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Relations and Functions



TOPIC 1

Relations, Domain, Codomain and Range of a Relation, Functions, Domain, Codomain and Range of a Function



- Let R_1 and R_2 be two relations defined as follows :
 $R_1 = \{(a, b) \in \mathbf{R}^2 : a^2 + b^2 \in \mathbf{Q}\}$ and
 $R_2 = \{(a, b) \in \mathbf{R}^2 : a^2 + b^2 \notin \mathbf{Q}\}$, where $\mathbf{Q}$ is the set of all rational numbers. Then : **[Sep. 03, 2020 (II)]**
 (a) Neither R_1 nor R_2 is transitive.
 (b) R_2 is transitive but R_1 is not transitive.
 (c) R_1 is transitive but R_2 is not transitive.
 (d) R_1 and R_2 are both transitive.
- The domain of the function $f(x) = \sin^{-1}\left(\frac{|x|+5}{x^2+1}\right)$ is $(-\infty, -a] \cup [a, \infty)$. Then a is equal to : **[Sep. 02, 2020 (I)]**
 (a) $\frac{\sqrt{17}}{2}$ (b) $\frac{\sqrt{17}-1}{2}$ (c) $\frac{1+\sqrt{17}}{2}$ (d) $\frac{\sqrt{17}}{2}+1$
- If $R = \{(x, y) : x, y \in \mathbf{Z}, x^2 + 3y^2 \leq 8\}$ is a relation on the set of integers $\mathbf{Z}$, then the domain of R^{-1} is : **[Sep. 02, 2020 (I)]**
 (a) $\{-2, -1, 1, 2\}$ (b) $\{0, 1\}$
 (c) $\{-2, -1, 0, 1, 2\}$ (d) $\{-1, 0, 1\}$
- Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = \frac{x}{1+x^2}$, $x \in \mathbf{R}$. Then the range of f is : **[Jan. 11, 2019 (I)]**
 (a) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (b) $\mathbf{R} - [-1, 1]$
 (c) $\mathbf{R} - \left[-\frac{1}{2}, \frac{1}{2}\right]$ (d) $(-1, 1) - \{0\}$
- The domain of the definition of the function $f(x) = \frac{1}{4-x^2} + \log_{10}(x^3 - x)$ is: **[April. 09, 2019 (II)]**
 (a) $(-1, 0) \cup (1, 2) \cup (3, \infty)$
 (b) $(-2, -1) \cup (-1, 0) \cup (2, \infty)$
 (c) $(-1, 0) \cup (1, 2) \cup (2, \infty)$
 (d) $(1, 2) \cup (2, \infty)$
- The range of the function $f(x) = \frac{x}{1+|x|}$, $x \in \mathbf{R}$, is **[Online May 7, 2012]**
 (a) $\mathbf{R}$ (b) $(-1, 1)$ (c) $\mathbf{R} - \{0\}$ (d) $[-1, 1]$
- The domain of the function $f(x) = \frac{1}{\sqrt{|x|-x}}$ is **[2011]**
 (a) $(0, \infty)$ (b) $(-\infty, 0)$
 (c) $(-\infty, \infty) - \{0\}$ (d) $(-\infty, \infty)$
- Domain of definition of the function $f(x) = \frac{3}{4-x^2} + \log_{10}(x^3 - x)$, is **[2003]**
 (a) $(-1, 0) \cup (1, 2) \cup (2, \infty)$ (b) $(a, 2)$
 (c) $(-1, 0) \cup (a, 2)$ (d) $(1, 2) \cup (2, \infty)$

TOPIC 2

Even and Odd Functions, Explicit and Implicit Functions, Greatest Integer Function, Periodic Functions, Value of a Function, Equal Functions, Algebraic Operations on Functions



- Let $[t]$ denote the greatest integer $\leq t$. Then the equation in x , $[x]^2 + 2[x+2] - 7 = 0$ has : **[Sep. 04, 2020 (I)]**
 (a) exactly two solutions
 (b) exactly four integral solutions
 (c) no integral solution
 (d) infinitely many solutions

