CHAPTER 12

Plastic Analysis and Design

12.1 Introduction

- Steel has a very unique property i.e. ductility because of which it is able to absorb large deformations beyond the elastic limit without getting fractured.
- Due to this unique property only, it has a reserve of strength beyond its yield point as shown in Fig. 1.4 and explained in Section-1.4 (section & fig. 1.3 are for local buckling).
- In plastic method of design, the design philosophy is the ultimate strength and thus behaviour of members beyond the yield stress in inelastic (or plastic) range is considered.
- The structure fails at a much higher load than working load and that is called as collapse load. The working loads are enhanced by specified factors known as load factors to get the ultimate loads at which the structure collapses. There after the maximum plastic moment is found. The plastic section modulus Z_{pz} is found by dividing the maximum plastic moment by the yield stress. i.e., $Z_{pz} = M_p/f_y$
- Plastic: The term plastic implies that at failure, parts of the member will be under very large strains so that the member is put into plastic range. When the entire cross section becomes plastic, infinite rotation takes place at constant bending moment and a sort of plastic hinge is formed. After the formation of sufficient number of plastic hinges in the member at the maximum stressed locations, a collapse mechanism develops. Since the working loads are less than the collapse load by a factor of safety called as load factor, the members designed will be safe beyond doubt.

12.2 Strength of Tensile Member

12.2.1 Single Tension Member

Under the action of load 'P' less then the yield load, (i.e. $P < P_y$), the member shown in Fig. 12.1 will elongate elastically. The deflection is given by

$$\Delta = \frac{PL}{AE}$$

where, P = Axial load in tension
A = Cross-sectional area of bar
E = Modulus of elasticity of material of bar (usually steel)

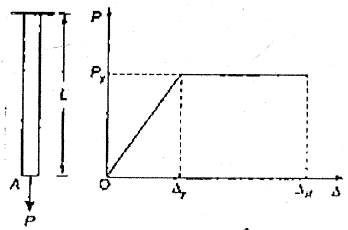


Fig. 12.1 Strength of single tension member

'Δ' will increase linearly with 'P' as shown in Fig. 12.1

As the load P approaches P_y, Δ will approach Δ_y

where, $P_y = f_y A$

and $\Delta_y = \epsilon_y L$

As soon as 'p' becomes equal to P_y, the deflection will increase instantaneously and even without increasing the load further.

i.e., $$\Delta_{st} = \epsilon_{st} L \quad ...(12.1)$$

To increase the deflection beyond Δ_{st}, more load will have to be applied. The real load causing fracture of the member will be $\sigma_{max} A$. But since the deformation at the stage of Δ_{st} is about 1.5% of the length of the member which will be found to be very large for most structures, the usefulness of the member is practically finished at the start of strain hardening. Hence for all practical purposes, the load $P_y = \sigma_y A$ may be taken as the ultimate load P_u for tension member.

$$P_u = P_y = f_y A \quad ...(12.2)$$

12.2.2 Tension Members in Parallel

Fig. 12.2 shows three tension rods arranged symmetrically in parallel to support a rigid bar which carries a load 'P' at its centre.

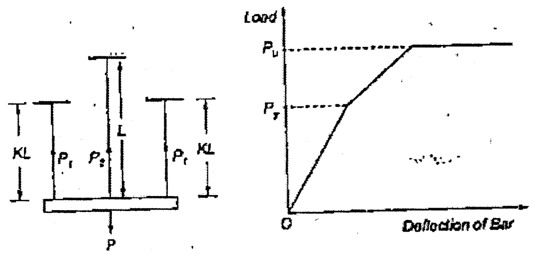


Fig. 12.2 Tension members in parallel

Let cross-sectional area of each rod is A. Suppose that the load P is being gradually increased. As long as the load remains elastic, the load in the rods can be determined by elastic theory.

Thus from vertical equilibrium

$$2P_1 + P_2 = P \quad ...(12.3)$$

and from compatibility of deflections

$$\Delta_1 = \Delta_2$$

$$\frac{P_1 KL}{AE} = \frac{P_2 L}{AE} \quad ...(12.4)$$

Solving Eq. (12.3) and (12.4)

$$P_1 = \frac{P}{2+K} \quad ...(12.5)$$

and $$P_2 = P_1 K$$

$$P_2 = \frac{K}{2+K} \cdot P \quad \text{...(12.6)}$$

If $K < 1$ then $P_1 > P_2$, i.e. the smaller rods will reach yield point earlier. At this stage the total load 'P' becomes equal to P_y which will be defined as the load causing first yield anywhere in the structure. The longer rod will be below its yield value.

Thus by putting $P_1 = \sigma_y A$

we get, $P_y = (2 + k)\,\sigma_y A$

The deflection at this stage is determined by the yielding of shorter rods ($\Delta = \epsilon_y kL$). Therefore, if we increase the load P further, shorter rod will take constant load ($\sigma_y A$) while the longer rod will take more load as it is still within elastic. At the time of ultimate load all the three rods is having equal stress (σ_y).

so, $P_u = 3\sigma_y A$

The deflection, when P_u has just reached the yield point will be

$$\Delta = \epsilon_y L$$

The curve of P v/s Δ is shown in Fig. 12.2. This shows that $P_u > P_y$ of the system. Further the deflection of the structure just before P_u is reached is less than $\epsilon_y L$ which is certainly permissible as for the single member of length L.

12.2.3 Reserve Strength

The load interval or margin between P_y and P_u is known as reserve strength.

Mathematically:

$$\text{Reserve strength } (\psi) = \frac{\text{Ultimate load (collapse load)}}{\text{Load causing first yield}} = \frac{P_u}{P_y}$$

12.2.4 Load Factor

It is defined as the ratio of ultimate load (collapse load) to working load.

Mathematically:

$$\text{Load factor } (\alpha) = \frac{\text{Ultimate load}}{\text{Working load}} = \frac{P_u}{P} \quad \text{...(12.7)}$$

12.3 Strength of Compression Member

Unlike tension members, the strength of axially loaded compression members may be limited by yielding or buckling depending upon its slenderness ($\lambda = l/r$). If l/r ratio is such that the critical stress is less than the limit of proportionality of steel, the critical stress is given by Euler's Formula for elastic bucking giving the failure stress of column.

12.4 Strength of Flexural Member

The plastic moment carrying capacity of a beam section is same as given in Section 6.3. The variation of flexural stress as load approaches the plastic load is shown in Fig. 6.3.

12.5 Theory of Plastic Bending

Assumptions:

(a) Plane section remain plane and normal to the axis of bending at all stages. Thus it is assumed that strains are proportional to the distance from the neutral axis.

(b) The stress-strain relationship is idealized to consist of two straight lines as shown in Fig. 12.3.

(c) The stress in any fibre can be found from its strain according to the stress-section curve idealized as above without reference to other fibres. The shearing strains are neglected.

(d) The deformations are assumed to be small, so the slope of the beam at any point may be assumed to be equal to its tangent.

(e) Steel is ductile, able to deform plastically without fracture.

(f) The properties of steel in compression are assumed to be the same in tension.

