

DAY FOURTY

Mock Test 3

(Based on Complete Syllabus)

Instructions ●

1. This question paper contains of 30 Questions of Mathematic, divided into two Sections :
Section A Objective Type Questions and **Section B** Numerical Type Questions.
2. Section A contains 20 Objective questions and all Questions are compulsory (**Marking Scheme** : Correct +4, Incorrect -1).
3. Section B contains 10 Numerical value questions out of which only 5 questions are to be attempted (**Marking Scheme** : Correct +4, Incorrect 0).

Section A : Objective Type Questions

- 1 Two circles in complex plane are

$$C_1 : |z - i| = 2$$

$$C_2 : |z - 1 - 2i| = 4. \text{ Then,}$$

- (a) C_1 and C_2 touch each other
- (b) C_1 and C_2 intersect at two distinct points
- (c) C_1 lies within C_2
- (d) C_2 lies within C_1

- 2 If $\int \frac{\sqrt{\cos 2x}}{\sin x} dx = -\log |\cot x + \sqrt{\cot^2 x - 1}| + A + C,$

then A is equal to

- (a) $\frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}} \right|$
- (b) $\frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \sec^2 x}}{\sqrt{2} - \sqrt{1 - \sec^2 x}} \right|$
- (c) $\frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \sin^2 x}}{\sqrt{2} - \sqrt{1 - \sin^2 x}} \right|$
- (d) None of the above

- 3 If m_1 and m_2 are the roots of the equation $x^2 + (\sqrt{3} + 2)x + \sqrt{3} - 1 = 0$, then the area of the triangle formed by the lines $y = m_1x$, $y = -m_2x$ and $y = 1$ is

- (a) $\frac{1}{2} \left(\frac{\sqrt{3} + 2}{\sqrt{3} - 1} \right)$
- (b) $\frac{1}{2} \left(\frac{\sqrt{3} + 2}{\sqrt{3} + 1} \right)$
- (c) $\frac{1}{2} \left(\frac{-\sqrt{3} + 2}{\sqrt{3} - 1} \right)$
- (d) None of these

- 4 The equation of the ellipse whose axes are coincident with the coordinate axes and which touches the straight lines $3x - 2y - 20 = 0$ and $x + 6y - 20 = 0$, is

- (a) $\frac{x^2}{5} + \frac{y^2}{8} = 1$
- (b) $\frac{x^2}{40} + \frac{y^2}{10} = 10$
- (c) $\frac{x^2}{40} + \frac{y^2}{10} = 1$
- (d) $\frac{x^2}{10} + \frac{y^2}{40} = 1$

- 5 A variable straight line through the point of intersection of the lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$ meets the coordinate axes in A and B.

I. The locus of mid-point of AB is the curve $2xy(a + b) = ab(x + y)$.

II. The locus of mid-point of AB is the curve $2xy(x + y) = ab(a + b)$.

- (a) Both I and II are true
- (b) Only I is true
- (c) Only II is true
- (d) Both I and II are false

- 6 If the function $f(x) = \frac{2x - \sin^{-1} x}{2x + \tan^{-1} x}$, $x \neq 0$ is continuous at each point of its domain, then the value of $f(0)$ is
 (a) 2 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $-\frac{1}{3}$
- 7 If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$, (where $pq \neq 0$) are bisected by the X-axis, then
 (a) $p^2 = q^2$ (b) $p^2 = 8q^2$
 (c) $p^2 < 8q^2$ (d) $p^2 > 8q^2$
- 8 If A and B are different matrices satisfying $A^3 = B^3$ and $A^2B = B^2A$, then
 (a) $\det(A^2 + B^2)$ must be zero
 (b) $\det(A - B)$ must be zero
 (c) $\det(A^2 + B^2)$ as well as $\det(A - B)$ must be zero
 (d) Atleast one of $\det(A^2 + B^2)$ or $\det(A - B)$ must be zero
- 9 A fair coin is tossed 100 times. The probability of getting tails 1, 3, ..., 49 times is
 (a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{16}$
- 10 The distance between the line
 $r = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - \hat{j} + 4\hat{k})$
 and the plane $r \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 5$ is
 (a) $\frac{10}{3}$ (b) $\frac{3}{10}$ (c) $\frac{10}{3\sqrt{3}}$ (d) $\frac{10}{9}$
- 11 The negation of the compound proposition $p \vee (\sim p \vee q)$ is
 (a) $(p \wedge \sim q) \wedge \sim p$ (b) $(p \wedge \sim q) \vee \sim p$
 (c) $(p \wedge \sim q) \vee \sim p$ (d) $(p \wedge q) \wedge q$
- 12 If $f(x) = \begin{cases} \frac{x-1}{2x^2-7x+5}, & \text{for } x \neq 1 \\ -1/2, & \text{for } x = 1 \end{cases}$, then $f'(1)$ is equal to
 (a) $-\frac{1}{9}$ (b) $-\frac{2}{9}$ (c) $-\frac{1}{3}$ (d) $\frac{1}{3}$
- 13 The value of $\tan \left\{ \cos^{-1} \left(\frac{-2}{7} \right) - \frac{\pi}{2} \right\}$ is
 (a) $\frac{2}{3\sqrt{5}}$ (b) $\frac{2}{3}$ (c) $\frac{1}{\sqrt{5}}$ (d) $\frac{4}{\sqrt{5}}$
- 14 A boat is being rowed away from a cliff of 150 m height. At the top of the cliff the angle of depression of boat changes from 60° to 45° in 2 min. Then, the speed of the boat (in m/h) is
 (a) $\frac{4500}{\sqrt{3}}$ (b) $\frac{4500}{\sqrt{3}}(\sqrt{3} - 1)$
 (c) $\frac{4300}{\sqrt{3}}$ (d) $\frac{4500}{\sqrt{3}}(\sqrt{3} + 1)$
- 15 The equation of a line of intersection of planes $4x + 4y - 5z = 12$ and $8x + 12y - 13z = 32$ can be written as
 (a) $\frac{x-1}{2} = \frac{y+2}{-3} = \frac{z}{4}$ (b) $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}$
 (c) $\frac{x}{2} = \frac{y+1}{3} = \frac{z-2}{4}$ (d) $\frac{x}{2} = \frac{y}{3} = \frac{z-2}{4}$

