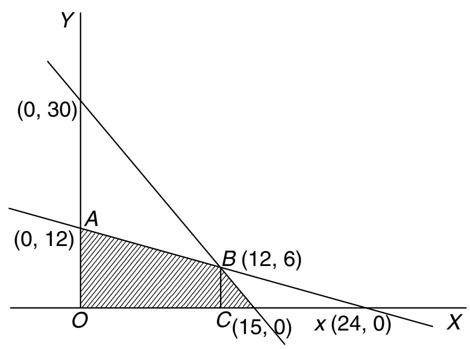
Example 2: A company produces two products P and Q. The articles are processed through two departments D_1 and D_2 their capacity being 60 hrs/week and 48 hrs/week respectively. One unit of 'P' requires 4 hrs in D_1 and 2 hrs in D_2. One unit of Q requires 2 hrs in D_1 and 4 hrs in D_2. Profits fetched by one unit of P and Q are ₹8 and ₹6 respectively. Optimising the production, maximum profit is
(A) ₹132 (B) ₹216 (C) ₹286 (D) ₹323.

Solution:
Tabulating the details we have, (assume 'x' units of P and 'y' units of Q are to be produced)

	Product P ₹8/unit	Product Q ₹6/unit	Hours available
D_1	4	2	60 hrs
D_2	2	4	48 hrs

$$4x + 2y \leq 60, 2x + 4y \leq 48$$
$$Z = 8x + 6y, x, y \geq 0$$

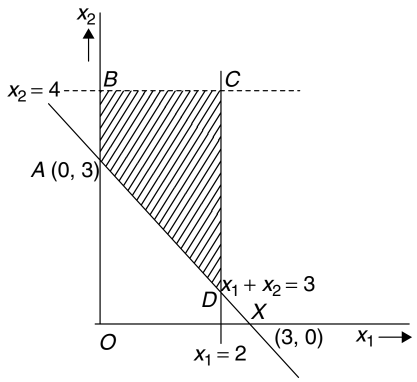
Considering the inequalities as equations and plotting,



Point B is
$4x + 2y = 60$
$2x + 4y = 48$
$4x + 2y = 60$
$4x + 8y = 96$
$6y = 36$
$y = 6$
$x = 12$
The feasible region is $OABC$
Z at A $8 \times 0 + 6 \times 12 = 72$
Z at B $8 \times 12 + 6 \times 6 = 132$
Z at C $8 \times 15 + 6 \times 0 = 120$
$Z_{max} =$ ₹132.

Example 3: Maximise the function $Z = 4x_1 + 6x_2$ subjected to the constraints
$x_1 \leq 2$
$x_2 \leq 4, x_1 + x_2 \geq 3$
$x_1, x_2 \geq 0$
The maximum value is
(A) 24 (B) 18 (C) 24 (D) 32.

Solution:



As $x_1 + x_2 \geq 3$,
the feasible region is $ABCD$; the corner points A, B, C, D are
(A) (0,3) (B) (0,4) (C) (2,4) (D) (2,1).

Value of Z at the corner points is

Z at $A = 4x_1 + 6x_2 = 18$

Z at $B = 4x_1 + 6x_2$
$= 0 + 6 \times 4 = 24$

Z at $C = 4x_1 + 6x_2$
$= 4 \times 2 + 6 \times 4 = 32$

Z at $D = 4x_1 + 6x_2$
$= 4 \times 2 + 6 \times 1 = 14$

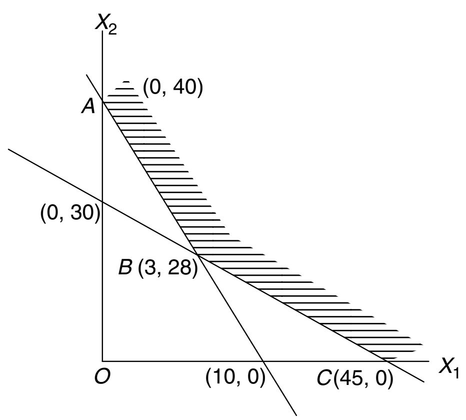
$\therefore Z_{max} = 32.$

Example 4: Minimum value of the function $5x_1 + 4x_2$ subjected to

$$4x_1 + x_2 \geq 40$$
$$2x_1 + 3x_2 \geq 90$$
$$x_1, x_2 \geq 0 \text{ is}$$

(A) 225 (B) 160 (C) 127 (D) 112.

Solution:



$$\frac{x_1}{10} + \frac{x_2}{40} \geq 1$$

$$\frac{x_1}{45} + \frac{x_2}{30} \geq 1$$

It is a minimisation problem and the feasible region is above AB and BC.

Required to find the minimum of Z. The corner points are (0, 40), (3, 28), (45, 0).

Z at $A = 5x_1 + 4x_2$
$= 4 \times 40 = 160.$

The points B is

$$= 4x_1 + x_2 = 40$$
$$= 2x_1 + 3x_2 = 90$$
$$x_1 + 3x_2 = 120$$
$$2x_1 + 3x_2 = 90$$
$$10x_1 = 30 \quad \therefore x_1 = 3,$$
$$x_2 = 28;\ (3, 28)$$

Z at $B = 5x_1 + 4x_2$
$= 5 \times 3 + 4 \times 28 = 15 + 112 = 127$

Z at $C = 5x_1 + 4x_2 = 5 \times 45 + 0 = 225$

The least value is $Z = 127$.

NOTE

This is an example of unbounded solution space. But the solution is bounded at (3, 28).

Had it been a problem to maximise, it would have been a case of unbounded solution space and unbounded optimal solution.

Example 5: The maximum value of the function $Z = 3x + 2y$ subjected to the constraints

$x - y \leq 15$

$2x - y \leq 40$, $x, y \geq 0$ is

(A) unbounded (B) ∞ (C) Zero (D) 225.

Solution:

Consider the constraints

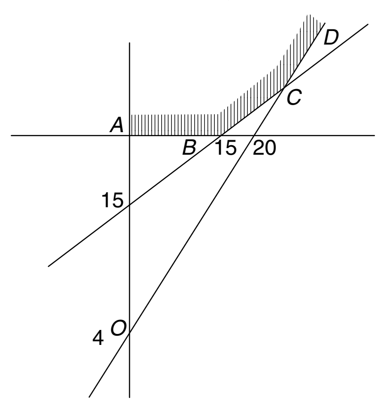
$$x - y = 15$$

$$\text{i.e. } \frac{x}{15} + \frac{y}{-15} = 1$$

$$2x - y = 40$$

$$\text{i.e. } \frac{x}{20} + \frac{y}{-40} = 1$$

$$x, y \geq 0$$

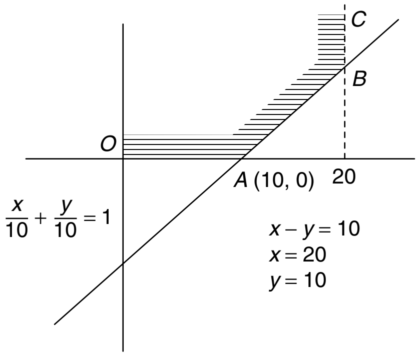


As $x, y \geq 0$, the feasible region is above $ABCD$; the solution space is unbounded. The optimal solution is also unbounded.

Example 6: The maximum value of $Z = 3x - y$ subjected to the constraints
$x - y \leq 10$
$x \leq 20, x, y \geq 0$

(A) 40 (B) 60 (C) 50 (D) 80.

Solution:

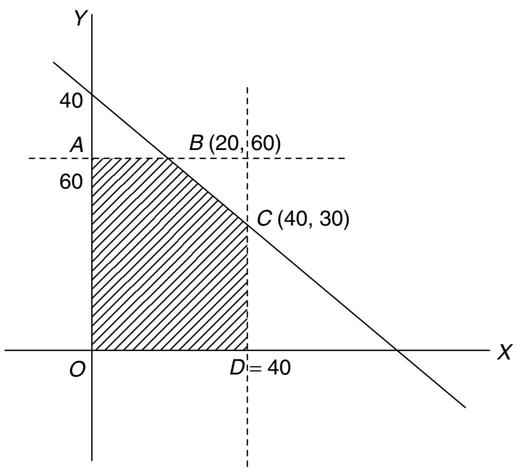


The solution space is *OABC,* corner points *B* (20, 10).
Z at $A = 30$
Z at $B = 3 \times 20 - 10 = 50.$
At any point along *BC*, other than *B*, $z < 50$.
Therefore, $Z_{max} = 50$.
This is a case of unbounded solution space, but bounded optimum solution.

Example 7: The value of Z_{max} when $Z = 3x + 2y$ subjected to the constraints
$x \leq 40$
$y \leq 60$
$3x + 2y \leq 180, x, y \geq 0$

(A) 120 (B) 160 (C) 180 (D) 240.

Solution:



$\frac{x}{60} + \frac{y}{90} \leq 1$. The feasible region is *OABCD*,
when $x = 40$

$$\frac{2}{3} + \frac{y}{90} = 1$$

$$\frac{y}{90} = \frac{1}{3} \quad y = 30$$

$$\therefore C(40, 30)$$

$$\frac{x}{60} + \frac{2}{3} = 1$$

$$x = 60 \times \frac{1}{3} = 20 \quad \therefore B\ (20, 60).$$

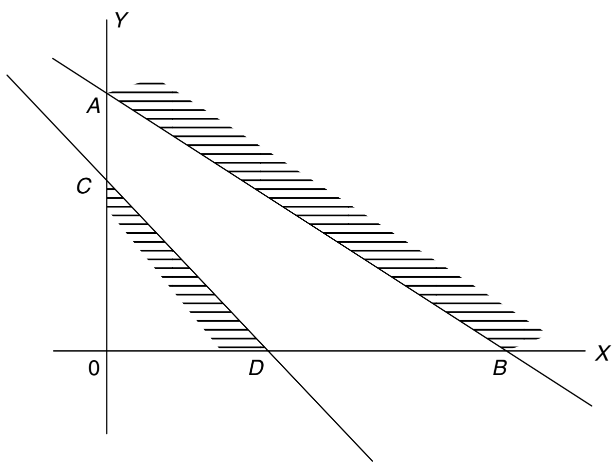
Z at $A = 3 \times 0 + 2 \times 60 = 120$
Z at $B = 3 \times 20 + 2 \times 60 = 180$
Z at $C = 3 \times 40 + 2 \times 30$
$= 120 + 60 = 180$
Z at $D = 3 \times 40 + 0 = 120$
This is a problem with multiple solutions.

Example 8: The maximum value of Z when $Z = 3x + 4y$ subjected to the constraints

$2x + y \leq 4$
$x + 2y \geq 12, x, y \geq 0$

(A) No feasible solution
(B) Though solution space is unbounded, optimal solution is bounded
(C) Multiple solutions
(D) Convex solution space, unbounded optimal solution.

Solution:



$$\frac{x}{2} + \frac{y}{4} = 1$$

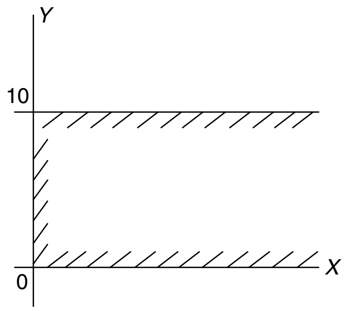
$$\frac{x}{12} + \frac{y}{6} = 1$$

The feasible region is below *CD* and at the same time, above *AB* also. There is no feasible solution.

Example 9: The maximum value of Z when $Z = 2x + 3y$, subjected to $x, y \geq 0$
$y \leq 10$ is

(A) 20 (B) 30
(C) 50 (D) unbounded solution.

Solution:



The solution space as well as the solution are both unbounded.

Example 10: The value of Z_{max} when $Z = 2x + 5y$, subjected to the constraints

$x + 3y \le 40$

$3x + y \le 24$

$x + y \le 10$, $x, y \ge 0$

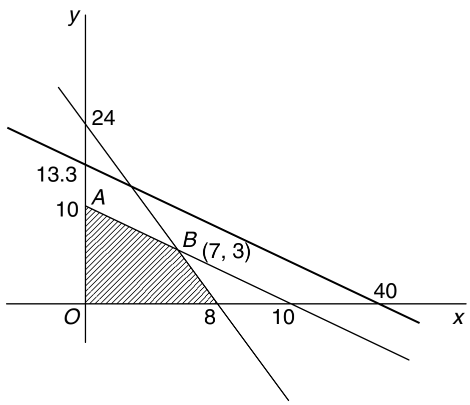
(A) 32 (B) 29 (C) 26 (D) 22.