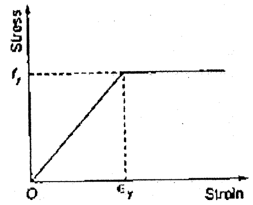


Fig. 12.3 Idealised Stress-Strain diagram

(g) The influence of axial and shearing force are neglected.

(h) Strain energy stored due to elastic bending is ignored.

(i) The connections provide full continuity so that plastic moment can be transmitted through them.

(j) Members are initially straight and prismatic and instability does not develop before plastic action develops.

(k) There is an axis of symmetry in the cross-section.

(e) Cross-section must be symmetrical w.r.t. the plane of loading (to avoid affect of twisting).

12.5.1 Equations of Equilibrium for Plastic Theory

(a) $\int_A f_y \cdot dA = 0$ i.e. the condition of equilibrium of axial force to be zero.

(b) $\int_A f_y \cdot y dA = M$ i.e. at the condition of equilibrium that internal moment balances the applied moment.

12.5.2 Calculation of Plastic Moment

Consider any section with one axis of symmetry. The neutral axis lies in the plane containing the symmetrical axis. Under fully plastic conditions, the stress at the section, both in tension and compression zone is assumed to be at yield point of the material as shown in Fig. 12.4.

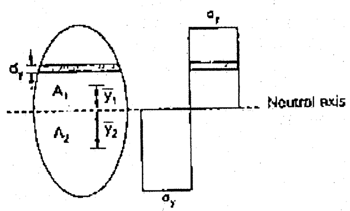


Fig. 12.4 Calculation of plastic moment

Equilibrium Condition-1: Axial thrust is zero at the section.

$$\int_A f_y dA = 0$$

$$\Rightarrow \quad f_y(A_2 - A_1) = 0 \Rightarrow A_2 = A_1$$

Hence the neutral axis under plastic condition divides the section into two equal areas.

Remember: Neutral axis under plastic condition or plastic neutral axis divides the section into two equal areas.

Equilibrium Condition-2: Internal moment is equal to applied moment.

$$\int_A f_y y dA = M_p$$

$$\Rightarrow \quad f_y(A_1\bar{y}_1 + A_2\bar{y}_2) = M_p \quad \text{...(12.8)}$$

It follows that the plastic moment of the section is given by the yield stress multiplied by the sum of moments of areas in tension and compression zones about the neutral axis.

Now the plastic moment M_p is given by $f_y Z_p$

where, $Z_p = A_1\bar{y}_1 + A_2\bar{y}_2$

Z_p is termed as the plastic modulus. As plastic neutral axis divides whole area into two equal halfs, so

$$A_1 = A_2 = \frac{A}{2}$$

$$\therefore \quad Z_p = \frac{A}{2}(\bar{y}_1 + \bar{y}_2) \qquad \text{...(12.9)}$$

12.5.3 Shape Factor

Shape factor is defined as the ratio of the plastic moment to yield moment. It is a function of the cross-section or shape and is represented by 'f'.

Mathematically:

$$f = \frac{M_p}{M_y} = \frac{f_y Z_{pz}}{f_y Z_{ez}} = \frac{Z_{pz}}{Z_{ez}}$$

where Z_{ez} = Elastic section modulus, Z_{pz} = Plastic section modulus

12.5.4 Relationship among Load Factor, Shape Factor and Factor of Safety

$$\text{Load Factor} = \frac{\text{Ultimate load}}{\text{Working load}} = \frac{\text{Collapse load}}{\text{Working load}} = \frac{P_u}{P_w}$$

$$\text{Shape Factor, } f = \frac{\text{Plastic moment}}{\text{Yield moment}} = \frac{M_P}{M_y} = \frac{f_y Z_{Pz}}{f_y Z_{ez}} = \frac{Z_{Pz}}{Z_{ez}}$$

$$\text{Factor of Safety, (FOS)} = \frac{\text{Yield Stress}}{\text{Permissible or working stress}} = \frac{f_y}{f}$$

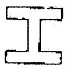
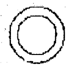
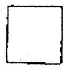
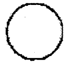
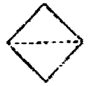
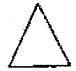
Now, $$\text{Load factor} = \frac{P_u}{P_w} = \frac{M_P}{M_y} = \left(\frac{f_y}{f}\right)\frac{Z_{Pz}}{Z_{ez}}$$

$\Rightarrow$ Load factor = Factor of safety × Shape factor ...(12.10)

Margin of Safety (Defined only for brittle material)

Margin of safety = FOS − 1 $\qquad$ where $\text{FOS} = \frac{\text{Ultimate load/stress}}{\text{Working load/stress}}$

Table 12.1 Shape factor for different sectional shapes

	Section		Shape Factor
1.	I-section		1.12 to 1.14
2.	Circular hollow section		1.27
3.	Rectangle/Square		1.5
4.	H-section		1.5
5.	Circular section		1.67
6.	Diamond		2.0
7.	Triangular		2.34

12.6 Plastic Hinge

- Flexural members remain elastic until the moment reaches a value M_p, the maximum moment of resistance of a fully yielded cross section.
- Any additional moment will cause the beam to rotate with little increase of stress.
- This rotation occurs at a constant moment M_p. This zone acts as a hinge except with a constant restraining moment M_p.
- In reality hinges do not form but large changes in slope occurs over very small lengths of the member at the location of maximum moment.
- Strain hardening occurs at these hinges so that large deflections are accompanied by a little increase of loads. This plastic hinge is represented by a black dot.
- A structure can sustain the computed ultimate loads due to successive formation of plastic hinges at critical locations. The plastic hinges are formed first at the the location of greatest deformation (i.e. curvature).

12.6.1 Possible Places of Plastic Hinge formation in a Structure with Prismatic Members/material

(a) Point of concentrated load

(b) At the ends of member meeting at a connection involving a change in geometry.

(c) Point of zero shear in structure subjected to uniformly distributed load.

(d) At the location of fixed ends.

12.6.2 Mechanism

When a structure is subjected to a system of loads, it is stable and functional until a sufficient number of plastic hinges have been formed to make the structure unstable. As soon as the structure reaches an unstable condition, it is considered to have been failed. The segments of the beam between the plastic hinges are able to move without any increase of load. This condition is called as **mechanism**. The concepts of virtual work are used in the plastic analysis and design of steel structures.

Remember

If an indeterminate structure has redundancy 'r', then insertion of 'r' plastic hinge makes it statically determinate and any further insertion of hinge converts this statically determinate structure into a mechanism. Hence for collapse the number of plastic hinges required are $(r + 1)$.

12.6.3 Types of Mechanism

(a) Beam mechanism
(b) Panel mechanism (Sway mechanism)
(c) Gable mechanism
(d) Combined mechanism

(a) Beam Mechanism

All the loaded spans behave as beam mechanism. A portal frame is shown in Fig. 12.5. The beam and column are of same cross-section.

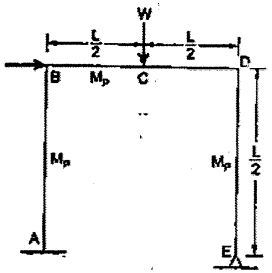


Fig. 12.5 Beam mechanism

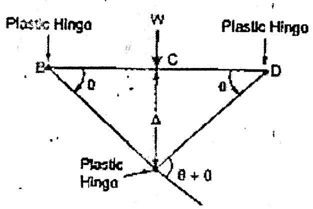


Fig. 12.6 Explanation of beam mechanism

The beam mechanism for the portal frame ABCDE is as shown in Fig. 12.6.

