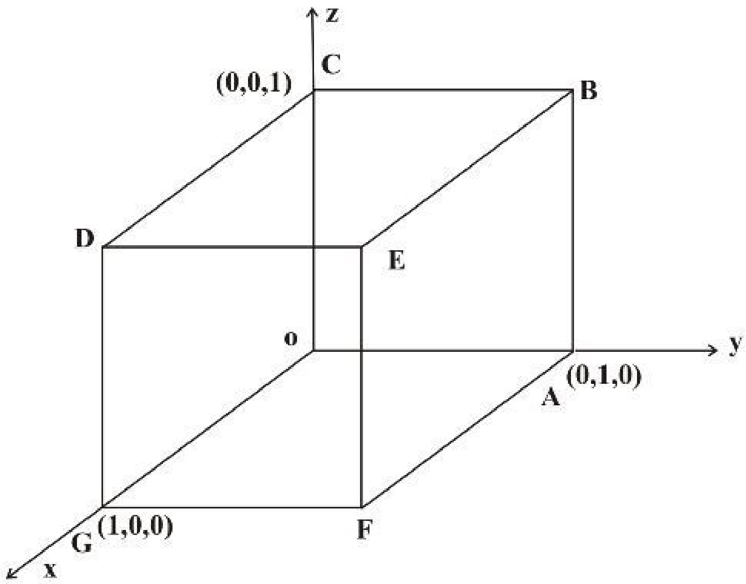
# Exercise 16.9

Chapter 16 Vector Calculus Exercise 16.9 1E




Now $\vec{F} = 3x\hat{i} + xy\,\hat{j} + 2xz\,\hat{k}$

Then $div\,\vec{F} = \vec{\nabla}.\vec{F}$

$= 3 + x + 2x$

$= 3 + 3x$

Then $\iiint_E \vec{\nabla}.\vec{F}\,dV = \iiint_E 3(1+x)\,dV$

Here $E = \{(x,y,z),\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

Then $\iiint_E \vec{\nabla}.\vec{F}\,dV = 3\int_0^1\int_0^1\int_0^1 (1+x)\,dx\,dy\,dz$

$= 3\left(x + \frac{x^2}{2}\right)_0^1 (y)_0^1 (z)_0^1$

$= 3\left(1 + \frac{1}{2}\right)(1)(1)$

$= \frac{(3)^2}{2}$

$= \frac{9}{2}$

We need to find $\iint_S \vec{F}.d\vec{s}$ or $\iint_S \vec{F}.\hat{n}\,ds$ over the six faces of the cube

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Over the face DEFG, $\hat{n} = \hat{i}$ and x = 1

Therefore $\iint_{DEFG} \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1 3x\,dy\,dz$

$= 3(y)_0^1(z)_0^1$

$= 3$

Over face ABCD, $n = -\hat{i}$, $x = 0$

Then $\iint_{ABCD} \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1 -3x\,dy\,dz$

$= \int_0^1\int_0^1 (0)\,dy\,dz$

$= 0$

Over the face ABEF, $\hat{n}=\hat{j},\ y=1$

Therefore $\displaystyle\iint_{ABEF} \vec{F}.\hat{n}\,ds=\int_0^1\int_0^1 (xy)\,dx\,dz$

$$=\int_0^1 x\,dx\int_0^1 dz$$

$$=\frac{1}{2}$$

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Over the face OGDC, $\hat{n}=-\hat{j},\ y=0$

Therefore $\displaystyle\iint_{BGDC} \vec{F}.\hat{n}\ ds=\int_0^1\int_0^1 (-xy)\,dx\,dz$

$$=0$$

Over the face BCDE, $\hat{n}=\hat{k}$ and $z=1$

Then $\displaystyle\iint_{BCDE} \vec{F}.\hat{n}\,ds=\int_0^1\int_0^1 (2xz)\,dx\,dy$

$$=2\int_0^1 x\,dx\int_0^1 dy$$

$$=2\times\frac{1}{2}$$

$$=1$$

And over the face AFGD, $\hat{n}=-\hat{k}$, and $z=0$

Then $\displaystyle\iint_{AFGD} \vec{F}.\hat{n}\,ds=\int_0^1\int_0^1 (-2xz)\,dx\,dy$

$$=0$$

Therefore $\displaystyle\iint_S \vec{F}.\hat{n}\,ds=3+0+\frac{1}{2}+0+1+0$

$$=\frac{9}{2}$$

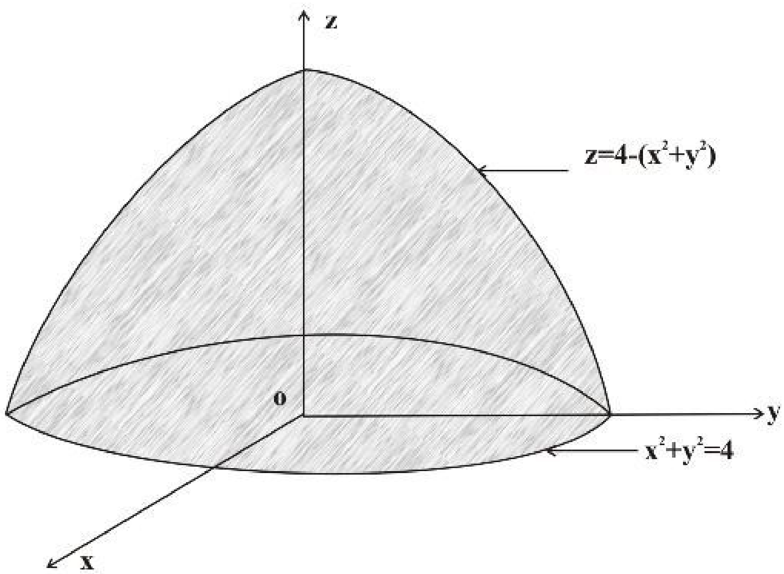
Hence $\displaystyle\iiint_E \vec{\nabla}.\vec{F}\,dV=\iint_S \vec{F}.d\vec{s}=\frac{9}{2}$

Which proves the divergence theorem

### Chapter 16 Vector Calculus Exercise 16.9 2E

$$\vec{F}(x,y,z)=x^2\hat{i}+xy\hat{j}+z\hat{k}$$

Where E is the solid bounded by the paraboloid $z=4-x^2-y^2$ and xy – plane



The plane xy meets paraboloid $z = 4 - x^2 - y^2$ in circular disk $x^2 + y^2 = 4$. Then E in cylindrical co – ordinates is

$$E = \{(r,\theta,z):\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi, 0 \le z \le 4 - r^2\}$$

Also $div\,\vec{F} = 2x + x + 1$
$= 3x + 1$

Then $$\iiint_E div\,\vec{F}\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} (3r\cos\theta + 1)\,r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 (3r\cos\theta+1)r(4-r^2)\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 (3r\cos\theta+1)(4r-r^3)\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 \cos\theta(12r^2-3r^4)+4r-r^3 dr\,d\theta$$
$$= \int_0^{2\pi} \cos\theta\left(32-\frac{96}{5}\right)+(8-4)\,d\theta$$
$$= (\sin\theta)_0^{2\pi}\left(32-\frac{96}{5}\right)+4(\theta)_0^{2\pi}$$
$$= (0)\left(32-{}^{96}\!/_5\right)+4(2\pi)$$
$$= 8\pi$$

i.e. $$\iiint_E div\,\vec{F}\,dV = 8\pi$$

Now the surface under E is the region below curve $z = 4 - x^2 - y^2$ and above the circle $x^2 + y^2 = 4$

Then $$\iint_S \vec{F}.d\vec{s} = \iint_S \vec{F}.\hat{n}\,ds$$
$$= \iint_D \left(-x^2\frac{\partial z}{\partial x} - xy\frac{\partial z}{\partial y} + z\right)dA$$
$$= \iint_D 3\left(-x^2(-2x) - xy(-2y) + 4 - x^2 - y^2\right)dA$$
$$= \iint_D \left[2x^3 + 2xy^2 + 4 - x^2 - y^2\right]dA$$
$$= \iint_D \left[2x(x^2+y^2) + 4 - (x^2+y^2)\right]dA$$

In polar co – ordinates, $D = \{(r,\theta):\ 0 \le \theta \le 2\pi,\ 0 \le r \le 2\}$

Then $$\iint_S \vec{F}.d\vec{s} = \int_0^{2\pi}\int_0^2 \left[2r\cos\theta.r^2 + 4 - r^2\right]r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 \left[2r^3\cos\theta + 4 - r^2\right]r\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[\frac{2}{5}r^5\cos\theta + 2r^2 - \frac{r^4}{4}\right]_{r=0}^{r=2} d\theta$$
$$= \int_0^{2\pi}\left[\frac{64}{5}\cos\theta + 8 - 4\right]d\theta$$
$$= \frac{64}{5}(\sin\theta)_0^{2\pi} + 4(\theta)_0^{2\pi}$$
$$= \frac{64}{5}(0) + 4(2\pi)$$
$$= 8\pi$$

Hence $$\iiint_E div\,\vec{F}\,dV = \iint_S \vec{F}.d\vec{s} = 8\pi$$

Which proves the divergence in theorem

## Chapter 16 Vector Calculus Exercise 16.9 3E

Consider the vector field,

$\mathbf{F}(x,y,z)=\langle z,y,x\rangle$.

The objective is to verify that the divergence theorem is true for the given vector field on a solid ball, *E*: $x^2+y^2+z^2\le 16$.

According to the Divergence Theorem,

$$\iint_S \mathbf{F}\cdot d\mathbf{S}=\iiint_E \text{div }\mathbf{F}\,dV$$

Since the solid *E* is $x^2+y^2+z^2\le 16$, it has one surface with associated normal vector.

$$S:\mathbf{r}(\theta,\phi)=\langle 4\sin\phi\cos\theta,4\sin\phi\cos\theta,4\cos\phi\rangle$$

$$\mathbf{n}=\mathbf{r}_\theta\times\mathbf{r}_\phi=\langle 16\sin^2\phi\cos\theta,16\sin\phi\sin\theta,16\sin\phi\cos\phi\rangle$$

$$0\le\theta\le 2\pi,0\le\phi\le\pi$$

The value of the surface integral is,

$$\begin{aligned}
&\iint_S \mathbf{F}\cdot d\mathbf{S}\\
&=\iint_S\langle 4\cos\phi,4\sin\phi\sin\theta,4\sin\phi\cos\theta\rangle\cdot\langle 16\sin^2\phi\cos\theta,16\sin\phi\sin\theta,16\sin\phi\cos\phi\rangle\,dS\\
&=\iint_S 64\sin^2\phi\cos\phi\cos\theta+64\sin^3\phi\sin^2\theta+64\sin^2\phi\cos\phi\cos\theta\,dS\\
&=\int_0^{2\pi}\int_0^{\pi}128\sin^2\phi\cos\phi\cos\theta+64\sin^3\phi\sin^2\theta\,d\phi\,d\theta\\
&=128\int_0^{2\pi}\cos\theta\,d\theta\int_0^{\pi}\sin^2\phi\cos\phi\,d\phi+64\int_0^{2\pi}\sin^2\theta\,d\theta\int_0^{\pi}\sin^3\phi\,d\phi\\
&=128\left[-\sin\theta\right]_{\theta=0}^{\theta=2\pi}+\int_0^{\pi}\sin^2\phi\cos\phi\,d\phi+64\int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta)\,d\theta\int_0^{\pi}\sin\phi\left(1-\cos^2\phi\right)d\phi\\
&=0+\left[\frac{1}{3}\sin^3\phi\right]_{\phi=0}^{\phi=\pi}+32\int_0^{2\pi}(1-\cos 2\theta)\,d\theta+\int_0^{\pi}\left(\sin\phi-\sin\phi\cos^2\phi\right)d\phi\\
&=0+32\left[\theta-\frac{1}{2}\sin 2\theta\right]_{\theta=0}^{\theta=2\pi}+\left[-\cos\phi+\frac{1}{3}\cos^3\phi\right]_{\phi=0}^{\phi=\pi}\\
&=32(2\pi-0)\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]\\
&=\boxed{\frac{256\pi}{3}}
\end{aligned}$$

The divergence of $\mathbf{F}(x,y,z)=\langle z,y,x\rangle$ is,

$$\begin{aligned}
\text{div }\mathbf{F}&=\frac{\partial(z)}{\partial x}+\frac{\partial(y)}{\partial y}+\frac{\partial(x)}{\partial z}\\
&=0+1+0\\
&=1
\end{aligned}$$

Now, calculate the flux of **F** across the surface *S* as follows, where the solid *E* is $x^2+y^2+z^2\le 16$:

$$\begin{aligned}
\iiint_E \text{div }\mathbf{F}\,dV&=\iiint_E 1\,dV\\
&=\frac{4}{3}\pi(4)^3\\
&=\boxed{\frac{256\pi}{3}}
\end{aligned}$$

Thus, the flux of **F** across the boundary surface *S* of *E* is equal to the triple integral of the divergence of **F** over *E*.