Solution:

The feasible region is *OABC;*

point *B,*

$$\begin{aligned} x + y &= 10 \\ 3x + y &= 24 \\ x &= 7 \\ y &= 3 \end{aligned}$$



Z at $O = 0$

Z at $A = 2 \times 10 + 5 \times 0$

$= 20$

Z at $B = 2 \times 7 + 5 \times 3$

$= 14 + 15 = 29$

Z at $C = 2 \times 8 + 0 = 16.$

Example 11: The maximum value of Z, when $Z = x + 2y$ subjected to the conditions

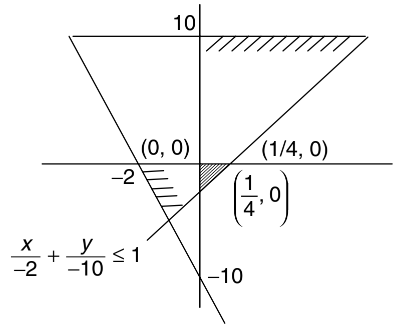
$4y - 4x \ge -1$, $-4y + 4x \le 1$

$5x + y \ge -10$

$y \le 10$, $x, y \ge 0$

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) 1.

Solution:



Only one point is feasible $\left(\frac{1}{4}, 0\right)$

$Z = \frac{1}{4}$.

Example 12: The maximum value of Z when $Z = 3x + 2y$ subjected to the constraints

$4x + y \le 60$

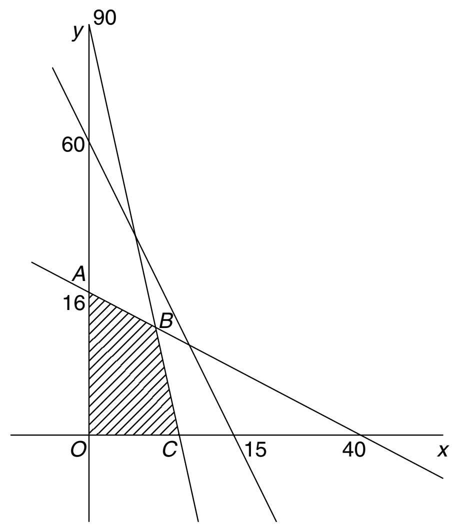
$8x + y \le 90$

$2x + 5y \le 80$

$x, y \ge 0$

(A) 80 (B) 108.5 (C) 102.5 (D) 112.5.

Solution:



The feasible region is *OABC*.

B (9.15, 16.84)

$$\begin{aligned} 8x + y &= 90 \\ 2x + 5y &= 80 \\ 8x + 20y &= 320 \\ 19y &= 230 \\ y &= 16.84 \\ 8x + 16.84 &= 90 \\ 8x &= 73.16 \\ x &= 9.15 \end{aligned}$$

$$\frac{x}{15} + \frac{y}{60} < 1, \quad \frac{x}{11.25} + \frac{y}{90} < 1$$

Z at $A = 5 \times 16 = 80$

Z at $B = 2 \times 9.15 + 5 \times 16.84 = 102.5$
Z at $C = 2 \times 11.25 = 22.5$
$Z_{max} = 102.5.$

Nature of Solution Space and Optimal Solution

1. If the solution space is a closed region, then the solution space and the optimal solution both are bounded.
2. If the feasible region formed by the constraints are without boundary and if the solution is at infinity, then the solution space and the optimal solution are both unbounded.
3. If the feasible region is without boundary and even then solution is possible, then the solution space is unbounded, but the optimal solution is bounded.
4. If separate regions are defined by constraints, then there is no feasible region and no solution.

Simplex Method

The following steps are required for solving a linear programming problem using the simplex method:

Step 1: Formulate the problem.
Step 2: Identify the objective function and constraints.
Step 3: Convert the inequalities to equations using slack variables.
Step 4: Develop the programme table.

Example 13: If $Z = 24x + 48y$,
subjected to the constraints
$x + 2y \leq 48$
$2x + y \leq 60$
$x, y \geq 0,$
Z_{max} is
(A) 1152 (B) 2304 (C) 576 (D) 1296.

Solution:
Simplex method
Applying slack variables, the problem becomes a maximisation problem

$$Z = 24x + 48y + 0.\,S_1 + 0.S_2$$

Subjected to

$$x + 2y + S_1 + 0.S_2 = 48$$
$$2x + y + 0.S_1 + S_2 = 60$$

Programme,

Table I

Slack Variables	Value of slack		24	48	0	0	
		Qty	x	y	S_1	S_2	bi
S_1	0	48	1	2	1	0	24✓
S_2	0	60	2	1	0	1	60
NER			24	48✓	0	0	

1. Take slack variables in the programme
2. Values of slack variables are zero
3. R.H.S of the constraints are denoted in the quantity column
4. express the constraints in the programme
5. Vector $Cj \rightarrow$ the coefficient of objective function.

NER is net evaluation row.
NER is $24 - \{1 \times 0 + 2 \times 0\} = 24$
$48 - \{2 \times 0 + 1 \times 0\} = 48$
$0 - \{1 \times 0 + 0 \times 0\} = 0$
$0 - \{0 \times 0 + 1 \times 0\} = 0.$

(a) NER denotes the penalty for not taking the variable into the programme. Consider the maximum value of NER. The column corresponding to the maximum penalty is key-column. The corresponding variable is to be taken to programme.
(b) To get the values for 'bi' vector, divide the quantity values by the corresponding key-column number.
(c) Consider the least 'bi' value. The corresponding row is known as the key row. The number at the intersection of the key column and key row is known as the key number. In this case, '2' is the key number.

Table II

Variable	Value		24	48	0	0	0
		Qty	x	y	S_1	S_2	bi
Y	48	24	$\frac{1}{2}$	1	$\frac{1}{2}$	0	
S_2	0	36	$\frac{3}{2}$	0	$-\frac{1}{2}$	1	
NER		0	0	–24	0		

In Table II, the outgoing variable is that denoted by the key row i.e. S_1

The incoming variable is that denoted by the key-column i.e.,'y'. The quantity for the newly entered variable is the coefficient of the variable in the objective function i.e., 48. Quantity onwards, divide by the key number and write the values. For variable 'S', the values from quantity can be written by the formula (qty onwards)

$$\text{Old row no} - \text{key row no} \times \text{fixed ratio.}$$

$$\text{Fixed ratio for a row containing '}S_2\text{'} = \frac{\text{Row No.}}{\text{Key row No.}}$$

$\therefore$ For S_2 it is $= \dfrac{1}{2}$

Quantity for S_2 is $60 - 48 \times \dfrac{1}{2} = 60 - 24 = 36$.

Subsequent values are

$$2 - 1 \times \frac{1}{2} = \frac{3}{2}$$
$$1 - 2 \times \frac{1}{2} = 0$$
$$0 - 1 \times \frac{1}{2} = -\frac{1}{2}$$
$$1 - 0 \times \frac{1}{2} = 1.$$

Next is NER,

$$24-\left[48\times\frac{1}{2}-\frac{3}{2}\times 0\right]=0$$

$$48-[1\times 48+0\times 0]=0$$

$$0-\left[\frac{1}{2}\times 48+\frac{1}{2}\times 0\right]=-24$$

$$0-[0-1\times 0]=0.$$

When all the values in NER are either negative or zero, the solution is optimal.

$\therefore\ y = 24;\ S_2 = 36$, but S_2 is a slack variable which is insignificant.

$\therefore\ y = 24, x = 0$

$Z_{\max} = 24x + 48y$

$= 24 \times 0 + 48 \times 24$

$= 1152.$

Condition	Maximisation	Minimisation
Optimality	$c_j - z_j \geq 0 \quad \forall j$	$c_j - z_j \leq 0 \quad \forall j$
Unique Optimal	$c_j - z_j \geq 0 \quad \forall j$	$c_j - z_j \leq 0 \quad \forall j$
	Number of zeroes in $(c_j - z_j)$ = Number of Constraints	
Alternative Optimal Solution	$c_j - z_j \geq 0 \quad \forall j$	$c_j - z_j \leq 0 \quad \forall j$
	Number of zeroes in $(c_j - z_j)$ > Number of Constraints	
Unbounded Solution	(1) All the key column elements ≤ 0 (or) (2) There exists an entering variable but no leaving variable	
Infeasible Solution	$c_j - z_j \geq 0 \quad \forall j$	$c_j - z_j \leq 0 \quad \forall j$
	There exists an artificial variable in the basis	

Here, $c_j - z_j$ = NER.

Example 14: A company produces two products P and Q. Both require processing in two machines M_1 and M_2. The time requirement in each machine and the total time availability of each machine are given in the following table.

	Product *P*	Product *Q*	Time availability
M/c M_1	1	2	6
M/c M_2	4	3	12

Profit fetched by one unit of P and one unit of Q are ₹7 and ₹5 respectively. The maximum profit obtained is

(A) 36 (B) 30 (C) 21 (D) 15.

Solution:

The problem is an LPP.

Let 'x' units of 'P' and 'y' units of 'Q' be produced to maximise the profit.

Then the profit function $Z = 7x + 5y$.

Constraints are $x + 2y \leq 6$

$4x + 3y \leq 12;\ x, y \geq 0$

$x + 2y + S_1 = 6$

$\therefore\ 4x + 3y + S_2 = 12.$

Taking this into programme.

Table 1

P	Value	*Z* =	7	5	0	0	
		Qty	*x*	*y*	S_1	S_2	*bi*
S_1	0	6	1	2	1	0	6
S_2	0	12	4	3	0	1	3 ✓
NER			7 ✓	5	0	0	

Table 2

P	Value	*Z* =	7	5	0	0	
		Qty	*x*	*y*	S_1	S_2	*bi*
S_1	0	3	0	$-\frac{5}{4}$	1	$-\frac{1}{4}$	
x	7	3	1	$\frac{3}{4}$	0	$\frac{1}{4}$	
NER			0	$-\frac{1}{4}$	0	$-\frac{1}{4}$	

Fixed ratio for S_1 is $\frac{1}{4}$

S_1 row values are

Old row No. – Key row no × Fixed ratio.

$$6-12\times\frac{1}{4}=3 \text{ and } 1-4\times\frac{1}{4}=0 \text{ and}$$

$$2-3\times\frac{1}{4}=\frac{5}{4} \text{ and } 1-0=1 \text{ and}$$

$$0-1\times\frac{1}{4}=\frac{-1}{4}$$

As NER is negative or zero, we arrive at the optimum solution.

$\therefore\ x = 3, S_1 = 3, S_1$ is the slack variable which is equal to zero.

$x = 3, y = 0$

$\therefore\ Z_{\max} = 7x + 5y$

$= 21 + 0 = 21.$

Example 15: The maximum value of $Z = 3x + 5y + 4z$

Subjected to

$2x + 3y \leq 8$

$2y + 5z \leq 10$

$3x + 2y + 4z \leq 15$

(A) $\frac{680}{41}$ (B) $\frac{765}{41}$ (C) $\frac{812}{41}$ (D) $\frac{869}{41}$

Solution:

Objective function is

$Z = 3x + 5y + 4z + 0.s_1 + 0.s_2 + 0.s_3$

Constraints are

$2x + 3y + s_1 + 0.s_2 + 0.s_3 = 8$

$0.x + 2y + 5z + 0.s_1 + s_2 + 0.s_3 = 10$

$3x + 2y + 4z + 0.s_1 + 0.s_2 + s_3 = 15.$

Taking these into programme.

Table I

Programme	Z - Value	Qty	3 x	5 y	4 z	0 S_1	0 S_2	0 S_3	bi
S_1	0	8	2	3	0	1	0	0	$\frac{8}{3}$ ✓
S_2	0	10	0	2	5	0	1	0	5
S_3	0	15	3	2	4	0	0	1	7.5
NER			3	5✓	4	0	0	0	

Table II

Programme	Z - Value	Qty	3 x	5 y	4 z	0 S_1	0 S_2	0 S_3	bi
y	5	$\frac{8}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	∞
S_2	0	14/3	−4/3	0	5	−2/3	1	0	$\frac{14}{15}$ ✓
S_3	0	29/3	5/3	0	4	−2/3	0	1	29/12
NER			−1/3	0	4✓	−5/3	0	0	

Fixed ratio for $s_2 = \frac{2}{3}$

Old ratio no. – key row no. × $F \times R$

$$10 - 8 \times \frac{2}{3} = \frac{14}{3}$$
$$0 - 2 \times \frac{2}{3} = \frac{-4}{3}$$
$$2 - 3 \times \frac{2}{3} = 0$$
$$5 - 0 \times \frac{2}{3} = 5$$
$$0 - \frac{2}{3} = \frac{-2}{3}$$
$$1 - 0 \times \frac{2}{3} = 1.$$

Fixed ratio for s_3 is also $\frac{2}{3}$

$$15 - 8 \times \frac{2}{3} = \frac{29}{3}$$
$$3 - 2 \times \frac{2}{3} = \frac{5}{3}$$
$$2 - 3 \times \frac{2}{3} = 0$$
$$4 - 0 \times \frac{2}{3} = 4$$
$$0 - 1 \times \frac{2}{3} = \frac{-2}{3}$$
$$0 - 0 \times \frac{2}{3} = 0$$
$$1 - 0 \times \frac{2}{3} = 1.$$

For the second table, z is the incoming variable and S_2 is the outgoing variable.