10. Let $f(x)$ be a quadratic polynomial such that $f(-1) + f(2) = 0$. If one of the roots of $f(x) = 0$ is 3, then its other root lies in :
[Sep. 02, 2020 (II)]
(a) $(-1, 0)$ (b) $(1, 3)$ (c) $(-3, -1)$ (d) $(0, 1)$
11. Let $f(1, 3) \rightarrow R$ be a function defined by $f(x) = \frac{x[x]}{1+x^2}$, where $[x]$ denotes the greatest integer $\leq x$. Then the range of f is:
[Jan. 8, 2020 (II)]
(a) $\left(\frac{2}{5}, \frac{3}{5}\right) \cup \left(\frac{3}{4}, \frac{4}{5}\right)$ (b) $\left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{3}{5}, \frac{4}{5}\right)$
(c) $\left(\frac{2}{5}, \frac{4}{5}\right)$ (d) $\left(\frac{3}{5}, \frac{4}{5}\right)$
12. If $f(x) = \log_e \left(\frac{1-x}{1+x} \right)$, $|x| < 1$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to :
[April 8, 2019 (I)]
(a) $2f(x)$ (b) $2f(x^2)$ (c) $(f(x))^2$ (d) $-2f(x)$
13. Let $f(x) = a^x$ ($a > 0$) be written as $f(x) = f_1(x) + f_2(x)$, where $f_1(x)$ is an even function and $f_2(x)$ is an odd function. Then $f_1(x+y) + f_1(x-y)$ equals :
[April. 08, 2019 (II)]
(a) $2f_1(x)f_1(y)$ (b) $2f_1(x+y)f_1(x-y)$
(c) $2f_1(x)f_2(y)$ (d) $2f_1(x+y)f_2(x-y)$
14. Let $f(n) = \left[\frac{1}{3} + \frac{3n}{100} \right] n$, where $[n]$ denotes the greatest integer less than or equal to n . Then $\sum_{n=1}^{56} f(n)$ is equal to:
[Online April 19, 2014]
(a) 56 (b) 689 (c) 1287 (d) 1399
15. Let f be an odd function defined on the set of real numbers such that for $x \geq 0$, $f(x) = 3 \sin x + 4 \cos x$.
Then $f(x)$ at $x = -\frac{11\pi}{6}$ is equal to: [Online April 11, 2014]
(a) $\frac{3}{2} + 2\sqrt{3}$ (b) $-\frac{3}{2} + 2\sqrt{3}$
(c) $\frac{3}{2} - 2\sqrt{3}$ (d) $-\frac{3}{2} - 2\sqrt{3}$
16. A real valued function $f(x)$ satisfies the functional equation $f(x-y) = f(x)f(y) - f(a-x)f(a+y)$ where a is a given constant and $f(0) = 0$, $f(2a-x)$ is equal to
[2005]
(a) $-f(x)$ (b) $f(x)$
(c) $f(a) + f(a-x)$ (d) $f(-x)$
17. The graph of the function $y = f(x)$ is symmetrical about the line $x = 2$, then
[2004]
(a) $f(x) = -f(-x)$ (b) $f(2+x) = f(2-x)$
(c) $f(x) = f(-x)$ (d) $f(x+2) = f(x-2)$
18. If $f : R \rightarrow R$ satisfies $f(x+y) = f(x) + f(y)$, for all x , $y \in R$ and $f(1) = 7$, then $\sum_{r=1}^n f(r)$ is
[2003]
(a) $\frac{7n(n+1)}{2}$ (b) $\frac{7n}{2}$
(c) $\frac{7(n+1)}{2}$ (d) $7n + (n+1)$



