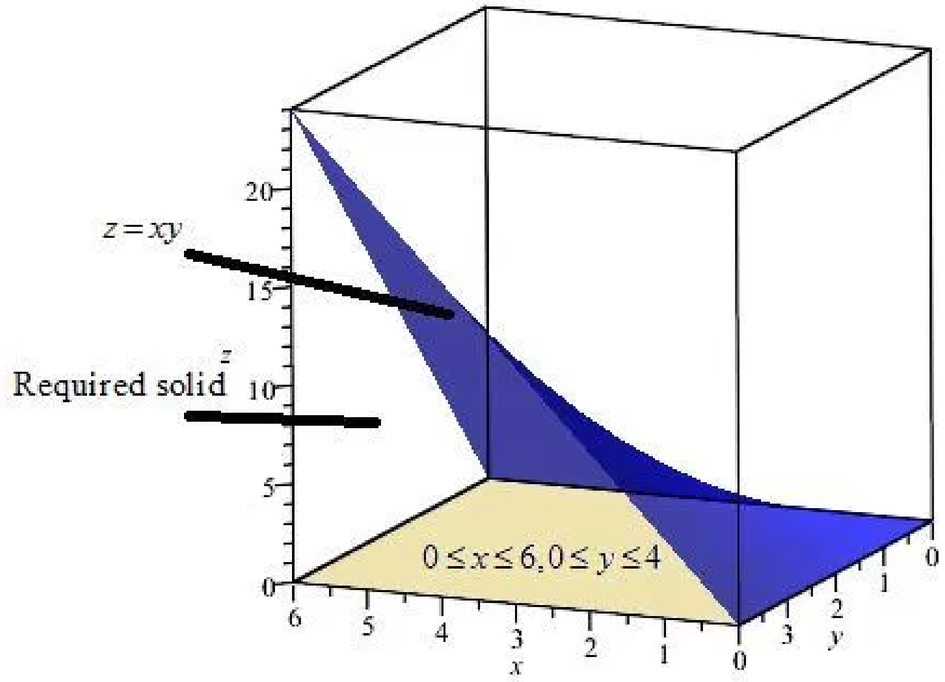
Exercise 15.1

Chapter 15 Multiple Integrals 15.1 1E

(a)

Consider the surface $z = xy$.

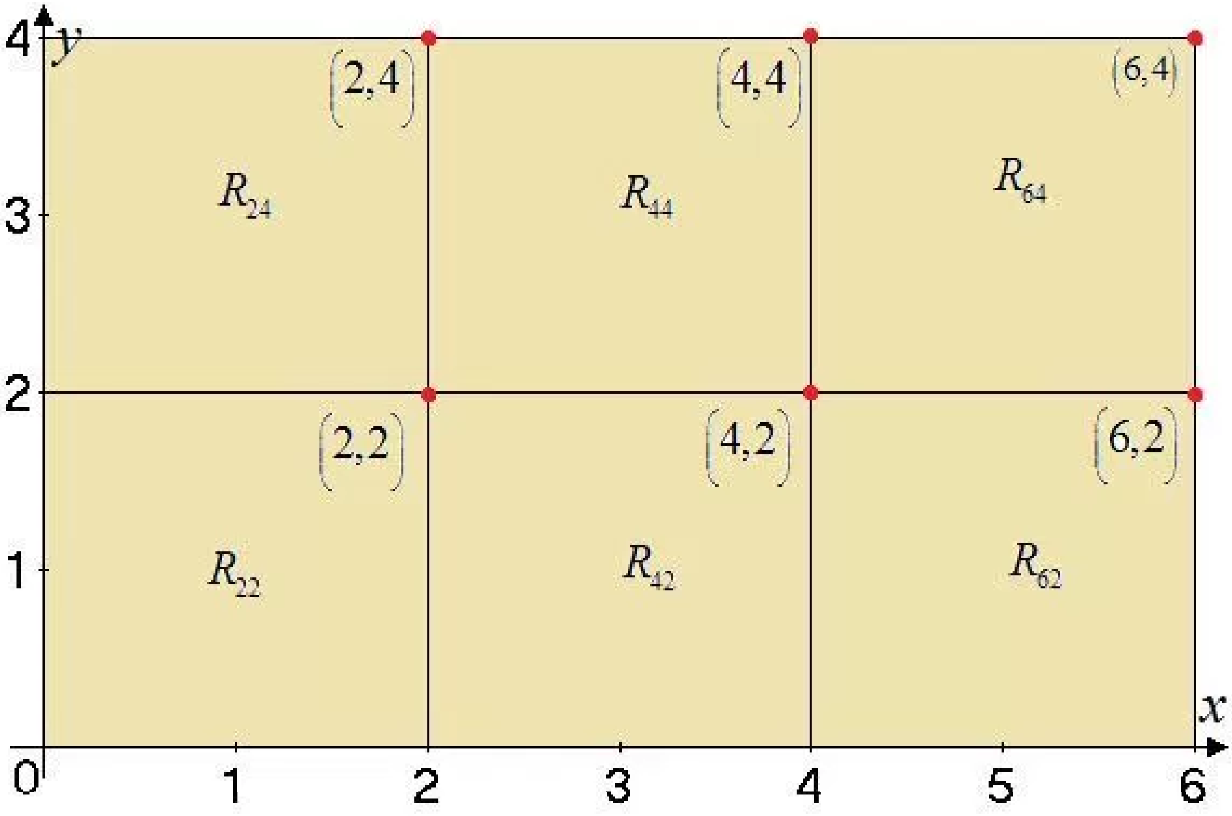
The solid that lies below the surface and above the rectangle $R = \{(x, y) \mid 0 \le x \le 6, 0 \le y \le 4\}$ is shown below:



Need to find the volume of the solid using Riemann sums.

Take $m=3, n=2.$

The squares are shown in the below figure:



Area of the square with side length 2 is $2^2 = 4.$

The upper right corner of each square is $(2,2),(2,4),(4,2),(4,4),(6,2),(6,4).$

And the function is $z = f(x,y) = xy.$

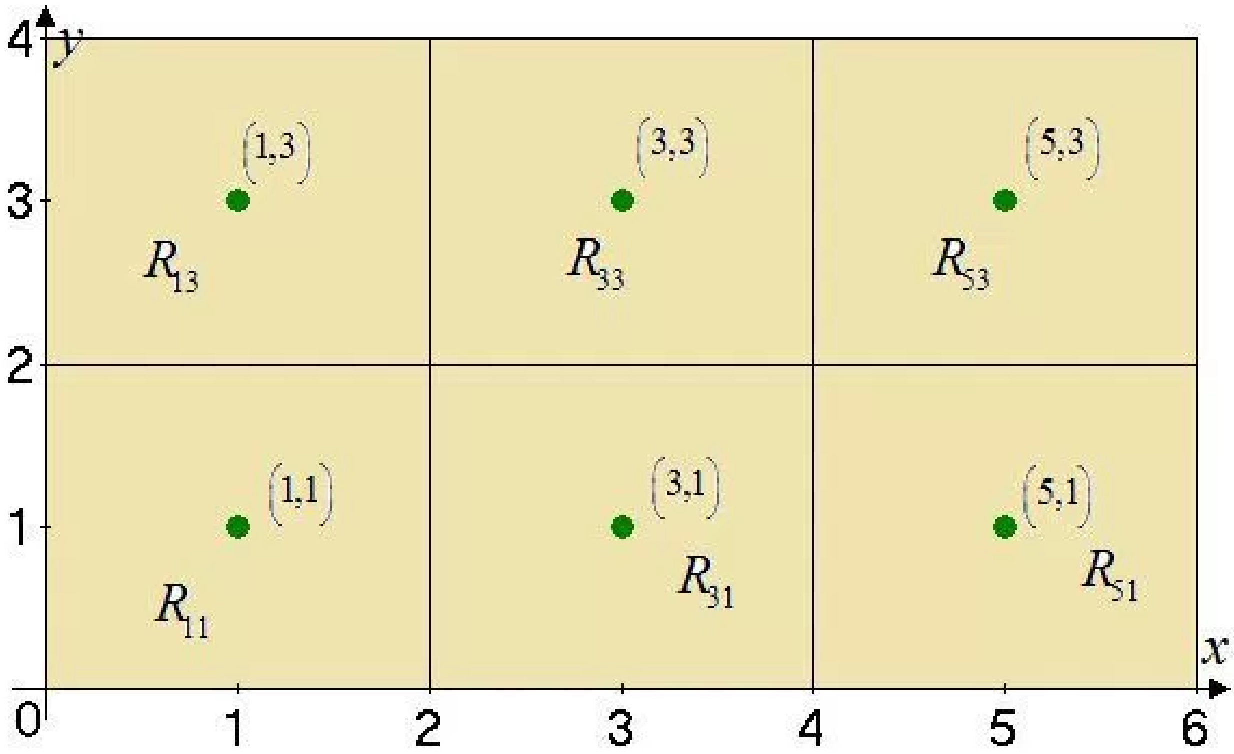
The volume of the solid using Riemann sum with $m=3, n=2$ is

$$
\begin{aligned}
V &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\Delta A \\
&= f(2,2)\Delta A + f(2,4)\Delta A + f(4,2)\Delta A + f(4,4)\Delta A + f(6,2)\Delta A + f(6,4)\Delta A \\
&= (2)(2)(4)+(2)(4)(4)+(4)(2)(4)+(4)(4)(4)+(6)(2)(4)+(6)(4)(4) \\
&= 16+32+32+64+48+96 \\
&= 288
\end{aligned}
$$

Therefore the required volume of the solid using Riemann sum is $\boxed{288}$.