Remember: The ends B and D of the beam BD are rigid joints. Therefore, the beam BD acts as fixed ended beam. The possible location of plastic hinge are B, D and C.

(b) Sway Mechanism

It is formed due to lateral loads. In this mechanism, motion occurs in the structure due to sidesway or horizontal displacement of the structure.

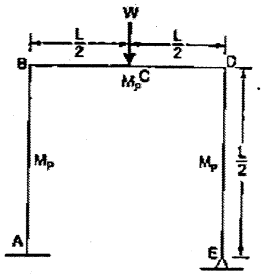


Fig. 12.7 Sway mechanism

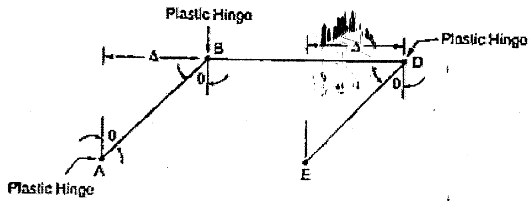


Fig. 12.8 Explanation of sway mechanism

Remember: The column AB and ED are of the same length so both deflect by the same amount Δ and the rotations at the ends are also same i.e. θ. Possible locations of plastic hinges are A, B and D.

(c) Gable Mechanism

It is exhibited in gable frames where column spread more at the top than at the base.

or

This mechanism is characteristic of gabled frames. The motion involves due to spreading of column tops with respect to the bases.

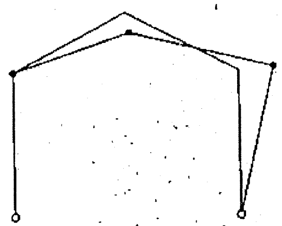


Fig. 12.9 Gable mechanism

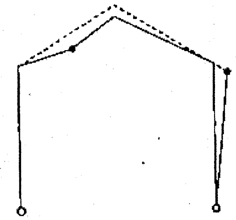


Fig. 12.10 Combined mechanism

(d) Combined Mechanism

Any of the two independent mechanisms may be combined to form a composite mechanism. A beam and gable mechanism are jointed to form a combined mechanism as shown in Fig. 12.10.

Do You Know?

Combined mechanism may be obtained by combining systematically the various independent mechanisms. During this mechanism, the frame is still indeterminate at failure or it may be a "complete" combined mechanism in which, the frame is determinate at failure.

In order to obtain the lowest possible ultimate load P_u combinations are selected in such a way that the external work is maximum or internal work is minimum. The procedure generally is to make combinations involving mechanism motions by as many loads as possible and the elimination of plastic hinges which reduces the internal work.

Number of Independent Mechanism

To avoid any possible omission of a combined mechanism it is essentials to have in advance a clear idea of the number of independent mechanism.

Let N = number of possible plastic hinges

r = number of redundancies

n = possible independent mechanism

Then $n = N - r$ ...(12.11)

After getting the number of independent mechanisms all the possible combinations are made in such a way that the external work is maximum or internal work minimum. This is done to obtain the lowest possible load.

12.6.4 Plastic Hinge and Mechanism

- The beam shown in Fig. 12.11 will fail when the central section becomes fully plastic. Then this section will go on rotating at constant moment and deflection will grow too large. In fact with simple supports at the ends and a plastic hinge at the centre, the beam will transform into a mechanism consisting of two links.
- From the bending moment diagram at ultimate load, it can be seen that a central length of the beam has moments in excess of the yield moment and hence the sections in this length are partially plastic.
- The length of the plastic zone depends upon the ratio of M_p to M_y. Greater this ratio, larger will be the length of the plastic zone. Each section will contribute towards beam rotations. Therefore, it follows that the plastic hinge is not a section but it is zone of yielding near the sections of fully plasticity.

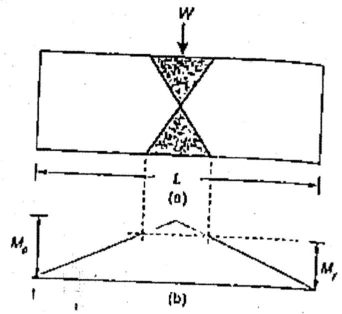


Fig. 12.11 Plastic hinge formation due to point load

12.6.5 Hinge Length

- As shown in Fig. 12.12, the value of moment at sections adjacent to the yield zone for a certain length is more than the yield moment. This little length is called as hinge length.
- This hinge length depends on the loading type and the geometry of the cross section.
- For simplified analyses, this hinge length is neglected and plastic hinge is considered to be formed at particular points of zero length.
- But this hinge length cannot be neglected in calculation of deflections and the design of bracings as the length over which yielding extends is quite important.

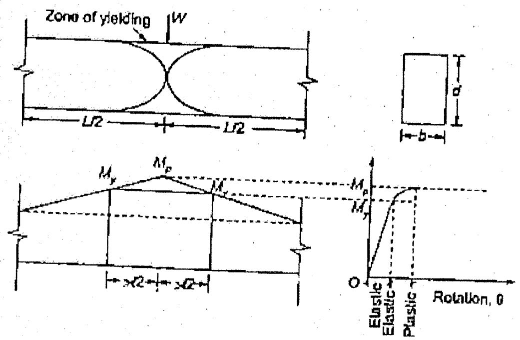


Fig. 12.12 Plastic hinge due to concentrated load at mid span

Let there be a simply supported beam which is subjected to a gradually increasing concentrated load W at the mid span. A plastic hinge will form at the center of the beam as shown in Fig. 12.12.

$$M_p = \frac{WL}{4} \quad ...(12.12)$$

$$M_y = f_y Z_e = f_y \frac{bd^2}{6} = f_y\left(\frac{1}{6}\right)\left(\frac{1}{4}\right)\frac{bd^2}{4} = \frac{2}{3} f_y Z_p = \frac{2}{3} M_p \quad ...(12.13)$$

From the bending moment diagram of Fig. 12.12,

$$\frac{M_p}{L/2} = \frac{M_y}{L/2 - x/2} \qquad \text{In general} \qquad \frac{M_p}{L/2} = \frac{M_y}{\frac{L}{2} - \frac{x}{2}}$$

$$\Rightarrow (L-x)\frac{M_p}{2} = \frac{L}{2}M_y \qquad \Rightarrow (L-x)\frac{M_p}{M_y} = L$$

$$\Rightarrow (L-x)M_p = LM_y \qquad \Rightarrow (L-x)\,\text{S.F.} = L$$

$$\Rightarrow (L-x)M_p = L\left(\frac{2}{3}\right)M_p \qquad \Rightarrow x = L - \frac{L}{\text{S.F.}}$$

$$\Rightarrow x = \frac{L}{3} \quad ...(12.14) \qquad \Rightarrow x = L\left(1 - \frac{1}{\text{S.F.}}\right)$$

Thus length of plastic hinge in a beam subjected to concentrated load at mid span is $L/3$ where L is the span of the beam.