- 16 If the value of the determinant $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$ is positive, then
 (a) $abc > 1$ (b) $abc > -8$
 (c) $abc < -8$ (d) $abc > -2$
- 17 A bag contains a white and b black balls. Two players A and B alternately draw a ball from the bag, replacing the ball each time after the draw. A begins the game. If the probability of A winning (that is drawing a white ball) is twice the probability of B winning (that is drawing a black ball), then the ratio $a : b$ is equal to
 (a) $1 : 2$ (b) $2 : 1$
 (c) $1 : 1$ (d) None of these
- 18 If $(5 + 2\sqrt{6})^n = l + f$; $n, l \in \mathbb{N}$ and $0 \leq f < 1$, then l equals
 (a) $\frac{1}{f} - f$ (b) $\frac{1}{1+f} - f$ (c) $\frac{1}{1+f} + f$ (d) $\frac{1}{1-f} - f$
- 19 The median of a set of 11 distinct observations is 20.5. If each of the last 5 observations of the set is increased by 4, then the median of the new set
 (a) is increased by 2
 (b) is decreased by 2
 (c) is two times the original a median
 (d) remains the same as that of the original set
- 20 Solution of the differential equation $\frac{\sqrt{x} dx + \sqrt{y} dy}{\sqrt{x} dx - \sqrt{y} dy} = \sqrt{\frac{y^3}{x^3}}$ is given by
 (a) $\frac{3}{2} \log \left(\frac{y}{x} \right) + \log \left| \frac{x^{3/2} + y^{3/2}}{x^{3/2}} \right| + \tan^{-1} \left(\frac{y}{x} \right)^{3/2} + c = 0$
 (b) $\frac{2}{3} \log \left(\frac{y}{x} \right) + \log \left| \frac{x^{3/2} + y^{3/2}}{x^{3/2}} \right| + \tan^{-1} \left(\frac{y}{x} \right) + c = 0$
 (c) $\frac{2}{3} \log \left(\frac{y}{x} \right) + \log \left(\frac{x+y}{x} \right) + \tan^{-1} \left(\frac{y^{3/2}}{x^{3/2}} \right) + c = 0$
 (d) None of the above

Section B : Numerical Type Questions

- 21 Sixteen metre of wire is available to fence off a flower bed in the form of a sector. If the flower bed has the maximum surface then radius is
- 22 A circle is drawn to pass through the extremities of the latusrectum of the parabola $y^2 = 8x$. It is given that, this circle also touches the directrix of the parabola. The radius of this circle is equal to
- 23 The number of permutations of the letters a, b, c, d such that b does not follow a , c does not follow b and d does not follow c , is
- 24 If $f'(x) = \sin(\log x)$ and $y = f\left(\frac{2x+3}{3-2x}\right)$ and $\frac{dy}{dx}$ at $x = 1$ is equal to $p \sin \log(q)$, then value of $p + q$ is equal to

25 If $x + y + z = 1$ and x, y, z are positive numbers such that $(1-x)(1-y)(1-z) \geq kxyz$, then k^2 equals

26 If $f(x)$ is a function satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{N}$ such that $f(1) = 3$ and $\sum_{x=1}^n f(x) = 120$, then the value of n^4 is