(b)

Midpoints of the six squares are shown in the below figure:



Area of the square with side length 2 is $2^2 = 4.$

The Midpoints of each square is $(1,1),(3,1),(5,1),(1,3),(3,3),(5,3).$

And the function is $z = f(x,y) = xy.$

The volume of the solid using Riemann sum with $m = 3, n = 2$ is

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\overline{x_i}, \overline{y_j}\right)\Delta A$$

$$= f(1,1)\Delta A + f(1,3)\Delta A + f(3,1)\Delta A + f(3,3)\Delta A + f(5,1)\Delta A + f(5,3)\Delta A$$

$$= (1)(1)(4) + (1)(3)(4) + (3)(1)(4) + (3)(3)(4) + (5)(1)(4) + (5)(3)(4)$$

$$= 4 + 12 + 12 + 36 + 20 + 60$$

$$= 144$$

Therefore the required volume of the solid using Riemann sum is $\boxed{144}$.

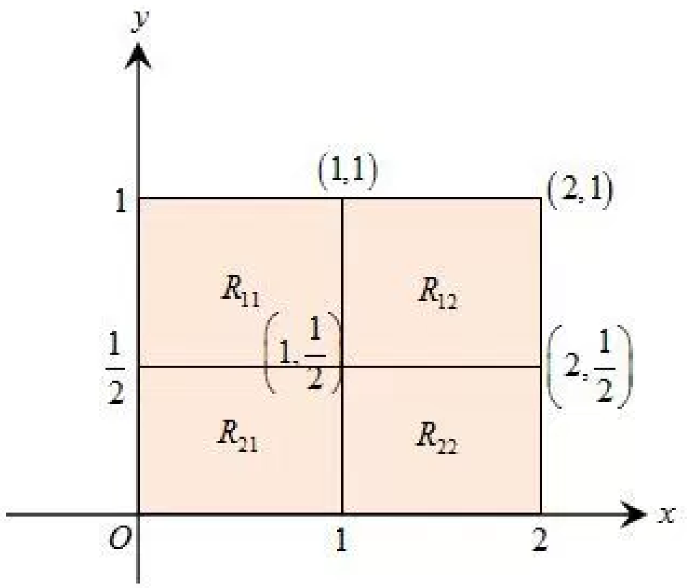
Chapter 15 Multiple Integrals 15.1 3E

Consider the surface $z = xe^{-xy}$.

The objective is to estimate the value of the integral $\iint_R xe^{-xy}\, dA$ using Riemann sum with $m = n = 2$.

Where $R = [0,2]\times[0,1]$.

Sketch the rectangular region R in xy - plane, as $m = n = 2$ divide the region R into 4 smaller equal rectangles, label the right upper corner points and then find the area of the each rectangle.



As it is observed from the above diagram that, length of the each rectangle is 1 and width is $\frac{1}{2}$ so the area of the each rectangle ΔA is $\frac{1}{2}$.

Write the Riemann sum formula to estimate the value of the integral.

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_i, y_j\right)\Delta A \qquad \text{......(1)}$$

Write the data observed from the above discussion.

$$\left.\begin{aligned} f(x,y) &= xe^{-xy} \\ m = n &= 2 \\ \Delta A &= \frac{1}{2} \end{aligned}\right\} \qquad \text{......(2)}$$

Write the upper right corner points of each rectangle (observe the diagram the above).

$$\left.\begin{aligned} \left(x_1, y_1\right) &= (1,1) \\ \left(x_2, y_2\right) &= (2,1) \\ \left(x_3, y_3\right) &= \left(1, \frac{1}{2}\right) \\ \left(x_4, y_4\right) &= \left(2, \frac{1}{2}\right) \end{aligned}\right\} \qquad \text{......(3)}$$

Use (2) and (3) in (1) and rewrite it.

$$\begin{aligned} \iint_R xe^{-xy}\,dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(x_i, y_j\right)\Delta A \\ &= f(1,1)\Delta A + f(2,1)\Delta A + f\left(1, \frac{1}{2}\right)\Delta A + f\left(2, \frac{1}{2}\right)\Delta A \\ &= (1)e^{(-1)(1)}\left(\frac{1}{2}\right) + (2)e^{(-2)(1)}\left(\frac{1}{2}\right) + (1)e^{(-1)\left(\frac{1}{2}\right)}\left(\frac{1}{2}\right) + (2)e^{(-2)\left(\frac{1}{2}\right)}\left(\frac{1}{2}\right) \end{aligned}$$

$$= \frac{1}{2}\left(e^{-1} + 2e^{-2} + e^{-\frac{1}{2}} + 2e^{-1}\right)$$

$$= \frac{1}{2}\left(3e^{-1} + 2e^{-2} + e^{-\frac{1}{2}}\right)$$

≈ 0.99041 Use calculator

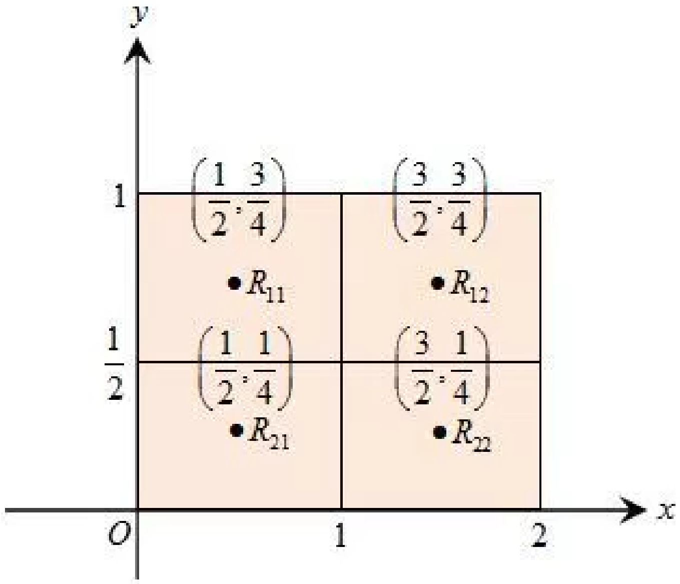
Therefore, the required volume of the solid using Riemann sum is $\boxed{0.99041}$.

(b).

Using midpoint rule:

Divide the region R into four equal squares and choose the sample point to the midpoint of each square.

The Midpoints are shown in the below figure.



As it is observed from the above diagram that, length of the each rectangle is 1 and width is $\frac{1}{2}$ x

so the area of the each rectangle ΔA is $\frac{1}{2}$.

Write the Riemann sum formula to estimate the value of the integral.

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\overline{x_i},\overline{y_j}\right)\Delta A \qquad ......(4)$$

Where $\overline{x_i}$ is the midpoint of $[x_{i-1},x_i]$ and $\overline{y_i}$ is the midpoint of $[y_{i-1},y_i]$.

Write the data observed from the discussion.

$$\left.\begin{aligned} f(x,y) &= xe^{-xy} \\ m = n &= 2 \\ \Delta A &= \frac{1}{2} \end{aligned}\right\} \qquad ......(5)$$

Write the midpoints. (Observe the diagram the above).

$$\left.\begin{aligned} \left(\overline{x_1},\overline{y_1}\right) &= \left(\frac{1}{2},\frac{3}{4}\right) \\ \left(\overline{x_2},\overline{y_2}\right) &= \left(\frac{3}{2},\frac{3}{4}\right) \\ \left(\overline{x_3},\overline{y_3}\right) &= \left(\frac{1}{2},\frac{1}{4}\right) \\ \left(\overline{x_4},\overline{y_4}\right) &= \left(\frac{3}{2},\frac{1}{4}\right) \end{aligned}\right\} \qquad ......(6)$$

Use (5) and (6) in (4) and rewrite it.

$$\begin{aligned} \iint_R xe^{-xy}\,dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\overline{x_i},\overline{y_j}\right)\Delta A \\ &= f\left(\frac{1}{2},\frac{3}{4}\right)\Delta A + f\left(\frac{3}{2},\frac{3}{4}\right)\Delta A + f\left(\frac{1}{2},\frac{1}{4}\right)\Delta A + f\left(\frac{3}{2},\frac{1}{4}\right)\Delta A \\ &= \left\{\begin{aligned} &\left(\frac{1}{2}\right)e^{\left(-\frac{1}{2}\right)\left(\frac{3}{4}\right)}\left(\frac{1}{2}\right) + \left(\frac{3}{2}\right)e^{\left(-\frac{3}{2}\right)\left(\frac{3}{4}\right)}\left(\frac{1}{2}\right) \\ &+ \left(\frac{1}{2}\right)e^{\left(-\frac{1}{2}\right)\left(\frac{1}{4}\right)}\left(\frac{1}{2}\right) + \left(\frac{3}{2}\right)e^{\left(-\frac{3}{2}\right)\left(\frac{1}{4}\right)}\left(\frac{1}{2}\right) \end{aligned}\right. \end{aligned}$$

$$= \frac{1}{2}\left(\frac{e^{-\frac{3}{8}}}{2} + \frac{3e^{-\frac{9}{8}}}{2} + \frac{e^{-\frac{1}{8}}}{2} + \frac{3e^{-\frac{3}{8}}}{2}\right)$$

$$\approx 1.1514 \qquad \text{Use calculator}$$

Hence $\iint_R xe^{-xy}\,dA \approx \boxed{1.1514}$ (Using Midpoint Rule).