Table III

Programme	Value	Qty	3 x	5 y	4 z	0 S_1	0 S_2	0 S_3	bi
y	5	$\frac{8}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	4
z	4	$\frac{14}{15}$	$\frac{-4}{15}$	0	1	$\frac{-2}{15}$	$\frac{1}{5}$	0	– ve
S_3	0	$\frac{89}{15}$	$\frac{41}{15}$	0	4	$\frac{-2}{15}$	$\frac{-4}{5}$	1	$\frac{89}{41}$ ✓
NER			$\frac{11}{15}$	0	0	$\frac{-17}{15}$	$\frac{-4}{5}$	0	

Fixed ratio for 'y' row is zero.

Fixed ratio for s_3 row $\frac{4}{5}$

$$\frac{29}{3} - \frac{14}{3} \times \frac{4}{5}$$
$$\frac{145 - 56}{15} = \frac{89}{15}$$
$$\text{and } \frac{5}{3} + \frac{4}{3} \times \frac{4}{5} = \frac{41}{15}$$

and 0,

$$\text{and } 4 - 5 \times \frac{4}{5} = 0.$$
$$\frac{-2}{3} + \frac{2}{3} \times \frac{4}{5} = \frac{-2}{3} + \frac{8}{15} = \frac{-2}{15}$$
$$0 - 1 \times \frac{4}{5} = \frac{-4}{5}$$
$$1 - 0 \times \frac{4}{5} = 1.$$

$$\text{NER } 3 - \left\{5 \times \frac{2}{3} - 4 \times \frac{4}{15} + 0\right\}$$
$$3 - \frac{10}{3} + \frac{16}{15} = \frac{16}{15} - \frac{1}{3} = \frac{11}{15}$$
$$5 - \{5 \times 1 + 0 + 0\} = 0$$
$$4 - (5 \times 0 + 4 \times 1 + 0) = 0$$
$$0 - \left[\frac{0}{3} - \frac{8}{15}\right] = \frac{-17}{15}$$
$$0 - \left\{0 + \frac{4}{5}\right\} = -\frac{4}{5}$$
$$0 - 0 = 0.$$

Table IV

Programme	Value	Qty	x	y	z	S_1	S_2	S_3
y	5	$\frac{50}{41}$	0	1	0	$\frac{15}{41}$	$\frac{8}{41}$	$\frac{-10}{41}$
z	4	$\frac{62}{41}$	0	0	1	$\frac{-6}{41}$	$\frac{5}{41}$	$\frac{4}{41}$
x	3	$\frac{89}{41}$	1	0	0	$\frac{-2}{41}$	$\frac{-12}{41}$	$\frac{15}{41}$
NER			0	0	0	$\frac{-45}{41}$	$\frac{-24}{41}$	$\frac{-11}{41}$

Fixed ratio for $Z = -\dfrac{4}{41}$

$$= \frac{14}{15} - \frac{89}{15} \times \frac{-4}{41} = \frac{14}{15} + \frac{356}{15 \times 41}$$

$$= \frac{62}{41}$$

and $-\dfrac{4}{15} - \dfrac{41}{15} \times \left(\dfrac{-4}{41}\right) = 0$

and = 0

$$\frac{-2}{15} + \frac{2}{15} \times \frac{-4}{41} = \frac{-74}{15 \times 41}$$

$$\frac{-2}{15} + \frac{8}{15 \times 41} = \frac{-82}{15 \times 41} = \frac{-90}{15 \times 41} x$$

$$\frac{-6}{41}$$

and $\dfrac{1}{5} + \dfrac{4}{5}\left(\dfrac{-4}{41}\right)$

and $\dfrac{1}{5} - \dfrac{16}{5 \times 41} = \dfrac{41}{41 \times 5} - \dfrac{16}{5 \times 41} = \dfrac{25}{5 \times 41} = \dfrac{5}{41}$

and $0 - \left(\dfrac{-4}{41}\right) = \dfrac{4}{41}$

The fixed ratio of 'y' row is $\dfrac{2}{3} \times \dfrac{15}{41} = \dfrac{5}{41} = \dfrac{10}{41}$

Quantity

$$= \frac{8}{3} - \frac{89}{15} \times \frac{10}{41} = \frac{8 \times 5 \times 41}{15 \times 41} - \frac{810}{15 \times 41}$$

$$= \frac{1640}{15 \times 41} - \frac{890}{15 \times 41} = \frac{750}{15 \times 41} = \frac{50}{41}$$

and $\dfrac{2}{3} - \dfrac{41}{15} \times \dfrac{10}{41} = 0$

and $\dfrac{1}{3} + \dfrac{2}{15} \times \dfrac{10}{41} = \dfrac{5 \times 41}{15 \times 41} + \dfrac{20}{15 \times 41}$

$$= \frac{225}{15 \times 41} = \frac{15}{41}$$

and $0 + \dfrac{4}{5} \times \dfrac{10}{41} = \dfrac{8}{41}$

Net evaluation row is totally negative or zero in Table IV.

$\therefore$ We arrived at optimum solution.

$$\therefore x = \frac{89}{41}, y = \frac{50}{41}, z = \frac{62}{41}$$

$Z_{max} = 3x + 5y + 4z$

$$= 3\left[\frac{89}{41}\right] + 5\left[\frac{50}{41}\right] + 4\left[\frac{62}{41}\right] = \frac{267}{41} + \frac{250}{41} + \frac{248}{41} = \frac{765}{41}$$

Duality in Linear Programming

A linear programming problem can be rewritten by transposing the rows and columns of the given problem. This results in the dual programme. The original problem is known as the primal programme. The solution to the dual problem is found in a similar manner to that of the primal problem and the optimal solution of the dual problem gives complete information of the primal problem optimal solution.

Duality in linear programming is used to solve problems in which the initial solution is infeasible.

Primal of Dual Problem

Every given linear programming problem is taken as a primal problem. For every primal problem, a dual problem can be generated.

1. If the primal problem is a maximisation problem, the dual will be a minimisation problem.
2. If the primal is a minimisation problem, the dual is a maximisation problem.
3. Inequalities given as $\le$ in the primal become $\ge$ in the dual problem.
4. Inequalities given as $\ge$ in the primal problem become $\le$ in the dual problem.
5. Values of the constraints in the primal problem become coefficients of the objective function in the dual.
6. The coefficient of the objective function in the primal problem will become the value of constraints of the dual problem.
7. If there are 'n' variables and 'm' constraints in the primal problem, they will become 'm' variables and 'n' constraints in the dual problem.

Example 16: A linear programming problem is given as follows.

Maximise z, when $z = 2x + y$ subjected to the constraints

$$x + 2y \le 18$$
$$2x + y = 15$$
$$x, y \ge 0.$$

The dual is:

(A) Minimise $18a + 15b$
Subjected to $a + 2b \ge 2$
$2a + b \ge 1$
$a, b \ge 0$

(B) Minimise $18a + 15b$
Subjected to $a + 2 \ge 2$
$2a + b = 1$
$a, b \ge 0$

(C) Minimise $18a + 15b$
Subjected to $a + 2b \le 1$
$2a + b \le 1$
$a, b \ge 0$

(D) Minimise $18a + 15b - 15c$
Subjected $a - 2b + 2c \ge 2$
$2a - b + c \ge 1$
a, b ≥ 0.

Solution:

The given problem is:

Maximise $Z = 2x + y$

Subjected to $x + 2y \leq 18$

$$2x + y = 15$$
$$x, y \geq 0.$$

The second constraint can be witten as

$2x + y \geq 15$ and $2x + y \leq 15$

i.e., $-2x - y \leq -15$.

$\therefore$ The constraints are

$$x + 2y \leq 18$$
$$-2x - y \leq -15$$
$$2x + y \leq 15.$$

There are 2 variables and 3 constraints.

The dual is

Minimise $18a + 15b - 15c$

Subjected to

$$a - 2b + 2c \geq 2$$
$$2a - b + c \geq 1.$$

Example 17: A linear programming problem is given as follows:

Maximise $Z = 50x_1 + 40x_2$

Subjected to the constraints

$2x_1 + 3x_2 \leq 3$

$8x_1 + 4x_2 \leq 5$

$x_1, x_2 \geq 0.$

The dual problem for this is,

(A) Minimise $3a + 5b$
Subjected to $50a + 40b \geq 50$
$a, b \geq 0$

(B) Minimise $Z = 3a + 5b$
Subjected to $2a + 8b \geq 50$
$3a + 4b \geq 40$
$a, b \geq 0$

(C) Minimise $Z = 3a + 5b$
Subjected to $2a + 8b \leq 50$
$3a + 4b \leq 40$
$a, b \geq 0$

(D) Minimise $Z = 3a + 5b$
Subjected to $2a + 8b \geq 50$
$3a + 40 \leq 40$
$a, b \geq 0.$

Solution:

Objective function is to minimise

$Z = 3a + 5b$

Subjected to $2a + 8b \geq 50$

$$3a + 4b \geq 40$$
$$a, b \geq 0.$$

Dual Simplex Method

This method is explained using the following example.

1. Solve the given LP problem by dual simplex method.
   Minimise $Z = 2x_1 + 3x_2 + 4x_3$
   Subject to $x_1 + 2x_2 + 3x_3 \geq 3$
   $x_1 + 3x_2 + 2x_3 \leq 5$
   $x_1 + x_2, x_3 \geq 0.$

Solution:

Form the dual simplex table:

Slack variables	Value of slack	–2	–3	–4	0	0	
		x_1	x_2	x_3	S_1	S_2	*bi*
S_1	0	–1	–2	(–3)	1	0	–3
S_2	0	1	3	2	0	1	5
NER		–2	–3	–4	0	0	

Step 1: The given minimisation problem is converted to a maximisation problem.

$$G = -Z = -2x_1 - 3x_2 - 4x_3$$

Step 2: The greater than or equal to sign is converted to less than or equal to.

$$-x_1 - 2x_2 - 3x_3 \leq -3$$

Step 3: Converting into the standard form.
Maximise $G = -2x_1 - 3x_2 - 4x_3$

Subject to $-x_1 - 2x_2 - 3x_3 + S_1 = -3$

$$x_1 + 3x_2 + 2x_3 + S_2 = 5.$$

By considering $x_1 = x_2 = x_3 = 0$, the initial basic solution is $S_1 = -3$ and $S_2 = 5$. As S_1 is negative, the solution is infeasible.

Step 4: All the NER values are negative or zero. Therefore, the solution is optimal. Proceed to step 5.

Step 5: As $b_1 = -3$, the first row is the key row and S_1 is the outgoing variable.

Step 6: Find the ratio of NER to the key row elements.

$$\frac{-2}{-1} = 2, \frac{-3}{-2} = \frac{3}{2} \quad \text{and} \quad \frac{-4}{3} = \frac{4}{3}$$

As 4/3 is the smallest ratio, x_3 column is the key column. x_3 is the incoming variable and (–3) is the key element.

Step 7: Replace S_1 by x_3

Variables	Value	–2	–3	–4	0	0	
		x_1	x_2	x_3	S_1	S_2	*bi*
x_3	–4	1/3	2/3	1	–1/3	0	1
S_2	0	1/3	5/3	0	2/3	1	3
NER		–2/3	–1/3	0	–4/3	0	

As all the NER values are negative and zero and all the *bi* values are positive, the solution is optimal.

$\therefore$ $x_1 = 0, x_2 = 0$ and $x_3 = 1$

$\therefore$ max. $G = -2x_1 - 3x_2 = -4x_3 = -4$

or min. $z = 4.$

Transportation Problem

Transportation problem is also a particular type of linear programming problem. This problem considers the transportation of various quantities of a commodity which are stored at several origins to a number of different destinations so as to satisfy the demand, with the availability at the origins. But the transportation should be in such a way that the cost of transportation is minimum.

For example, consider an industry having '*x*' plants located at different cities in the country. There are '*y*' retail shops situated in different cities in the country which can absorb all the quantity of the commodity produced by the industries. Here, the transportation problem is to decide a transportation schedule to distribute the quantities produced in '*x*' plants to the retail shops in '*y*' cities so that the cost of transportation is minimum.