27 Number of solutions of the equation $\sin x = [x]$, where $[.]$ denotes the largest integer function, is

28 If the value of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$ is $\frac{m}{n} - \pi$, then value of mn is equal to

29 If the area inside the parabola $5x^2 - y = 0$ but outside the parabola $2x^2 - y + 9 = 0$ is $p\sqrt{3}$, then value of p is equal to

30 If $\hat{i} - 3\hat{j} + 5\hat{k}$ bisects the angle between \hat{a} and $-\hat{i} + 2\hat{j} + 2\hat{k}$ when \hat{a} is a unit vector, then $\hat{a} = \frac{\lambda\hat{i} + \mu\hat{j} + \gamma\hat{k}}{105}$, then the value of $\lambda^2 - 2\mu + \nu$ must be

Hints and Explanations

1 (c) Given equations can be rewritten as

$$x^2 + (y-1)^2 = 2^2$$

and $(x-1)^2 + (y-2)^2 = 4^2$

Here, centres are $C_1(0, 1)$ and $C_2(1, 2)$ and radii are

$$r_1 = 2, r_2 = 4$$

Now, $C_1C_2 = \sqrt{(1-0)^2 + (2-1)^2}$

$$= \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\therefore |r_1 - r_2| = |2 - 4| = 2$$

$$C_1C_2 < |r_1 - r_2|$$

2 (a) Let $I = \int \frac{\cos 2x}{\sin^2 x} dx$

$$= \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x} dx$$

$$= \int \sqrt{\cot^2 x - 1} dx$$

On putting $\cot x = \sec \theta$

$$\Rightarrow -\operatorname{cosec}^2 x dx = \sec \theta \tan \theta d\theta,$$

we get

$$I = \int \sqrt{\sec^2 \theta - 1} \times \frac{\sec \theta \tan \theta}{-\operatorname{cosec}^2 x} d\theta$$

$$= - \int \frac{\sec \theta \tan^2 \theta}{1 + \sec^2 \theta} d\theta$$

$$= - \int \frac{\sin^2 \theta}{\cos \theta + \cos^3 \theta} d\theta$$

$$= - \int \frac{1 - \cos^2 \theta}{\cos \theta + \cos^3 \theta} d\theta$$

$$= - \int \frac{(1 + \cos^2 \theta) - 2\cos^2 \theta}{\cos \theta (1 + \cos^2 \theta)} d\theta$$

$$= - \int \sec \theta d\theta + 2 \int \frac{d(\sin \theta)}{1 + \cos^2 \theta}$$

$$= - \log |\sec \theta + \sqrt{\sec^2 \theta - 1}|$$

$$+ \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \cos^2 \theta}}{\sqrt{2} - \sqrt{1 - \cos^2 \theta}} \right| + C$$

$$= - \log |\cot x + \sqrt{\cot^2 x - 1}|$$

$$+ \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}} \right| + C$$

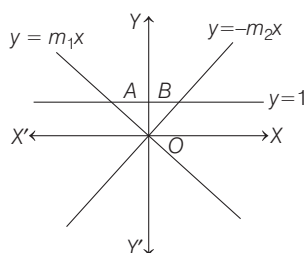
But $I = - \log |\cot x + \sqrt{\cot^2 x - 1}|$

$$+ A + C \text{ [given]}$$

$$\therefore A = \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}} \right|$$

3 (a) Since, m_1 and m_2 are the roots of the equation

$$x^2 + (\sqrt{3} + 2)x + (\sqrt{3} - 1) = 0.$$



$$\therefore m_1 + m_2 = -(2 + \sqrt{3})$$

$$\text{and } m_1 m_2 = \sqrt{3} - 1$$

$$\Rightarrow m_1 < 0, m_2 < 0$$

So, the point of intersections are

$$A\left(\frac{1}{m_1}, 1\right) \text{ and } B\left(-\frac{1}{m_2}, 1\right)$$

and $O(0, 0)$.