Note: Following same procedure as above, for S.S. beam subjected to UDL plastic hinge length $x = L\sqrt{1 - \frac{1}{\text{S.F.}}}$

Remember: Hinge length for a simply supported rectangular beam subjected to central concentrated load is $L/3$ and that due to uniformly distributed load over the entire span is $L/\sqrt{3}$.

12.7 Redistribution of Moments and Reserve of Strength

- Flexural members can sustain the ultimate loads due to redistribution of moments.
- Under gradually increasing load on the structure, the plastic moment is reached at a section that is most highly stressed.
- With further increase of load, the value of plastic moment is maintained and the section rotates. A plastic hinge is formed at this section.
- Other less highly stressed sections maintain equilibrium with the increased load by a proportionate increase in the moment and successive formation of plastic hinges at other sections. This process of formation of plastic hinges continues till the ultimate load is reached.

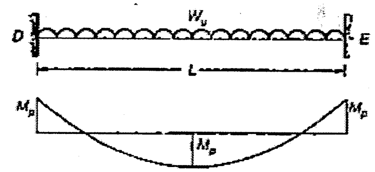


Fig. 12.13 Redistribution of moments in fixed beam

Let there be a fixed beam of span L subjected to a uniformly distributed load over the entire span as shown in Fig. 12.13.

For equilibrium,

$$\frac{W_u L}{8} = 2M_p \quad \Rightarrow \quad W_u = \frac{16M_p}{L} \quad ...(12.15)$$

At the elastic load, the mid span moment is half of the end moment i.e.

$$\frac{W_y L}{8} = M_y + \frac{M_y}{2} = \frac{3}{2}M_y \quad \Rightarrow \quad W_y = \frac{12M_y}{L} \quad ...(12.16)$$

Assuming a shape factor of 1.12 for an I-section, from Eqs. (12.15) and (12.16),

$$\frac{W_u}{W_y} = \frac{16M_p/L}{12M_y/L} = \left(\frac{4}{3}\right)\frac{M_p}{M_y} = \left(\frac{4}{3}\right)\times 1.12 = 1.493$$

Thus the ultimate load is 49.3% higher than the load at first yield. This reserve of strength is generally disregarded in usual elastic design practice.

12.8 Plastic Collapse

The following points are worth noticeable for plastic collapse:

(a) The plastic collapse of a structure depends upon its redundancy.

(b) When sufficient number of plastic hinges are formed to convert a structure into a mechanism, the structure collapses.

(c) The collapse of a structure can be partial, complete or over complete.

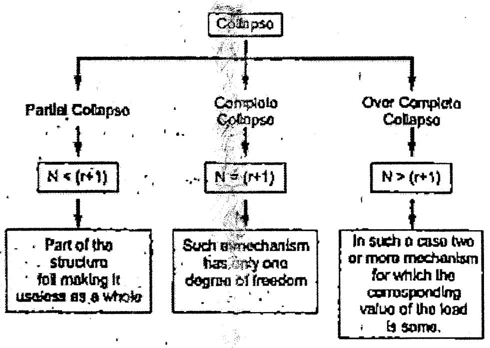


Fig. 12.14 Plastic collapse

12.9 Ultimate Load Analysis Fundamentals

(a) Structure must be in static equilibrium.

(b) Moment any where must not exceed the yield moment (structure should remain elastic).

(c) Continuity of slopes and deflection must not be violated (compatability condition).

12.10 Fundamental Conditions in Plastic Analysis

There are three fundamental conditions underlying the plastic analysis as follows:

1. **Conditions of Static Equilibrium or Equilibrium Condition**

   At the equilibrium condition, summation of all the forces and moments should be equal to zero.

   Mathematically: $\Sigma F_x = 0;\ \Sigma F_y = 0;\ \Sigma F_z = 0$

   $\Sigma M_x = 0;\ \Sigma M_y = 0;\ \Sigma M_z = 0$

2. **Mechanism Condition or Continuity Condition**

   The structure at collapse must be capable of deforming as a mechanism due to the formation of plastic hinges or there must be a enough plastic hinges that a mechanism is obtained.

3. **Plastic Moment Condition or Yield Condition**

   The plastic moment M_P must not exceed any where in the structure.

12.11 Theorem of Plastic Analysis

Generally, it is not possible to satisfy all the three conditions of plastic analysis in one operation. The equilibrium condition has to be satisfied in both the cases i.e., static as well as kinematic. Depending upon the method chosen to obtain a solution to the problem, one may obtain an upper limit and the other may obtain lower limit.

12.11.1 Static or Lower Bond Theorem

For a given frame and loading, if there exists any distribution of bending moments throughout the frame which is safe and statically admissible with a set of loads P, then the value of the load P must be less than or equal to the collapse load $P_u (P \le P_u)$. A load computed on the basis of an assumed equilibrium moment diagram in which the moments are not greater than M_P, is less than or equal to the true ultimate load and has a maximum factor of safety.

Static theorem: Equilibrium condition + Yield condition

12.11.2 Kinematic or Upper Bond Theorem

For a given frame subjected to a set of loads P, the value of P which is found corresponding to any assumed mechanism, must be either greater than or equal to the collapse load $P_u (P \ge P_u)$. A load computed on the basis of this mechanism will always be greater than or at least equal to the true ultimate load. Hence, the kinematic method represents an upper limit to the true ultimate load and has a small factor of safety.

Kinematic theorem: Equilibrium condition + Mechanism

12.11.3 Uniqueness Theorem

For a given frame and loading at least one safe and statically admissible bending moment distribution can be found and if in this distribution the bending moment is equal to the fully plastic moment at sufficient cross-section to cause failure of the frame as a mechanism due to plastic hinges, then the corresponding load will be equal to the collapse load ($P = P_u$).

Uniqueness Theorem : Equilibrium condition + Yield condition + Mechanism

12.12 Static Method

This consists of selecting the redundant force. The free and redundant bending moment diagrams are drawn for the structure. A combined bending moment diagram is drawn in such a way that a mechanism is formed. The collapse load is found by working out the equilibrium equation. It is checked that bending moment is not more than the fully plastic moment at any section. The objective of the static method is to satisfy the three fundamental conditions.

(a) Equilibrium (b) Yield (c) $M \le M_P$

12.12.1 Steps for Static Method of Plastic Analysis

Step-1. Calculate degree of static indeterminary.

Step-2. Remove redundant forces.

Step-3. Draw bending moment diagram on the statically determinate structure.

Step-4. Draw moment diagram at failure due to redundant forces.

Step-5. Superimpose the two moment diagrams so that a mechanism is formed sketch the mechanism.

Step-6. Solve the equation of equilibrium to find ultimate load.

Step-7. Check the moments, $M \le M_P$ everywhere. If so, the true ultimate load is obtained. If $M > M_P$ Somewhere, repeat steps (3) to (6) so that a plastic hinge is formed at the section of maximum moment and continue till the three basis conditions are satisfied.

12.13 Kinematic Method (Mechanism Method)

This consists of locating the possible places of plastic hinges. The possible independent and combined mechanism are ascertained. The collapse load is found by applying the **principle of virtual work**. A bending moment diagram corresponding to the collapse mechanism is drawn and it is checked that the bending moment is not more than the fully plastic moment at any section.