Hints & Solutions



1. (a) For R_1 let $a = 1 + \sqrt{2}$, $b = 1 - \sqrt{2}$, $c = 8^{1/4}$
 $aR_1b \Rightarrow a^2 + b^2 = (1 + \sqrt{2})^2 + (1 - \sqrt{2})^2 = 6 \in Q$
 $bR_1c \Rightarrow b^2 + c^2 = (1 - \sqrt{2})^2 + (8^{1/4})^2 = 3 \in Q$
 $aR_1c \Rightarrow a^2 + c^2 = (1 + \sqrt{2})^2 + (8^{1/4})^2 = 3 + 4\sqrt{2} \notin Q$
 $\therefore R_1$ is not transitive.
 For R_2 let $a = 1 + \sqrt{2}$, $b = \sqrt{2}$, $c = 1 - \sqrt{2}$
 $aR_2b \Rightarrow a^2 + b^2 = (1 + \sqrt{2})^2 + (\sqrt{2})^2 = 5 + 2\sqrt{2} \notin Q$
 $bR_2c \Rightarrow b^2 + c^2 = (\sqrt{2})^2 + (1 - \sqrt{2})^2 = 5 - 2\sqrt{2} \notin Q$
 $aR_2c \Rightarrow a^2 + c^2 = (1 + \sqrt{2})^2 + (1 - \sqrt{2})^2 = 6 \in Q$
 $\therefore R_2$ is not transitive.

2. (c) $\because f(x) = \sin^{-1} \left(\frac{|x|+5}{x^2+1} \right)$

$$\therefore -1 \leq \frac{|x|+5}{x^2+1} \leq 1$$

$$\Rightarrow |x|+5 \leq x^2+1 \quad [\because x^2+1 \neq 0]$$

$$\Rightarrow x^2 - |x| - 4 \geq 0$$

$$\Rightarrow \left(|x| - \frac{1-\sqrt{17}}{2} \right) \left(|x| - \frac{1+\sqrt{17}}{2} \right) \geq 0$$

$$\Rightarrow x \in \left(-\infty, -\frac{1+\sqrt{17}}{2} \right) \cup \left[\frac{1+\sqrt{17}}{2}, \infty \right)$$

$$\therefore a = \frac{1+\sqrt{17}}{2}$$

3. (d) Since, $R = \{(x, y) : x, y \in \mathbf{Z}, x^2 + 3y^2 \leq 8\}$
 $\therefore R = \{(1, 1), (2, 1), (1, -1), (0, 1), (1, 0)\}$
 $\Rightarrow D_{R^{-1}} = \{-1, 0, 1\}$

4. (a) $f(x) = \frac{x}{1+x^2}, x \in R$

$$\text{Let, } y = \frac{x}{1+x^2}$$

$$\Rightarrow yx^2 - x + y = 0 \Rightarrow x = \frac{1 \pm \sqrt{1-4y^2}}{2}$$

$$\Rightarrow 1 - 4y^2 \geq 0$$

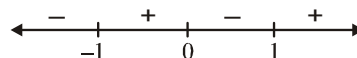
$$\Rightarrow 1 \geq 4y^2$$

$$\Rightarrow |y| \leq \frac{1}{2}$$

$$\Rightarrow -\frac{1}{2} \leq y \leq \frac{1}{2}$$

$$\Rightarrow \text{The range of } f \text{ is } \left[-\frac{1}{2}, \frac{1}{2} \right].$$

5. (e) To determine domain, denominator $\neq 0$ and $x^3 - x > 0$
 i.e., $4 - x^2 \neq 0 \Rightarrow x \neq \pm 2 \dots (1)$
 and $x(x-1)(x+1) > 0$



$$x \in (-1, 0) \cup (1, \infty) \dots (2)$$

Hence domain is intersection of (1) & (2).

$$\text{i.e., } x \in (-1, 0) \cup (1, 2) \cup (2, \infty)$$

6. (b) $f(x) = \frac{x}{1+|x|}, x \in R$

$$\text{If } x > 0, |x| = x \Rightarrow f(x) = \frac{x}{1+x}$$

which is not defined for $x = -1$

$$\text{If } x < 0, |x| = -x \Rightarrow f(x) = \frac{x}{1-x}$$

which is not defined for $x = 1$

Thus $f(x)$ defined for all values of R except 1 and -1

Hence, range = $(-1, 1)$.