$$\therefore \text{Area of } \triangle OAB = -\frac{1}{2} \begin{vmatrix} \frac{1}{m_1} & 1 & 1 \\ -\frac{1}{m_2} & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \left| \frac{1}{m_1} + \frac{1}{m_2} \right| = \frac{1}{2} \left| \frac{m_1 + m_2}{m_1 m_2} \right|$$

$$= \frac{1}{2} \left(\frac{2 + \sqrt{3}}{\sqrt{3} - 1} \right)$$

4 (c) The general equation of the tangent to the ellipse is

$$y = mx \pm \sqrt{a^2 m^2 + b^2} \quad \dots(i)$$

Since, $y = \frac{3}{2}x - 10$ is a tangent to the

ellipse, therefore its comparing with Eq. (i), we get

$$m = \frac{3}{2} \text{ and } a^2 m^2 + b^2 = 100$$

$$\Rightarrow 9a^2 + 4b^2 = 400 \quad \dots(ii)$$

Similarly, line

$$y = -\frac{1}{6}x + \frac{10}{3}$$

is a tangent to the ellipse, therefore its comparing with Eq. (i), we get

$$m = -\frac{1}{6}$$

$$\text{and } a^2 m^2 + b^2 = \frac{100}{9}$$

$$\Rightarrow a^2 + 36b^2 = 400 \quad \dots(iii)$$

On solving Eqs. (ii) and (iii), we get

$$a^2 = 40$$

$$\text{and } b^2 = 10$$

Hence, required equation of the ellipse is

$$\frac{x^2}{40} + \frac{y^2}{10} = 1.$$

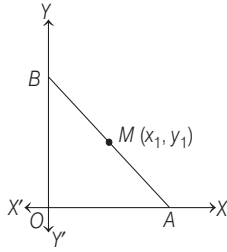
5 (b) Any line through the point of intersection of given lines is

$$\left(\frac{x}{a} + \frac{y}{b} - 1\right) + \lambda \left(\frac{x}{b} + \frac{y}{a} - 1\right) = 0$$

$$x \left(\frac{1}{a} + \frac{\lambda}{b}\right) + y \left(\frac{1}{b} + \frac{\lambda}{a}\right) = (1 + \lambda)$$

$$\Rightarrow x \left(\frac{b + a\lambda}{ab}\right) + y \left(\frac{a + b\lambda}{ab}\right) = (1 + \lambda)$$

$$\Rightarrow \frac{x}{\left\{\frac{ab(1 + \lambda)}{b + a\lambda}\right\}} + \frac{y}{\left\{\frac{ab(1 + \lambda)}{a + b\lambda}\right\}} = 1$$



This meets the X-axis at

$$A \equiv \left(\frac{ab(1+\lambda)}{b+a\lambda}, 0 \right)$$

and meets the Y-axis at

$$B \equiv \left(0, \frac{ab(1+\lambda)}{a+b\lambda} \right)$$

Let the mid-point of AB is M(x₁, y₁). Then,

$$x_1 = \frac{ab(1+\lambda)}{2(b+a\lambda)}$$

$$\text{and } y_1 = \frac{ab(1+\lambda)}{2(a+b\lambda)}$$

$$\therefore \frac{1}{x_1} + \frac{1}{y_1} = \frac{2(b+a\lambda)}{ab(1+\lambda)} + \frac{2(a+b\lambda)}{ab(1+\lambda)}$$

$$= \frac{2}{ab(1+\lambda)} (b+a\lambda + a+b\lambda)$$

$$= \frac{2}{ab(1+\lambda)} (b+a)(1+\lambda)$$

$$\Rightarrow \frac{(x_1 + y_1)}{x_1 y_1} = \frac{2(a+b)}{ab}$$

$$\Rightarrow 2x_1 y_1 (a+b) = ab(x_1 + y_1)$$

Hence, the locus of mid-point of AB is $2xy(a+b) = ab(x+y)$

6 (b) Since, f(x) is continuous

$$\therefore \lim_{x \rightarrow 0} \left(\frac{2x - \sin^{-1} x}{2x + \tan^{-1} x} \right) = f(0)$$

$$\therefore f(0) = \lim_{x \rightarrow 0} \frac{\left(2 - \frac{1}{\sqrt{1-x^2}} \right)}{2 + \frac{1}{1+x^2}}$$

$$= \frac{2 - \frac{1}{\sqrt{1}}}{2 + \frac{1}{1}} = \frac{1}{3}$$

[∵ apply L'Hospital's Rule]

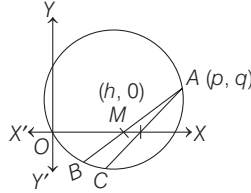
7 (d) Let AB be a chord of the circle through A(p, q) and M(h, 0) be the mid-point of AB. Therefore, the coordinates of B are (-p + 2h, -q).