The statical method works well with simpler structure like the continuous beam or single story single bay hinged frames. But, when the structure is having many degree of redundancy the mechanism method turns out to be easier. The objective of the kinematic method is to satisfy the three fundamental conditions:

(a) Equilibrium (b) Mechanism (c) $M \leq M_p$

12.13.1 Steps for Mechanism Method of Plastic Analysis

Step-1.Determine degree of static indeterminacy.

Step-2.Determine the location of possible plastic hinges. Such locations are concentrated load points, support points, joints between members and points of maximum moments.

Step-3.Sketch possible mechanisms. The mechanisms are termed as beam, panel or gable type which may occur independently or the failure of the structure as a whole or the mechanism may be combined type which may result in the failure of a part or the whole of the given structure.

The correct mechanisms is that which satisfies the other two fundamental conditions namely equilibrium and $M \leq M_p$.

Step-4.Determine the ultimate load by solving the equilibrium equations. The virtual displacement method is best suited for this calculation.

Step-5.Check the moments at all points to see that moment is everywhere less than the plastic moment.

NOTE: Since the mechanism method or kinematic method gives an upper bound to the ultimate load we should look for the lowest value of the load for any mechanism.

12.14 Principle of Virtual Work

The principle of virtual work states that if a system of forces acting on a elastic body is subjected to a virtual displacement, then work done by the external forces equals the work done by the internal forces.

Mathematically: $W_e = W_i$

Where W_e = external workdone, W_i = internal workdone

For rigid body, $W_e = 0$

12.15 Advantages and Disadvantages of Plastic Design over Elastic Design

12.15.1 Advantages

(a) It is more economical as it makes full use of the material strength beyond the elastic limit.

(b) The design procedures are much simpler and rational in case of plastic method.

NOTE: The very important point in plastic design is that, it gives some idea of the collapse load and strength of the structure.

12.15.2 Disadvantages

(a) The fabrication is to be done with ductile steel.

(b) The loads are carried by bending and the effects of axial load and shear force on a member are neglected. The theory of simple plastic analysis can not be applied to trusses, as the members are subjected to axial forces only.

(c) Strength is assumed to be the main criteria.

(d) It is very difficult to obtain the collapse mode if the structure is reasonably complicated.

(e) There is little savings in column design.

(f) It is difficult to design for fatigue.

(g) Lateral bracing requirements are more stringent than the elastic design.

12.16 Some Important Aspects of Plastic Design and Elastic Design

(a) The structure designed by the plastic method is economical. The saving in steel may be about 10-15%.

(b) For indeterminate and complicated structures the plastic method is more conveniently applicable and avoids analytical approximations.

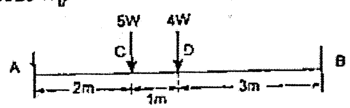
(c) In the elastic method of design, the design process is repeated several times to obtain an optimum section, which consumes more time. The plastic method of design produces a optimum section in a single attempt and thereby, saving the computational time.

(d) The factor of safety is same in both plastic and elastic design of indeterminate structure. The concept of this factor in the plastic design is more realistic.

(e) Stresses produced by settlement, erection etc., can be determined more correctly by the plastic method as the calculations are based on plastic deformations.

Illustrative Examples

Example 12.1 If the uniform beam as shown in figure below has a plastic moment capacity M_p than determine the ultimate load W_u.



Solution:

Step 1 :

Degree of static indeterminacy, $r = 2$

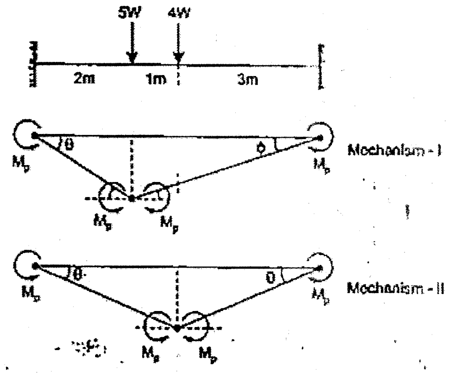
$\therefore$ Number of plastic hinges required for mechanism or collapse $= r + 1 = 3$

Step 2 : Determine the location of plastic hinges.

Plastic hinge will from at fixed end, i.e. at A and B and under the concentrated load i.e., at C and D.

$\therefore$ There are '4' possible locations namely A, B, C and D are available for the formation of plastic hinges in the given structure.

Step 3 : Sketch possible mechanism:



Step 4 : Determine the ultimate load by solving the equilibrium equations using principle of virtual work

For mechanism-I

$$2\theta = 4\phi \Rightarrow \phi = \theta/2$$

Internal work done $= M_p\theta + M_p(\theta+\phi) + M_p\phi$

and external work done $= 5W_u \times 2\theta + 4W_u \times 3\phi$

∴ From principle of virtual work,

Internal work done = external work done.

$$\Rightarrow \quad M_p\theta + \frac{3}{2}M_p\theta + M_p\frac{\theta}{2} = (10W_u + 6W_u)\theta$$

$$\Rightarrow \quad W_u = \frac{3}{16}M_p$$

For mechanism-II

Internal work done = external work done

$$\Rightarrow \quad M_p\theta + M_p(\theta+\theta) + M_p\theta = 5W_u(2\theta) + 4W_u(3\theta)$$

$$\Rightarrow \quad 4M_p\theta = (10W_u + 12W_u)\theta$$

$$\therefore \quad W_u = \frac{2}{11}M_p$$

Comparing W_u from (i) and (ii), it is seen that the second mechanism gives lower ultimate load.

Step 5 : Check bending moment

Check moment at 'C'

$$R_A = \frac{4W_u \times 3 + 5W_u \times 4}{6} + \frac{M_P - M_P}{6} = \frac{16}{3}W_u$$

$$\therefore \quad M_C = -M_P + \left(\frac{16}{3}\right)W_u = 0.939M_P < M_P$$

Hence $\quad W_u = \frac{2}{11}M_P$

Example 12.2 A propped cantilever carries an uniformly distributed load W over span L. Determine the ultimate collapse load W_u, if the plastic moment capacity of the beam is M_p.

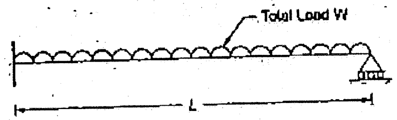
Solution:

Static method: **Figure (II)** shows the combined bending moment diagram of the determinate structure and the redundant. At collapse the plastic hinge will from at the fixed end at a distance of x from the free end where the difference of parabolic and triangular bending moment diagram is maximum. From **Figure (II)** at a distance x from hinged end.