Therefore, the Divergence Theorem is verified.

Chapter 16 Vector Calculus Exercise 16.9 4E

**Verify Gauss divergence theorem is true for the vector field $\mathbf{F}$ on the region $E$.**

**The given vector field:** $\mathbf{F}(x, y, z) = \langle x^2, -y, z \rangle$

**The given region is a solid cylinder** $E : y^2 + z^2 \le 9,\ 0 \le x \le 2.$

Write the gauss divergence theorem,

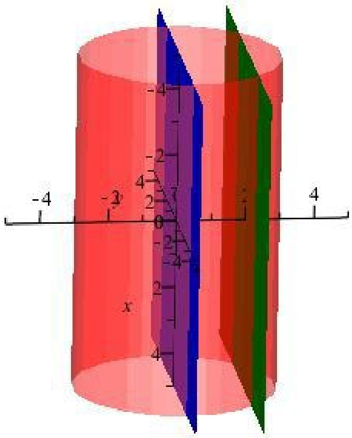
Let $E$ be a simple solid region and let $S$ be the boundary surface of $E$, given with positive (outward orientation). Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains $E$ then

Then, $\iint_S \mathbf{F} \cdot ds = \iiint_E \text{div}\mathbf{F}\, dV$ ......(1)

At first calculate the right hand side triple integral.

$$\begin{aligned} \text{div}\mathbf{F} &= \nabla \cdot \mathbf{F} \\ &= \nabla \cdot \langle x^2, -y, z \rangle \\ &= \frac{\partial(x^2)}{\partial x} + \frac{\partial(-y)}{\partial y} + \frac{\partial(z)}{\partial z} \\ &= 2x - 1 + 1 \\ &= 2x \end{aligned}$$

Sketch the solid $E : y^2 + z^2 \le 9,\ 0 \le x \le 2$ to find the limits of the variables $x, y$ and $z$.



Observe the above figure,

The given region is enclosed by a solid cylinder and the planes $x = 0, x = 2.$

over the solid region the variable *y* varies from $-\sqrt{9-z^2}$ to $\sqrt{9-z^2}$ , *z* varies from -3 to 3 and *x* varies from 0 to 2.

So the solid region can be expressed as

$$E = \left\{ (x, y, z) \mid 0 \le x \le 2, -3 \le z \le 3, -\sqrt{9-z^2} \le y \le \sqrt{9-z^2} \right\}$$

Now evaluate the triple integral:

$$\begin{aligned}\iiint_E \operatorname{div}\mathbf{F}\,dV &= \int_{-3}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}}\int_0^2 2x\,dx\,dy\,dz \\ &= \int_{-3}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}}\left[x^2\right]_0^2 dy\,dz \\ &= 4\int_{-3}^{3}\left[\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} dy\right]dz \\ &= 4\int_{-3}^{3}\left[y\right]_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} dz \\ &= 8\int_{-3}^{3}\sqrt{9-z^2}\,dz \\ &= 8\left[\frac{z}{2}\sqrt{9-z^2}+\frac{9}{2}\sin^{-1}\frac{z}{3}\right]_{-3}^{3} \\ &= 8\left[0+\frac{9}{2}\sin^{-1}\frac{3}{3}-0-\frac{9}{2}\sin^{-1}\frac{-3}{3}\right] \\ &= 8\left[\frac{9}{2}\cdot\frac{\pi}{2}+\frac{9}{2}\cdot\frac{\pi}{2}\right] \\ &= 36\pi\end{aligned}$$

Hence $\iiint_E \operatorname{div}\mathbf{F}\,dV = \boxed{36\pi}$ .......(2)

Next evaluate the left hand side (double integral) in (1).

The cylinder is bounded by three surfaces $S_1 : x = 0,\ y^2+z^2 \le 9$, $S_2 : x = 2,\ y^2+z^2 \le 9$, and the curved surface $S_3 : 0 \le x \le 2,\ y^2+z^2 = 9$.

On $S_1$, $x=0, y=u\cos v, z=u\sin v$ where $0 \le u \le 3,\ 0 \le v \le 2\pi$.

Then, $\mathbf{r}_u = \langle 0, \cos v, \sin v\rangle$ And $\mathbf{r}_v = \langle 0, -u\sin v, u\cos v\rangle$. Also, $\mathbf{r}_v \times \mathbf{r}_u = \langle u,0,0\rangle$.

But the outward normal is in the negative *x*-axis. Thus, $\mathbf{r}_v \times \mathbf{r}_u = -\langle u,0,0\rangle$.

Then,

$$\begin{aligned}\iint_{S_1}\mathbf{F}\cdot ds &= \iint_{S_1}\left[0(-u)+(-u\cos v)(0)+(u\sin v)(0)\right] dA \\ &= \iint_{S_1}[0]\,dA \\ &= 0\end{aligned}$$

On $S_2$, $x=2, y=u\cos v, z=u\sin v$ where $0 \le u \le 3,\ 0 \le v \le 2\pi$.

Then, $\mathbf{r}_u = \langle 0, \cos v, \sin v\rangle$ And $\mathbf{r}_v = \langle 0, -u\sin v, u\cos v\rangle$. Also, $\mathbf{r}_v \times \mathbf{r}_u = \langle u,0,0\rangle$.

Then,

$$\begin{aligned}\iint_{S_2}\mathbf{F}\cdot ds &= \int_0^{2\pi}\int_0^3\left[2^2(u)+(-u\cos v)(0)+(u\sin v)(0)\right] dudv \\ &= \int_0^{2\pi}\int_0^3[4u]\,dudv \\ &= \int_0^{2\pi}\left[2u^2\right]_0^3 dv \\ &= 18\int_0^{2\pi} dv \\ &= 18\left[v\right]_0^{2\pi} \\ &= 36\pi\end{aligned}$$

On $S_3$, $x = u, y = 3\cos v, z = 3\sin v, \quad 0 \le u \le 2, 0 \le v \le 2\pi$. Then,

Then, $\mathbf{r}_u = \langle 1, 0, 0 \rangle$ And $\mathbf{r}_v = \langle 0, -3\sin v, 3\cos v \rangle$. Also, $\mathbf{r}_v \times \mathbf{r}_u = \langle 0, -3\cos v, -3\sin v \rangle$.

$$\begin{aligned}\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) &= \langle u^2, -3\cos v, 3\sin v \rangle \cdot \langle 0, -3\cos v, -3\sin v \rangle \\ &= 9\left(\cos^2 v - \sin^2 v\right) \\ &= 9\cos 2v\end{aligned}$$

Find $\iint\limits_{S_3} \mathbf{F} \cdot ds$.

$$\begin{aligned}\iint\limits_{S_3} \mathbf{F} \cdot ds &= \int_0^2 \int_0^{2\pi} 9\cos 2v\,dv\,du \\ &= 9[u]_0^2 \left[\frac{\sin 2v}{2}\right]_0^{2\pi} \\ &= 18 \times 0 \\ &= 0\end{aligned}$$

Hence the total flux over the surface *S* is

$$\begin{aligned}\iint\limits_{S} \mathbf{F} \cdot ds &= \iint\limits_{S_1} \mathbf{F} \cdot ds + \iint\limits_{S_2} \mathbf{F} \cdot ds + \iint\limits_{S_3} \mathbf{F} \cdot ds \\ &= 0 + 36\pi + 0 \\ &= 36\pi\end{aligned}$$

$$\Rightarrow \iint\limits_{S} \mathbf{F} \cdot ds = \boxed{36\pi} \quad .......(3)$$

From $(2)$ and $(3)$, Gauss divergence theorem is verified.

## Chapter 16 Vector Calculus Exercise 16.9 5E

We have $\mathbf{F} = xye^z\,\mathbf{i} + xy^2z^3\,\mathbf{j} - ye^z\,\mathbf{k}$.

Then, $\nabla \cdot \mathbf{F} = ye^z + 2xyz^3 - ye^z$ or $\nabla \cdot \mathbf{F} = 2xyz^3$.

Thus, $\iiint \nabla \cdot \mathbf{F}\,dv = \int_0^3 \int_0^2 \int_0^1 2xyz^3\,dz\,dy\,dx$.

Simplify.

$$\begin{aligned}\iiint \nabla \cdot \mathbf{F}\,dv &= \int_0^3 \int_0^2 \left[xy\frac{z^4}{2}\right]_0^1 dy\,dx \\ &= \int_0^3 \left[\frac{xy^2}{4}\right]_0^2 dx \\ &= \left[\frac{x^2}{2}\right]_0^3 \\ &= \frac{9}{2}\end{aligned}$$

Thus, we get $\boxed{\iiint \nabla \cdot \mathbf{F}\,dv = \frac{9}{2}}$.

## Chapter 16 Vector Calculus Exercise 16.9 6E

We have $\mathbf{F} = x^2yz\,\mathbf{i} + xy^2z\,\mathbf{j} + xyz^2\,\mathbf{k}$. Then, $\nabla \cdot \mathbf{F} = 2xyz + 2xyz + 2xyz$ or $\nabla \cdot \mathbf{F} = 6xyz$.

Thus, $\iiint \nabla \cdot \mathbf{F}\,dv = \int_0^a\int_0^b\int_0^c 6xyz\ dzdydx$.

Simplify.

$$\begin{aligned}\iiint \nabla \cdot \mathbf{F}\,dv &= \int_0^a\int_0^b\int_0^c 6xyzdz\ dydx \\ &= \int_0^a\int_0^b \left[3xyz^2\right]_0^c\ dydx \\ &= 3c^2\int_0^a \left[\frac{xy^2}{2}\right]_0^b dx\end{aligned}$$

Evaluate the outer integral.

$$\begin{aligned}\iiint \nabla \cdot \mathbf{F}\,dv &= \frac{3}{2}c^2b^2\left[\frac{x^2}{2}\right]_0^a \\ &= \frac{3}{4}a^2b^2c^2\end{aligned}$$

Thus, we get $\boxed{\iiint \nabla \cdot \mathbf{F}\,dv = \frac{3}{4}a^2b^2c^2}$.