Chapter 15 Multiple Integrals 15.1 4E

(a)

The objective is to evaluate $z = f(x,y) = 1 + x^2 + 3y$ at the lower left corners of four rectangles shown in the figure.

Here, $\Delta x = 0.5$, $\Delta y = 1.5$ and $\Delta A = \frac{3}{4}$.

If $f(x, y) \geq 0$, then the volume of the solid that lies above the rectangle *R* and below the surface $z = f(x,y)$ is $V = \iint_R f(x, y)\,dA$.

Then,

$$V = \iint_R (1 + x^2 + 3y)\,dA.$$

From the figure, we get the sample points at the lower left corners as,

$(1,0),(1.5,0),(1,1.5)$ and $(1.5,1.5)$.

Then, the volume of the solid is given by

$$V = \frac{3}{4}\left\{\left[1+1^2+3(0)\right]+\left[1+(1.5)^2+3(0)\right]+\left[1+1^2+3(1.5)\right]+\left[1+(1.5)^2+3(1.5)\right]\right\}$$

$$V = \frac{3}{4}(2 + 3.25 + 6.5 + 7.75)$$

$$= \frac{3}{4}(19.5)$$

$$= 14.625$$

Therefore, the volume of the solid is given by $\boxed{14.625}$.

(b)

Approximating the volume by using Midpoint Rule with $m = n = 2$ we evaluate $f(x,y) = 1 + x^2 + 3y$ at the centers of four rectangles shown in the figure.

So, $x_1 = 1.25$, $x_2 = 1.75$, $y_1 = 0.75$, $y_2 = 2.25$.

The area of each sub rectangle $\Delta A = 0.75$.

From the figure shown in part (a), we have the mid points of the rectangle as

$(1.25,0.75),(1.75,0.75),(1.25,2.25)$ and $(1.75,2.25)$.

Then, the volume of the solid is given by

$$V = \frac{3}{4}\left\{\begin{array}{l}\left[1+(1.25)^2+3(0.75)\right]+\left[1+(1.75)^2+3(0.75)\right]+\left[1+(1.25)^2+3(2.25)\right] \\ \quad +\left[1+(1.75)^2+3(2.25)\right]\end{array}\right\}$$

$$\begin{aligned} V &= \frac{3}{4}(4.8125+6.3125+9.3125+10.8125) \\ &= \frac{3}{4}(31.25) \\ &= 23.4375 \end{aligned}$$

Therefore, the volume of the solid is given by $\boxed{23.4375}$.

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A swimming pool dimensions are 20×30 feet.

Consider that the depth is measured from one corner of the pool at 5 feet intervals.

The values are recorded in the table as follows:

	0	5	10	15	20	25	30
0	2	3	4	6	7	8	8
5	2	3	4	7	8	10	8
10	2	4	6	8	10	12	10
15	2	3	4	5	6	8	7
20	2	2	2	2	3	4	4

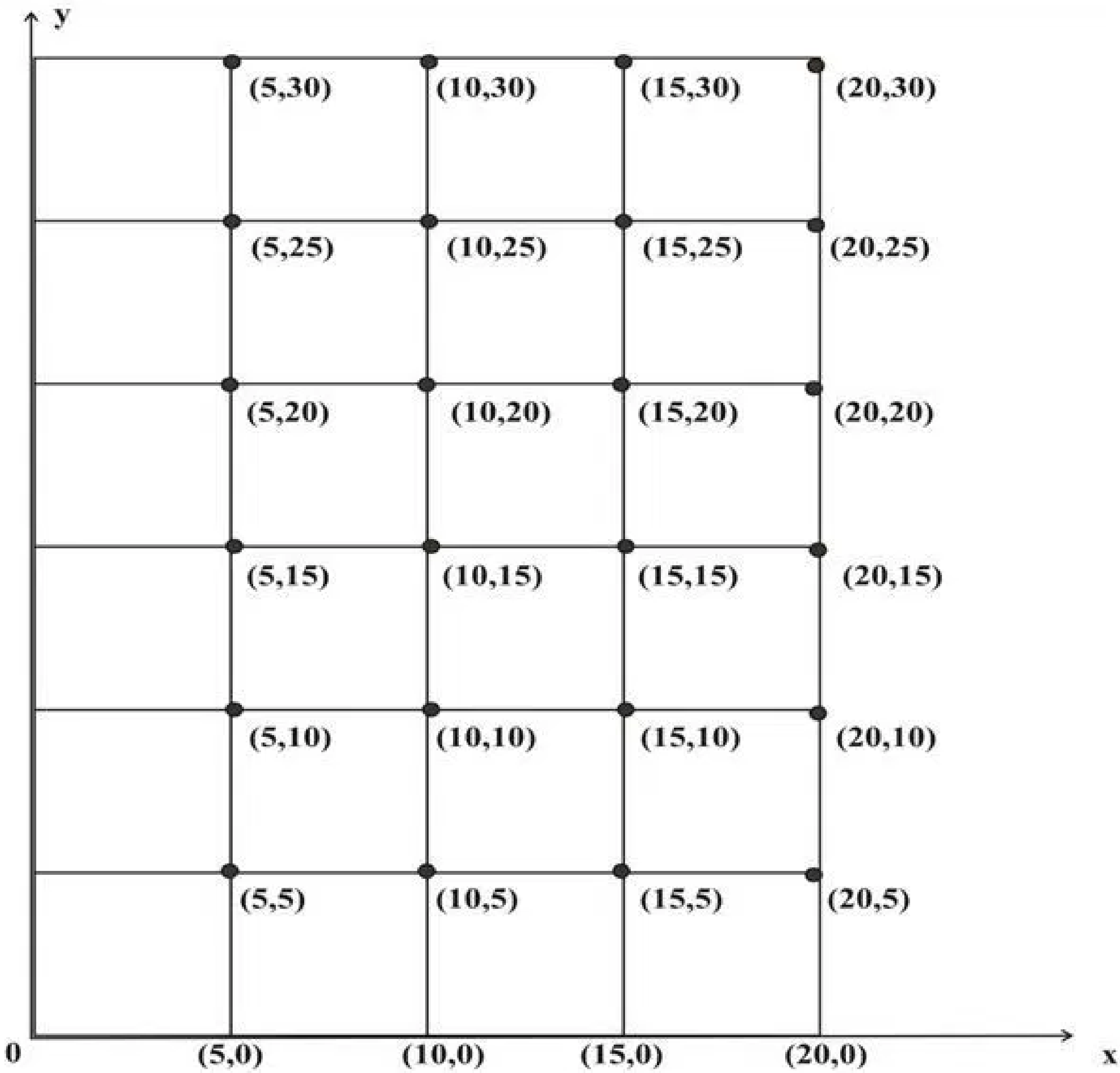
Objective is to estimate the volume of water in the pool.

$$R=[0,20]\times[0,30]$$

Then, the volume of the swimming pool is as follows:

$$V \approx \sum_{i=1}^{20}\sum_{j=1}^{30} f(x_i, y_i)\Delta A.$$

Approximate the volume by Riemann sum with $m = n = 5$.



$$\Delta x = \frac{20-0}{4}$$
$$= 5$$

$$\Delta y = \frac{30-0}{6}$$
$$= 5$$

Then, the area of each rectangle is as follows:

$$\Delta A = \Delta x \times \Delta y$$

$$= 5 \times 5$$

$$= 25$$

The volume of the swimming pool is as follows:

$$V \approx \sum_{i=1}^{20}\sum_{j=1}^{30} f(x_i, y_i)\Delta A$$

$$= f(5,5)\Delta A + f(10,5)\Delta A + f(15,5)\Delta A + f(20,5)\Delta A$$

$$+ f(5,10)\Delta A + f(10,10)\Delta A + f(15,10)\Delta A + f(20,10)\Delta A$$

$$+ f(5,15)\Delta A + f(10,15)\Delta A + f(15,15)\Delta A + f(20,15)\Delta A$$

$$+ f(5,20)\Delta A + f(10,20)\Delta A + f(15,20)\Delta A + f(20,20)\Delta A$$

$$+ f(5,25)\Delta A + f(10,25)\Delta A + f(15,25)\Delta A + f(20,25)\Delta A$$

$$+ f(5,30)\Delta A + f(10,30)\Delta A + f(15,30)\Delta A + f(20,25)\Delta A$$

$$= (3+4+3+2+4+6+4+2+7+8+5+2+8+10+6+3+10$$
$$+12+8+4+8+10+7+4)25$$

$$= 140 \times 25$$

$$= 3500$$

Hence, the volume of the swimming pool is $V = \boxed{3500 \text{ cubic ft.}}$.