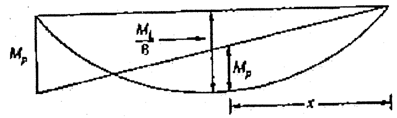
$$M_p + \frac{x}{L}M_p = \frac{W_u}{2}x - \frac{W_u}{L} \times \frac{x^2}{2}$$

or, $$W_u = 2M_p\frac{L+x}{x(L-x)} \quad \text{...(i)}$$

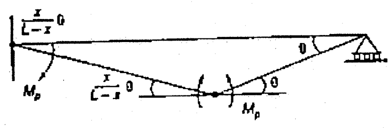
For maximum value of W_u, $$\frac{dW_u}{dx} = 0$$



(I) Beam



(II) Composite B.M. Diagram



(III) Beam Mechanism

or, $$2M_p \times \frac{[x(L-x)1 - (L+x)(L-2x)]}{x^2(L-x)^2} = 0$$

or, $$x^2 + 2Lx - L^2 = 0$$

or, $$x = 0.414L$$

Substituting x in the equilibrium equation (i).

or, $$W_u = \frac{2M_p}{L}\frac{L+x}{x(L-x)}$$

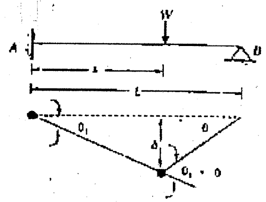
For maximum W_u, $x = 0.414L$ (as calculated earlier). Thus

$$W_u = 11.66\,M_p/L$$

Example 12.3 A beam of uniform cross section and span L is built-in at one end and simply supported at the other. It carries a concentrated load at a distance x from the built-in end. Show that the collapse load has the value $\frac{2L-x}{x(L-x)}M_p$, where M_p is the fully plastic moment. If both the ends had been built-in, show that the load at collapse would have increased in the ratio $\frac{2L}{2L-x}$.

Solution:

Since end A of the beam is fixed, a plastic hinge will develop at A. A plastic hinge will also be formed below the concentrated load and a mechanism is formed as shown in figure



$$\Delta = x\phi = (L-x)\theta$$

or,
$$\phi = \frac{L-x}{x}\theta$$

External work done = load × deflection

$$= W_u \times (L-x)\theta$$

Internal work done = moment × rotation

$$= M_p\phi + M_p(\phi+\theta)$$

$$= M_p\left(\frac{L-x}{x}\right)\theta + M_p\left(\frac{L-x}{x}\theta+\theta\right) = M_p\frac{2L-x}{x}\theta$$

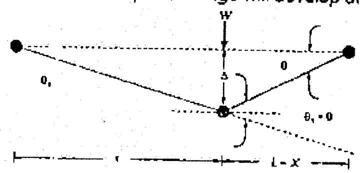
By the principle of virtual work,

External work done = Internal work done

$$W_u(L-x)\theta = M_p\frac{2L-x}{x}\theta$$

or,
$$W_u = \frac{2L-x}{x(L-x)}M_p$$

If both the ends are fixed than one more plastic hinge will develop at B



External work done = $W_u(L-x)\theta$

Internal work done = $M_p\phi + M_p(\phi+\theta)\ M_p\theta$

$$= M_p\frac{(L-x)}{x}\theta + M_p\left(\frac{L-x}{x}\theta+\theta\right) + M_p\theta = M_p\frac{2L}{x}\theta$$

by the principle of virtual work,

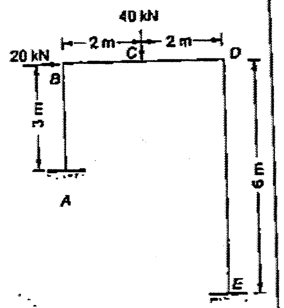
External work done = Internal work done

$$W_u(L-x)\theta = M_p\frac{2L}{x}\theta$$

or,
$$W_u = \frac{2L}{x(L-x)}M_p$$

Increase in the ratio of collapse load = $\frac{2L}{x(L-x)} \times \frac{x(L-x)}{2L-x} = \frac{2L}{2L-x}$

Example 12.4 Find out the fully plastic moment in the portal frame shown in figure. The frame has a uniform cross-section throughout



Solution:

The possible locations of plastic hinges are A, B, C, D and E.

Number of possible plastic hinges, $N = 5$

Degree of redundancy, $r = 6 - 3 = 3$

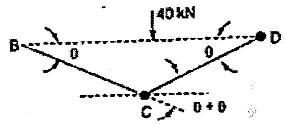
Number of possible independet mechanisms,

$$n = N - r = 5 - 3 = 2$$

The two independent mechanisms are:

1. Beam mechanism
2. Sway mechanism

Beam mechanism: The ends B and D of the beam BD are fixed joints. Therefore, the beam BD acts as a fixed end beam. The possible locations of the plastic hinge are B, D and C (below the concentrated load) as shown in figure.



External work done = load × deflection

$$= 40 \times 2 \times \theta = 80\theta$$

Internal work done = moment × rotation

$$= M_p\theta + M_p(\theta+\theta) + M_p\theta = 4M_p\theta$$

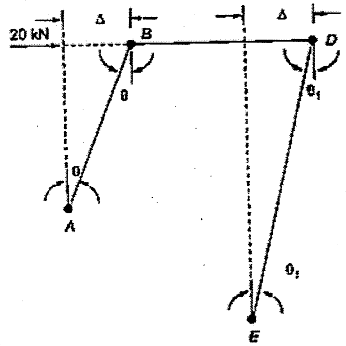
By the principle of virtual work done,

Internal work done = External work done

$$4M_p\theta = 80\theta$$

or,
$$M_p = 20 \text{ kNm}$$

Sway mechanism : The column AB and ED are of different lengths. These deflect by the same amount. Therefore, the rotations at the ends of the columns will be different.



$$\Delta = 3\theta = 6\phi$$

or,
$$\phi = \theta/2$$

Internal work done = moment × rotation

$$= M_p\theta + M_p\theta + M_p\phi + M_p\phi$$

$$= M_p\theta + M_p\theta + M_p\theta/2 + M_p\theta/2$$

$$= 3M_p\theta$$

External workdone = $20 \times 3\theta = 60\theta$

By the principle of virtual work,

Internal work done = External work done

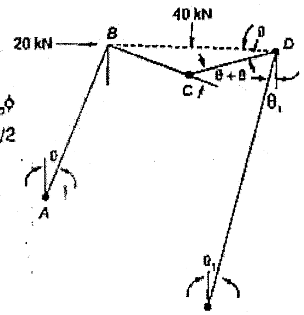
$$3M_p\theta = 60\theta$$

or,
$$M_p = 20 \text{ kNm}$$

Combined mechanism : The possible locations of plastic hinges are A, C, D and E only.

External work done = Σ(load × deflection)

$= 40 \times 2 \times \theta + 20 \times 3 \times \theta = 1400$



Internal work done = Σ(plastic moment × rotation)

$= M_p\theta + M_p(\theta + \theta) + M_p\theta + M_p\phi + M_p\phi$

$= M_p\theta + 2M_p\theta + M_p\theta + M_p\theta/2 + M_p\theta/2$

$= 5M_p\theta$

By the principle of virtual work,

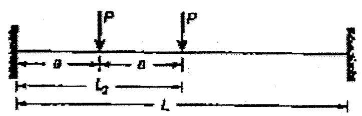
Internal work done = External work done

$5M_p\theta = 1400$

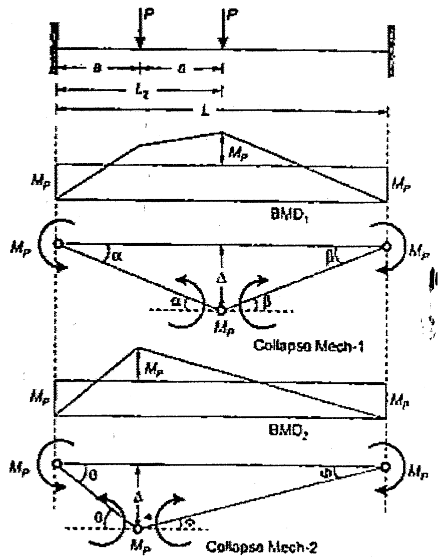
or, $M_p = \frac{140}{5} = 28$ kNm

The plastic moment for the frame is the maximum of the three plastic moments found. Therefore, the value of the plastic moment is 28 kNm.