## Chapter 16 Vector Calculus Exercise 16.9 7E

$$\vec{F}(x,y,z) = 3xy^2\hat{i} + xe^z\hat{j} + z^3\hat{k}$$

Where S is the surface of the solid bounded by the cylinder $y^2+z^2=1$ and the plane x = -1 and x = 2.

In cylindrical co – ordinates the region E bounded by surface S is given by

$$E = \{(x,r,\theta):\ -1 \le x \le 2,\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$$

By divergence theorem we know $\iint_S \vec{F}.d\vec{s} = \iiint_E div\,\vec{F}\,dV$

Here $\quad div\,\vec{F} = 3y^2 + 0 + 3z^2$

$$= 3(y^2+z^2)$$

Then

$$\begin{aligned}\iiint_E div\,\vec{F}\,dV &= \int_{-1}^2\int_0^{2\pi}\int_0^1 3r^2.r\,dr\,d\theta\,dx \\ &= 3\int_{-1}^2 dx\int_0^{2\pi} d\theta\int_0^1 r^3\,dr \\ &= 3(x)_{-1}^2(\theta)_0^{2\pi}\left(\frac{r^4}{4}\right)_0^1 \\ &= 3(3)(2\pi)\left(\frac{1}{4}\right) \\ &= 9\pi/2\end{aligned}$$

Hence flux of $\vec{F}$ across $S = \iint_S \vec{F}.d\vec{s}$

$$= \boxed{\frac{9\pi}{2}}$$

## Chapter 16 Vector Calculus Exercise 16.9 8E

We have $\mathbf{F} = (x^3+y^3)\mathbf{i}+(x^3+y^3)\mathbf{j}+(x^3+y^3)\mathbf{k}$.

Then, $\nabla\cdot\mathbf{F} = 3x^2+3y^2+3z^2$.

Thus, $\iiint \nabla\cdot\mathbf{F}\,dv = 3\iiint_S (x^2+y^2+z^2)\,dzdydx$.

Now, the equation of the sphere with centre at origin and radius 2 is given by $x^2+y^2+z^2=4$.

Let $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$ where $0 \le \rho \le 2$, $0 \le \theta \le 2\pi$, and $0 \le \varphi \le \pi$.

$$\iiint \nabla\cdot\mathbf{F}\,dv = 3\int_0^2\int_0^{2\pi}\int_0^{\pi}(\rho^2\cdot\rho^2\sin\varphi)\,d\varphi d\theta d\rho$$
$$= 3\left[\frac{\rho^5}{5}\right]_0^2[\theta]_0^{2\pi}[-\cos\varphi]_0^{\pi}$$
$$= 3\times\frac{2^5}{5}\times 2\pi\times 2$$
$$= \frac{384}{5}\pi$$

Therefore, we get $\boxed{\iiint \nabla\cdot\mathbf{F}\,dv = \frac{384}{5}\pi}$.

## Chapter 16 Vector Calculus Exercise 16.9 9E

We have $\mathbf{F} = x^2\sin y\mathbf{i} + x\cos y\mathbf{j} - xz\sin y\mathbf{k}$.

Then, $\nabla\cdot\mathbf{F} = 2x\sin y - x\sin y - x\sin y$ or $\nabla\cdot\mathbf{F} = 0$

Therefore, we get $\boxed{\iiint \nabla\cdot\mathbf{F}\,dv = 0}$.

## Chapter 16 Vector Calculus Exercise 16.9 10E

We have $\mathrm{F} = z\mathrm{i}+y\mathrm{j}+zx\mathrm{k}$. Then, $\nabla\cdot\mathbf{F} = 1+x$.

Thus, $\iiint \nabla\cdot\mathbf{F}\,dv = \int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}\int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)}(1+x)\,dzdydx$.

Simplify.

$$\iiint \nabla\cdot\mathbf{F}\,dv = \int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}\left[(1+x)z\right]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)}dydx$$
$$= c\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}\left[(1+x)\left(1-\frac{x}{a}-\frac{y}{b}\right)\right]dydx$$
$$= c\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}\left[(1+x)\left(1-\frac{x}{a}\right)-(1+x)\left(\frac{y}{b}\right)\right]dydx$$

Evaluate the outer integral.

$$\iiint \nabla\cdot\mathbf{F}\,dv = c\int_0^a\left[(1+x)\left(1-\frac{x}{a}\right)y-(1+x)\left(\frac{y^2}{2b}\right)\right]_0^{b\left(1-\frac{x}{a}\right)}dx$$
$$= cb\int_0^a\left[(1+x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a}\right)-(1+x)\frac{\left(1-\frac{x}{a}\right)^2}{2b}\right]dx$$
$$= cb\int_0^a(1+x)\frac{\left(1-\frac{x}{a}\right)^2}{2}dx$$
$$= \frac{bc}{2}\int_0^a\left[1+x+\frac{x^2}{a^2}+\frac{x^3}{a^2}-\frac{2x}{a}-\frac{2x^2}{a}\right]dx$$

Now, evaluate the outermost integral.

$$\iiint \nabla \cdot \mathbf{F}\,dv = \frac{bc}{2}\left[x + \frac{x^2}{2} + \frac{x^3}{3a^2} + \frac{x^4}{4a^2} - \frac{x^2}{a} - \frac{2x^3}{3a}\right]_0^a$$

$$= \frac{bc}{2}\left[a + \frac{a^2}{2} + \frac{a}{3} + \frac{a^2}{4} - a - \frac{2a^2}{3}\right]$$

$$= \frac{abc}{2}\left(\frac{1}{3} + \frac{a}{12}\right)$$

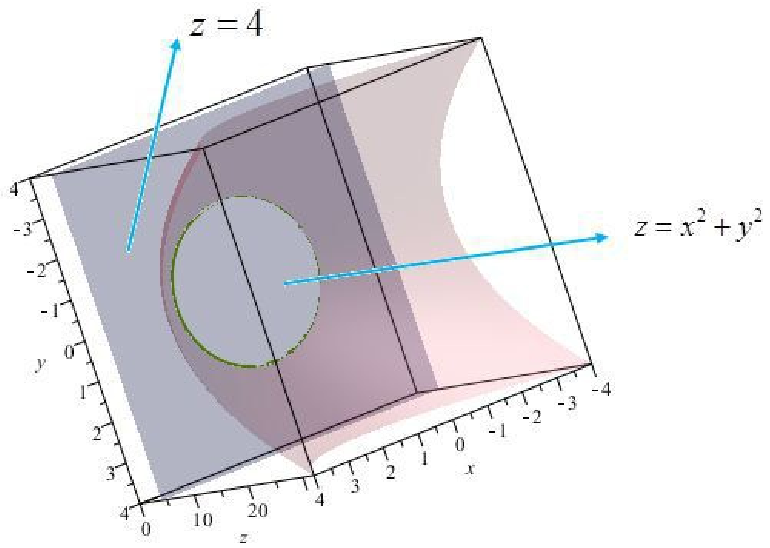
Thus, we get $\boxed{\iiint \nabla \cdot \mathbf{F}\,dv = \frac{abc}{2}\left(\frac{1}{3} + \frac{a}{12}\right)}$.

### Chapter 16 Vector Calculus Exercise 16.9 11E

Consider the vector field

$$\mathbf{F}(x,y,z) = \left(\cos z + xy^2\right)\mathbf{i} + xe^{-z}\mathbf{j} + \left(\sin y + x^2 z\right)\mathbf{k}$$

Where *S* is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and plane $z = 4$




From the figure,

The plane z =4 intersects paraboloid $z = x^2 + y^2$ in the shape of $x^2 + y^2 = 4$.

Clearly $x^2 + y^2 = 4$ represents a circle.

And the radius of the circle is 2.

Then in cylindrical co – ordinates the region *E* bounded by S can be given as

$$E = \left\{(r,\theta,z) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\quad r^2 \le z \le 4\right\}$$

Now calculate the surface integral using the divergence theorem.

Divergence theorem:

Let ***E*** be a simple solid region and let *S* be the boundary surface of *E*, given with positive (outward) orientation. Let ***F*** be a vector field whose component functions have continuous partial derivatives on an open region that contains *E*. then

$$\iint_S \mathbf{F}.d\mathbf{S} = \iiint_E div\,\mathbf{F}\,dV$$

From this

First calculate $div\,\mathbf{F}$

$$div\,\mathbf{F} = \frac{\partial}{\partial x}\left(\cos z + xy^2\right) + \frac{\partial}{\partial y}\left(xe^{-z}\right) + \frac{\partial}{\partial z}\left(\sin y + x^2 z\right)$$

$$= y^2 + 0 + x^2$$

$$= x^2 + y^2$$

$$= r^2$$

Now apply the integration:

$$\iiint_E div\,\vec{F}\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 \left(x^2+y^2\right).rdz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2\int_{r^2}^4 (r)^2 .rdz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^3 (z)_{r^2}^4\,dr\,d\theta$$ Apply integration on z

$$= \int_0^{2\pi}\int_0^2 r^3\left(4-r^2\right)dr\,d\theta$$ Apply the limits for z

$$= \int_0^{2\pi}\int_0^2 \left(4r^3-r^5\right)dr\,d\theta$$ Do multiplication

$$= \int_0^{2\pi}\left[r^4-\frac{r^6}{6}\right]_0^2 d\theta$$ Apply integration on r

$$= \int_0^{2\pi}\left[16-\frac{64}{6}\right]d\theta$$ Apply the limits for r

Continuation to the above steps:

By divergence theorem

$$\iint_S \mathbf{F}.d\mathbf{S} = \iiint_E div\,\mathbf{F}\,dV$$

$$= \int_0^{2\pi}\left(16-\frac{32}{3}\right)d\theta$$

$$= \frac{16}{3}\times(\theta)_0^{2\pi}$$ Apply integration on $\theta$

$$= \frac{16}{3}\times 2\pi$$ Apply the limits for $\theta$

$$= \frac{32\pi}{3}$$ Do Multiplication

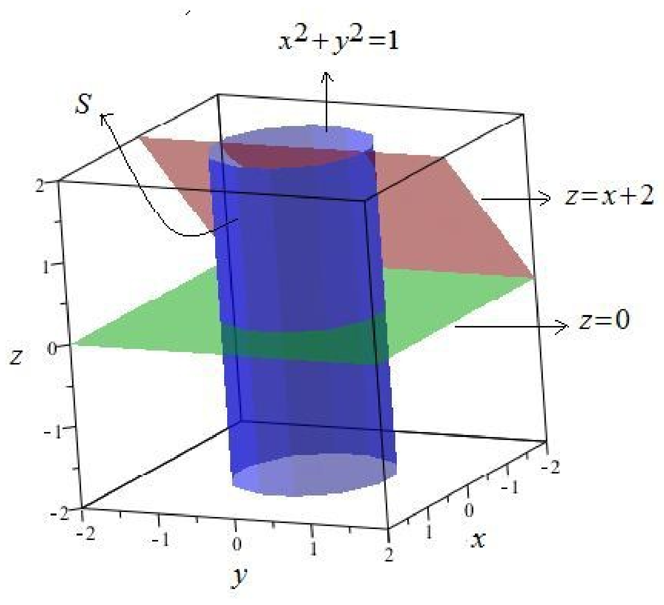
Hence $\iint_S \mathbf{F}.d\mathbf{S} = \boxed{\frac{32}{3}\pi}$

## Chapter 16 Vector Calculus Exercise 16.9 12E

Consider the following vector field:

$$\mathbf{F}(x,y,z) = x^4\mathbf{i} - x^3z^2\mathbf{j} + 4xy^2z\mathbf{k}$$

The surface $S$ of the solid bounded by the cylinder $x^2+y^2=1$ and the planes $z=x+2$ and z = 0 is shown below:



Use Divergence Theorem, to calculate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$:

To convert from cylindrical to rectangular coordinates, we use the equations.