Since, B lies on the circle $x^2 + y^2 = px + qy$, then

$$(-p+2h)^2 + (-q)^2 = p(-p+2h) + q(-q)$$

$$\Rightarrow 2p^2 + 2q^2 - 6ph + 4h^2 = 0$$

$$\Rightarrow 2h^2 - 3ph + (p^2 + q^2) = 0 \quad \dots(i)$$



As, there are two distinct chords AB and AC from A(p, q) which are bisected on X-axis there must be two distinct values of h satisfying Eq. (i), then $D = (b^2 - 4ac) > 0$, we have $(-3p)^2 - 4(2)(p^2 + q^2) > 0$
 $\Rightarrow p^2 > 8q^2$

8 (c) Since, $A^3 = B^3$ and $A^2 B = B^2 A$

$$\therefore A^3 - A^2 B = B^3 - B^2 A$$

$$\Rightarrow (A^2 + B^2)(A - B) = 0$$

$$\Rightarrow \det(A^2 + B^2) \det(A - B) = 0$$

9 (b) Here, $p = \frac{1}{2}$ and $q = \frac{1}{2}$

$$\therefore P(X=1) + P(X=3) + \dots + P(X=49)$$

$$= {}^{100}C_1 \left(\frac{1}{2} \right)^{100} + {}^{100}C_3 \left(\frac{1}{2} \right)^{100}$$

$$+ \dots + {}^{100}C_{49} \left(\frac{1}{2} \right)^{100}$$

$$= \left(\frac{1}{2} \right)^{100} ({}^{100}C_1 + {}^{100}C_3 + \dots + {}^{100}C_{49})$$

$$\left[\begin{array}{l} \therefore ({}^{100}C_1 + {}^{100}C_3 + \dots + {}^{100}C_{99}) = 2^{99} \\ \text{but } {}^{100}C_{99} = {}^{100}C_1 \\ {}^{100}C_{97} = {}^{100}C_3, \dots, {}^{100}C_{51} = {}^{100}C_{49} \\ \therefore 2({}^{100}C_1 + {}^{100}C_3 + \dots + {}^{100}C_{49}) = 2^{99} \end{array} \right]$$

$$= \left(\frac{1}{2} \right)^{100} \times 2^{98} = \frac{1}{4}$$

10 (c) Line is parallel to plane as

$$(\hat{i} - \hat{j} + 4\hat{k}) \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 0$$

General point on the line is $(\lambda + 2, -\lambda - 2, 4\lambda + 3)$. For $\lambda = 0$ point on this line is (2, -2, 3) and distance from $r \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 5$ or

$$x + 5y + z = 5, \text{ is}$$

$$d = \frac{|2 + 5(-2) + 3 - 5|}{\sqrt{1 + 25 + 1}}$$

$$\Rightarrow d = \frac{|-10|}{3\sqrt{3}} = \frac{10}{3\sqrt{3}}$$

11 (a) Negation of $p \vee (\sim p \vee q)$

$$\Rightarrow \sim [p \vee (\sim p \vee q)] \equiv \sim p \wedge \sim (\sim p \vee q)$$

$$\equiv \sim p \wedge (\sim (\sim p) \wedge \sim q)$$

$$\equiv \sim p \wedge (p \wedge \sim q)$$

$$\equiv (p \wedge \sim q) \wedge \sim p$$

12 (b) Given,

$$f(x) = \begin{cases} \frac{x-1}{2x^2-7x+5}, & x \neq 1 \\ -\frac{1}{3}, & x = 1 \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{2x-5}, & x \neq 1 \\ -\frac{1}{3}, & x = 1 \end{cases}$$

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{2(1+h)-5} - \left(-\frac{1}{3}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{2h-3} + \frac{1}{3}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3+2h-3}{3h(2h-3)} = -\frac{2}{9}$$

$$Lf'(1) = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{2(1-h)-5} - \left(-\frac{1}{3}\right)}{-h}$$

$$= \lim_{h \rightarrow 0} -\frac{2}{3(2h+3)} = -\frac{2}{9}$$

$$\therefore f'(1) = -\frac{2}{9}$$

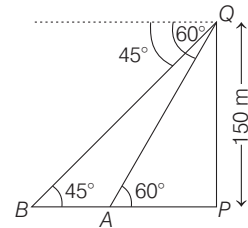
13 (a) $\tan \left\{ \cos^{-1} \left(-\frac{2}{7} \right) - \frac{\pi}{2} \right\}$

$$= \tan \left\{ \pi - \cos^{-1} \left(\frac{2}{7} \right) - \frac{\pi}{2} \right\}$$

$$= \tan \left\{ \sin^{-1} \left(\frac{2}{7} \right) \right\}$$

$$= \tan \left\{ \tan^{-1} \left(\frac{2}{3\sqrt{5}} \right) \right\} = \frac{2}{3\sqrt{5}}$$

14 (b) Let PQ = 150 m



$$\text{In } \triangle APQ, \tan 60^\circ = \frac{PQ}{AP}$$

$$\Rightarrow AP = \frac{150}{\sqrt{3}} \quad \dots(i)$$

and in $\triangle BPQ$,

$$\tan 45^\circ = \frac{PQ}{AB + AP}$$

$$\Rightarrow AB + \frac{150}{\sqrt{3}} = 150$$

$$\Rightarrow AB = \frac{150}{\sqrt{3}} (\sqrt{3} - 1)$$

$$\therefore \text{Speed of boat} = \frac{AB}{2}$$

$$= \frac{1}{2} \times \frac{150}{\sqrt{3}} (\sqrt{3} - 1) \times 60$$

$$= \frac{4500}{\sqrt{3}} (\sqrt{3} - 1) \text{ m/h}$$

15 (b) Given equation of planes are

$$4x + 4y - 5z = 12 \quad \dots(i)$$

$$\text{and } 8x + 12y - 13z = 32 \quad \dots(ii)$$

Let DR's of required line be (l, m, n) .