A transportation network can be represented by the following matrix:

Let O_1, O_2, O_3…………………. be the origins and D_1, D_2, D_3………………be the destinations.

	D_1	D_2	D_3	D_4	……	D_y	a_1
O_1	C_{11}	C_{12}	C_{13}	C_{14}		C_{1y}	a_2
O_2	C_{21}	C_{22}	C_{23}	C_{24}		C_{2y}	a_3
O_3	C_{31}	C_{32}	C_{33}	C_{34}		C_{3y}	a_4
O_4	–	–	–	–	–	–	–
O_5	–	–	–	–	–	–	–
	–	–	–	–	–	–	–
O_x	C_{x1}	C_{x2}	C_{x3}	C_{x4}	……	C_{xy}	a_x
	b_1	b_2	b_3	b_4		b_y	

a_1, a_2, a_3…………a_x are the capacities at the origin and b_1, b_2, b_3…………………b_y are the demand requirements of the destinations. '*C*' values are the cost of transportation; C_{11} denotes the cost of transportation; of one unit of the material from the origin O_1 to the destination D_1. The objective of the transportation problem is to minimise $\Sigma\Sigma\, a_{ij}\, c_{ij}$.

where, a_{ij} is the quantity transferred from the i^{th} origin to the j^{th} destinations and *Cij* being its cost.

Feasible Solution

It is the set of non – negative allocations which satisfy the origin capacities and destination demands. That means the sum of allocations along the row must be equal to the availability in that row. Similarly, the sum of allocations along the column must be equal to the total demand of the destination.

Basic Feasible Solution

A feasible solution becomes a basic feasible solution in an $m \times n$ matrix when the number of allocated cells are

$$m + n - 1.$$

Optimal Solution

The basic feasible solution becomes an optimal solution when the allocated material corresponds to the minimum cost of transportation.

There are different methods to arrive at the basic feasible solution. The various methods are:

1. North – West corner rule
2. Lowest cost entry method
3. Vogels approximation method.

Among the three, Vogels approximation method is considered to be the one by which we can arrive at a basic feasible solution which is very close to the optimal solution.

The Vogels Approximation Method (VAM) has been discussed in detail.

Consider the problem shown in the following tables:

	D_1	D_2	D_3	D_4	D_5	Capacity
O_1	12	4	9	5	9	55
O_2	8	1	6	6	7	45
O_3	1	12	4	7	7	30
O_4	10	15	6	9	1	50
Demand	40	20	50	30	40	180

The capacity of each origin is given at the end of each row.

The demand of each of the destination is given at the end of each column.

The top – right corner of every cell, which is the cost incurred for the transportation of unit quantity is given. For example, in the cell O_2D_3, the value is 6. This means for transporting one unit from O_2 to D_3, the transportation cost is ₹6.

Vogels Approximation Method can be illustrated as follows:

	D_1	D_2	D_3	D_4	D_5	Capacity	
O_1	12	4	9	5	9	55	1
O_2	8	1	6	6	7	45	5
O_3	1 (30)	12	4	7	7	~~30~~	3
O_4	10	15	6	9	1	50	5
Demand	10 ~~40~~	20	50	30	40	180	
	7✓	3	2	1	6		

Step 1: Find the penalty numbers for each row. It is the difference between the lowest and the next higher cost. If there are two lowest costs, the penalty number is zero. Similarly, the penalty for each column can be found.

Step 2: Among the penalty numbers, identify the maximum penalty. In this case, '7' is the maximum penalty. Give a tick mark. Consider the corresponding column.

Step 3: Take the low cost cell in the corresponding column (or row). Here it is a column. The low cost cell is O_3D_1 origin capacity in 30, but the requirement is 40. Allocate '30' units in that cell. The origin capacity is satisfied and therefore, delete that origin. Subsequenty, the demand reduces to 10 from 40.

The new matrix is

	D_1	D_2	D_3	D_4	D_5	Capacity	
O_1	12	4	9	5	9	55	(1)
O_2	8	1	6	6	7	45	(5)
O_4	10	15	6	9	1 40	10 ~~50~~	(5)
Demand	10	20	50	30	~~40~~	150	
	(2)	(3)	(0)	(1)	(6)✓		

Step 4: Again, find the penalty to numbers for rows and columns; the maximum penalty is '6'. 1 corresponds to the fifth column. The low cost cell is the fifth column which is O_4 D_5. The requirement is 40 and the availability is 50. Allocate 40 units in the cell and delete the column.

The row matrix is

	D_1	D_2	D_3	D_4		
O_1	12	4	9	5	55	(1)
O_2	8	1 20	6	6	~~45~~ 25	(5)✓
O_4	10	15	6	9	10	(3)
	10	~~20~~	50	30	110	
	(2)	(3)	(0)	(1)		

Step 5: Again, find the penalty numbers for rows and columns and repeat. Allocate 20 units in the cell O_2D_2. Delete column '2'; the capacity of O_2 reduces to 25 from 45. The new matrix becomes

	D_1	D_3	D_4		
O_1	12	9	5 30	~~55~~ 25	(4)✓
O_2	8	6	6	25	(0)
O_4	10	6	9	10	(3)
	10	50	~~30~~	90	
	(2)	(0)	(1)		

Step 6: Again, find the penalty numbers for rows and columns and repeat. Allocate '30' units in cell O_1D_4 and delete the column D_4. The capacity of the origin O_1 reduces to 25 from 55. The new matrix is

	D_1	D_3		
O_1	12	9	25	(3)
O_2	8	6	25	(2)
O_4	10	6 10	10	(4)✓
	10	~~50~~ 40	60	
	(2)	(0)		

Step 7: Again, find the penalty numbers for rows and columns. Allocate 10 units in the cell O_3D_3 and delete the row. The demand of destination D_3 reduces to 40 from 50. Now, the new matrix becomes

	D_1	D_3		
O_1	12	9 25	25	(3)
O_2	8 10	6 15	~~25~~ 15	(2)
	~~10~~	40	50	
	(4)✓	(3)		

Step 8: Again, find the penalty numbers for rows and columns and repeat. Allocate '10' units to the cell O_2D_1. Then on inspection, the remaining allocations are '25' units to O_1D_3 and 15 units to O_2D_3.

Step 9: With all the allocations, re-draw the matrix.

	D_1	D_2	D_3	D_4	D_5	Cap
O_1			25	30		55
O_2	10	20	15			45
O_3	30					30
O_4			10		40	50
	40	20	50	30	40	

This is the basic feasible solution.

Now, we have to find the optimal solution. First of all, we will check whether the solution arrived at, is optimal or not.

Check of Optimality (MODI Method)

1. In the basic feasible solution, the number of occupied cells must be $m + n - 1$

   where, 'm' is the total number of rows in the matrix and 'n' the total number of columns in the matrix. In the above problem

$$m = 4, n = 5,$$

   $m + n - 1 = 8$. So, the number of occupied cells must be '8'. Here it is satisfied.

	D_1	D_2	D_3	D_4	D_5	
O_1	12	4	9 (25)	5 (30)	9	0
O_2	8 (10)	1 (20)	6 (15)	6	7	–3
O_3	1 (30)	12	4	7	7	–10
O_4	10	15	6 (10)	9	1 (40)	–3
	11	4	9	5	4	

Step 1: Put the row numbers and column numbers. The row numbers are 'u' and the column numbers are 'v'. Put the first row number as zero. Consider the occupied cells in the row. The cost indicated in the occupied cells must be $(u + v)$, which is the sum of the column numbers and row numbers. Once we know the row number, the column number can be found.

Initially, we have put the row number as zero, we have then found the column numbers for columns '3' and '4' because in the first row, the third and fourth cells are occupied.

$\therefore$ For the third cell (O_1, D_3) cost is 9.

$$\therefore u + v = 9$$

$$O + v = 9$$

$$\therefore\ v = 9.$$

Therefore, 9 becomes the column number of the third column. Similarly, the fourth cell is occupied; (O_1, D_4): the cost is 5.

Therefore, column number of fourth column is also obtained = 5, because $0 + 5 = u + v = 5$.

Step 2: Now, column numbers for the third and fourth columns are obtained. In the third column, apart from (O_1, D_3) which we have considered, there are two more occupied cells. They are O_2D_3 and O_4D_3

In O_2D_3, the cost is 6. The column number which we have already found is 9. So, we can find the row number of O_2 from

$$u + v = 6$$

$$u + 9 = 6$$

$$\therefore u = -3.$$

Step 3: Now, we have the row number for O_2. But in the second row, there are two occupied cells other than O_2D_3. They are O_2D_1 and O_2D_2. For O_2D_1 the cost is '8'; for O_2, we have obtained the row no. as '–3'. So, the column number for D_1 can be found from

$$u + v = 8$$

$$-3 + v = 8 \qquad \therefore\ v = 11.$$

Also, for O_2D_2 the cost is 1.

$$u + v = 1$$

$$-3 + v = 1 \qquad \therefore\ v = 4.$$

Column number for D_2 is also obtained.

Step 4: Similarly, O_3D_1 is an occupied cell with cost 1. We have already seen that the column number for D_1 is 11.

$$u + v = 1$$

$$u + 11 = 1 \qquad \therefore\ u = -10$$

$\therefore$ Row number of O_3 is –10.

Step 5: Now, we have to get the column number of D_5. O_4D_5 is an occupied cell with the cost = 1. But we have already found the row number for $O_4 = -3$

$$u + v = 1$$

$$-3 + v = 1 \qquad \therefore\ v = 4$$

Column number for D_5 is 4.

Now, all the row numbers and column numbers are obtained. Next, we have to find the $(u + v)$ values of all the unoccupied cells. These $(u + v)$ values for unoccupied cells can be taken as impressed costs.

Opportunity cost = impressed cost – actual cost.

If the opportunity cost is positive, the meaning is that, there is a penalty for not taking that cell for occupation. The solution is not optimal. But if the opportunity costs are zero or negative for all the unoccupied cells, then the solution is optimal.

	D_1	D_2	D_3	D_4	D_5	
O_1	12	4	9 (25)	5 (30)	9	0
O_2	8 (10)	1 (20)	6 (15)	6	7	–3
O_3	1 (30)	12	4	7	7	–10
O_4	10	15	6 (10)	9	1 (40)	–3
	11	4	9	5	4	

Step 6: Find the impressed cost for all the unoccupied cells ($u + v$) and insert it at the bottom left corner.

	D_1	D_2	D_3	D_4	D_5	
O_1	12, –, 11	4, 0, 4	9 (25)	5 (30)	9, –, 4	0
O_2	8 (10)	1 (20)	6 (15)	6, –, 2	7, –, 1	–3
O_3	1 (30)	12, –, –6	4, –, –1	7, –, –5	7, –, –6	–10
O_4	10, –, 8	15, –, 1	6 (10)	9, –, 2	1 (40)	–3
	11	4	9	5	4	

Step 7: Find the impressed cost – actual cost and check whether they are negative or zero. All unoccupied cells have a penalty cost which is negative or zero. The solution is optimum.

The minimum cost of transportation is

$$= 25 \times 9 + 30 \times 5 + 10 \times 8 + 20 \times 1 + 15 \times 6 + 30 \times 1 + 10 \times 6 + 40 \times 1$$
$$= 225 + 150 + 80 + 20 + 90 + 30 + 60 + 40$$
$$= 375 + 100 + 120 + 100 = 695.$$

Example 18: The least cost for the transportation problem given below is

	A_1	A_2	A_3	Supply
P_1	2	7	4	5
P_2	3	3	1	8
P_3	5	4	7	7
P_4	1	6	2	14
Demand	7	9	18	34

(A) ₹83 (B) ₹76 (C) ₹72 (D) ₹68.

The problem can be rewritten as

Solution:

Iteration I

	A_1	A_2	A_3	Supply	
P_1	2	7	4	5	(2)
P_2	3	3	1 (8)	~~8~~	(2)
P_3	5	4	7	7	(1)
P_4	1	6	2	14	(1)
Demand	7	9	~~18~~ 10	26	
	(1)	(1)	(1)	(1)	

Initial feasible solution can be obtained by Vogels Approximation Method.

By proving penalty numbers row wise and column wise, maximum penalty occurs in the first and second rows. Choose the row having the least cost cell and make allocation.

Iteration II

	A_1	A_2	A_3		
P_1	2	7	4	5	2
P_2	5	4	7	7	1
P_3	1	6	2 (10)	~~14~~ 4	1
	7	9	~~10~~	16	
	(1)	(2)	(2)		

Maximum penalty occurs in three cases, choose any one of them.