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Consider the function,

$$f(x,y) = \sqrt{52 - x^2 - y^2}$$

And the rectangle $2 \le x \le 4, 2 \le y \le 6.$

The objective is to arrange *V*, *L*, and *U* in increasing order without calculating them.

Recall the mid-point rule for double integrals,

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\overline{x_i}, \overline{y_j}\right)\Delta A$$

where $\overline{x_i}$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y_j}$ is the midpoint of $[y_{j-1}, y_j]$.

Use midpoint rule with $m = n = 2$ to evaluate the integral.

From the table, R is a rectangle defined as $R = [2,4] \times [2,6]$.

Divide this rectangle in to sub rectangles $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$. To divide this find $\Delta x, \Delta y$ where $\Delta x = \frac{b-a}{m}, \Delta y = \frac{d-c}{n}$.

As R is of the form $[a,b] \times [c,d]$, $a = 2, b = 4, c = 2, d = 6$

$$\Delta x = \frac{b-a}{m} \quad \Delta y = \frac{d-c}{n}$$
$$= \frac{4-2}{2} \qquad = \frac{6-2}{2}$$
$$= 1 \qquad\qquad = 2$$

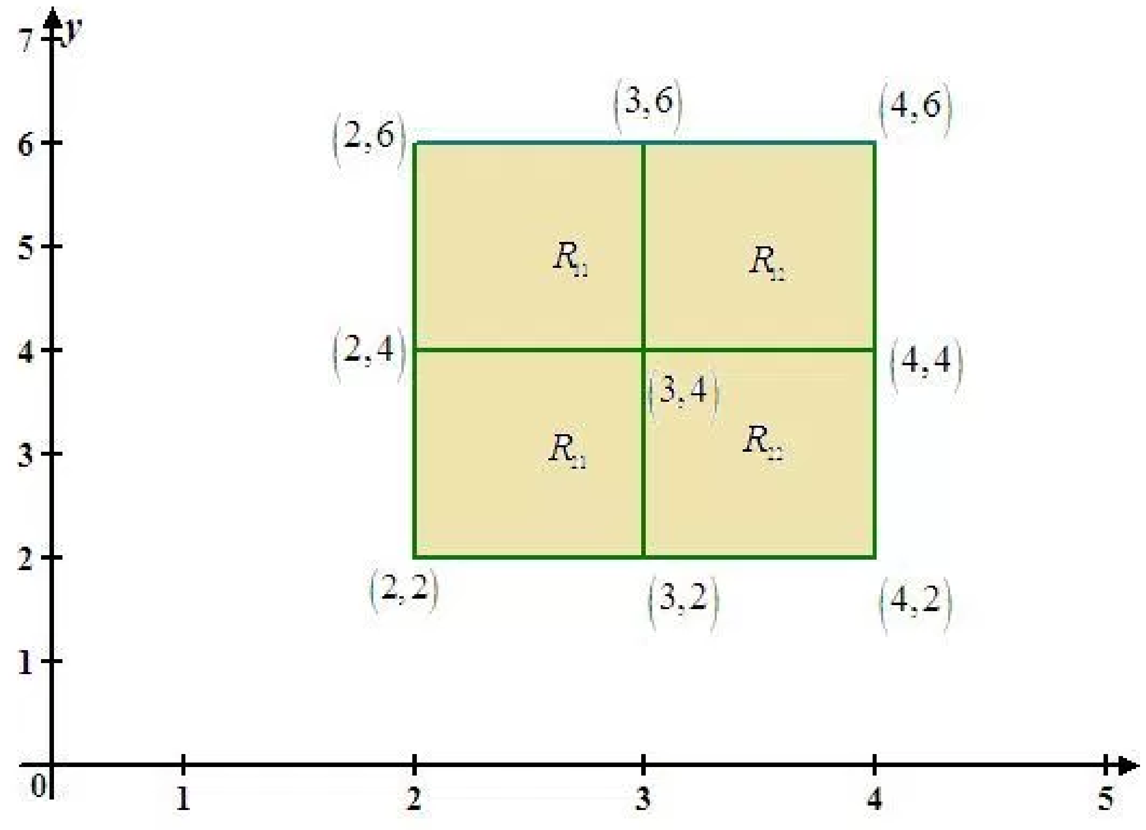
The lines $x = 3, y = 4$ divide the rectangle into four sub rectangles.

The line $x = 3$ divides the interval $[2,4]$ to $[2,3],[3,4]$.

The line $y = 4$ divides the interval $[2,6]$ to $[2,4],[4,6]$.

Name them as R_{11}, R_{12}, R_{21} and R_{22}

Sketch the figure representing the sub rectangles.



The centers of the intervals $[2,3],[3,4]$

$$\bar{x}_1 = \frac{2+3}{2} = \frac{5}{2}, \quad \bar{x}_2 = \frac{3+4}{2} = \frac{7}{2}$$

The centers of the intervals $[2,4],[4,6]$

$$\bar{y}_1 = \frac{2+4}{2} = 3, \quad \bar{y}_2 = \frac{4+6}{2} = 5$$

The area of each sub rectangle,

$$\begin{aligned}\Delta A &= \Delta x \Delta y \\ &= 1 \cdot 2 \\ &= 2\end{aligned}$$

Thus, the volume of the solid is,

$$V = f(\bar{x}_1, \bar{y}_1)\Delta A + f(\bar{x}_1, \bar{y}_2)\Delta A + f(\bar{x}_2, \bar{y}_1)\Delta A + f(\bar{x}_2, \bar{y}_2)\Delta A$$

$$= f(2.5, 3)\Delta A + f(3.5, 3)\Delta A + f(2.5, 5)\Delta A + f(3.5,5)\Delta A$$

$$= [f(2.5, 3) + f(3.5, 3) + f(2.5, 5) + f(3.5, 5)] \cdot 2$$ Substitute $\Delta A = 2$

From the figure, the lower left corners are $(2,2),(3,2),(2,4)$ and $(3,4)$ and the upper right corners are $(3,4),(4,4),(3,6),$ and $(4,6)$

The Riemann sum using lower left corners of the sub rectangles is,

$$L = f(2,2)\Delta A + f(3, 2)\Delta A + f(2,4)\Delta A + f(3,4)\Delta A$$

$$= f(2,2)\cdot 2 + f(3, 2)\cdot 2 + f(2,4)\cdot 2 + f(3,4)\cdot 2$$ Substitute $\Delta A = 2$

$$= [f(2,2) + f(3,2) + f(2,4) + f(3,4)]\cdot 2$$ Plug out 2

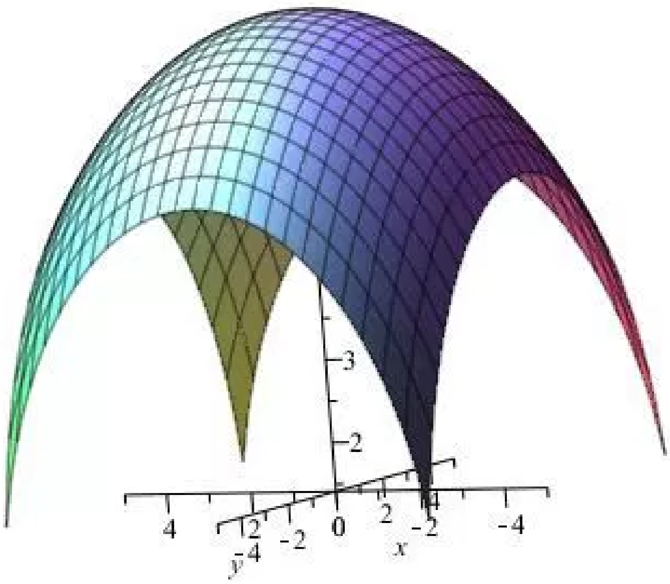
And the Riemann sum using upper right corners is,

$$U = f(3,4)\Delta A + f(4,4)\Delta A + f(3,6)\Delta A + f(4,6)\Delta A$$

$$= f(3,4)\cdot 2 + f(4,4)\cdot 2 + f(3,6)\cdot 2 + f(4,6)\cdot 2$$ Substitute $\Delta A = 2$

$$= [f(3,4) + f(4,4) + f(3,6) + f(4,6)]\cdot 2$$

Sketch the graph of the function $f(x,y)=\sqrt{52-x^2-y^2}$ as shown below:



Clearly *f* is a decreasing function within the rectangle.