From Eqs. (i) and (ii), we get

$$4l + 4m - 5n = 0$$

$$\text{and } 8l + 12m - 13n = 0$$

$$\Rightarrow \frac{l}{8} = \frac{m}{12} = \frac{n}{16}$$

$$\Rightarrow \frac{l}{2} = \frac{m}{3} = \frac{n}{4}$$

Now, we take intersection point with $z = 0$ is given by

$$4x + 4y = 12 \quad \dots(iii)$$

$$\text{and } 8x + 12y = 32 \quad \dots(iv)$$

On solving Eqs. (i) and (ii), we get the point $(1, 2, 0)$.

$$\therefore \text{Required line is } \frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}$$

16 (b) Let $\Delta' = \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$

$$= abc + 2 - a - b - c > 0$$

$$\text{or } abc + 2 > a + b + c \quad \dots(i)$$

$$\therefore AM > GM \Rightarrow \frac{a+b+c}{3} > (abc)^{1/3}$$

$$a + b + c > 3(abc)^{1/3} \quad \dots(ii)$$

From Eqs. (i) and (ii),

$$abc + 2 > 3(abc)^{1/3}$$

$$\text{Let } (abc)^{1/3} = x$$

$$\text{Then, } x^3 + 2 > 3x$$

$$\Rightarrow (x-1)^2(x+2) > 0$$

$$\therefore x+2 > 0 \Rightarrow x > -2$$

$$\Rightarrow x^3 > -8 \Rightarrow abc > -8$$

17 (c) Here, $P(W) = \frac{a}{a+b}$

$$\text{and } P(B) = \frac{b}{a+b}$$

\therefore Probability of A winning

$$= P(W) + P(\bar{W})P(\bar{B})P(W) + \dots$$

$$= \frac{P(W)}{1 - P(\bar{W})P(\bar{B})}$$

$$= \frac{\frac{a}{a+b}}{1 - \frac{b}{a+b} \cdot \frac{a}{a+b}}$$

$$= \frac{a(a+b)}{a^2 + b^2 + ab} = P_1 \quad [\text{say}]$$

and probability of B winning

$$= 1 - P_1 = 1 - \frac{a^2 + ab}{a^2 + b^2 + ab}$$

$$= \frac{b^2}{a^2 + b^2 + ab} = P_2 \quad [\text{say}]$$

Given, $P_1 = 2P_2$

$$\Rightarrow \frac{a^2 + ab}{a^2 + b^2 + ab} = \frac{2b^2}{a^2 + b^2 + ab}$$

$$\Rightarrow a^2 + ab - 2b^2 = 0$$

$$\Rightarrow (a-b)(a+2b) = 0$$

$$\Rightarrow a - b = 0 \quad [\because a + 2b \neq 0]$$

$$\Rightarrow a = b$$

$$\therefore a : b = 1 : 1$$

18 (d) $I + f + f'$

$$= (5 + 2\sqrt{6})^n + (5 - 2\sqrt{6})^n$$

$$= 2k \quad [\text{even integer}]$$

$$\therefore f + f' = 1$$

Now,

$$(I + f)f' = (5 + 2\sqrt{6})^n (5 - 2\sqrt{6})^n$$

$$= (1)^n = 1$$

$$\Rightarrow (I + f)(1 - f) = 1$$

$$\Rightarrow I = \frac{1}{1-f} - f$$

19 (d) Since, $n = 11$, then median term

$$= \left(\frac{11+1}{2} \right) \text{th term} = 6 \text{th term}$$

As, last five observations are increased by 4. Hence, the median of the 6th observations will remain same.