$x = r\cos\theta, y = r\sin\theta, z = z$

So, in cylindrical coordinates the cylinder $x^2 + y^2 = 1$ is,

$$\begin{aligned} r^2\cos^2\theta + r^2\sin^2\theta &= 1 \\ r^2\left(\cos^2\theta + \sin^2\theta\right) &= 1 \\ r^2 &= 1 \\ r &= 1 \end{aligned}$$

From the figure observe that, $\theta$ varies from 0 to $2\pi$.

And the planes $z = x + 2$ and z = 0 becomes,

$z = r\cos\theta + 2$, and $z = 0$

So, from this information we can write the region *E* bounded by the surface *S* as,

$$E = \left\{(r,\theta,z):\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le z \le r\cos\theta + 2\right\} \quad \text{...... (1)}$$

By divergence theorem we have,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV \quad \text{...... (2)}$$

Now,

$$\begin{aligned} \operatorname{div}\mathbf{F} &= \nabla\cdot\left(x^4\mathbf{i} - x^3z^2\mathbf{j} + 4xy^2z\mathbf{k}\right) \\ &= \frac{\partial}{\partial x}\left(x^4\right) - \frac{\partial}{\partial y}\left(x^3z^2\right) + \frac{\partial}{\partial z}\left(4xy^2z\right) \\ &= 4x^3 + 0 + 4xy^2 \\ &= 4x\left(x^2 + y^2\right) \end{aligned}$$

Use cylindrical coordinates,

$$\begin{aligned} \operatorname{div}\mathbf{F} &= 4r\cos\theta\left(r^2\cos^2\theta + r^2\sin^2\theta\right) \\ &= 4r\cos\theta\cdot r^2 \\ &= 4r^3\cos\theta \end{aligned}$$

And in cylindrical coordinates, the volume of the small portion of the required solid $E$ is,

$dx\,dy\,dz = r\,dz\,dr\,d\theta$

Therefore,

$$\begin{aligned} \iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV \\ &= \iiint_E \operatorname{div}\mathbf{F}\,dx\,dy\,dz \\ &= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{r\cos\theta+2} 4r^3\cos\theta\left(r\,dz\,dr\,d\theta\right) \\ &= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\left(\int_{z=0}^{r\cos\theta+2} dz\right)4r^4\cos\theta\,dr\,d\theta \end{aligned}$$

$$= \int_0^{2\pi}\int_0^1 4r^4\cos\theta\left(z\right)_0^{r\cos\theta+2} dr\,d\theta$$

$$= 4\int_0^{2\pi}\int_0^1 r^4\cos\theta\left(r\cos\theta + 2\right)dr\,d\theta$$

$$= 4\int_0^{2\pi}\int_0^1 \left(r^5\cos^2\theta + 2r^4\cos\theta\right)dr\,d\theta$$

By using $\int r^n dr = \frac{r^{n+1}}{n+1}, n \neq -1$, the surface integral of **F** over $S$ is,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = 4\int_0^{2\pi}\left(\int_0^1 \left(r^5\cos^2\theta + 2r^4\cos\theta\right)dr\right)d\theta$$

$$= 4\int_0^{2\pi}\left(\left(\frac{r^6}{6}\right)\cos^2\theta + 2\left(\frac{r^5}{5}\right)\cos\theta\right)_{r=0}^{r=1} d\theta$$

$$= 4\int_0^{2\pi}\left(\frac{\cos^2\theta}{6} + \frac{2\cos\theta}{5}\right)d\theta$$

Use the following formulas:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\int \cos a\theta \, d\theta = \frac{1}{a}\sin a\theta$$

Therefore,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = 4\int_0^{2\pi}\left(\frac{1}{6}\left(\frac{1+\cos 2\theta}{2}\right) + \frac{2}{5}(\cos\theta)\right)d\theta$$

$$= 4\left[\frac{1}{6}\left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right) + \frac{2}{5}\sin\theta\right]_0^{2\pi}$$

$$= 4\left[\left(\frac{1}{6}\left(\frac{2\pi}{2} + \frac{1}{4}\sin 4\pi\right) + \frac{2}{5}\sin 2\pi\right) - \left(\frac{1}{6}\left(\frac{0}{2} + \frac{1}{4}\sin 0\right) + \frac{2}{5}\sin 0\right)\right]$$

$$= 4\left[\left(\frac{1}{6}(\pi + 0) + 0\right) - \left(\frac{1}{6}(0+0) + \frac{2}{5}(0)\right)\right]$$

$$= 4\left(\frac{\pi}{6}\right)$$

$$= \frac{2}{3}\pi$$

Hence the surface integral of **F** over *S* (the flux of **F** across *S*) is,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \boxed{\frac{2}{3}\pi}$$

## Chapter 16 Vector Calculus Exercise 16.9 13E

**Use divergence theorem to calculate the surface integral** $\iint_S \mathbf{F}\cdot d\mathbf{S}$

The given function is $\mathbf{F} = |\mathbf{r}|\mathbf{r}$.

We have

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ And $|\mathbf{r}| = \sqrt{x^2+y^2+z^2}$

Rewrite the function as

$$\mathbf{F} = |\mathbf{r}|\mathbf{r}$$

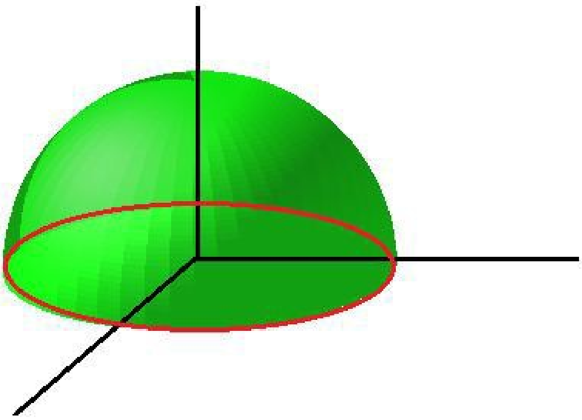
$$= \sqrt{x^2+y^2+z^2}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right)$$

$$= x\sqrt{x^2+y^2+z^2}\,\mathbf{i} + y\sqrt{x^2+y^2+z^2}\,\mathbf{j} + z\sqrt{x^2+y^2+z^2}\,\mathbf{k}$$

Now find the divergence for the vector function $\mathbf{F}$

$$\begin{aligned}
\text{div}\mathbf{F} &= \nabla \cdot \mathbf{F} \\
&= \begin{cases} \dfrac{\partial}{\partial x}\left(x\sqrt{x^2+y^2+z^2}\right)+\dfrac{\partial}{\partial y}\left(y\sqrt{x^2+y^2+z^2}\right) \\ +\dfrac{\partial}{\partial z}\left(z\sqrt{x^2+y^2+z^2}\right) \end{cases} \\
&= \begin{cases} \sqrt{x^2+y^2+z^2}+\dfrac{x^2}{\sqrt{x^2+y^2+z^2}}+\sqrt{x^2+y^2+z^2}+\dfrac{y^2}{\sqrt{x^2+y^2+z^2}} \\ +\sqrt{x^2+y^2+z^2}+\dfrac{z^2}{\sqrt{x^2+y^2+z^2}} \end{cases} \\
&= 3\sqrt{x^2+y^2+z^2}+\frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}} \\
&= 4\sqrt{x^2+y^2+z^2}
\end{aligned}$$

The surface $\mathbf{S}$ consists of the hemisphere $z=\sqrt{1-x^2-y^2}$ and the disk $x^2+y^2\le 1$ in the $xy$ -plane. It is shown below.



On the solid (shown in figure) the variable *x* varies from $-1$ to 1, the variable *y* varies from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$ and the variable *z* varies from 0 to $\sqrt{1-x^2-y^2}$.

So the region can be expressed as

$$E=\left\{(x,y,z)\mid -1\le x\le 1,-\sqrt{1-x^2}\le y\le\sqrt{1-x^2},0\le z\le\sqrt{1-x^2-y^2}\right\}$$

Use divergence theorem to transform the given integral into a triple integral.

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \nabla\cdot\mathbf{F}\,dV$$
$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} \left(4\sqrt{x^2+y^2+z^2}\right)dzdydx$$

It is difficult to solve this triple integral directly, change the coordinates to spherical coordinates to evaluate the triple integral in easier way.

Make the substitutions $x=\rho\sin\phi\cos\theta, y=\rho\sin\phi\sin\theta, z=\rho\cos\phi$ and

$\rho^2 = x^2+y^2+z^2$

Then the region $E$ (hemi sphere) can be expressed as

$$E=\left\{(\rho,\theta,\phi)\mid 0\le\rho\le 1, 0\le\theta\le 2\pi, 0\le\phi\le\frac{\pi}{2}\right\}$$

Thus,

$$\iiint_E \nabla\cdot\mathbf{F}\,dV = 4\int_0^1\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left(\rho\cdot\rho^2\sin\varphi\right)d\varphi d\theta d\rho$$

Simplify.

$$\iiint_E \nabla\cdot\mathbf{F}\,dV = 4\int_0^1\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left(\rho\cdot\rho^2\sin\varphi\right)d\varphi d\theta d\rho$$
$$= 4\left[\frac{\rho^4}{4}\right]_0^1 [\theta]_0^{2\pi}[-\cos\varphi]_0^{\frac{\pi}{2}}$$
$$= 4\times\frac{1}{4}\times 2\pi\times 1$$
$$= 2\pi$$

Therefore the flux of $\mathbf{F}$ across $\mathbf{S}$ is $\boxed{2\pi}$.

## Chapter 16 Vector Calculus Exercise 16.9 14E

We have $\mathbf{F} = |\mathbf{r}|^2\vec{\mathbf{r}}$. Then, $\nabla\cdot\mathbf{F} = \nabla\cdot\left(|\mathbf{r}|^2\vec{\mathbf{r}}\right)$.

Simplify.

$$\nabla\cdot\mathbf{F} = \left(\nabla\cdot\vec{\mathbf{r}}\right)\left|\vec{\mathbf{r}}\right|^2 + \nabla\left(|\mathbf{r}|^2\right)\cdot\vec{\mathbf{r}}$$
$$= 3\left|\vec{\mathbf{r}}\right|^2 + 2\mathbf{r}\left(\frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{r}}}{|\mathbf{r}|}\right)$$
$$= 5\left|\vec{\mathbf{r}}\right|^2$$

Thus, $\iiint(\nabla\cdot\mathbf{F})dv = 5\int_0^R\int_0^{2\pi}\int_0^{\pi}\left(\rho^2\cdot\rho^2\sin\varphi\right)d\varphi d\theta d\rho$

$$\iiint\nabla\cdot\mathbf{F}\,dv = 5\left[\frac{\rho^5}{5}\right]_0^R[2\pi][-\cos\varphi]_0^{\pi}$$
$$= R^5\times 2\pi\times 2$$
$$= 4\pi R^5$$

Thus, we get $\boxed{4\pi R^5}$.

Chapter 16 Vector Calculus Exercise 16.9 15E

Consider the vector field,

$$\mathbf{F}(x,y,z)=e^{y}\tan z\,\mathbf{i}+y\sqrt{3-x^{2}}\,\mathbf{j}+x\sin y\,\mathbf{k}$$

And $S$ is a surface of the solid that lies above the $xy$-plane and below the surface $z=2-x^{4}-y^{4},-1\le x\le 1,-1\le y\le 1$.