Iteration III

	A_1	A_2		
P_1	2	7	5	(5)
P_2	5	4	7	(1)
P_4	1 (4)	6	4	(5)✓
	~~7~~ 3	9	12	
	(1)	(2)		

The next iteration is,

Iteration IV

	A_1	A_2		
P_1	2 (3)	7 (2)	5	(5)✓
P_3	5	4 (7)	7	(1)
	3	9		

Now, the allocation is completed by VAM.

Iteration V

	A_1	A_2	A_3	
P_1	2 3	7 2	4	5
P_2	3	3	1 8	8
P_3	5	4 7	7	7
P_4	1 4	6	2 10	14
	7	9	18	34

Checking optimality,

1. No. of occupied cells
   $m + n - 1 = 4 + 3 - 1 = 6.$
   It is satisfied. Next, try optimality with row numbers and column numbers

Iteration VI

	A_1	A_2	A_3		
P_1	2 3	7 2	4 – 3	5	(0)
P_2	3 – 0	3 + 5	1 8	8	(–2)
P_3	5 – –1	4 7	7 – 0	7	(–3)
P_4	1 4	6 0 6	2 10	14	(–1)
	7 (2)	9 (7)	18 (3)		

Iteration VII

	A_1	A_2	A_3	
P_1	2 3 +	7 2 –	4	5
P_2	3	3 +	1 8 –	8
P_3	5	4 7	2	7
P_4	1 4 –	6	2 10 +1	14
	7	9	18	

Iteration VIII

	A_1	A_2	A_3		
P_1	2 5	7 – 5	4 – 3	5	(0)
P_2	3 – 0	3 2	1 6	8	(–2)
P_3	5 – 1	4 7	2 0 2	7	(–1)
P_4	1 2	6 – 4	2 12	14	(–1)
	7 (2)	9 (7)	18 (3)		

In Iteration VII, we have formed a closed loop using occupied cells, a corner being at the cell P_2 A_2, where the opportunity cost is positive. Then, shift the least among the corner values to the cell P_2 A_2, and add or subtract this quantity from the other corner values so that row availability and column requirements are satisfied. Then, we get iteration VIII.

In iteration VIII, all the unoccupied cells have zero or negative opportunity cost.

$\therefore$ The solution is optimum. The least cost is

$$= 2 \times 5 + 2 \times 3 + 6 \times 1 + 7 \times 4 + 12 \times 2 + 2 \times 1$$
$$= 10 + 6 + 6 + 28 + 24 + 2$$
$$= 22 + 28 + 26 = 50 + 26 = 76 = ₹76.$$

Example 19: For the transportation problem given below, the least cost of transportation is

	Q_1	Q_2	Q_3	Q_4	Q_5	Cap
P_1	3	4	6	8	9	20
P_2	2	10	1	5	8	30
P_3	7	11	20	40	3	15
P_4	2	1	9	14	16	13
	40	6	8	18	6	78

[P_1, P_2, P_3, P_4 are production centres having capacity as shown. Q_1, Q_2, Q_3, Q_4, Q_5 are warehouses with a demand as shown.]

(A) 347 (B) 312 (C) 286 (D) 267.

Solution:

Iteration I

	Q_1	Q_2	Q_3	Q_4	Q_5		
P_1	3	4	6	8	9	20	(1)
P_2	2	10	1, 8	5	8	30	(1)
P_3	7	11	20	40	3	15	(4)
P_4	2	1	9	14	16	13	(0)
	40	6	~~8~~	18	6	70	
	(0)	(3)	(5)✓	(3)	(5)		

Iteration II

	Q_1	Q_2	Q_4	Q_5		
P_1	3	4	8	9	20	(1)
P_2	2	10	5	8	22	(3)
P_3	7	11	40	3, 6	9 ~~15~~	(4)
P_4	2	1	14	16	13	(1)
	40	6	18	~~6~~	64	
	(0)	(3)	(3)	(5)✓		

Iteration III

	Q_1	Q_2	Q_4		
P_1	3	4	8	20	(1)
P_2	2	10	5	22	(3)
P_3	7, 9	11	40	~~9~~	(4)✓
P_4	2	1	14	13	(1)
	31 ~~40~~	6	18	55	
	(0)	(3)	(3)		

Iteration IV

	Q_1	Q_2	Q_4		
P_1	3	4	8	20	(1)
P_2	2	10	5	22	(3)
P_4	2	1, 6	14	7 ~~13~~	(4)✓
	31	~~6~~	18	49	(1)
	(0)	(3)	(3)		

Iteration V

	Q_1	Q_4		
P_1	3	8	20	(5)
P_2	2	5	22	(3)
P_4	2, 7	14	~~7~~	(12)✓
	24 ~~31~~	18	42	
	(0)	(3)		

Iteration VI

	Q_1	Q_4		
P_1	3, 20	8	~~20~~	(5)✓
P_2	2, 4	5, 18	22	(3)
	~~24~~ 4	18	~~42~~ 22	
	(0)	(3)		

The allocation is complete by Vogels Approximation Method. There are six iterations required. Consolidating the allocations,

	Q_1	Q_2	Q_3	Q_4	Q_5	Cap	
P_1	3, 20	4, 2	6, 2	8, 6	9, –1	20	(0)
P_2	2, 4	10, 5	1, 8	5, 18	8	30	(–1)
P_3	7, 9	11, 6	20, 6	40, 10	3, 6	15	(4)
P_4	2, 7	1, 6	9, –1	14, 5	16, –2	13	(–1)
Dem	40	6	8	18	6		
	(3)	(2)	(2)	(6)	(–1)		

Optimality test

1. No. of occupied cells

$$m + n - 1 = 4 + 5 - 1 = 8$$

It is satisfied.

Providing row numbers and column numbers, we find the penalty cost

= (impressed cost − actual cost)

All are negative or zero in the unoccupied cells. Therefore, the solution is optimum.

The least cost is

$$= 20 \times 3 + 4 \times 2 + 8 \times 1 + 18 \times 5 + 9 \times 7 + 6 \times 3 + 7 \times 2 + 6 \times 1$$
$$= 60 + 8 + 8 + 90 + 63 + 18 + 14 + 6$$
$$= 76 + 90 + 63 + 32 + 6$$
$$= 166 + 95 + 6 = 166 + 101 = 267$$

Least cost is ₹ = 267.

Example 20: In an industry, 8 loads of its products are to be transported to 8 godowns situated in different cities. The production centers C_1, C_2, C_3 are also in different cities. The kilometer distance from the production centeres to the godowns are as per the matrix given below. The cost per km transportation is ₹10.

	G_1	G_2	G_3	
C_1	50	30	220	1
C_2	90	45	170	3
C_3	250	200	50	4
	4	2	2	

The least cost of transportation will be

(A) 9200 (B) 8800 (C) 8200 (D) 7800.

Solution:

Giving allocations according to the Vogels Approximation Method, we have

Iteration I

	G_1	G_2	G_3	
C_1	50	30	220	1 (20)
C_2	90	45	170	3 (45)
C_3	250	200	50 (2)	~~4~~ 2 (150)✓
	4 (40)	2 (15)	~~2~~ (120)	

Iteration II

	G_1	G_2	
C_1	50 (1)	30	1 (20)
C_2	90 (3)	45	3 (45)
C_3	250	200 (2)	~~2~~ (50)✓
	4 (40)	~~2~~ (15)	

Consolidating the allocations,

	G_1	G_2	G_3	
C_1	50 (1)	30	220	1
C_2	90 (3)	45	170	3
C_3	250	200 (2)	50 (2)	4
	4	2	2	8

Optimality test,

1. $m + n - 1 = 3 + 3 - 1 = 5$
   No. of occupied cells = 4
   The optimality condition is not satisfied; allocation does not satisfy the condition. One more occupied cell is required for satisfying the condition.
   When we start giving row numbers and column numbers, a break occurs. To overcome come the break,

	G_1	G_2	G_3		
C_1	50 (1)	30, 2, –	220, 2, –	1	(0)
C_2	90 (3)	45 (ε)	170, 42, –	3	(40)
C_3	250, 8	200 (2)	50 (2)	4	(195)
	4	2	2	8	
	(50)	(5)	(–145)		

We may provide a very small quantity (ε) at C_2 G_2 cell, row numbers and column numbers can now be completed as shown $\varepsilon \approx 0$.

The impressed cost can be calculated and entered at the left bottom corner in unoccupied cells. Now, penalty cost = [impressed cost − actual cost] is negative or zero in all unoccupied cells. Therefore, the solution is optimum.

The least cost is

$$= ₹10\{1\times50+3\times90+2\times200+2\times50\}$$

$$=10\{50+270+400+100\}$$

$$=10\{320+500\}=₹8200.$$

Example 21: A transportation problem is given below. Products from 3 plants P_1, P_2, and P_3 are transported to four warehouses. The demand of the warehouses, availability from the plants, the cost of transportation etc. are given in the following chart.

	W_1	W_2	W_3	W_4	Cap
P_1	11	20	7	8	50
P_2	21	16	10	12	40
P_3	8	12	18	9	70
Dem	30	25	35	40	

The least cost of transportation is

(A) 1250 (B) 1150 (C) 1050 (D) 950.

Solution:

The given problem is an unbalanced one because the total availability does not match with the total demand.

Total availability = 160

Total demand = 130.

To balance, we have to introduce a fictious warehouse W_5 having a capacity '30' so that the demand and capacity matches. The cost of transportation from all the plants may be taken as zero. Then the problem becomes,

	W_1	W_2	W_3	W_4	W_5	
P_1	11	20	7	8	0	50
P_2	21	16	10	12	0	40
P_3	✓ 8	12	18	9	0	70
	30	25	35	40	30	160

Now, the problem becomes a balanced one; the allocations can be made according to VAM.

Iteration I

	W_1	W_2	W_3	W_4	W_5		
P_1	11	20	7	8	0	50	(7)
P_2	21	16	10	12	0 30	10 ~~40~~	(10)✓
P_3	✓ 8	12	18	9	0	70	(8)
	30	25	35	40	~~30~~	130	
	(3)	(4)	(3)	(1)	(0)		

Iteration II

	W_1	W_2	W_3	W_4		
P_1	11	20	7	8	50	(1)
P_2	21	16	10	12	10	(2)
P_3	✓ 8	12 25	18	9	~~70~~ 45	(1)
	30	~~25~~	35	40	105	
	(3)	(4)✓	(3)	(1)	(0)	

Iteration III

	W_1	W_3	W_4		
P_1	11	7 35	8	~~50~~ 15	(1)
P_2	21	10	12	10	(2)
P_3	8	18	9	45	(1)
	30	~~35~~	40	70	
	(3)	(3)	(1)		

Iteration IV

	W_1	W_4		
P_1	11	8	15	(3)
P_2	21	12 10	~~10~~	(9)
P_3	8	9	45	(1)
	30	~~40~~ 30	60	
	(3)	(1)		

Iteration V

	W_1	W_4		
P_1	$\boxed{11}$ 15	$\boxed{8}$	15	(3)
P_3	$\boxed{8}$ 15	$\boxed{9}$ 30	45	(1)
	30	30	60	
	(3)	(1)		

Consolidating the allocations,

	W_1	W_2	W_3	W_4	W_5		
P_1	$\boxed{11}$ 15	$\boxed{20}$ – 15	$\boxed{7}$ 35	$\boxed{8}$ + 12	$\boxed{0}$ – 0	50	(0)
P_2	$\boxed{21}$ – 11	$\boxed{16}$ – 15	$\boxed{10}$ – 7	$\boxed{12}$ 10	$\boxed{0}$ 30	40	(0)
P_3	$\boxed{8}$ 15	$\boxed{12}$ 25	$\boxed{18}$ – 4	$\boxed{9}$ 30	$\boxed{0}$ – –3	70	(–3)
	30	25	35	40	30	160	
	(11)	(15)	(7)	(12)	(0)		

Check for optimality,

1. $m + n - 1 = 3 + 5 - 1 = 7$,
   it is satisfied.
   All the cells except (P_1W_4) are indicating negative or zero penalty cost.

	W_1	W_2	W_3	W_4	W_5	
P_1	$\boxed{11}$ 15	$\boxed{20}$	$\boxed{7}$ 35	$\boxed{8}$ +	$\boxed{0}$	50
P_2	$\boxed{21}$	$\boxed{16}$	$\boxed{10}$	$\boxed{12}$ 10	$\boxed{0}$ 30	40
P_3	$\boxed{8}$ 15 +	$\boxed{12}$ 25	$\boxed{18}$ –	$\boxed{9}$ 30	$\boxed{0}$	70

	W_1	W_2	W_3	W_4	W_5	
P_1	$\boxed{11}$ 7	$\boxed{20}$ 11	$\boxed{7}$ 35 –	$\boxed{8}$ 15 +	$\boxed{0}$ – –1	(0)
P_2	$\boxed{21}$ 11	$\boxed{16}$ 15	$\boxed{10}$ + 11	$\boxed{12}$ 10 –	$\boxed{0}$ 30 3	(4)
P_3	$\boxed{8}$ 30	$\boxed{12}$ 25	$\boxed{18}$ 8	$\boxed{9}$ 15	$\boxed{0}$ 0	(1)
	7	11	(7)	(8)	–1	

All cells except P_2W_3 show negative or zero penalty cost.