In the sub rectangle R_{11} ,

$$f(2,4)>f(2.5,5)>f(3,6)$$

In the sub rectangle R_{12},

$$f(3,4)>f(3.5,5)>f(4,6)$$

In the sub rectangle R_{21},

$$f(2,2)>f(2.5,3)>f(3,4)$$

In the sub rectangle R_{22},

$$f(3,2)>f(3.5,3)>f(4,4)$$

Add the above inequalities to obtain that,

$$\left[f(2,2)+f(3,2)+f(2,4)+f(3,4)\right]$$
$$>\left[f(2.5,3)+f(3.5,3)+f(2.5,5)+f(3.5,5)\right]$$
$$>\left[f(3,4)+f(4,4)+f(3,6)+f(4,6)\right]$$

$$\left[f(2,2)+f(3,2)+f(2,4)+f(3,4)\right]\cdot 2$$
$$>\left[f(2.5,3)+f(3.5,3)+f(2.5,5)+f(3.5,5)\right]\cdot 2 \quad \text{Multiply by 2}$$
$$>\left[f(3,4)+f(4,4)+f(3,6)+f(4,6)\right]\cdot 2$$

That is,

$$L > V > U$$

or

$$U < V < L$$

Thus, arrange the sums in increasing order as $\boxed{U < V < L}$.

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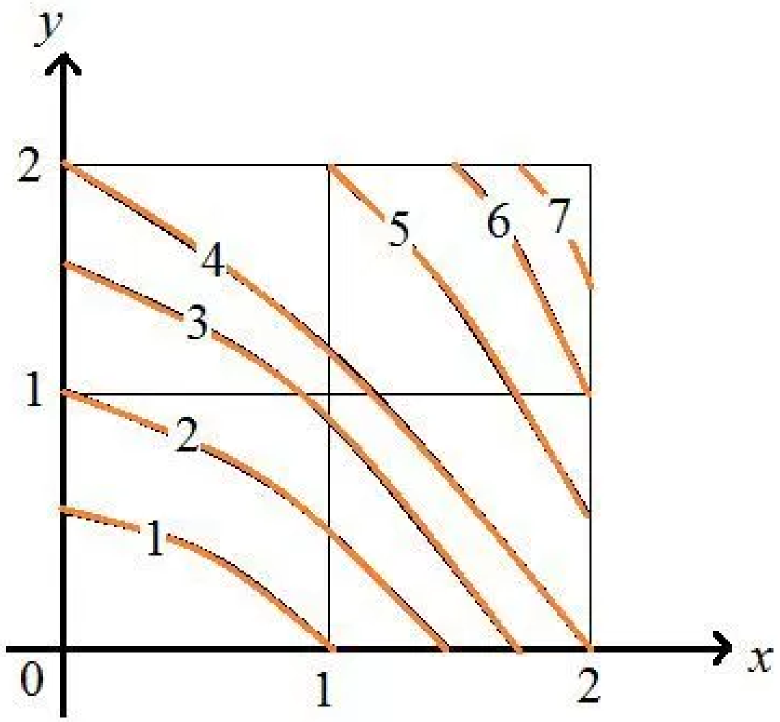
Consider the following figure:



The figure shows level curves of a function f in the square $R=[0,2]\times[0,2]$.

The objective is to estimate the value of integral $\iint_R f(x,y)\,dA$, use the midpoint rule with $m=n=2$.

To approximate the integral, use a double Riemann sum is as follows:

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij},y_{ij}\right)\Delta A_{ij}$$

The variable m represents the number of partitions along the x interval $[a,b]$.

$$a=x_0<x_1<\cdots<x_{i-1}<x_i<\cdots<x_m=b$$

The variable n represents the number of partitions along the y interval $[c,d]$.

$$c=y_0<y_1<\cdots<y_{i-1}<y_i<\cdots<y_n=d$$

And $\Delta A_{ij}=\Delta x_{ij}\Delta y_{ij}$.

In this particular problem the x interval is $[0,2]$ and Δx_{ij} is as follows:

$$\begin{aligned}\Delta x_{ij} &= \frac{2-0}{2}\\ &= \frac{2}{2}\\ &= 1\end{aligned}$$

Similarly, say that the y interval is $[0,2]$ and Δy_{ij} is as follows:

$$\begin{aligned}\Delta y_{ij} &= \frac{2-0}{2}\\ &= \frac{2}{2}\\ &= 1\end{aligned}$$

Find area ΔA_{ij}:

$$\begin{aligned}\Delta A_{ij} &= \Delta x_{ij}\cdot\Delta y_{ij}\\ &= (1)(1)\\ &= 1\end{aligned}$$

This allows us to write the integral as the double Riemann sum.

$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(x_{ij},y_{ij}\right)1\\ &\approx 1\sum_{i=1}^{2}\sum_{j=1}^{2} f\left(x_{ij},y_{ij}\right)\end{aligned}$$

In any manner choose the sample points $f\left(x_{ij}, y_{ij}\right)$.

In this particular problem to apply the midpoint rule which yields:

$$x_1 = 0.5,\ x_2 = 1.5$$
$$y_1 = 0.5,\ y_2 = 1.5$$

Take the sample points from the given figure as follows:

$$f\left(0.5, 0.5\right) = 1$$
$$f(0.5, 1.5) = 3.5$$
$$f\left(1.5, 0.5\right) = 3$$
$$f(1.5, 1.5) = 5$$

Then the integral becomes:

$$\begin{aligned}\iint_D f\left(x, y\right)\ dA &= 1\sum_{i=1}^{2}\sum_{j=1}^{2} f\left(x_{ij}, y_{ij}\right) \\ &= \left[f\left(0.5, 0.5\right) + f\left(1.5, 0.5\right) + f\left(0.5, 1.5\right) + f\left(1.5, 1.5\right)\right]1 \\ &= 1\left(1 + 4 + 3.5 + 5\right)\end{aligned}$$

$$= 13.5$$

Therefore, the required estimated value of the integral is $\iint_D f\left(x, y\right)\ dA = \boxed{13.5}$.

Chapter 15 Multiple Integrals 15.1 9E

(a) Consider the square $R=[0,4]\times[0,4]$.

Take $m=n=2$, then the midpoint rule field

$$\iint_R f(x,y)\Delta A \approx [f(1,1)+f(3,1)+f(1,3)+f(3,3)]\times\Delta A$$

Now $\Delta x=\frac{4-0}{2}=2$, $\Delta y=\frac{4-0}{2}=2$.

Therefore area of the each square with side length 2 is

$$\begin{aligned}\Delta A &= \Delta x\times\Delta y\\ &=4\end{aligned}$$

Then the integral becomes

$$\begin{aligned}\iint_R f(x,y)\Delta A &= [f(1,1)+f(3,1)+f(1,3)+f(3,3)]\times\Delta A\\ &=[f(1,1)+f(3,1)+f(1,3)+f(3,3)](4)\end{aligned}$$

$$\approx(27+15+15+5)(4)$$
$$=(62)(4)$$
$$=248$$

Hence $\iint_R f(x,y)\Delta A=\boxed{248}$.

(b) Estimate the average value of $f(x,y)$.

The area of the region R is

$$\begin{aligned}A(R)&=4\times4\\ &=16\end{aligned}$$

The average value of $f(x,y)$ is

$$f_{ave}=\frac{1}{A(R)}\iint_R f(x,y)\Delta A$$

$$=\frac{1}{16}(248)$$
$$=15.5$$

Hence the required average value of $f(x,y)$ is $\boxed{15.5}$.