Example 12.5 **Find collapse for the beam as shown. The beam is of uniform section with plastic moment capacity M_p.**



Solution:



Degree of static indeterminacy $(n) = 2$

∴ No. of hinge required for collapse $= (n + 1) = 2 + 1 = 3$

Two of these hinges will form at the ends and the remaining third hinge will from either at C or at D.

Let third hinge forms at D as shown in collapse mechanism-1.

$$W_{ext} = P\Delta + \frac{P\Delta}{2} = \frac{3P\Delta}{2}$$

$$W_{int} = M_p\alpha + M_p(\alpha+\beta) + M_p\beta = 2M_p(\alpha-\beta)$$

But $\alpha = \beta = \frac{\Delta}{L/2} = \frac{2\Delta}{L}$

∴ $W_{int} = 2M_p\left(\frac{2\Delta}{L} + \frac{2\Delta}{L}\right) = \frac{8M_p\Delta}{L}$

∴ $W_{ext} = W_{int}$

⇒ $\frac{3P\Delta}{2} = \frac{8M_p\Delta}{L}$

⇒ $P = \frac{16M_p}{3L}\left(= \frac{5.33M_p}{L}\right)$ ...(i)

If third hinge gets formed at C then collapse mechanism-2 will four.

∴ $W_{ext} = P\Delta + \frac{P.2\Delta}{3} = \frac{5P\Delta}{3}$

$W_{int} = M_p\theta + M_p(\theta+\phi) + M_p\phi = 3M_p(\theta+\phi)$

But $q = 3f$

∴ $W_{int} = 2M_p(\phi + 3\phi) = 8M_p\phi$

∴ $W_{ext} = W_{int}$

⇒ $\frac{5P\Delta}{3} = 8M_p\phi$

But $\phi = \frac{\Delta}{(3L/4)}$

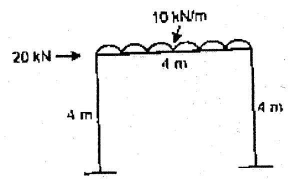
⇒ $5P\Delta = 32M_p\frac{\Delta}{L}$

⇒ $P = \frac{32M_p}{5L}\left(= \frac{6.4M_p}{L}\right)$ ...(ii)

From (i) and (ii), collapse load will be the lesser of two i.e.,

$$P = \frac{16M_p}{3L}$$

Example 12.6 Determine the value of fully plastic moment M_p for the frame shown in figure below. Plastic moment capacity of beam = 1/2 × plastic moment capacity of column.



Solution:

Number of possible plastic hinges, $n = 5$

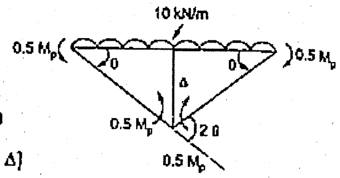
Degree of static indeterminacy, $r = 6 - 3 = 3$

Number of independent mechanism, $i = n - r = 5 - 3 = 2$

Beam Mechanism :

Applying principle of virtual work, we get

External workdone = Internal workdone



$$10\times\frac{1}{2}\times4\times\Delta = 0.5\,M_p\theta + 0.5\,M_p(\theta+\theta) + 0.5\,M_p\theta$$

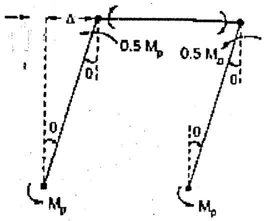
$\Rightarrow$ $20\times\Delta = 2\,M_p\theta$ $[\because 2\theta = \Delta]$

$\Rightarrow$ $M_p = 20$ kN-m

Sway Mechanism :

Applying principle of virtual work, we get

External workdone = Internal workdone



$$20\times\Delta = M_p\theta + 0.5M_p\theta + 0.5M_p\theta + M_p\theta \quad ..[\Delta = 4\theta]$$

$\Rightarrow$ $20\times4\theta = 3M_p\theta$

$\Rightarrow$ $M_p = 26.67$ kN-m

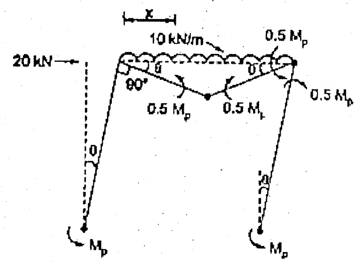
Combined Mechanism :

Applying principle of virtual work, we get

External workdone = Internal workdone

$$20\times4\theta + 10\times\frac{1}{2}\times x\theta\times4 = M_p\theta + M_p\theta + 0.5\,M_p(\theta+\phi) + 0.5\,M_p(\theta+\phi)$$

$\Rightarrow$ $80\theta + 20x\theta = 3\,M_p\theta + M_p\phi$



$\Rightarrow$ $(80+20x)\theta = 3M_p\theta + M_p\times\dfrac{\theta x}{4-x}$

$\Rightarrow$ $80+20x = 3M_p + \dfrac{M_p x}{4-x}$

$\Rightarrow$ $M_p\left(3+\dfrac{x}{4-x}\right) = 80+20x$

$\Rightarrow$ $M_p\left(\dfrac{12-3x+x}{4-x}\right) = 80+20x$

$\Rightarrow$ $M_p = \dfrac{(4-x)(40+10x)}{(6-x)} = \dfrac{160-10x^2}{60-x}$

For maximum value of M_p, $\dfrac{dM_p}{dx} = 0$

$\Rightarrow$ $(160-10x^2)\times(-1) - [(6-x)\times(-20x)] = 0$

$\Rightarrow$ $10x^2 - 160 + 120x - 20x^2 = 0$

$\Rightarrow$ $x^2 - 12x + 16 = 0$

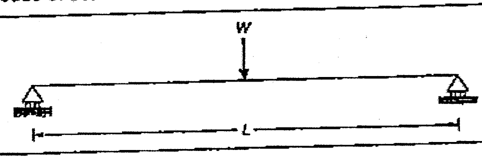
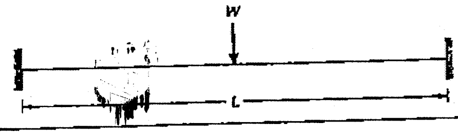
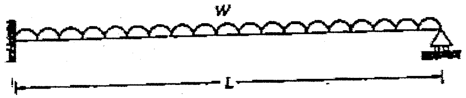
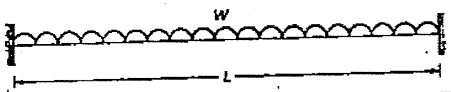
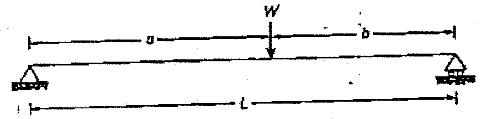
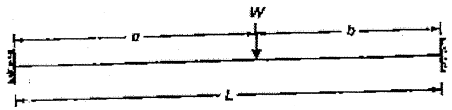
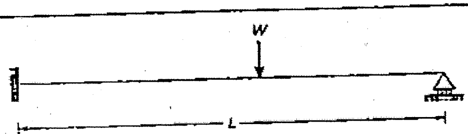
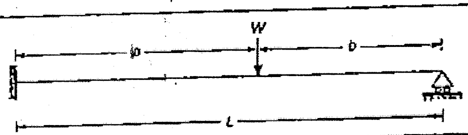
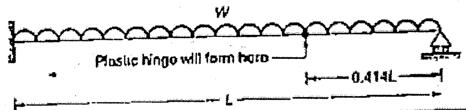
$\Rightarrow$ $x = 1.53$ m