7. (b) $f(x) = \frac{1}{\sqrt{|x|-x}}, f(x)$ is define if $|x| - x > 0$
 $\Rightarrow |x| > x, \Rightarrow x < 0$
 Hence domain of $f(x)$ is $(-\infty, 0)$

8. (a) $f(x) = \frac{3}{4-x^2} + \log_{10}(x^3 - x)$

$$4 - x^2 \neq 0; x^3 - x > 0;$$

$$x \neq \pm\sqrt{4} \text{ and } -1 < x < 0 \text{ or } 1 < x < \infty$$



$$\therefore D = (-1, 0) \cup (1, \infty) - \{\sqrt{4}\}$$

$$D = (-1, 0) \cup (1, 2) \cup (2, \infty).$$

9. (d) The given equation

$$[x]^2 + 2[x] + 4 - 7 = 0$$

$$\Rightarrow [x]^2 + 2[x] - 3 = 0$$

$$\Rightarrow [x]^2 + 3[x] - [x] - 3 = 0$$

$$\Rightarrow ([x] + 3)([x] - 1) = 0 \Rightarrow [x] = 1 \text{ or } -3$$

$$\Rightarrow x \in [-3, -2) \cup [1, 2)$$

$\therefore$ The equation has infinitely many solutions.

10. (a) Let $f(x) = ax^2 + bx + c$

$$\text{Given: } f(-1) + f(2) = 0$$

$$a - b + c + 4a + 2b + c = 0$$

$$\Rightarrow 5a + b + 2c = 0 \quad \dots(i)$$

$$\text{and } f(3) = 0 \Rightarrow 9a + 3b + c = 0 \quad \dots(ii)$$

From equations (i) and (ii),

$$\frac{a}{1-6} = \frac{b}{18-5} = \frac{c}{15-9} \Rightarrow \frac{a}{-5} = \frac{b}{13} = \frac{c}{6}$$

$$\text{Product of roots, } \alpha\beta = \frac{c}{a} = \frac{-6}{5} \text{ and } \alpha = 3$$

$$\Rightarrow \beta = \frac{-2}{5} \in (-1, 0)$$

$$11. (b) f(x) = \begin{cases} \frac{x}{x^2+1}; & x \in (1, 2) \\ \frac{2x}{x^2+1}; & x \in [2, 3) \end{cases}$$

$$f'(x) = \begin{cases} \frac{1-x^2}{1+x^2}; & x \in (1, 2) \\ \frac{1-2x^2}{1+x^2}; & x \in [2, 3) \end{cases}$$

$\therefore f(x)$ is a decreasing function

$$\therefore y \in \left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{6}{10}, \frac{4}{5}\right)$$

$$\Rightarrow y \in \left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{3}{5}, \frac{4}{5}\right)$$

12. (a) $f(x) = \log\left(\frac{1-x}{1+x}\right), |x| < 1$

$$f\left(\frac{2x}{1+x^2}\right) = \log\left(\frac{1-\frac{2x}{1+x^2}}{1+\frac{2x}{1+x^2}}\right)$$

$$= \log\left(\frac{1+x^2-2x}{1+x^2+2x}\right) = \log\left(\frac{1-x}{1+x}\right)^2$$

$$= 2 \log\left(\frac{1-x}{1+x}\right) = 2f(x)$$

13. (a) Given function can be written as

$$f(x) = a^x = \left(\frac{a^x + a^{-x}}{2}\right) + \left(\frac{a^x - a^{-x}}{2}\right)$$

$$\text{where } f_1(x) = \frac{a^x + a^{-x}}{2} \text{ is even function}$$

$$f_2(x) = \frac{a^x - a^{-x}}{2} \text{ is odd function}$$

$$\Rightarrow f_1(x+y) + f_1(x-y)$$

$$= \left(\frac{a^{x+y} + a^{-x-y}}{2}\right) + \left(\frac{a^{x-y} + a^{-x+y}}{2}\right)$$