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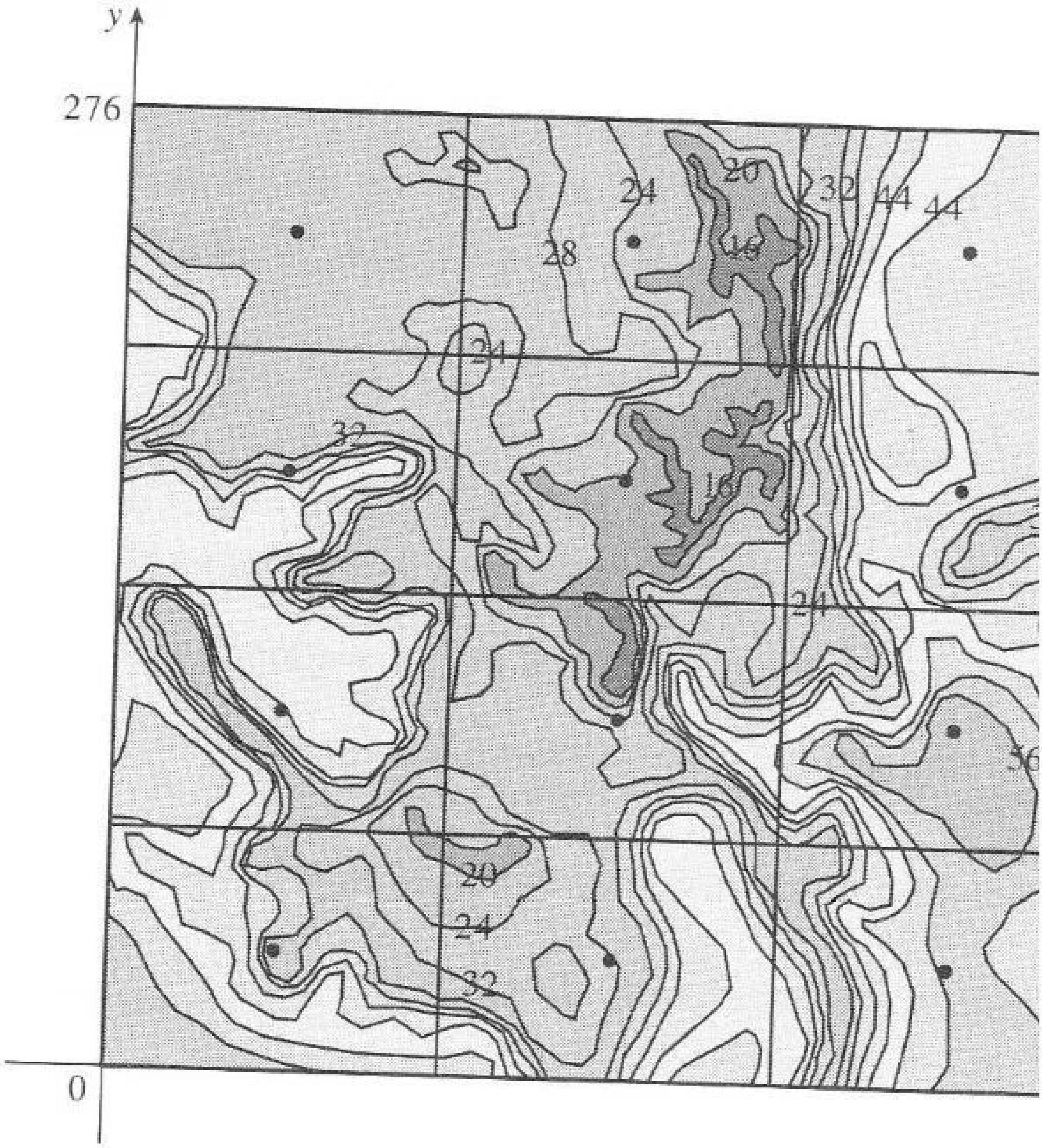
As in Example 4, we place the origin at the southwest corner of the state. Then R = [0, 388] x [0, 276](in miles) is the rectangle corresponding to Colorado and we define f(x,y) to be the temperature at the location (x,y).

The average temperature is given by

$$f_{\text{avg}} = \frac{1}{A(R)} \iint_R f\left(x, y\right) \text{dA} = \frac{1}{388 \cdot 276} \iint_R f\left(x, y\right) \text{dA}$$

To use the Midpoint Rule with m = n = 4, we divide R into 16 regions of equal size, as shown in the figure, with the center of each subrectangle indicated.



The area of each subrectangle is $\Delta A = \frac{388}{4} \cdot \frac{276}{4} = 6693$, so using the contour map to estimate the function values at each midpoint, we have

$$\iint_R f\left(x, y\right) \mathrm{dA} \sum_{i=1}^{4} \sum_{j=1}^{4} f\left(\bar{x}_i, \bar{y}_j\right) \Delta A$$

$$\approx \Delta A[31 + 28 + 52 + 43 + 43 + 25 + 57 + 46 + 36 + 20 + 42 + 45 + 30 + 23 + 4$$

$$= 6693(605)$$

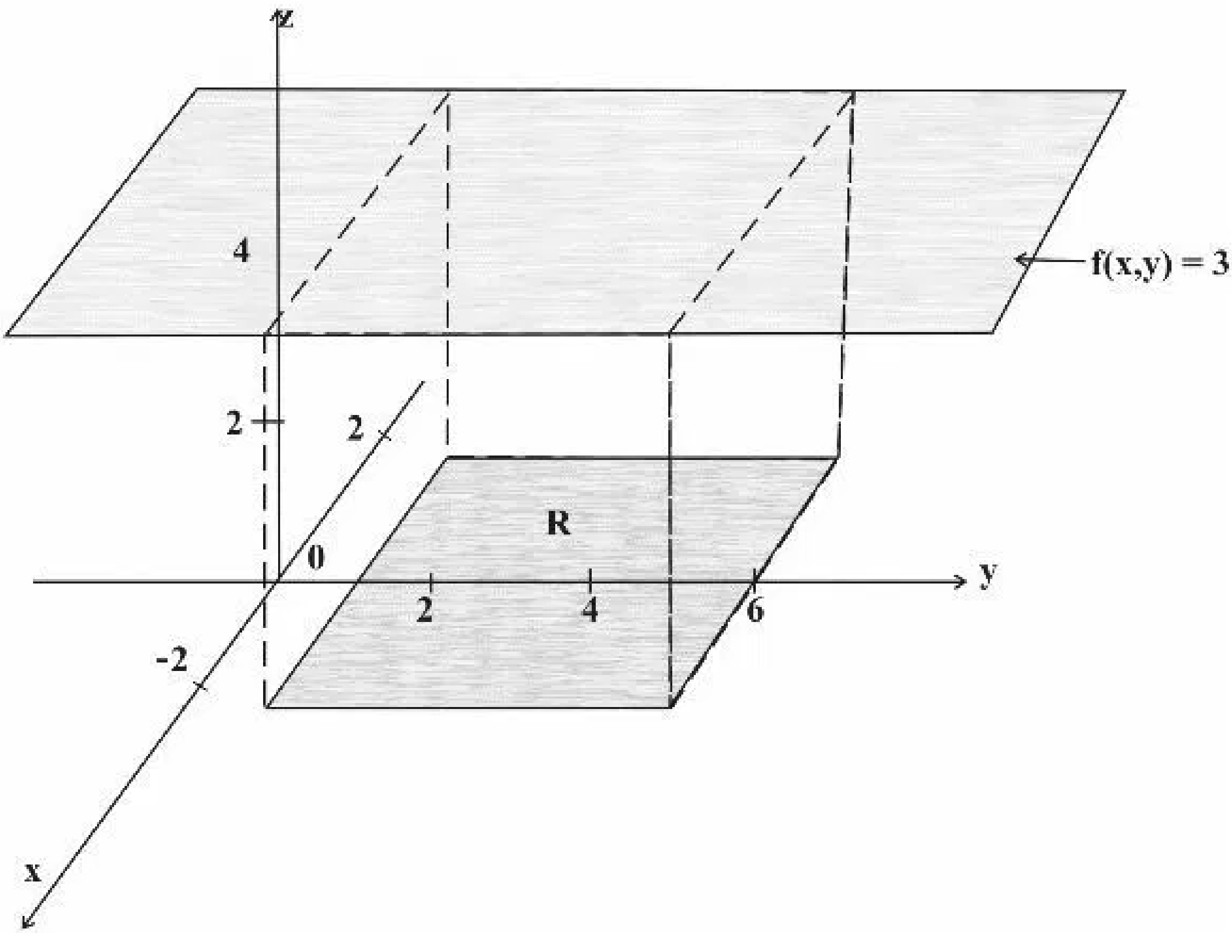
Therefore, $f_{\text{avg}} \approx \frac{6693 \cdot 605}{388 \cdot 276} \approx 37.8$, so the average temperature in Colorado at 4:00 PM on February 26, 2007, was approximately 37.8oF.

Chapter 15 Multiple Integrals 15.1 11E

202-16.1-11E SA: 1848
SR: 1848

We evaluate the integral $\iint_R 3\,dA$ by interpreting it as a volume.

The integral $\iint_R f(x, y)\,dA$, where $R = \{(x, y): -2 \le x \le 2,\ 1 \le y \le 6\}$ is the volume of the solid that lies above the rectangle $R = [-2, 2] \times [1, 6]$ and below the function $f(x, y) = 3$



So the given double integral represents the volume of the solid that lies above the rectangle $R=[-2,\ 2]\times[1,\ 6]$ and below the function $f(x,y)=3$.