Use divergence theorem to determine the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$:

Recollect, **the Divergence Theorem .**

That is,

$$\iint_S \mathbf{F}\cdot d\mathbf{S}=\iiint_E \operatorname{div}\mathbf{F}\,dV$$

Describe the solid $E$:

Here, $S$ is a surface of the solid that lies above the $xy$-plane and below the surface $z=2-x^{4}-y^{4},-1\le x\le 1,-1\le y\le 1$.

This express $E$ as

$$E=\{(x,y,z)\,|-1\le x\le 1,-1\le y\le 1,0\le z\le 2-x^{4}-y^{4}\}$$

Use maple software to find the $\operatorname{div}\mathbf{F}$:

**Step 1**: Open the worksheet mode in the Maple.

Next, upload the linear algebra package by giving the command

with(linalg);

The output will be displayed as shown:

> *with*(*linalg*)

[*BlockDiagonal, GramSchmidt, JordanBlock, LUdecomp, QRdecomp, Wronskian, addcol, addrow, adj, adjoint, angle, augment, backsub, band, basis, bezout, blockmatrix, charmat, charpoly, cholesky, col, coldim, colspace, colspan, companion, concat, cond, copyinto, crossprod, curl, definite, delcols, delrows, det, diag, diverge, dotprod, eigenvals, eigenvalues, eigenvectors, eigenvects, entermatrix, equal, exponential, extend, ffgausselim, fibonacci, forwardsub, frobenius, gausselim, gaussjord, geneqns, genmatrix, grad, hadamard, hermite, hessian, hilbert, htranspose, ihermite, indexfunc, innerprod, intbasis, inverse, ismith, issimilar, iszero, jacobian, jordan, kernel, laplacian, leastsqrs, linsolve, matadd, matrix, minor, minpoly, mulcol, mulrow, multiply, norm, normalize, nullspace, orthog, permanent, pivot, potential, randmatrix, randvector, rank, ratform, row, rowdim, rowspace, rowspan, rref, scalarmul, singularvals, smith, stackmatrix, submatrix, subvector, sumbasis, swapcol, swaprow, sylvester, toeplitz, trace, transpose, vandermonde, vecpotent, vectdim, vector, wronskian*]

**Step 2**: To find the $\operatorname{div}\mathbf{F}$ use the command *diverge* as follows:

diverge([e^y*tan(z),y*sqrt(3-x^2),x*sin(y)], [x,y,z]);

The output will be displayed as shown:

> $diverge\left(\left[e^{y}\cdot\tan(z),y\cdot\sqrt{3-x^{2}},x\cdot\sin(y)\right],[x,y,z]\right);$

$\sqrt{-x^{2}+3}$

Then, the surface integral becomes

$$\iint_S \mathbf{F}\cdot d\mathbf{S}=\iiint_E \operatorname{div}\mathbf{F}\,dV$$
$$=\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{2-x^{4}-y^{4}}\sqrt{3-x^{2}}\,dz\,dy\,dx$$

Finally, to evaluate the above triple integral use the maple command as follows:

int(int(int(sqrt(3-x^2),z=0..2-x^4-y^4),y=-1..1),x=-1..1);

> $int\left(int\left(int\left(\mathrm{sqrt}(3-x^{2}),z=0..2-x^{4}-y^{4}\right),y=-1..1\right),x=-1..1\right)$

$\frac{341}{60}\sqrt{2}+\frac{81}{20}\arcsin\left(\frac{1}{3}\sqrt{3}\right)$

Therefore, flux of **F** across *S* is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1}\int_{0}^{2-x^4-y^4}\sqrt{3-x^2}\,dz\,dy\,dx$$

$$= \frac{341}{60}\sqrt{2}+\frac{81}{20}\arcsin\left(\frac{1}{3}\sqrt{3}\right)$$

Hence, the value of $\iint_S \mathbf{F}\cdot d\mathbf{S}$ is $\boxed{\frac{341}{60}\sqrt{2}+\frac{81}{20}\arcsin\left(\frac{1}{3}\sqrt{3}\right)}$.

## Chapter 16 Vector Calculus Exercise 16.9 16E

Consider the vector field,

$$\mathbf{F}(x,y,z) = \sin x\cos^2 y\,\mathbf{i} + \sin^3 y\cos^4 z\,\mathbf{j} + \sin^5 z\cos^6 x\,\mathbf{k}$$

Use a Maple to plot the vector field **F** in the cube cut from the first octant by the planes $x = \pi/2, y = \pi/2,$ and $z = \pi/2.$

**Step 1**: Open the worksheet mode in the Maple.

Next, upload the plots package by giving the command

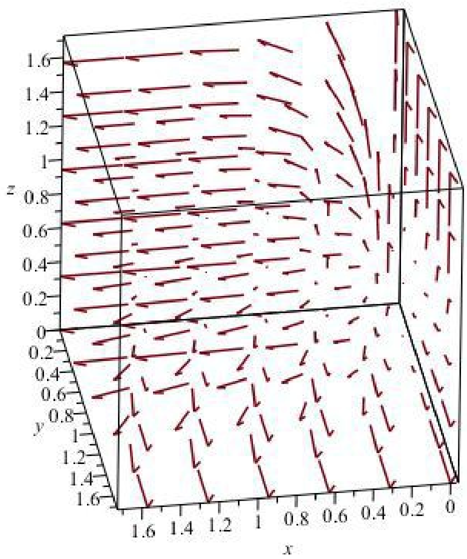
with(plots);

The output will be displayed as shown:

```
> with(plots);
[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d,
  conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot,
  display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot,
  implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot,
  listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple,
  odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d,
  polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions,
  setoptions3d, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot]
```

**Step 2**: Use the filedplot3d command to plot the vector field **F** as follows:

$$> fieldplot3d\left(\left[\sin(x)\cdot(\cos(y))^2, (\sin(y))^3\cdot(\cos(z))^4, (\sin(z))^5\cdot(\cos(x))^6\right], x=0..\frac{\pi}{2}, y=0..\frac{\pi}{2}, z=0..\frac{\pi}{2}, grid=[6,6,6], axes=boxed\right);$$




Determine the flux across the surface of the cube.

Recollect, **the Divergence Theorem**

That is,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \text{div}\,\mathbf{F}\,dV$$

Describe the solid *E*:

Here, *S* is a surface of the cube from the first octant by the planes $x=\pi/2, y=\pi/2$, and $z=\pi/2$.

This express *E* as

$$E=\{(x,y,z)\,|\,0\le x\le \pi/2,\,0\le y\le \pi/2,\,0\le z\le \pi/2\}$$

Use maple software to find the $\operatorname{div}\mathbf{F}$:

**Step 1**: Open the worksheet mode in the Maple.

Next, upload the linear algebra package by giving the command

with(linalg);

The output will be displayed as shown:

```
> with(linalg)
```

[*BlockDiagonal, GramSchmidt, JordanBlock, LUdecomp, QRdecomp, Wronskian, addcol, addrow, adj, adjoint, angle, augment, backsub, band, basis, bezout, blockmatrix, charmat, charpoly, cholesky, col, coldim, colspace, colspan, companion, concat, cond, copyinto, crossprod, curl, definite, delcols, delrows, det, diag, diverge, dotprod, eigenvals, eigenvalues, eigenvectors, eigenvects, entermatrix, equal, exponential, extend, ffgausselim, fibonacci, forwardsub, frobenius, gausselim, gaussjord, geneqns, genmatrix, grad, hadamard, hermite, hessian, hilbert, htranspose, ihermite, indexfunc, innerprod, intbasis, inverse, ismith, issimilar, iszero, jacobian, jordan, kernel, laplacian, leastsqrs, linsolve, matadd, matrix, minor, minpoly, mulcol, mulrow, multiply, norm, normalize, nullspace, orthog, permanent, pivot, potential, randmatrix, randvector, rank, ratform, row, rowdim, rowspace, rowspan, rref, scalarmul, singularvals, smith, stackmatrix, submatrix, subvector, sumbasis, swapcol, swaprow, sylvester, toeplitz, trace, transpose, vandermonde, vecpotent, vectdim, vector, wronskian*]

**Step 2**: To find the $\operatorname{div}\mathbf{F}$ use the command *diverge* as follows:

The output will be displayed as shown:

> $diverge([\sin(x)\cdot(\cos(y))^2, (\sin(y))^3\cdot(\cos(z))^4, (\sin(z))^5\cdot(\cos(x))^6], [x,y,z]);$

$\cos(x)\cos(y)^2 + 3\sin(y)^2\cos(z)^4\cos(y) + 5\sin(z)^4\cos(x)^6\cos(z)$

Then, the surface integral becomes

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV\\ &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2}\begin{pmatrix}\cos x\cos^2 y+3\sin^2 y\cos^4 z\cos y\\ +5\sin^4 z\cos^6 x\cos z\end{pmatrix}dz\,dy\,dx\end{aligned}$$

Finally, to evaluate the above triple integral use the maple command as follows:

> $int\left(int\left(int\left(\cos(x)\cdot(\cos(y))^2 + 3\cdot(\sin(y))^2\cdot(\cos(z))^4\cdot\cos(y) + 5\cdot(\sin(z))^4\cdot(\cos(x))^6\cdot\cos(z), z=0..\frac{\pi}{2}\right), y=0..\frac{\pi}{2}\right), x=0..\frac{\pi}{2}\right)$

$\frac{19}{64}\pi^2$

Therefore, flux of **F** across *S* is $\boxed{\dfrac{19\pi^2}{64}}$.

Chapter 16 Vector Calculus Exercise 16.9 17E

Consider the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$,

where $\mathbf{F}(x,y,z) = z^2 x\,\mathbf{i} + \left(\frac{1}{3}y^3 + \tan z\right)\mathbf{j} + \left(x^2 z + y^2\right)\mathbf{k}$ and $S$ is the top half of the sphere $x^2 + y^2 + z^2 = 1$.

The Divergence Theorem states that the flux of **F** across the boundary surface *S* of *E* is equal to the triple integral of the divergence of **F** over *E*:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

Before calculating this integral, we must describe the solid *E* and determine the divergence of **F**.

If **F** is a continuous defined on an oriented surface *S* with unit normal vector **n**, then the surface integral (flux) of **F** over *S* is

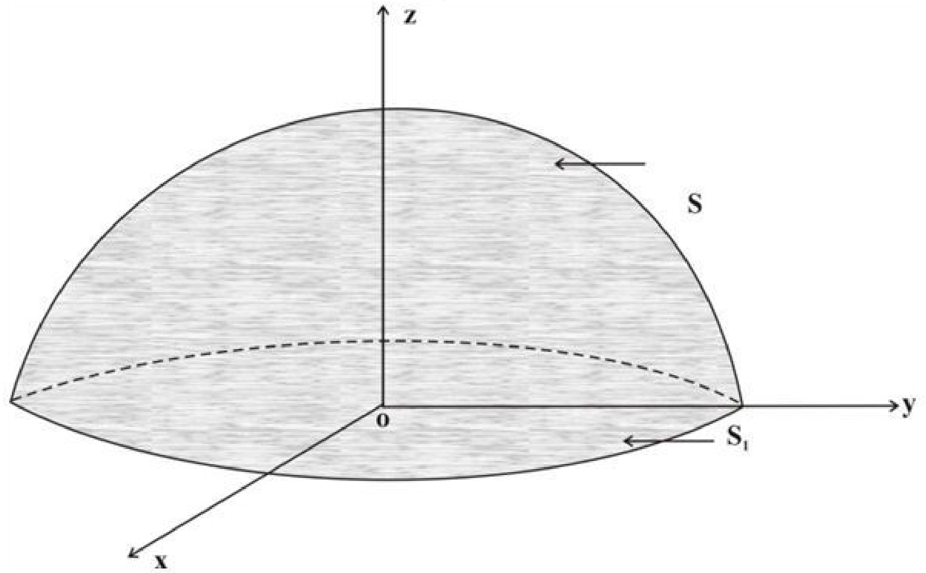
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

Recall from the fundamental theorem of calculus that

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where *F(x)* is the antiderivative of *f(x)*.