20 (d) We have, $\frac{\sqrt{x} dx + \sqrt{y} dy}{\sqrt{x} dx - \sqrt{y} dy} = \frac{\sqrt{y^3}}{\sqrt{x^3}}$

$$\Rightarrow \frac{d(x^{3/2}) + d(y^{3/2})}{d(x^{3/2}) - d(y^{3/2})} = \frac{y^{3/2}}{x^{3/2}}$$

$$\Rightarrow \frac{du + dv}{du - dv} = \frac{v}{u}$$

where $u = x^{3/2}$ and $v = y^{3/2}$

$$\Rightarrow \frac{u du + v dv}{u du + v dv} = \frac{v du - u dv}{v du - u dv}$$

$$\Rightarrow \frac{u^2 + v^2}{u^2 + v^2} = \frac{v du - u dv}{v du - u dv}$$

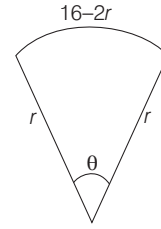
$$\Rightarrow \frac{d(u^2 + v^2)}{u^2 + v^2} = -2d \tan^{-1} \left(\frac{v}{u} \right) + c$$

On integrating, we get

$$\log(u^2 + v^2) = -2 \tan^{-1} \left(\frac{v}{u} \right) + c$$

$$\Rightarrow \frac{1}{2} \log(x^3 + y^3) + \tan^{-1} \left(\frac{y}{x} \right)^{3/2} = \frac{c}{2}$$

21 (4) Let r be the radius of the sector and angle subtended at the centre be θ .



Then, S = surface area of sector

$$= \frac{\theta}{360} \times \pi r^2$$

We know, $\theta = \frac{\text{length of arc}}{\text{radius}}$

$$= \frac{16-2r}{r}$$

$$\therefore S = \frac{\theta}{2\pi} \pi r^2 = \frac{16-2r}{2r} \cdot r^2$$

$$\Rightarrow S = (8-r) \cdot r = 8r - r^2$$

$$\Rightarrow \frac{dS}{dr} = 8 - 2r$$

Now, for area to be maximum,

$$\frac{dS}{dr} = 0$$

$$\Rightarrow r = 4$$

22 (4) Extremities of the latusrectum of the parabola are $(2, 4)$ and $(2, -4)$.

Since, any circle drawn with any focal chord at its diameter touches the directrix, thus equation of required circle is

$$(x-2)^2 + (y-4)(y+4) = 0$$

$$\Rightarrow x^2 + y^2 - 4x - 12 = 0$$

$$\therefore \text{Radius} = \sqrt{(2)^2 + 12} = 4$$

23 (11) We have the following cases.

Case I

a			
---	--	--	--

In this case we have only two

possibilities, namely, **a c b d** and **a d c b**

Case II

b			
---	--	--	--

In this case we have only three

possibilities, namely, **b a d c**, **b d a c** and **b d c a**

Case III

c			
---	--	--	--

In this case we have only three

possibilities, namely, **c a d b**, **c b d a** and **c b a d**

Case IV

d			
---	--	--	--

In this case we have only three possibilities, namely, **d c b a**, **d a c b** and **d b a c**

Hence, the total number of ways
 $= 2 + 3 + 3 + 3 = 11$

24 (17) Given that, $y = f\left(\frac{2x+3}{3-2x}\right)$

$$\Rightarrow \frac{dy}{dx} = f'\left(\frac{2x+3}{3-2x}\right) \frac{d}{dx}\left(\frac{2x+3}{3-2x}\right)$$

$$= \sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]$$

$$\left[\frac{(3-2x)(2)-(2x+3)(-2)}{(3-2x)^2}\right]$$

$$= \frac{12}{(3-2x)^2} \sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x=1)} = \frac{12}{(3-2)^2} \sin \log(5)$$

$$= 12 \sin \log(5)$$

Here, $p = 12$, $q = 5$
 $\therefore p + q = 17$

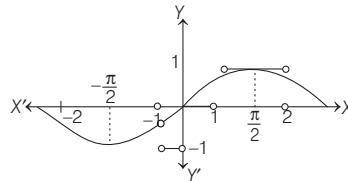
25 (64) Clearly, $\frac{x+y}{2} \geq \sqrt{xy}$; $\frac{y+z}{2} \geq \sqrt{yz}$
and $\frac{x+z}{2} \geq \sqrt{xz}$
 $\therefore \frac{(x+y)}{2} \cdot \frac{(y+z)}{2} \cdot \frac{(x+z)}{2} \geq \sqrt{xy} \cdot \sqrt{yz} \cdot \sqrt{xz}$
 $\Rightarrow (1-z)(1-x)(1-y) \geq 8xyz$
 $[\because x + y + z = 1]$

$\therefore k = 8$
Hence, $k^2 = 64$

26 (256) $f(x) = f(1 + 1 + 1 + \dots + x$
times)
 $= f(1)f(1)f(1) \dots x \text{ times}$
 $= [f(1)]^x = 3^x$
 $\therefore \sum_{x=1}^n f(x) = \sum_{x=1}^n 3^x = 3^1 + 3^2 + \dots + 3^n$
 $= \frac{3^1 - 3^{n+1}}{1-3}$
 $= \frac{3^{n+1} - 3}{2} \left[\because \text{sum} = \frac{a - lr}{1-r} \right]$
 $\therefore \frac{3^{n+1} - 3}{2} = 120 \Rightarrow 3^{n+1} = 243 = 3^5$
 $\Rightarrow n + 1 = 5$
 $\Rightarrow n = 4$
 $\therefore n^4 = 256$

27 (2) $y = \sin x = [x]$

Graphs of $y = \sin x$ and $y = [x]$ are as shown.