The volume of solid is the area of the rectangle with sides 4 *and* 5.

$$\therefore \iint_R 3\ dA = 3\iint_R dA$$

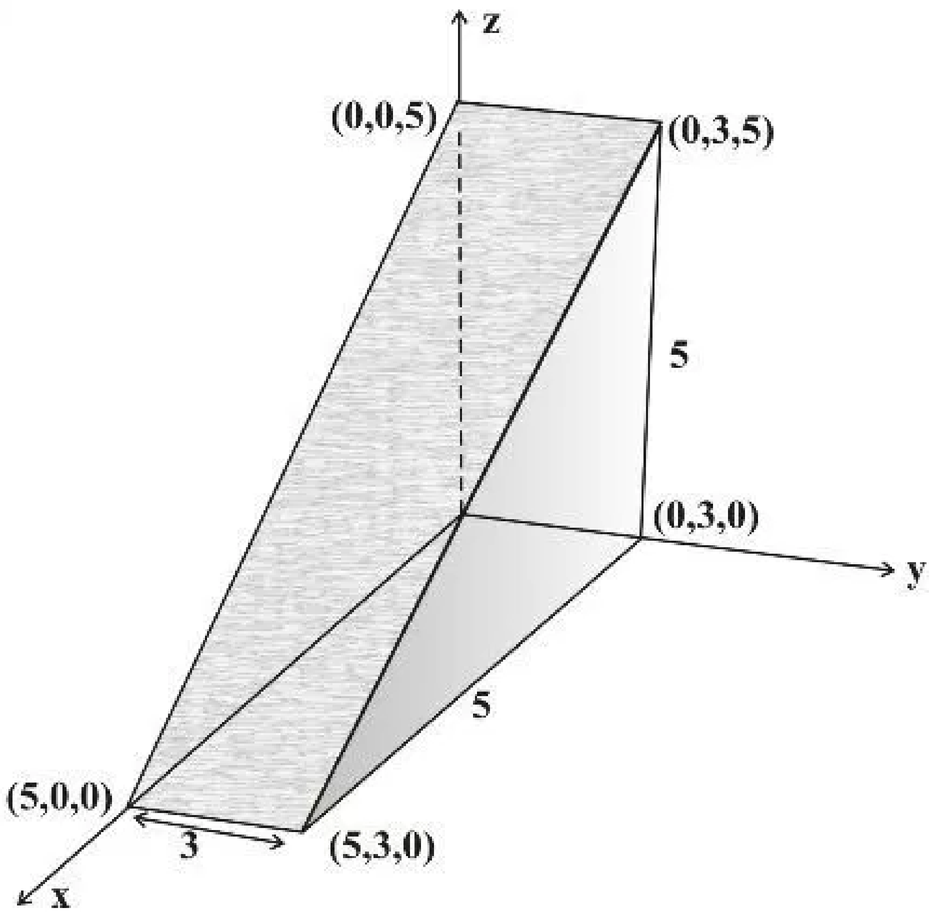
$$= 3(\text{Area of the rectangle with sides 4 and 5})$$

$$= 3(4\times 5)$$

$$= \boxed{60}$$

Chapter 15 Multiple Integrals 15.1 12E

The integral $\iint_R (5-x)\,dA$, where $R=\{(x,y): 0\le x\le 5,\ 0\le y\le 3\}$ is the volume of the solid that lies above the rectangle $R=[0,\ 5]\times[0,\ 3]$ and below the plane $f(x,y)=5-x$ or $z=5-x$



The solid is a triangular cylinder whose volume is 3 times the area of triangle

Then $\iint_R (5-x)\, dA = 3\left(\frac{1}{2}\times 5\times 5\right)$

$= 3\times\frac{25}{2}$

$= \boxed{37.5}$

Chapter 15 Multiple Integrals 15.1 13E

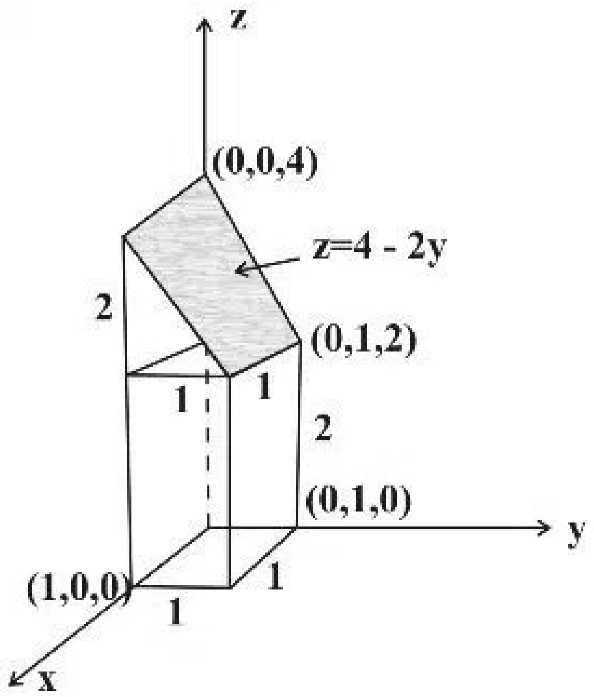
The integral $\iint_R (4-2y)\, dA$ where $R=[0,1]\times[0,1]$ is the volume of the solid that lies above the rectangle $R=[0,1]\times[0,1]$ and below the plane

$$z = f(x,y)$$
$$= 4-2y$$

Since $z = f(x,y) = 4-2y \ge 0$, for $0 \le y \le 1$, thus the given integral represents the volume of that part portion of rectangular solid.

$$S = \{(x,y,z): 0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 4\}$$

That lies below the plane $z = 4-2y$



Thus $\iint_R (4-2y)\,dA = (1)(1)(2)+1\left(\frac{1}{2}\times 1\times 2\right)$

$= 2+1$

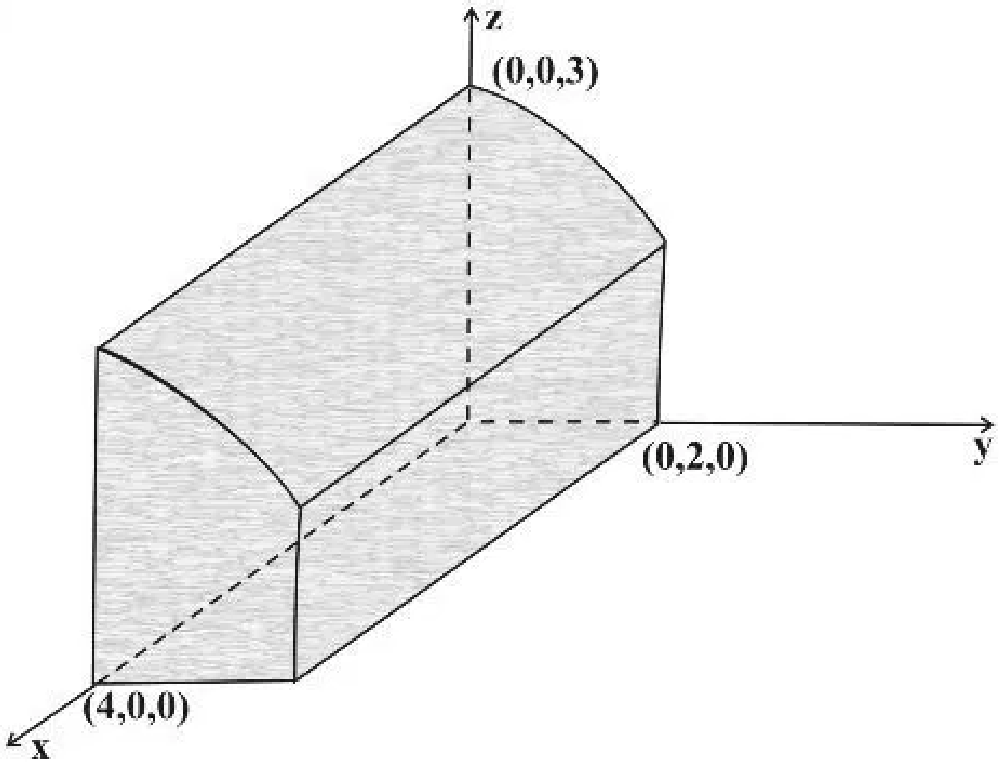
$= \boxed{3}$

Chapter 15 Multiple Integrals 15.1 **14E**

The given integral is $\iint_R \sqrt{9-y^2}\,dA$, $R=[0,4]\times[0,2]$

The given function $f(x, y) = \sqrt{9-y^2}$ is always positive when $0 \le y \le 2$. Then if $z = \sqrt{9-y^2}$, then $y^2 + z^2 = 9$ and $z \ge 0$

So the given double integral represents the volume of the solid that lies below the circular cylinder $y^2 + z^2 = 9$ and above the rectangle R.