Rearranging the cell P_2W_3 using quantity shift through closed loop.

	W_1	W_2	W_3	W_4	W_5		
P_1	$\boxed{11}$ – 7	$\boxed{20}$ – 11	$\boxed{7}$ 25	$\boxed{8}$ 25	$\boxed{0}$ – –3	50	(0)
P_2	$\boxed{21}$ – 10	$\boxed{16}$ – 14	$\boxed{10}$ 10	$\boxed{12}$ – 11	$\boxed{0}$ 30	40	
P_3	$\boxed{8}$ 30	$\boxed{12}$ 25	$\boxed{18}$ – 8	$\boxed{9}$ 15	$\boxed{0}$ – –3	70	(1)
	30	25	35	40	30	160	
	(7)	(11)	(7)	(8)	(–3)		

All the cells show zero or negative penalty cost. Therefore, the solution is optimum.

(O) The least cost is

$$25 \times 7 + 25 \times 8 + 10 \times 10 + 30 \times 0 + 30 \times 8 + 25 \times 12 + 15 \times 9$$
$$= 175 + 200 + 100 + 0 + 240 + 300 + 135$$
$$= 375 + 340 + 435$$
$$= ₹1150.$$

Assignment Problem

Assignment problem assigns a number of origins to a number of destinations (equal number) at a minimum cost. It can be the assignment of persons towards machines so that the cost of operation is a minimum. It is a particular case of transportation problem.

Let C_{ij} be the cost incurred for assigning the ith person with the jth job. Then, assignment problem is the system of assignments in which the total cost for performing the job is a minimum.

	J_1	J_2	J_3	J_n
1	C_{11}	C_{12}	C_{13}	C_{1n}
2	C_{21}	C_{22}	C_{23}	C_{2n}
3	C_{31}	C_{32}	C_{33}	C_{3n}
4	C_{41}	C_{42}	C_{43}	C_{4n}
an	C_{n1}	C_{n2}	C_{n3}	C_{nn}

Above, we get a cost matrix for the assignment. By re-arranging the values in iterations. We can arrive at an arrangement in between (1, 2, 3, ………n) and (J_1, J_2, J_3 …….. J_n) so that the cost is a minimum.

Method for Solving Assignment Problem

It is known as the assignment algorithm. The method of solution is known as the Hungarian method. It is also known as Floods technique.

The method consists of three steps.

The first step involves the derivation of a 'total opportunity cost matrix' from the given pay – off matrix of the problem.

This is done by subtracting the lowest number of each column of the given pay-off matrix from all the other numbers in that column, and then subtracting the lowest number of each row of the matrix (already obtained by column substraction), from all the other numbers in that row.

The total opportunity cost matrix thus obtained will have at least one zero in each row or column. Any cell having 'zero' in the 'total opportunity cost' matrix is considered to be a candidate for assignment.

The second step is to determine whether an optimal assignment can be made. We test it. If the test shows that it can be made, the problem is solved. Otherwise, we proceed to step three.

The third step is to revise the current total opportunity cost matrix to obtain better assignment pattern. It is done by the re-distribution of zeros or by creating one or more zero cells.

Procedure of Solution

Step 1:

1. Arrive at a column opportunity – cost matrix by subtracting the lowest entry of each column of the given pay-off matrix from all the other entries of that column.
2. Then subtract the lowest entry of each row of the matrix obtained in (1) from all the entries of that row.

The result of (2) gives the total – opportunity – cost matrix.

Step 2:

1. Cover all zeros of the total opportunity cost matrix with a minimum possible number of horizontal and vertical lines.
2. If the number of lines drawn is the number of rows or columns of the matrix, the problem is solved. Give assignment to the zeros.
3. If the number of lines drawn are less than the number of rows or columns proceed to step 3.

Step 3:

1. Subtract the lowest entry in the uncovered cells of the current total opportunity – cost matrix from the entries of all the uncovered cells.
2. Add the same lowest entry to only those cells in which the covering lines cross.

The resulting matrix is the revised opportunity cost matrix.

Repeat steps 2 and 3 till optimum solution is arrived at.

Example 22: Three machines are to be assigned to three persons, the costs involved for the various combinations are given.

	Machine		
	M_1	M_2	M_3
P_1	20	27	30
P_2	10	18	16
P_3	14	16	12

Total minimum cost for the optimum assignment is
(A) 37 (B) 40 (C) 45 (D) 49.

Solution:
To find the total opportunity cost matrix, subtract the lowest value of each column from all the other values of the column. Then from the resulting matrix, subtract the least value of each row from the other values of the row.

$$\begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array}\begin{array}{c} \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \left[\begin{array}{ccc} 10 & 11 & 18 \\ 0 & 2 & 4 \\ 4 & 0 & 0 \end{array}\right] \end{array} \longrightarrow \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array}\begin{array}{c} \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \left[\begin{array}{ccc} 0 & 1 & 18 \\ 0 & 2 & 4 \\ 4 & 0 & 0 \end{array}\right] \end{array}$$

We can cover all the zeros by two lines (minimum).

$$\begin{array}{c|ccc|} & M_1 & M_2 & M_3 \\ \hline P_1 & 0 & 1 & 8 \\ P_2 & 0 & 2 & 4 \\ P_3 & 4 & 0 & 0 \\ \hline \end{array}$$

Subtract the lowest (i.e., 1) from all the uncovered values and add it to the value at the function,

$$\begin{array}{c|ccc|} & M_1 & M_2 & M_3 \\ \hline P_1 & 0 & 0 & 7 \\ P_2 & 0 & 1 & 3 \\ P_3 & 5 & 0 & 0 \\ \hline \end{array}$$

Now, the minimum number of lines required to cover all the zeros is 3. This is equal to the number of rows or columns. Therefore, optimum assignment can be made.

Identify a row or column consisting of one zero. Give the assignment there, delete the row and column corresponding to that cell. Identify another row or column having only one zero from the remaining matrix. Give assignment there and it can be thus repeated.

$$\begin{array}{c|ccc|} & M_1 & M_2 & M_3 \\ \hline P_1 & 0 & \boxed{0} & 7 \\ P_2 & \boxed{0} & 1 & 3 \\ P_3 & 5 & 0 & \boxed{0} \\ \hline \end{array}$$

$P_1 \rightarrow M_2$
$P_2 \rightarrow M_1$
$P_3 \rightarrow M_3$

The minimum cost is
$= 27 + 10 + 12 = 49$
₹49.

Example 23: Four jobs are to be assigned to 4 workers. The cost involved in various combinations is given below.

Jobs \ Workers	1	2	3	4
1	12	30	11	15
2	18	33	9	31
3	44	25	24	21
4	23	30	28	14

Arrive at the assignment for the cost to be least and the least cost is

(A) ₹60 (B) ₹52 (C) ₹50 (D) ₹46.

Solution:

12	30	11	15
18	33	9	31
44	25	24	21
23	30	28	14

0	5	2	1
6	8	0	17
32	0	15	7
11	5	19	0

→

0	5	2	1
6	8	0	17
32	0	15	7
11	5	19	0

The minimum number of lines required to cover all the zeros is '4'.

Therefore, optimum solution is possible.

$\boxed{0}$	5	2	1
6	8	$\boxed{0}$	17
32	$\boxed{0}$	15	7
11	5	19	$\boxed{0}$

$$\therefore J_1 \to W_1$$
$$J_2 \to W_3$$
$$J_3 \to W_2$$
$$J_4 \to W_4$$

Cost (minimum) = 12 + 9 + 25 + 14
= 21 + 25 + 14
= ₹60.

Example 24: Four sales men A, B, C, D are to be assigned to four territories for sales operation. The cost involved for various assignments is given below. The cost values are in ten thousands. The minimum cost is

(A) 1.5 lakhs (B) 2 lakhs
(C) 2.5 lakhs (D) 2.8 lakhs.

Solution:

Salesmen	T_1	T_2	T_3	T_4
A	2	3	4	5
B	4	5	6	7
C	7	8	9	8
D	3	5	8	4

	T_1	T_2	T_3	T_4
A	0	1	2	3
B	0	1	2	3
C	0	1	2	1
D	0	2	5	1

0	0	0	2
0	0	0	2
0	0	0	0
0	1	3	0

The minimum number of lines required to cover all the zeros is '4'.

Therefore, optimal solution can be obtained. As there exists no row or column having a single zero, search for the column or row containing two zeros and give assignments.

$\boxed{0}$	0	0	2
0	0	$\boxed{0}$	2
0	0	0	$\boxed{0}$
$\boxed{0}$	1	3	0

$A \to T_2$	or	$A \to T_3$
$B \to T_3$		$B \to T_2$
$C \to T_4$		$C \to T_4$
$D \to T_1$		$D \to T_1$
Cost in		Cost in
3 + 6 + 8 + 3		4 + 5 + 8 + 3
= 20		= 20

There are two solutions and the least cost is ₹200000.

Example 25: In the assignment problem shown below, the least cost for optimal assignment is

	I	II	III	IV	V
A	1	3	2	3	6
B	2	4	3	1	5
C	5	6	3	4	6
D	3	1	4	2	2
E	1	5	6	5	4

(the matrix gives the cost value for each association)
(A) 10 units (B) 14 units
(C) 16 units (D) 18 units.

Solution:

0	2	1	2	5
1	3	2	0	4
2	3	0	1	3
2	0	3	1	1
0	4	5	4	3

The minimum no. of lines required to cover all the zeros is only '4'.

Therefore, optimum solution cannot be arrived at this stage. Subtracting 2 from all the involved values and adding 2 to the values at the intersections, we have

0	2	1	2	5
1	3	2	0	4
2	3	0	1	3
2	0	3	1	1
0	4	5	4	3

0	0	1	2	3
1	1	2	0	2
2	1	0	1	1
4	0	5	3	1
0	2	5	4	1

$\boxed{0}$	0	0	1	2
2	2	2	$\boxed{0}$	2
3	2	$\boxed{0}$	1	1
4	$\boxed{0}$	4	2	0
0	2	4	3	$\boxed{0}$

$A - 1, B \to 4, C \to 3, D \to 2, E \to 5$

The cost is 1 + 1 + 3 + 1 + 4 = 10 units.

Example 26: When five numbers of machines are assigned to five different operators, the profit obtained for various combinations is given in the matrix below.

	O_1	O_2	O_3	O_4	O_5
M_1	30	37	40	28	40
M_2	40	24	27	21	36
M_3	40	32	33	30	35
M_4	25	38	40	36	36
M_5	29	28	41	34	39

Maximum profit for the optimal association is
(A) 190 (B) 188 (C) 176 (D) 172.

Solution:
This is a maximisation problem unlike the previous ones. Therefore, it is to be converted to a minimisation problem. Subtract all the elements from the maximum value in the matrix. Then, it will become an opportunity – loss matrix which is to be minimised. The maximum value is 41. The opportunity loss matrix is,

11	4	1	13	1
1	17	14	20	5
1	9	8	11	6
16	3	1	5	5
12	13	0	7	2

Then, adopt the usual procedure

10	3	0	12	0
0	16	132	19	4
0	8	7	10	5
15	2	0	4	4
12	13	0	7	2

10	1	0	8	0
0	14	132	15	4
0	6	7	6	5
15	0	0	0	4
12	11	0	3	2

Four lines are required to cover all the zeros. Optimal solution cannot be arrived at this stage. Revising,

12	1	2	8	0
0	12	13	13	2
0	4	7	4	3
17	0	2	0	4
12	9	0	1	0

12	0	1	7	0
0	11	12	12	2
0	3	6	3	3
18	0	2	0	5
13	9	0	1	1

Now, 5 lines are required to cover all the zeros. Optimum solution can be arrived at. Give allocations.

14	$\boxed{0}$	1	7	10
0	9	10	10	$\boxed{0}$
$\boxed{0}$	1	4	1	1
20	0	2	$\boxed{0}$	5
15	9	$\boxed{0}$	1	1

$M_1 - O_2$
$M_2 - O_5$
$M_3 - O_1$
$M_4 - O_4$
$M_5 - O_3.$

Total profit
= 37 + 36 +40 +36 +41
= 190.