Hence, two solutions are $x = 0$ and
 $x = \frac{\pi}{2}$

28 (154) Let $I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$

$$= \int_0^1 x^4 \left(x^2 - 4x + 5 - \frac{4}{1+x^2} \right) dx$$

$$= \int_0^1 (x^6 - 4x^5 + 5x^4) dx - 4 \int_0^1 \frac{x^4}{(1+x^2)} dx$$

$$= \int_0^1 (x^6 - 4x^5 + 5x^4) dx - 4 \int_0^1 \left(x^2 - 1 + \frac{1}{1+x^2} \right) dx$$

$$= \left(\frac{x^7}{7} - \frac{4x^6}{6} + x^5 \right)_0^1 - 4 \left(\frac{x^3}{3} - x + \tan^{-1} x \right)_0^1$$

$$= \left(\frac{1}{7} - \frac{4}{6} + 1 \right) - 4 \left(\frac{1}{3} - 1 + \frac{\pi}{4} \right)$$

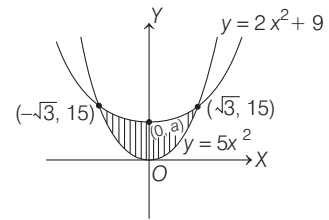
$$= \frac{22}{7} - \pi$$

$$m = 22, n = 7$$

Hence, $m \times n = 22 \times 7 = 154$

29 (12) Given parabolas are
 $5x^2 - y = 0$
and $2x^2 - y + 9 = 0$
Now, eliminating y from above equations, we get
 $5x^2 - (2x^2 + 9) = 0$
 $\Rightarrow 3x^2 = 9 \Rightarrow x = \pm \sqrt{3}$

Given parabolas intersect at $(\sqrt{3}, 15)$
and $(-\sqrt{3}, 15)$.



The two parabolas are

$$x^2 = \frac{1}{5}y, x^2 = \frac{1}{2}(y-9)$$

$$\therefore \text{Area} = 2 \int_0^{\sqrt{3}} (y_1 - y_2) dx$$

$$= 2 \int_0^{\sqrt{3}} [(2x^2 + 9) - 5x^2] dx$$

$$= 2 \int_0^{\sqrt{3}} (9 - 3x^2) dx$$

$$= 2[9x - x^3]_0^{\sqrt{3}}$$

$$\therefore \text{Area} = 12\sqrt{3}$$

$$\therefore p = 12$$

30 (1817) We must have
 $\alpha(\hat{i} - 3\hat{j} + 5\hat{k}) = \hat{a} + \frac{2\hat{k} + 2\hat{j} - \hat{i}}{3}$
 $\Rightarrow 3\hat{a} = \alpha(3\hat{i} - 9\hat{j} + 15\hat{k}) - 2\hat{k} - 2\hat{j} + \hat{i}$
 $\Rightarrow 3\hat{a} = (3\alpha + 1)\hat{i} - (9\alpha + 2)\hat{j} + (15\alpha - 2)\hat{k}$
 $\therefore 3|\hat{a}| = |(3\alpha + 1)\hat{i} - (9\alpha + 2)\hat{j} + (15\alpha - 2)\hat{k}|$
 $\Rightarrow 3 = \sqrt{(3\alpha + 1)^2 + (9\alpha + 2)^2 + (15\alpha - 2)^2}$
 $\Rightarrow 9 = 135\alpha^2 - 18\alpha + 9$
 $\therefore \alpha = 0, \frac{2}{35}$
For $\alpha = 0, \hat{a} = \frac{\hat{i} - 2\hat{j} - 2\hat{k}}{3}$ (not acceptable)
For $\alpha = \frac{2}{35}, \hat{a} = \frac{2}{35}(\hat{i} - 3\hat{j} + 5\hat{k}) - \frac{(2\hat{k} + 2\hat{j} - \hat{i})}{3}$
 $= \frac{1}{105}(41\hat{i} - 88\hat{j} - 40\hat{k})$
On comparing, we get
 $\lambda = 41, \mu = -88, v = -40$
 $\therefore \lambda^2 - 2\mu + v = (41)^2 + 2(88) - 40$
 $= 1681 + 176 - 40 = 1817$