$\therefore$ $M_p = \dfrac{160-10\times(1.53)^2}{6-1.53}$

$\Rightarrow$ $M_p = 30.56$ kN-m

Thus the maximum value of M_p is chosen as the plastic moment capacity of the portal frame i.e $M_p = 30.56$ kN-m.

Collapse loads of some standard arrangements:

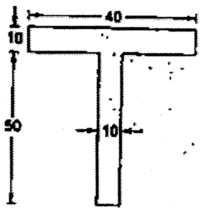
1.		$W_u = \dfrac{8M_p}{L}$
2.		$W_u = \dfrac{8M_p}{L}$
3.		$W_u = \dfrac{8M_p}{L^2}$
4.		$W_u = \dfrac{16M_p}{L^2}$
5.		$W_u = \dfrac{M_pL}{ab}$
6.		$W_u = \dfrac{2M_pL}{ab}$
7.		$W_u = \dfrac{6M_p}{L}$
8.		$W_u = \dfrac{M_p(L+b)}{ab}$
9.		$W_u = \dfrac{11.656M_p}{L^2}$

Objective Brain Teasers

Q.1 Which one of the following is the correct maximum shear capacity of a prismatic beam under plastic design of steel structures?

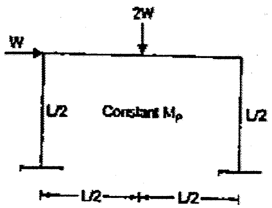
(a) $0.5A_wF_y$ (b) $0.55A_wF_y$
(c) $0.75A_wF_y$ (d) A_wF_y

Q.2 In the T-section shown in figure (all dimensions in mm), the distance of plastic neutral axis from top is



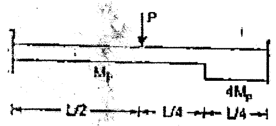
(a) 10 mm (b) 15 mm
(c) 20 mm (d) 30 mm

Q.3 Given figure shows a portal frame with loads. All members of frame have same plastic moment of resistance M_p. The collapse load W is



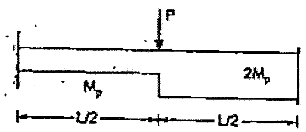
(a) $\frac{2M_p}{L}$ (b) $\frac{4M_p}{L}$
(c) $\frac{6M_p}{L}$ (d) $\frac{8M_p}{L}$

Q.4 For the beam shown in the given figure, the collapse load P is given by



(a) $\frac{16M_p}{L}$ (b) $\frac{14M_p}{L}$
(c) $\frac{12M_p}{L}$ (d) $\frac{10M_p}{L}$

Q.5 A fixed beam made of steel is shown in the figure below. At collapse, the value of load P will be equal to



(a) $\frac{10M_p}{L}$ (b) $\frac{12M_p}{L}$
(c) $\frac{16M_p}{L}$ (d) $\frac{20M_p}{L}$

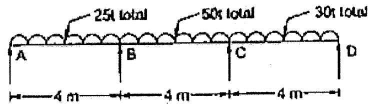
Q.6 Hinge length cannot be taken simply as point while
(a) computing deflections
(b) designing the bracings
(c) Both (a) and (b)
(d) None of (a) or (b)

Answers

1. (b) 2. (b) 3. (b) 4. (c) 5. (a)
6. (c)

Conventional Practice Questions

Q.1 A continuous beam rests on four supports A, B, C, D; AB = BC = CD = 4m. Each span carrier a uniformly distributed load; 25t on AB, 50 t on BC, and 30 t on CD. The beam is to be of uniform section between supports but the section of the centre span is to be heavier than that of either span. Find the required value of the full plastic moment for each span if a load factor of 2 at collapse is to be provided.



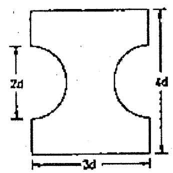
[*Ans.*: For spam AB M_p = 17.148 m.t; For spam BC M_p = 30.9 mt; For spam CD M_p =2058 m.t]

Q.2 Obtain the length and profile of a plastic hinge for simply supported beam of length '*l*' subjected to uniformly distributed load 'W' per unit run.

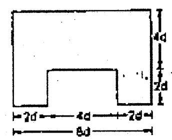
[*Ans.* $l/\sqrt{3}$]

Q.3 Deduce the shape factors for the following sections:

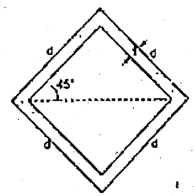
(a)



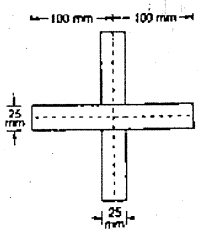
(b)



(c)

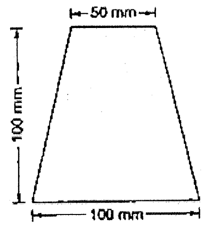


(d)



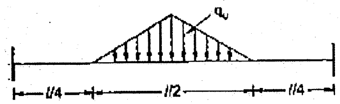
[*Ans.* *(a)* 1.402 *(b)* 2.036 *(c)* 1.5, *(d)* 1.64]

Q.4 For the beam section shown in figure determine the plastic modulus and the fully plastic moment. (Take f_y = 250 N/mm²)



[*Ans.*:182362.5 mm³, 45.59 kNm]

Q.5 A fixed beam of span '*l*' is subjected to a loading as shown in figure. Compute the ultimate load by the mechanism method.

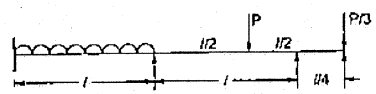


[*Ans.*: $\frac{192M_P}{5l^2}$]

Q.6 A simply supported beam of span '*l*' overhangs by a distance a. It is subjected to a *u.d.l* of *w* per unit run. Find the critical value of 'a' so that collapse occurs simultaneously in the cantilever portion and in the interior span. Compute the value of load in terms of M_p and *l*.

[*Ans.*: 0.415*l*, $\frac{11.656M_P}{l^2}$]

Q.7 The continuous beam shown in Figure is having uniform section throughout. It is subjected to loadings as shown in figure. Compute the ultimate load. Sketch BMD at collapse.



[*Ans.* $\frac{7.2M_P}{l}$]