Exercises

Practice Problems I

1. 'Shadow price' is the term used in linear programming to denote
   (A) Optimum cost assigned to the variable
   (B) Value assigned to one unit of capacity
   (C) Maximum cost/unit
   (D) Minimum cost/unit.
2. When linear programming problem is solved graphically, every corner value in the feasible region shows
   (A) A basic feasible solution
   (B) An optimum solution
   (C) Solution not depending on constraints
   (D) Alternate solutions.
3. In graphical solution of linear programming problem, the optimum unique solution will be
   (A) Anywhere in the feasible region
   (B) Only at a corner value
   (C) In the feasible region but away from the origin
   (D) In the feasible region rearer to the origin.
4. In simplex method of solution of linear programming problem, at the optimum level of solution, the values of the stack variables are
   (A) Infinitely large
   (B) Positive or negative
   (C) Zeros
   (D) Can take any value.
5. Modified distribution method is used in
   (A) Linear programming problem
   (B) Transportation problem
   (C) Assignment problem
   (D) Queuing problems.
6. Consider a linear programming problem with two variables. The objective function is $x_1 + x_2$ It has two constraints. The corner points of the feasible region are (0, 0), (0, 3), (3, 0), (4, 4). The problem is to maximise the objective function. If one more constraint is added namely $x_1 + x_2 = 7$, the new maximum value will be
   (A) 8 (B) 15 (C) 7 (D) $\frac{14}{3}$
7. In a company, two types of products are produced: A and B. The levels of production are x and y, respectively. The profit function is $5x + 3y$. Constraints are given by
   $X + 3y \le 4$
   $X + y \le 10$
   $3x + y \le 24, x > 0$
   $y > 0.$
   The maximum profit obtainable is
   (A) 56 (B) 44
   (C) 40 (D) 32.
8. The maximum values of $Z = 5x + 2y$ subjected to
   $x \le 30$ $y \le 50$
   $3x + 2y \le 90$, $x, y \ge 0$ is
   (A) 135 (B) 150
   (C) 162 (D) 175.
9. The dual problem of the primal LPP Maximise $Z = 2x + y$ subjected to
   $x + y \le 12$
   $2x + y \le 18$
   $x, b \ge 0$ is
   (A) Minimise $Z = 12x + 18y$ subjected to
   $x + 2y \le 2$
   $x + y \le 1$ $x, y \ge 0.$
   (B) Minimise $Z = 12x + 8y$ subjected to
   $2x + y \le 12$
   $y + 2x \le 18, x, y > 0.$

(C) Minimise $Z = 12x + 8y$ subjected to
$x + 2y \geq 12$
$x + y \geq 18,\ x, y \geq 0.$

(D) Minimise $Z = 12x + 18y$ subjected to
$x + 2y \geq 2$
$x + y \geq 1,\ x, y \geq 0.$

10. Hungarian method is applicable to
(A) Linear programming problem
(B) Transportation problem
(C) Assignment problem
(D) Degenerated LPP.

11. If $Z = x + 4y$, the value of Z_{max} subjected to
$2x + 3y \leq 12$
$x + y \geq 8,\ x, y \geq$ is
(A) 40 (B) 16
(C) 24 (D) no solution.

12. Z is given as $60x_1 + 40x_2$. Maximum value of Z subjected to the constraints
$2x_1 + x_2 \leq 60$
$x_1 \leq 25$
$x_2 \leq 35,\ x_1, x_2 \geq 0$ is
(A) 2150 (B) 1900
(C) 1400 (D) 750.

13. Given $Z = 8x + 6y$, Z_{max} subjected to
$4x + 2y \leq 60$
$2x + 4y \leq 48$
$x, y \geq 0$ is
(A) 138 (B) 132
(C) 128 (D) 122.

14. The cost matrix for the transported problem is given below:

	X	Y	Z	
P	5	3	2	30
Q	2	5	4	40
R	6	1	5	80
	50	10	90	

The least cost of transportation is
(A) 740 (B) 620
(C) 510 (D) 460.

15. From three firms, materials are transported to 4 cities. The cost matrix gives the transportation cost/unit. For optimal arrangement, the least cost is

	M_1	M_2	M_3	M_4	
F_1	4	2	8	7	70
F_2	3	5	4	1	80
F_3	2	3	5	6	50
	20	60	60	40	

(A) ₹700 (B) ₹620 (C) ₹540 (D) ₹500

16. Given below is the cost matrix for transporting goods from factories F_1, F_2 and F_3 to the godowns G_1, G_2, G_3, G_4

	G_1	G_2	G_3	G_4	
F_1	13	11	16	20	2
F_2	17	14	12	13	6
F_3	18	18	15	12	7
	3	3	4	5	15

The least cost of transportation is
(A) 196 (B) 182 (C) 172 (D) 168.

17. Given $Z = 2x + 3y$. The maximum value of $2x + 3y$, subjected to $x + y \leq 30$
$Y \geq 3$
$0 \leq y \leq 12$
$x - y \geq 0$
$0 \leq x \leq 20$ is
(A) 80 (B) 72 (C) 68 (D) 65.

18. Given $Z = 8x + 8y$. The minimum value of the function subjected to
$x + y \geq 9$
$3x + y \geq 9\ x, y \geq 0$ is
(A) 38 (B) 36 (C) 32 (D) 30.

19. Given is the following matrix in a transportation problem. Availabilities of the origin $P + Q$ are 300 and 230, respectively. The demand from A and B are 420 and 110, respectively.

	A	B	Available
P	40 / 300	15	300
Q	10 / 120	12 / 110	230
Demand	420	110	530

Optimising the arrangement, the least cost is

(A) 7600 (B) 9850
(C) 11150 (D) 12500.

20. Three jobs J_1 J_2 and J_3 are to be assigned to three persons P_1, P_2 and P_3 so that the operational cost is a minimum. The matrix showing the operational cost for each combination is as shown below.

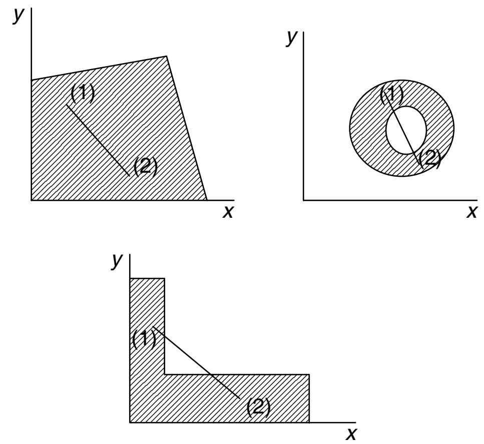
	P_1	P_2	P_3
J_1	2	20	11
J_2	8	23	9
J_3	34	15	14

The least cost for optimal assignment is

(A) ₹31 (B) ₹34 (C) ₹38 (D) ₹40.

Practice Problems 2

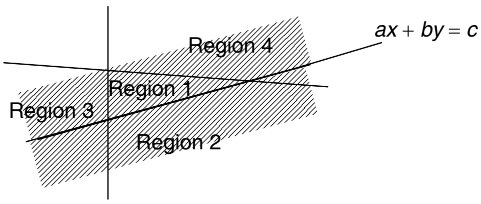
1. Three solution spaces are given.



Convex solution space is

(A) 1 only
(B) 2 only
(C) 3 only
(D) 2 + 3 only.

2. In a LPP, one of the constraints is $ax + by \geq c$. In the graph $ax + by = c$, the line 'l' showing the inequality $ax + by \geq c$ is



(A) Region (1) (B) Region (3)
(C) Region (2) (D) Region (4).

3. Consider the LPP:
Maximise $Z = x + y$ subjected to
$3x + 7y \geq 21$
$4x + 3y \geq 12, x, y > 0$

(A) Solution space is bounded unique solution
(B) Solution space is unbounded, with no optimal solution
(C) Solution space is unbounded, with infinite solutions
(D) Solution space is bounded, with a number of solutions

4. A linear programming problem is given as: Maximise $Z = 2x - y$ subjected to
$x - 2y \leq 10$
$x \leq 12, x, y \geq 0$
The maximum value of Z is

(A) 27 (B) 25 (C) 23 (D) 24.

5. Consider the linear programming problem, maximise $4x + 3y$ subjected to the constraints
$5x + 7y \geq 35$
$4x + y \leq 24, x, y \geq 0.$

(A) It has a unique solution
(B) It has a multiple solution
(C) It can have a unique or multiple solution
(D) It does not have a solution.

6. For the linear programming problem given as, maximise $Z = 2x_1 + 4x_2$ subjected to
$x_1 + 2x_2 \leq 20$
$2x_1 + x_2 \leq 12$
$x_1 + x_2 \leq 8 \;\; x_1 > 0, x_2 > 0$
the maximum value of Z is

(A) 24 (B) 32 (C) 36 (D) 42.

7. There exists a feasible region, but there is no closed boundary and the solution is possible. Then the problem is said to have

(A) Unbounded solution space, unbounded solution
(B) Unbounded solution space with bounded solution
(C) Bounded solution space with bounded solution
(D) Bounded solution space with bounded solution.

8. Given that, for a linear programming problem, slope of the objective function is the same as the slope of a constraint which forms a boundary of the feasible region. Then, the problem will have

(A) Unique solution (B) Multiple solution
(C) No solutions (D) Negative solution.

9. A linear programming problem is given as:

Maximise $Z = 3x_1 + 2x_2$

Subjected to

$6x_1 + x_2 \leq 30$

$2x_1 + 3x_2 \leq 15$

$x_1 ; x_2 \geq 0.$

The dual of this problem is

(A) Minimise $A = 30a + 15b$ subjected to
$6a + 2b \geq 3$
$a + 3b \geq 2$
$a, b \geq 0$

(B) Minimise $A = 2a + b$ subjected to
$a + 6b \geq 30$
$3a + 2b \geq 15, a, b \geq 0$

(C) Minimise $A = b + 2a$ subjected to
$a + 6b \geq 30$
$2a + 2b \geq 15, a, b \geq 0$

(D) Minimise $A = 3a + 2b$ subjected to
$6a + 3b \geq 30$
$2a + b \geq 15, a, b \geq 0.$

10. Two products $A + B$ produced in a firm require to pass through two sections one after another for their competition. The number of hours available/month in section 1 and 2 are respectively 200 and 220. A requires 2 hrs in section 1 and 6 hrs in section 2. B requires 4 hrs in section 1 and 3 hrs in section 2. If the profit/unit of A is ₹12 and the profit/unit of B is ₹14, then to maximise the profit, the formulated LPP is

(A) Maximise $Z = 12x + 14y$
Subjected to
$200x + 220y \leq 3$
$3x + 2y \leq 200$
$x, y \geq 0$

(B) Maximise $Z = 12x + 14y$
Subjected to
$2x + 4y \leq 200$
$6x + 3y \leq 220$
$x, y \geq 0$

(C) Maximise $12x + 4y$
Subjected to
$2x + 6y \leq 200$
$4x + 3y \leq 200, x, y \geq 0$

(D) Maximise $12x + 4y$
Subjected to
$2x + 3y \leq 200$
$4x + 6y \leq 220, x, y \geq 0.$

11. A transportation problem is said to be balanced, if
(A) the total capacity is equal to the total demand.
(B) the number of origins are numerically equal to the number of destinations.
(C) the problem does not degenerate.
(D) the problem can be a maximisation or a minimisation problem.

12. For optimality, a transportation problem with 'm' origins and 'n' destinations, in the allocated matrix, the number of occupied cells shall be
(A) $(m - 1) + (n - 1)$
(B) $m + n$
(C) $m + n + 1$
(D) $m + n - 1$.

13. 'MODI' method is applicable to
(A) Linear programming problem
(B) Assignment problem
(C) Degenerated LPP
(D) Transportation problem.

14. Vogels approximation method is used in transportation problem to find the
(A) Initial feasible solution
(B) Basic feasible solution
(C) Optimal solution
(D) To solve degeneracy in the problem.

15. In transportation problem, a feasible solution is said to be basic feasible solution, if
(A) Rows and columns satisfy the condition of $m + n - 1$ occupied cells.
(B) Rows and columns satisfy the condition $m + n -$ occupied cells.
(C) The total cost becomes least
(D) The problem does not get degenerated.

16. A transportation problem is said to degenerate, if
(A) The number of occupied cells are greater than $m + n - 1$
(B) The number of occupied cells are less than $m + n - 1$
(C) The number of occupied cells are $\neq m + n - 1$
(D) If a solution cannot be obtained.

17. When a transportation problem degenerates, to overcome this, we may
(A) Introduce a new origin
(B) Introduce a new destination
(C) Allocate a small quantity in an unoccupied cell
(D) Overcome degeneracy by generating column numbers and row numbers.