Sketch the graph of half of the sphere $x^2 + y^2 + z^2 = 1$




First compute integrals over $S_1$ and $S_2$, where $S_1$ is the disk $x^2 + y^2 \le 1$, oriented downward, and $S_2 = S \cup S_1$

Since the top half of the sphere is not a closed surface, we apply the divergence theorem to a closed hemi-sphere and subtract the surface integral of the missing bottom disk

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$$

*S* is the surface of the top hemi-sphere of radius 1 centered at the origin, which immediately allows us to describe the solid *E* is in spherical coordinates by

$$E = \{(\rho,\theta,\phi) : 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}\}$$

The surface of the missing bottom disk of the hemisphere and its associated normal vector is

$$S_1 : x^2 + y^2 \le 1, z = 0,\ \mathbf{n} = -\mathbf{k};$$

**F** is the vector field $\mathbf{F}(x,y,z) = z^2x\,\mathbf{i} + \left(\frac{1}{3}y^3 + \tan z\right)\mathbf{j} + \left(x^2z + y^2\right)\mathbf{k}$. Recall that the divergence of **F** is defined as:

$$\text{div}\,\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Calculating the div **F** we have:

$$P = z^2x, \frac{\partial P}{\partial x} = z^2$$

$$Q = \frac{1}{3}y^3 + \tan z, \frac{\partial Q}{\partial y} = y^2$$

$$R = x^2z + y^2, \frac{\partial R}{\partial z} = x^2$$

and conclude $\text{div}\,\mathbf{F} = x^2 + y^2 + z^2$, or in spherical coordinates, $\text{div}\,\mathbf{F} = \rho^2$.

Use the Divergence Theorem to calculate the flux of **F** across the closed hemisphere as follows, where $E = \{(\rho,\theta,\phi): 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}\}$, 0 and 1 are the lower and upper limits of integration with respect to $\rho$, 0 and $2\pi$ are the lower and upper limits of integration with respect to $\theta$, and 0 and $\frac{\pi}{2}$ are the lower and upper limits of integration with respect to $\phi$.

Applying the Divergence Theorem and rules of integration,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div}\,\mathbf{F}\, dV$$

$$= \iiint_E \left(x^2 + y^2 + z^2\right) dV$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 p^2\, p^2\, \sin\phi\, dp\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 p^4\, \sin\phi\, dp\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \left([\theta]_{\theta=0}^{\theta=2\pi}\right)\left(\left[\frac{1}{5}p^5\right]_{p=0}^{p=1}\right)\sin\phi\, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \left([2\pi - 0]\right)\left(\left[\frac{1}{5} - 0p^5\right]\right)\sin\phi\, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \frac{2\pi}{5}\sin\phi\, d\phi$$

$$= \frac{2\pi}{5}\int_0^{\frac{\pi}{2}} \sin\phi\, d\phi$$

$$= \frac{2\pi}{5}\left[-\cos\phi\right]_{\phi=0}^{\phi=\frac{\pi}{2}}$$

$$= \frac{2\pi}{5}\left[-\cos\frac{\pi}{2} + \cos 0\right]$$

$$= \frac{2\pi}{5}[0 + 1]$$

$$= \frac{2\pi}{5}$$

Compute the surface integral of the missing disk

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \left\langle 0, \frac{1}{3}y^3, y^2 \right\rangle \cdot (-\mathbf{k})\ dS$$
$$= \iint_{S_1} \left(-y^2\right) dS$$
$$= \int_0^{2\pi} \int_0^1 -r^2 \sin^2\theta\ r\ dr\ d\theta$$
$$= \int_0^{2\pi} \int_0^1 -r^3 \sin^2\theta\ dr\ d\theta$$
$$= \left[-\frac{1}{4}r^4\right]_{r=0}^{r=1} \int_0^{2\pi} \frac{1}{2}(1-\cos 2\theta)\ d\theta$$
$$= -\frac{1}{8}\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\theta=0}^{\theta=2\pi}$$
$$= -\frac{1}{8}[2\pi]$$
$$= -\frac{\pi}{4}$$

Now, subtract the surface integral of the missing disk from the flux of the solid hemisphere

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\ dV - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$$
$$= \frac{2\pi}{5} - \left(-\frac{\pi}{4}\right)$$
$$= \boxed{\frac{13\pi}{20}}.$$

## Chapter 16 Vector Calculus Exercise 16.9 18E

Consider the vector field $\mathbf{F}(x,y,z) = z\tan^{-1}\left(y^2\right)\mathbf{i} + z^3 \ln\left(x^2+1\right)\mathbf{j} + z\mathbf{k}$.

The surface $S$ is the part of the paraboloid $x^2+y^2+z=2$ that lies above the plane $z=1$.

Find the flux of the given vector field over the surface $S$.

Here, the orientation is upward.

Note that $S$ is not a closed surface.

Find the intersection of the paraboloid $x^2+y^2+z=2$ and the plane $z=1$.

$$x^2+y^2+z=2$$
$$x^2+y^2+1=2$$
$$x^2+y^2=2-1$$
$$x^2+y^2=1$$

So the paraboloid and the plane intersects in a circle of radius 1.

Take $S_1$ is the disk $x^2+y^2 \le 1$ and is oriented downward, so its unit normal vector is $\mathbf{n} = -\mathbf{k}$.

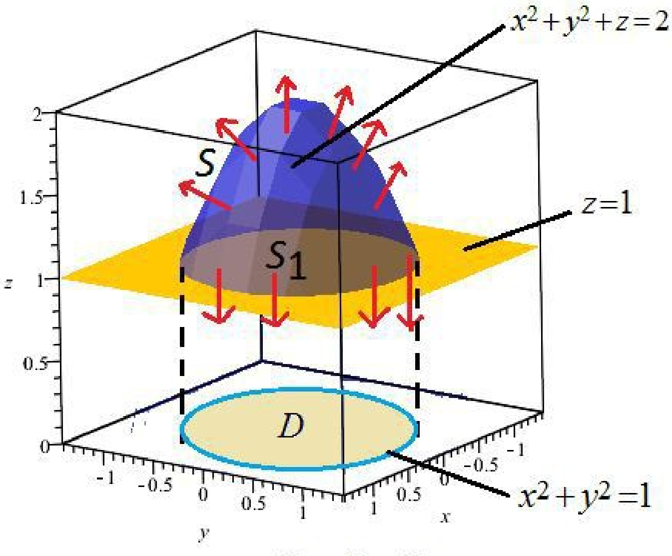
First find the flux of the vector field over the disk $S_1$.

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \left(z\tan^{-1}\left(y^2\right)\mathbf{i} + z^3\ln\left(x^2+1\right)\mathbf{j} + z\mathbf{k}\right) \cdot (-\mathbf{k})\, dS$$
$$= \iint_{S_1} (-z)\, dS$$
$$= \iint_{S_1} (-1)\, dS \qquad \left[\text{Since on } S_1, z=1.\right]$$
$$= (-1)\iint_{S_1} dS$$
$$= (-1)\left(\pi\left(1^2\right)\right) \qquad \left[\begin{array}{l}\iint_{S_1} dS \text{ represents the area of the surface } S_1. \\ \text{Here, } S_1 \text{ is the disk of radius 1.}\end{array}\right]$$
$$= -\pi$$

Take $S_2$, another surface, which constists of parabolic top surface $S: x^2+y^2+z=2$ and a circular bottom surface $S_1: x^2+y^2 \le 1$.

Here, $S_2$ is a closed surface. So use divergence theorem to find the flux of the vector field over the surface $S_2$.

The surfaces are shown in the below figure:




$S_2 = S \cup S_1$

Use cylindrical coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$dxdy = rdrd\theta$$
$$x^2+y^2 = r^2$$

From the graph observe that $z$ enters at $z=1$ and leaves at $z=2-x^2-y^2$.

In cylindrical coordinates, $z$ enters at $z=1$ and leaves at $z=2-r^2$.

The projection $D$ of the surface $S_2$ on the *xy*-plane is a circle of radius 1.

The description of the region $D$ in *xy*-plane is $D=\{(x,y)\,|\,0\le r\le 1, 0\le\theta\le 2\pi\}$.

The description of the region $S_2$ is $E=\{(r,\theta,z)\,|\,0\le r\le 1, 0\le\theta\le 2\pi, 1\le z\le 2-r^2\}$.

Since $\mathbf{F} = M\mathbf{i}+N\mathbf{j}+P\mathbf{k} = z\tan^{-1}(y^2)\mathbf{i}+z^3\ln(x^2+1)\mathbf{j}+z\mathbf{k}$. Then

$$\begin{aligned}\text{div }\mathbf{F} &= \frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}\\ &= \frac{\partial}{\partial x}\left(z\tan^{-1}(y^2)\right)+\frac{\partial}{\partial y}\left(z^3\ln(x^2+1)\right)+\frac{\partial}{\partial z}(z)\\ &= 0+0+1\\ &= 1\end{aligned}$$

By the Divergence Theorem, the Flux across $S_2$ is

$$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iiint_{S_2} \text{div } \mathbf{F} \, dV$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=1}^{2-r^2} (1) r dz \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r[z]_1^{2-r^2} \, dr d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r\left[2-r^2-1\right] dr d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r\left[1-r^2\right] dr d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left[r-r^3\right] dr d\theta$$

$$= [\theta]_0^{2\pi} \left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1$$

$$= [2\pi]\left[\frac{1^2}{2}-\frac{1^4}{4}\right]$$

$$= \frac{\pi}{2}$$

Therefore $\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \frac{\pi}{2}$.

Since $S_2 = S \cup S_1$, the Flux across $S$ is,

$$\iint_{S} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} - \iint_{S_1} \mathbf{F}\cdot d\mathbf{S}$$

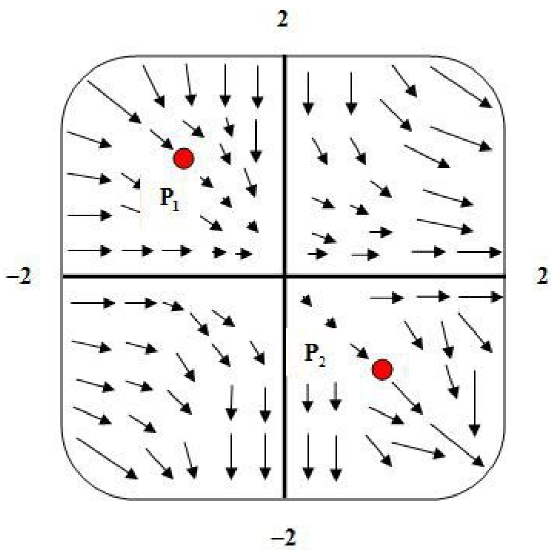
$$= \frac{\pi}{2} - (-\pi)$$

$$= \frac{3\pi}{2}$$

Hence, the required flux of the vector field $\mathbf{F}$ across the part of the paraboloid $x^2+y^2+z=2$ that lies above the plane $z=1$ is $\iint_{S} \mathbf{F}\cdot d\mathbf{S} = \boxed{\frac{3\pi}{2}}$.