$$= \frac{1}{2} [a^x(a^y + a^{-y}) + a^{-x}(a^y + a^{-y})]$$

$$= \frac{(a^x + a^{-x})(a^y + a^{-y})}{2} = 2f_1(x) \cdot f_1(y)$$

14. (d) Let $f(n) = \left[\frac{1}{3} + \frac{3n}{100}\right]n$

where $[n]$ is greatest integer function,

$$= \left[0.33 + \frac{3n}{100}\right]n$$

For $n = 1, 2, \dots, 22$, we get $f(n) = 0$

and for $n = 23, 24, \dots, 55$, we get $f(n) = 1 \times n$

For $n = 56, f(n) = 2 \times n$

$$\text{So, } \sum_{n=1}^{56} f(n) = 1(23) + 1(24) + \dots + 1(55) + 2(56)$$

$$= (23 + 24 + \dots + 55) + 112$$

$$= \frac{33}{2} [46 + 32] + 112$$

$$= \frac{33}{2} (78) + 112 = 1399.$$

15. (c) Given f be an odd function

$$f(x) = 3 \sin x + 4 \cos x$$

Now,

$$f\left(\frac{-11\pi}{6}\right) = 3 \sin\left(\frac{-11\pi}{6}\right) + 4 \cos\left(\frac{-11\pi}{6}\right)$$

$$f\left(\frac{-11\pi}{6}\right) = 3\sin\left(-2\pi + \frac{\pi}{6}\right) + 4\cos\left(-2\pi + \frac{\pi}{6}\right)$$

$$f\left(\frac{-11\pi}{6}\right) = 3\sin\left\{-\left(2\pi - \frac{\pi}{6}\right)\right\} + 4\cos\left\{-\left(2\pi - \frac{\pi}{6}\right)\right\}$$

$$\left\{ \begin{array}{l} \text{For odd functions } \sin(-\theta) = -\sin \theta \\ \text{and } \cos(-\theta) = -\cos \theta \end{array} \right\}$$

$$\therefore f\left(\frac{-11\pi}{6}\right) = -3\sin\left(2\pi - \frac{\pi}{6}\right) - 4\cos\left(2\pi - \frac{\pi}{6}\right)$$

$$\Rightarrow f\left(\frac{-11\pi}{6}\right) = +3\sin\left(\frac{\pi}{6}\right) - 4\cos\frac{\pi}{6}$$

$$\Rightarrow f\left(\frac{-11\pi}{6}\right) = 3 \times \frac{1}{2} - 4 \times \frac{\sqrt{3}}{2}$$

$$\text{or } f\left(\frac{-11\pi}{6}\right) = \frac{3}{2} - 2\sqrt{3}$$

16. (a) Given that $f(0) = 0$ and put

$$x = 0, y = 0,$$

$$f(0) = f^2(0) - f^2(a)$$

$$\Rightarrow f^2(a) = 0 \Rightarrow f(a) = 0$$

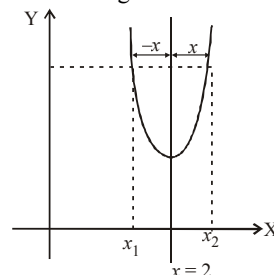
$$f(2a - x) = f(a - (x - a))$$

$$= f(a)f(x - a) - f(0)f(x)$$

$$= f(a)f(x - a) - f(x) = -f(x)$$

$$\Rightarrow f(2a - x) = -f(x)$$

17. (b) Given that a graph symmetrical with respect to line $x = 2$ as shown in the figure.



From the figure

$$f(x_1) = f(x_2), \text{ where } x_1 = 2 - x \text{ and } x_2 = 2 + x$$

$$\therefore f(2 - x) = f(2 + x)$$

18. (a) $f(x + y) = f(x) + f(y)$.

$$\therefore f(1) = 7$$

$$f(2) = f(1 + 1) = f(1) + f(1) = 14$$

$$f(3) = f(1 + 2) = f(1) + f(2) = 21$$

$$\text{-----}$$

$$\therefore \sum_{r=1}^n f(r) = 7(1 + 2 + 3 + \dots + n)$$

$$= \frac{7n(n+1)}{2}$$