18. 'Modi' method is used in transportation problem to
(A) Find the basic feasible solution
(B) Overcome degeneracy
(C) Optimise the solution
(D) To balance the problem.

19. The difference between the impressed cost and the actual cost for an unoccupied cell is known as
(A) The least cost (B) The pay off cost
(C) The penalty cost (D) The optimum cost.

20. For the transportation shown as follows with usual notations, the least cost of transportation in the optimum case is

	D_1	D_2	D_3	D_4	Capacity
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	15	4	35

(A) 128 (B) 125
(C) 114 (D) 112.

21. The lowest cost for the transportation of 80 loads of materials from stations *A, B, C* and *D* to destinations D_1, D_2, D_3 and D_4 is when the cost matrix is given as follows:

	D_1	D_2	D_3	D_4	Supply
A	1	5	3	3	34
B	3	3	1	2	15
C	0	2	2	2	12
D	2	7	2	4	19
Demand	21	25	17	17	80

(A) 169 (B) 172 (C) 182 (D) 195.

22. Steel is transported from the furnaces F_1, F_2, F_3 to 5 rolling mills M_1 M_2 M_3 M_4 and M_5. Capacities of the furnaces and requirements of the rolling mills along with the transportation costs are given in the cost matrix below.
The lowest cost of transportation is

	M_1	M_2	M_3	M_4	M_5	
F_1	4	2	3	2	6	8
F_2	5	4	5	2	1	12
F_3	6	5	4	7	3	14
	4	4	6	8	8	

(A) 120 (B) 104 (C) 80 (D) 60.

23. Hungarian method is applicable to
(A) Linear programming problem
(B) Transportation problem
(C) Transportation problem with degeneracy
(D) Assignment problem.

24. The Net Evaluation row in the linear programming problem (simple algorithm) means
(A) The set of penalty values for not taking the variable into the programme
(B) The optimum values for not taking the variable into the programme.
(C) Set of values which are proportional to the profit index.
(D) Proportional to the allocations given to the slack variables.

25. In an assignment problem, 'x' jobs are assigned to 'x' machines. The optimum assignment for least cost is obtained when the minimum number of straight lines required to cover the zeros are
(A) n (B) $2n - 1$
(C) $n + 1$ (D) n^2

26. A computer centre has three expert programmers. The centre requires 3 application programmes to the developed. The manager determines the computer time (min) required by the experts.

Programmes	Application programmes 1	2	3
A	120	100	80
B	80	90	110
C	110	140	120

When the programmers are assigned with the programmes to have the least computer time, the least time is
(A) 310 min (B) 300 min
(C) 280 min (D) 240 min.

27. Given $Z = 5x_1 - 2x_2$ subjected to the constraints
$2x_1 + 3x_2 \geq 1$
$x_1, x_2 \geq 0$. The minimum value of Z is

(A) $\frac{-1}{2}$ (B) $\frac{5}{2}$

(C) $+\frac{2}{3}$ (D) $-\frac{2}{3}$.

28. Given $Z = 3x_1 + 4x_2$. The maximum value of Z subjected to $x_1 - x_2 \leq -1$
$-x_1 + x_2 \leq 0$
$x_1\ x_2 \geq 0$ is
(A) Unbounded solution
(B) No solution
(C) $Z = -1$
(D) $Z = +1$.

29. A company has three plants A, B, C and three warehouses X, Y, Z. The number of units available with the plant are 60, 70, 80 units, respectively. Demand at X, Y, Z is 50, 80, 80 respectively. The unit cost of transportation from the plants to warehouses are given by the following matrix.

Plants \ Warehouses	X	Y	Z
A	8	7	3
B	3	8	9
C	11	3	5

For optimum arrangement, the least cost of transportation is

(A) ₹820 (B) ₹750
(C) ₹725 (D) ₹700.

30. The cost matrix for an assignment problem is given below. The '0' variables are to be assigned to D variables. For optimum assignment, the least cost is

	D_1	D_2	D_3
0_1	20	27	30
0_2	10	18	16
0_3	14	16	12

(A) 49 (B) 52 (C) 42 (D) 40.

Previous Years' Questions

1. A company produces two types of toys: P and Q. Production time of Q is twice that of P and the company has a maximum of 2000 time units per day. The supply of raw material is just sufficient to produce 1500 toys (of any type) per day. Toy type Q requires an electric switch which is available at 600 pieces per day only. The company makes a profit of ₹3 and ₹5 on type P and Q respectively. For maximisation of profits, the daily production quantities of P and Q toys should respectively be **[2004]**

(A) 1000, 500 (B) 500, 100
(C) 800, 600 (D) 1000, 1000.

2. A company has two factories $S1$, $S2$ and two ware houses $D1$, $D2$. The supplies from $S1$and $S2$ are 50 and 40 units respectively. Warehouse $D1$ requires a minimum of 20 units and a maximum of 40 units. Warehouse $D2$ requires a minimum of 20 units and over and above, it can take as much as can be supplied. A balanced transportation problem is to be, formulated for th above situation. The number of supply points, the number of demand points and the total supply (or total demand) in the balanced transportation problem respectively are **[2005]**

(A) 2, 4, 90 (B) 2, 4, 100
(C) 3, 4, 90 (D) 3, 4, 110.

Direction for questions 3 and 4: Consider a linear programming problem with two variables and two constraints. The objective function is: Maximise $x_1 + x_2$. The corner points of the feasible region are (0, 0), (0, 2), (2, 0) and $\left[\frac{4}{3}, \frac{4}{3}\right]$

3. If an additional constraint $x_1 + x_2 \leq 5$ is added, the optimal solution is **[2005]**

(A) $\left(\frac{5}{3}, \frac{5}{3}\right)$ (B) $\left(\frac{4}{3}, \frac{4}{3}\right)$
(C) $\left(\frac{5}{2}, \frac{5}{2}\right)$ (D) (5, 0).

4. Let Y_1 and Y_2 be the decision variables of the dual and v_1 and v_2 be the slack variables of the dual of the given linear programming problem. The optimum dual variables are **[2005]**

(A) Y_1 and Y_2 (B) Y_1 and v_1
(C) Y_1 and v_2 (D) v_1 and v_2

5. A firm is required to procure three items (P, Q and R). The prices quoted for these items (in ₹) by supplier $S1$, $S2$ and $S3$ are given in table. The management policy requires that each item has to be supplied by only one supplier and one supplier supply only one item. The minimum total cost (in ₹) of procurement to the firm is: **[2006]**

Item	Suppliers		
	S1	**S2**	**S3**
P	110	120	130
Q	115	140	140
R	125	145	165

(A) 350 (B) 360 (C) 385 (D) 395.

6. For standard transportation linear programme with m sources and n destinations and total supply equalling total demand, an optimal solution (lowest cost) with the smallest number of non-zero x_{ij} values (amounts from source i to destination j) is desired. The best upper bound for this number is **[2008]**

(A) mn (B) $2(m + n)$
(C) $m + n$ (D) $m + n - 1$.

Direction for questions 7 and 8:

Consider the Linear Programme (LP)

Maximise $4x + 6y$

Subject to

$3x + 2y \leq 6$

$2x + 3y \leq 6$

$x, y \geq 0$

7. After introducing slack variables s and t, the initial basic feasible solution is represented by the table below (basic variables are $s = 6$ and $t = 6$, and the objective function value is 0) **[2008]**

	–4	–6	0	0	0
s	3	2	1	0	6
t	2	3	0	1	6
	x	y	s	t	RHS

After some simplex iterations, the following table is obtained.

	0	0	0	2	12
s	5/3	0	1	–1/3	2
y	2/3	1	0	1/3	2
	x	y	s	T	RHS

From this, one can conclude that

(A) The LP has a unique optimal solution
(B) The LP has an optimal solution that is not unique
(C) The LP is infeasible
(D) The LP is unbounded.

8. The dual for the LP in Q 74 is **[2008]**

(A) Minimise $6u + 6v$ subject to
$3u + 2v \geq 4$
$2u + 3v \geq 6$
$u, v \geq 0$

(B) Maximise $6u + 6v$ subject to
$3u + 2v \leq 4$
$2u + 3v \leq 6$
$u, v \geq 0$

(C) Maximise $4u + 6v$ subject to
$3u + 2v \geq 6$
$2u + 3v \geq 6$
$u, v \geq 0$

(D) Minimise $4u + 6v$ subject to
$3u + 2v \leq 6$
$2u + 3v \leq 6$
$u, v \geq 0.$

9. Consider the following Linear Programming Problem (LPP):

Maximise $z = 3x_1 + 2x_2$

Subject to $x_1 \leq 4$

$x_2 \leq 6$

$3x_1 + 2x_2 \leq 18$

$x_1 \geq 0, x_2 \geq 0.$ **[2009]**

(A) The LPP has a unique optimal solution
(B) The LPP is infeasible
(C) The LPP is unbounded
(D) The LPP has multiple optimal solutions.

10. Simplex method of solving linear programming problem uses **[2010]**

(A) all the points in the feasible region
(B) only the corner points of the feasible region
(C) intermediate points within the infeasible region
(D) only the interior points in the feasible region.

Direction for questions 11 and 12: One unit of product P_1 requires 3 kg of resource R_1 and 1 kg of resource R_2. One unit of product P_2 requires 2 kg of resource R_1 and 2 kg of resource R_2. The profits per unit by selling product P_1 and P_2 are ₹2000 and ₹3000 respectively. The manufacturer has 90 kg of resource R_1 and 100kg of resource R_2.

11. The unit worth of resource R_2, i.e. dual price of resource R_2 in ₹ per kg is **[2011]**

(A) 0 (B) 1350
(C) 1500 (D) 2000.

12. The manufacturer can make a maximum profit of ₹ **[2011]**

(A) 60000 (B) 135000
(C) 150000 (D) 200000.

13. A linear programming problem is shown below.

Maximise $3x + 7y$
Subject to $3x + 7y \leq 10$
$4x + 6y \leq 8$
$x, y \geq 0$

It has **[2013]**

(A) an unbounded objective function
(B) exactly one optimal solution
(C) exactly two optimal solutions
(D) infinitely many optimal solutions.

14. If there are m sources and n destinations in a transportation matrix, the total number of basic variables in a basic feasible solution is **[2014]**

(A) $m + n$ (B) $m + n + 1$
(C) $m + n - 1$ (D) m.

15. The total number of decision variables in the objective function of an assignment problem of size $n \times n$ (n jobs and n machines) is **[2014]**

(A) n^2 (B) $2n$
(C) $2n - 1$ (D) n.

16. For the linear programming problem:

Maximize $Z = 3X_1 + 2X_2$

Subject to

$-2X_1 + 3X_2 \leq 9$

$X_1 - 5X_2 \geq -20$

$X_1, X_2 \geq 0$

The above problem has: **[2015]**

(A) Unbounded solution
(B) Infeasible solution
(C) Alternative optimum solution
(D) Degenerate solution

17. Maximize $Z = 15X_1 + 20X_2$ subject to

$$12X_1 + 4X_2 \geq 36$$
$$12X_1 - 6X_2 \leq 24$$
$$X_1, X_2 \geq 0$$

The above linear programming problem has **[2016]**

(A) infeasible solution
(B) unbounded solution
(C) alternative optimum solutions
(D) degenerate solution

18. A firm uses a turning center, a milling center and a grinding machine to produce two parts. The table below provides the machining time required for each part and the maximum machining time available on each machine. The profit per unit on parts I and II are ₹40 and ₹100, respectively. The maximum profit per week of the firm is ₹ ________. **[2016]**

Type of machine	Machining time required for the machine part (minutes)		Maximum machining time available per week (minutes)
	I	II	
Turning Center	12	6	6000
Milling Center	4	10	4000
Grinding Machine	2	3	1800

Answer Keys

Exercises

Practice Problems 1

1. B **2.** A **3.** C **4.** C **5.** B **6.** A **7.** B **8.** B **9.** D **10.** C
11. D **12.** A **13.** B **14.** C **15.** D **16.** A **17.** B **18.** B **19.** C **20.** A

Practice Problems 2

1. A **2.** D **3.** B **4.** D **5.** A **6.** B **7.** B **8.** B **9.** A **10.** B
11. A **12.** D **13.** D **14.** A **15.** A **16.** B **17.** C **18.** C **19.** C **20.** C
21. A **22.** C **23.** D **24.** A **25.** A **26.** C **27.** D **28.** A **29.** B **30.** C

Previous Years' Questions

1. A **2.** C **3.** B **4.** D **5.** C **6.** D **7.** B **8.** A **9.** D **10.** B
11. C **12.** B **13.** B **14.** C **15.** A **16.** A **17.** B **18.** 40000