## Chapter 16 Vector Calculus Exercise 16.919E

Consider the vector field **F** as shown:



Use the interpretation of divergence to find the divergence of a vector field is positive or negative at $P_1$ and $P_2$.

In general, the divergence of a vector field can be used to determine the net rate of outward flux per unit volume. That is,

1. If the net flow is outward near *P*, then *P* is called the source and divergence is positive.

2. If the net flow is inward near *P*, then *P* is called the sink and divergence is negative.

From the above vector field, observe that

1. At the point $P_1$ , the incoming arrows are longer than the outgoing arrows.

That means the net flow is inward.

Therefore, divergence is negative at $P_1$ or $\boxed{\operatorname{div}(P_1) < 0}$.

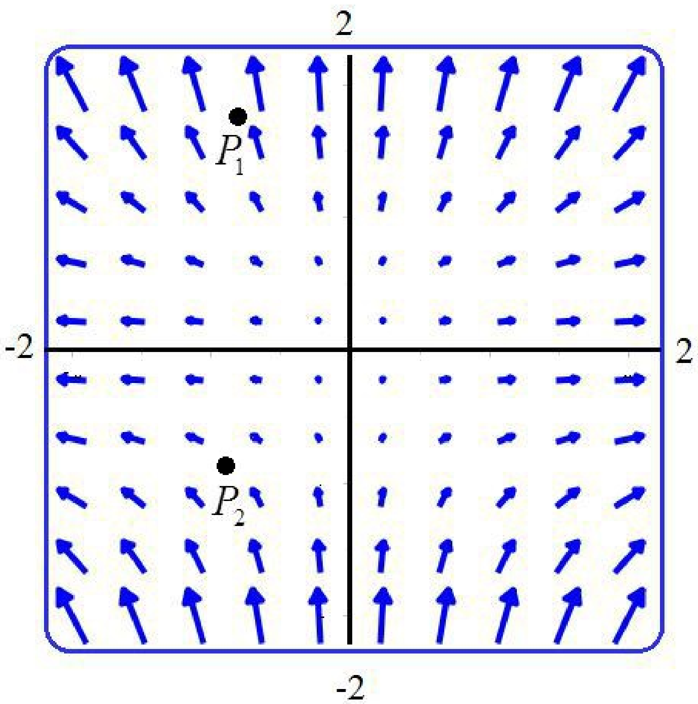
2. At the point $P_2$ , the incoming arrows are shorter than the outgoing arrows.

This means the net flow is outward.

Therefore, divergence is positive at $P_2$ or $\boxed{\operatorname{div}(P_2) > 0}$.

## Chapter 16 Vector Calculus Exercise 16.9 20E

Consider the vector field $\mathbf{F}(x, y) = \langle x, y^2 \rangle$ as shown:



In general, the divergence of a vector field can be used to determine the net rate of outward flux per unit volume. That is,

1. If divergence is positive at a point *P*, then the net flow is outward near *P*, and *P* is called the source.

2. If divergence is negative at a point *P*, then the net flow is inward near *P*, and *P* is called the sink.

(a)

Determine whether $P_1$ and $P_2$ are the sources or sinks for vector field **F** or not.

From the above vector field, observe that

1. At the point $P_1$, the incoming arrows are shorter than the outgoing arrows.

That means the net flow is outward.

Therefore, divergence is positive at $P_1$ or $\boxed{\text{div } \mathbf{F}(P_1) > 0}$ and $P_1$ is a **source**.

2. At the point $P_2$, the incoming arrows are longer than the outgoing arrows.

This means the net flow is inward.

Therefore, divergence is negative at $P_2$ or $\boxed{\text{div } \mathbf{F}(P_2) < 0}$ and $P_2$ is a **sink**.

(b)

Use the definition of divergence to verify the answer in the part (a):

Recollect the divergence of two-dimensional vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is defined as

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

Find $\text{div}\left(\mathbf{F}(x, y)\right)$:

$$\begin{aligned}\text{div } \mathbf{F} &= \frac{\partial (x)}{\partial x} + \frac{\partial \left(y^2\right)}{\partial y} \\ &= 1 + 2y\end{aligned}$$

Observe that the divergence of this vector field is only dependent upon the *y* coordinate.

Take,

$$\begin{aligned}\text{div } \mathbf{F} &> 0 \\ 1 + 2y &> 0 \\ 2y &> -1 \\ y &> -\frac{1}{2}\end{aligned}$$

That means $\text{div } \mathbf{F}$ is positive when $y > -\frac{1}{2}$.

So, the points which lie above the line $y = -\frac{1}{2}$ are sources and the points which lie below are sinks.

Here, the point $P_1$ lies above the line $y = -\frac{1}{2}$ and the point $P_2$ lies below the line.

Therefore, from using the definition of divergence,

$\boxed{P_1 \text{ is a source and } P_2 \text{ is a sink}}$.

## Chapter 16 Vector Calculus Exercise 16.921E

Consider the vector field **F**

$\mathbf{F}(x, y) = \langle xy + x + y^2 \rangle$ .

Use Maple software to plot the vector field **F**:

**Step 1**: Open the worksheet mode in the Maple.

Next, upload the plots package by giving the command

The output will be displayed as shown:

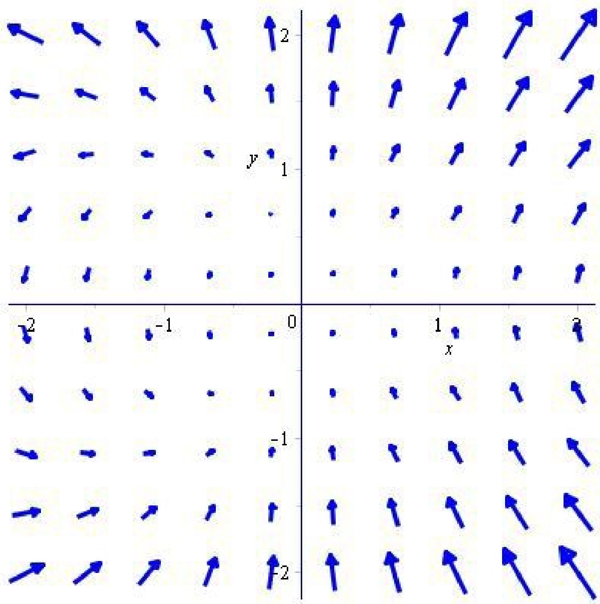
> *with*(*plots*);

[*animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot*]

**Step 2**: Use the filedplot3d command to plot the vector field **F** as follows:

Fieldplot([x*y,x+y^2],x=-2..2,y=-2..2,arrows=slim,grid=[10,10],colour=blue);

> $fieldplot([x \cdot y, x + y^2], x = -2..2, y = -2..2, arrows = slim, grid = [10, 10], colour = blue)$;




Determine where $\text{div}\,\mathbf{F} > 0$ and where $\text{div}\,\mathbf{F} < 0$.

In general, the divergence of a vector field can be used to determine the net rate of outward flux per unit volume. That is,

1. The divergence is positive at a point *P*, when the net flow is outward near *P*, and *P* is called the source.

2. The divergence is negative at a point *P*, when the net flow is inward near *P*, and *P* is called the sink.

Observe that,

1. For any point above the *x*-axis or in the I and II quadrants, the incoming arrows are shorter than the outgoing arrows.

That means the net flow is outward.

Therefore, divergence is positive or $\text{div}\,\mathbf{F} > 0$ at $\boxed{\text{I and II}}$ quadrants.

2. For any point below the *x*-axis or in the III and IV quadrants the incoming arrows are longer than the outgoing arrows.

This means the net flow is inward.

Therefore, divergence is negative or $\text{div}\,\mathbf{F} < 0$ at $\boxed{\text{III and IV}}$ quadrants.

Use the definition of divergence to verify the above answer.

Recollect the divergence of two-dimensional vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is defined as

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

Find $\text{div}(\mathbf{F}(x, y))$:

$$\begin{aligned}\text{div } \mathbf{F} &= \frac{\partial(xy)}{\partial x} + \frac{\partial(x + y^2)}{\partial y} \\ &= y + (2y) \\ &= 3y\end{aligned}$$

Observe that the divergence of this vector field is only dependent upon the *y* coordinate.

Also, clearly $\text{div } \mathbf{F} > 0$ when $y > 0$ and $\text{div } \mathbf{F} < 0$ when $y < 0$.

In general, *y* is greater than 0 in I and II quadrants and *y* is less than zero in III and IV quadrants.

Therefore, $\text{div } \mathbf{F} > 0$ at $\boxed{\text{I and II}}$ quadrants and $\text{div } \mathbf{F} < 0$ at $\boxed{\text{III and IV}}$ quadrants

## Chapter 16 Vector Calculus Exercise 16.9 22E

Consider the vector field **F**

$$\mathbf{F}(x, y) = \langle x^2, y^2 \rangle.$$

Use Maple software to plot the vector field **F**:

**Step 1**: Open the worksheet mode in the Maple.

Next, upload the plots package by giving the command

with(plots);

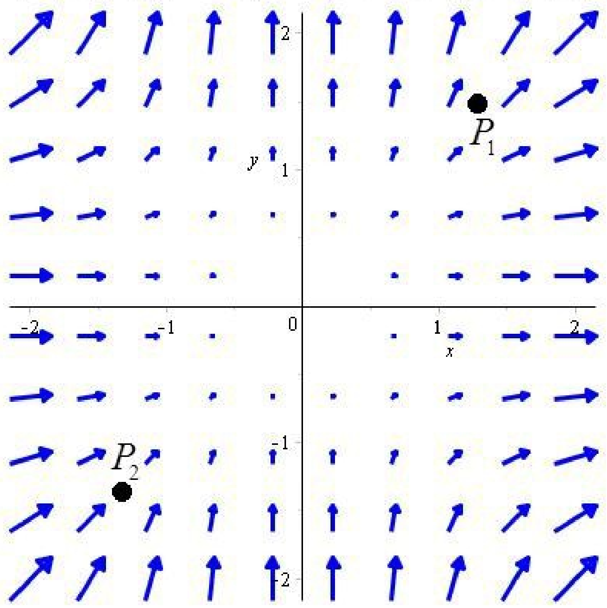
The output will be displayed as shown:

```
> with(plots);
[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d,
   conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot,
   display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot,
   implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot,
   listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple,
   odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d,
   polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions,
   setoptions3d, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot]
```

**Step 2**: Use the filedplot3d command to plot the vector field **F** as follows:

Fieldplot([x^2,y^2],x=-2..2,y=-2..2,arrows=slim,grid=[10,10],colour=blue);

> $fieldplot\left(\left[x^2, y^2\right], x=-2..2, y=-2..2, arrows = \text{slim}, \text{grid} = [10, 10], colour = blue\right)$




Determine where $\text{div}\ \mathbf{F} > 0$ and where $\text{div}\ \mathbf{F} < 0.$

Recollect, the divergence of a vector field can be used to determine the net rate of outward flux per unit volume. That is,

1. The divergence is positive at a point *P*, when the net flow is outward near *P*, and *P* is called the source.

2. The divergence is negative at a point *P*, when the net flow is inward near *P*, and *P* is called the sink.

Take two points $P_1$ and $P_2$ in the vector filed.

Observe that,

1. At the point $P_1$ , the incoming arrows are shorter than the outgoing arrows.

That means the net flow is outward.

Therefore, divergence is positive at $P_1$ or $\boxed{\text{div}\ \mathbf{F}(P_1) > 0}$ and $P_1$ is a **source**.

2. At the point $P_2$ , the incoming arrows are longer than the outgoing arrows.

This means the net flow is inward.

Therefore, divergence is negative at $P_2$ or $\boxed{\text{div}\ \mathbf{F}(P_2) < 0}$ and $P_2$ is a **sink**.

Use the definition of divergence to verify the above answer.

Recollect the divergence of two-dimensional vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is defined as

$$\text{div}\ \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

Find $\text{div}(\mathbf{F}(x,y))$:

$$\begin{aligned}\text{div}\ \mathbf{F} &= \frac{\partial(x^2)}{\partial x} + \frac{\partial(y^2)}{\partial y}\\ &= 2x + 2y\end{aligned}$$

Observe that the divergence of this vector field is only dependent upon the *y* coordinate.

Suppose $\text{div}\ \mathbf{F} > 0$, then

$$\begin{aligned}\text{div}\ \mathbf{F} &> 0\\ 2x + 2y &> 0\\ x + y &> 0\\ y &> -x\end{aligned}$$

That means $\text{div}\ \mathbf{F}$ is positive when $y > -x$.

So, the points which lie above the line $y = -x$ are sources and the points which lie below are sinks.

That means $\text{div}\ \mathbf{F} > 0$ above the line $y = -x$ and $\text{div}\ \mathbf{F} < 0$ below the line $y = -x$

## Chapter 16 Vector Calculus Exercise 16.9 23E

$$\begin{aligned}\vec{E}(\vec{x}) &= EQ\frac{\vec{x}}{|\vec{x}|^3}\\ &= EQ\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2+y^2+z^2)^{3/2}}\end{aligned}$$

Then $\quad div\,\vec{E} = \vec{\nabla}.\vec{E}$

$$= \frac{\partial}{\partial x}\frac{x}{|x|^3} + \frac{\partial}{\partial y}\frac{y}{|x|^3} + \frac{\partial}{\partial z}\frac{z}{|x|^3}$$

Now $\quad \frac{\partial}{\partial x}\frac{x}{|x|^3} = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)$

$$\begin{aligned}&= \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\\ &= \frac{(x^2+y^2+z^2-3x^2)(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\\ &= \frac{z^2+y^2-2x^2}{(x^2+y^2+z^2)^{5/2}}\end{aligned}$$

Similarly $\quad \frac{\partial}{\partial y}\frac{y}{|x|^3} = \frac{x^2+z^2-2y^2}{(x^2+y^2+z^2)^{5/2}}$

And $\quad \frac{\partial}{\partial z}\frac{z}{|x|^3} = \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}$

Then $\quad div\,\vec{E} = \frac{2x^2+2y^2+2z^2-2x^2-2y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}$

$$= 0$$

Hence $\quad div\,\vec{E} = 0$

## Chapter 16 Vector Calculus Exercise 16.9 24E

Consider the following integral:

$$\iint_S \left(2x+2y+z^2\right)dS$$

Where, $S$ is the sphere $x^2+y^2+z^2=1$.

To evaluate the given integral by divergence theorem, put the given integral in the form $\iint_S \vec{F}.\hat{n}dS$, where $\hat{n}$ is the unit normal to $S$.

Let,

$$f(x,y,z)=x^2+y^2+z^2-1$$

Then, $S$ is a level curve of $f(x,y,z)=0$. Now, it is known that $\vec{\nabla}f$ is always normal to surface $S$.

Then the unit normal vector is:

$$\begin{aligned}\hat{n} &= \frac{\vec{\nabla}f}{\left\|\vec{\nabla}f\right\|} \\ &= \frac{<2x,2y,2z>}{\sqrt{4\left(x^2+y^2+z^2\right)}} \\ &= \frac{2<x,y,z>}{2} \\ &=<x,y,z>\end{aligned}$$

Now we choose $\vec{F}$ such that

$$\begin{aligned}\vec{F}.\hat{n} &= \vec{F}.\left(x\hat{i}+y\hat{j}+z\hat{k}\right) \\ &= 2x+2y+z^2\end{aligned}$$

When compared, this will give:

$$\vec{F}=2\hat{i}+2\hat{j}+2\hat{k}$$

So, the integral becomes:

$$\iint_S \left(2x+2y+z^2\right)ds = \iint_S \left(2\hat{i}+2\hat{j}+2\hat{k}\right).d\vec{s}$$

Now, by divergence theorem:

$$\iint_S \vec{F}.d\vec{s} = \iiint_E div\vec{F}dV$$

Evaluate $div\vec{F}$:

$$\begin{aligned}div\vec{F} &= \vec{\nabla}.\left(2\hat{i}+2\hat{j}+2\hat{k}\right) \\ &= 1\end{aligned}$$

Then, the integral is evaluated as follows:

$$\begin{aligned}\iiint_E div\vec{F}dV &= \iiint_E 1.dV \\ &= V(E) \\ &= \frac{4}{3}\pi(1)^3\end{aligned}$$

Hence,

$$\iint_S \vec{F}.d\vec{s} = \frac{4}{3}\pi$$

Therefore the value of required integral is:

$$\boxed{\iint_S \left(2x+2y+z^2\right)ds = 4\pi/3}$$

## Chapter 16 Vector Calculus Exercise 16.925E

By divergence theorem we know

$$\iint_S \vec{a}.\hat{n}\,ds = \iiint_E div\,\vec{a}\,dV$$

Where E is the region bounded by surface S. Now $\vec{a}$ is a constant vector. As we know the divergence of a constant vector is zero, therefore

$$\iiint_E div\,\vec{a}\,dV = 0$$

And hence $\iint_S \vec{a}.\hat{n}\,ds = 0$

## Chapter 16 Vector Calculus Exercise 16.9 26E

By divergence theorem we know

$$\iint_S \vec{F}.d\vec{s} = \iiint_E div\,\vec{F}\,dV$$

Where E is the region bounded by surface S.

Now $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $div\,\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$

$= 1+1+1$

$= 3$

Then $\iiint_E div\,\vec{F}\,dV = \iiint_E 3dV$

$= 3\iiint_E 1.dV$

$= 3V(E)$

And therefore $\iint_S \vec{F}.d\vec{s} = 3V(E)$

Or $V(E) = \frac{1}{3}\iint_S \vec{F}.d\vec{s}$

## Chapter 16 Vector Calculus Exercise 16.9 27E

By divergence theorem we know

$$\iint_S \vec{F}.d\vec{s} = \iiint_E div\,\vec{F}\,dV$$

Where E is the region boundary by surface S

Then $\iint_S \left(curl\,\vec{F}\right).d\vec{s} = \iiint_E div\left(curl\,\vec{F}\right)dV$

We know $div\left(curl\,\vec{F}\right) = 0$

Then $\iiint_E div\left(curl\,\vec{F}\right)dV = 0$

And hence $\iint_S \left(curl\,\vec{F}\right).d\vec{s} = 0$

## Chapter 16 Vector Calculus Exercise 16.9 28E

By divergence theorem we know

$$\iint_S \vec{F}.d\vec{s} = \iiint_E div\,\vec{F}\,dV$$

Where E is the region bounded by surface S

Also we know $\vec{D}_n f = \vec{\nabla} f.\hat{n}$

Where $\hat{n}$ unit vector

Then $\iint_S D_n f\,ds = \iiint_E div\left(\vec{\nabla} f\right)dV$

$= \iiint_E \vec{\nabla}^2 f\,dV$

## Chapter 16 Vector Calculus Exercise 16.9 29E

By divergence theorem we know

$$\iint_S \vec{F}.d\vec{s} = \iiint_E div\ \vec{F}\ dV$$

Where E is the region bounded by surface S

## Chapter 16 Vector Calculus Exercise 16.9 30E

By divergence theorem we know

$$\iint_S \vec{F}.d\vec{s} = \iint_S \vec{F}.\hat{n}\,ds = \iiint_E div\,\vec{F}\,dV$$

Where E is the region bounded by surface S

Then $\iint_S \left(f\,\vec{\nabla}g - g\vec{\nabla}f\right).\hat{n}\ ds$

$$= \iiint_E div\left(f\,\vec{\nabla}\,g - g\,\vec{\nabla}\,f\right)dV$$

$$= \iiint_E \left[div\left(f\vec{\nabla}\,g\right) - div\left(g\ \vec{\nabla}f\right)\right]dV$$

$$= \iiint_E \left[f\vec{\nabla}^2 g \_ \vec{\nabla}f.\vec{\nabla}g - g\vec{\nabla}^2 f - \vec{\nabla}f.\vec{\nabla}g\right]dV$$

$$\left(as\ div\left(f\vec{F}\right) = f\ div\,\vec{F} - \vec{F}.\vec{\nabla}f\right)$$

$$= \iiint_E \left[f\,\vec{\nabla}^2 g - g\,\vec{\nabla}^2 f\right]dV$$

## Chapter 16 Vector Calculus Exercise 16.9 31E

If E is the volume bounded by a closed surface S and $\vec{F}$ is a vector point function with continuous derivatives, then from divergence theorem, we have

$$\iint_S \vec{F}.\vec{n}\,ds = \iiint_E div\,\vec{F}\,dV$$

Where $\vec{n}$ is unit outward drawn normal vector to the surface S.

Putting $\vec{F} = f\,\vec{c}$ where $\vec{c}$ is any arbitrary constant vector.

Then

$$\iint_S f\,\vec{c}.\vec{n}\,ds = \iiint_E div\left(f\,\vec{c}\right)dV$$

$$\Rightarrow \quad \iint_S \vec{c}.f\,\vec{n}\,ds = \iiint_E \nabla.(f\vec{c})dv$$

$$\Rightarrow \quad \vec{c}.\iint_S f\,\vec{n}\,ds = \vec{c}.\iiint_E \nabla f\,dv \qquad \text{Since } \vec{c} \text{ is a constant vector.}$$

$$\Rightarrow \quad \vec{c}.\iint_S f\,\vec{n}\,ds - \vec{c}\iiint_E \nabla f\,dv = 0$$

$$\Rightarrow \quad \vec{c}.\left[\iint_S f\,\vec{n}\,ds - \iiint_E \nabla f\,dv\right] = 0$$

$$\Rightarrow \quad \iint_S f\,\vec{n}\,ds - \iiint_E \nabla f\,dv = 0 \text{ since } \vec{c} \text{ is arbitrary}$$

$$\Rightarrow \quad \boxed{\iint_S f\,\vec{n}\,ds = \iiint_E \nabla f\,dv}$$

## Chapter 16 Vector Calculus Exercise 16.9 32E

Consider $\iint_S p\,\hat{n}\,ds$

As we know $\iint_S f\,\hat{n}\,ds = \iiint_E \vec{\nabla}f\,dV$

Where E is the region bounded by surface S

Then $\iint_S p\,\hat{n}\,ds = \iiint_E \vec{\nabla}\,p\,dV$

$$= \iiint_E \vec{\nabla}\rho\,gz\,dV$$

(As $p = \rho gz$)

$$= \rho g\iiint_E \vec{\nabla}z\,dV$$

(As $\rho, g$ are constant)

i.e. $\iint_S p\,\hat{n}\,ds = \rho g \iiint_E \hat{k}\,dV$

$$= \rho g\,\hat{k} \iiint_E dV$$

$$= \rho g\,\hat{k} V(E)$$

$$= \left[\rho V(E)\right] g\,\hat{k}$$

(As $M = \rho\, V(E)$ and $W = Mg$)

$$= w\hat{k}$$

Hence $\iint_S p\,\hat{n}\,ds = w\hat{k}$

Since $\vec{F} = -\iint_S p\,\hat{n}\,ds$, then $\boxed{\vec{F} = -w\hat{k}}$