Chapter 15 Multiple Integrals 15.1 15E

Consider the integral:

$$\iint_R \sqrt{1+xe^{-y}}\,dA$$

Where $R = [0,1] \times [0,1]$

Apply Mid-point formula over $R=[0,1]\times[0,1]$ with n^2 squares.

$$\iint_R f(x,y)\,dA=\sum_{i=1}^{n}\sum_{j=1}^{n} f\left(\overline{x_i},\overline{y_j}\right)\Delta A \quad \ldots\ldots (1)$$

Where,

$$\overline{x_i}=\frac{i}{n}-\frac{1}{2n}$$
$$\overline{y_i}=\frac{j}{n}-\frac{1}{2n}$$

Now, substitute $\overline{x_i}=\frac{i}{n}-\frac{1}{2n}$, $\overline{y_i}=\frac{j}{n}-\frac{1}{2n}$, and $\Delta A=\frac{1}{n^2}$ in the equation (1).

$$\iint_R f(x,y)\,dA=\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f\left(\frac{i}{n}-\frac{1}{2n},\frac{j}{n}-\frac{1}{2n}\right)$$

For $n=1$:

$$\begin{aligned}\iint_R f(x,y)\,dA&=\frac{1}{1^2}\sum_{i=1}^{1}\sum_{j=1}^{1} f\left(\frac{i}{1}-\frac{1}{2(1)},\frac{j}{1}-\frac{1}{2(1)}\right)\\&=\sum_{i=1}^{1}\sum_{j=1}^{1} f(i-0.5,\,j-0.5)\\&=\sum_{i=1}^{1}\sum_{j=1}^{1}\sqrt{1+(i-0.5)e^{-j+0.5}}\end{aligned}$$

Use Ti-Nspire to evaluate.

$$\sum_{i=1}^{1}\left(\sum_{j=1}^{1}\left(\sqrt{1+\frac{i-0.5}{1}\cdot e^{\frac{1}{1}\cdot(0.5-j)}}\right)\right)$$

1.14161

Hence, the value of $\iint_R \sqrt{1+xe^{-y}}\,dA$ for $n=1$ is $\boxed{1.14161}$.

For $n=2$:

$$\iint_R f(x,y)\,dA = \frac{1}{2^2}\sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\frac{i}{2}-\frac{1}{2(2)},\frac{j}{2}-\frac{1}{2(2)}\right)$$
$$= \frac{1}{2^2}\sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\frac{1}{2}(i-0.5),\frac{1}{2}(j-0.5)\right)$$
$$= \frac{1}{2^2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sqrt{1+\frac{(i-0.5)}{2}e^{\frac{1}{2}(0.5-j)}}$$

Use Ti-Nspire to evaluate.

$$\frac{1}{2^2}\cdot\sum_{i=1}^{2}\left(\sum_{j=1}^{2}\left(\sqrt{1+\frac{i-0.5}{2}\cdot \mathbf{e}^{\frac{1}{2}\cdot(0.5-j)}}\right)\right)$$

1.14319

Hence, the value of $\iint_R \sqrt{1+xe^{-y}}\,dA$ for $n=2$ is $\boxed{1.14319}$.

For $n=4$:

$$\iint_R f(x,y)\,dA = \frac{1}{4^2}\sum_{i=1}^{4}\sum_{j=1}^{4} f\left(\frac{i}{4}-\frac{1}{2(4)},\frac{j}{4}-\frac{1}{2(4)}\right)$$
$$= \frac{1}{4^2}\sum_{i=1}^{4}\sum_{j=1}^{4} f\left(\frac{1}{4}(i-0.5),\frac{1}{4}(j-0.5)\right)$$
$$= \frac{1}{4^2}\sum_{i=1}^{4}\sum_{j=1}^{4}\sqrt{1+\frac{(i-0.5)}{4}e^{\frac{1}{4}(0.5-j)}}$$

Use Ti-Nspire to evaluate.

$$\frac{1}{4^2}\cdot\sum_{i=1}^{4}\left(\sum_{j=1}^{4}\left(\sqrt{1+\frac{i-0.5}{4}\cdot \mathbf{e}^{\frac{1}{4}\cdot(0.5-j)}}\right)\right)$$

1.14353

Hence, the value of $\iint_R \sqrt{1+xe^{-y}}\,dA$ for $n=4$ is $\boxed{1.14319}$.

For $n=8$:

$$\iint_R f(x,y)\,dA = \frac{1}{8^2}\sum_{i=1}^{8}\sum_{j=1}^{8} f\left(\frac{i}{8}-\frac{1}{2(8)}, \frac{j}{8}-\frac{1}{2(8)}\right)$$
$$= \frac{1}{8^2}\sum_{i=1}^{8}\sum_{j=1}^{8} f\left(\frac{1}{8}(i-0.5), \frac{1}{8}(j-0.5)\right)$$
$$= \frac{1}{8^2}\sum_{i=1}^{8}\sum_{j=1}^{8} \sqrt{1+\frac{(i-0.5)}{8}e^{\frac{1}{8}(0.5-j)}}$$

Use Ti-Nspire to evaluate.

$$\frac{1}{8^2}\cdot\sum_{i=1}^{8}\left(\sum_{j=1}^{8}\left(\sqrt{1+\frac{i-0.5}{8}\cdot \mathbf{e}^{\frac{1}{8}\cdot(0.5-j)}}\right)\right)$$

1.14362

Hence, the value of $\iint_R \sqrt{1+xe^{-y}}\,dA$ for $n=8$ is $\boxed{1.14362}$.

For $n=16$:

$$\iint_R f(x,y)\,dA = \frac{1}{16^2}\sum_{i=1}^{16}\sum_{j=1}^{16} f\left(\frac{i}{16}-\frac{1}{2(16)}, \frac{j}{16}-\frac{1}{2(16)}\right)$$
$$= \frac{1}{16^2}\sum_{i=1}^{16}\sum_{j=1}^{16} f\left(\frac{1}{16}(i-0.5), \frac{1}{16}(j-0.5)\right)$$
$$= \frac{1}{16^2}\sum_{i=1}^{16}\sum_{j=1}^{16} \sqrt{1+\frac{(i-0.5)}{16}e^{\frac{1}{16}(0.5-j)}}$$

Use Ti-Nspire to evaluate.

$$\frac{1}{16^2}\cdot\sum_{i=1}^{16}\left(\sum_{j=1}^{16}\left(\sqrt{1+\frac{i-0.5}{16}\cdot \mathbf{e}^{\frac{1}{16}\cdot(0.5-j)}}\right)\right)$$

1.14364

Hence, the value of $\iint_R \sqrt{1+xe^{-y}}\,dA$ for $n=16$ is $\boxed{1.14364}$.

For $n = 32$:

$$\iint_R f(x,y)\,dA = \frac{1}{32^2}\sum_{i=1}^{32}\sum_{j=1}^{32} f\left(\frac{i}{32} - \frac{1}{2(32)}, \frac{j}{32} - \frac{1}{2(32)}\right)$$

$$= \frac{1}{32^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f\left(\frac{1}{32}(i-0.5), \frac{1}{32}(j-0.5)\right)$$

$$= \frac{1}{32^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{1+\frac{(i-0.5)}{32}e^{\frac{1}{32}(0.5-j)}}$$

Use Ti-Nspire to evaluate.

$$\frac{1}{32^2}\cdot\sum_{i=1}^{32}\left(\sum_{j=1}^{32}\left(\sqrt{1+\frac{i-0.5}{32}\cdot e^{\frac{1}{32}\cdot(0.5-j)}}\right)\right)$$

1.14364

Hence, the value of $\iint_R \sqrt{1+xe^{-y}}\,dA$ for $n = 32$ is $\boxed{1.14364}$.

Chapter 15 Multiple Integrals 15.1 16E

The Midpoint Rule states that the volume under a surface $f(x,y)$ over a region R can be approximated as follows:

$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta A$ where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Applying the Midpoint Rule with the region R divided into 1 square.

$$\iint_R \sin\left(x+\sqrt{y}\right)dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i,\overline{y}_j)\Delta A$$
$$=\sum_{i=1}^{1}\sum_{j=1}^{1} f(\overline{x}_i,\overline{y}_j)\Delta A$$
$$= f(\overline{x}_1,\overline{y}_1)\Delta A$$

If $R=[0,1]\times[0,1]$ is a region of 1 square, then it is a square that is $[0,1]\times[0,1]$, and the x and y midpoints of the intervals are both 1/2. The area of R is (1)(1) = 1. Therefore:

$$\iint_R \sin\left(x+\sqrt{y}\right)dA \approx f(\overline{x}_1,\overline{y}_1)\Delta A$$
$$= f\left(\frac{1}{2},\frac{1}{2}\right)(1)$$
$$=\sin\left(\frac{1}{2}+\sqrt{\frac{1}{2}}\right)$$
$$\approx \boxed{.9345907}$$

Apply the Midpoint Rule with the region R divided into 4 squares, the division is looked like [0, ½], [1/2, 1] both along x - axis and along y – axis. So, the midpoints of these squares are $\overline{x_1}=\frac{1}{4}, \overline{x_2}=\frac{3}{4}, \overline{y_1}=\frac{1}{4}, \overline{y_2}=\frac{3}{4}$

Further, each edge of the square is of length $\frac{1}{2}$, the corresponding area of the square is $\frac{1}{4}$. i.e., $\Delta A=\frac{1}{4}$. Using these things, we get

$$\iint_R \sin\left(x+\sqrt{y}\right)dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i,\overline{y}_j)\Delta A$$
$$=\sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x}_i,\overline{y}_j)\Delta A$$
$$= f(\overline{x}_1,\overline{y}_1)\Delta A + f(\overline{x}_1,\overline{y}_2)\Delta A + f(\overline{x}_2,\overline{y}_1)\Delta A + f(\overline{x}_2,\overline{y}_2)\Delta A$$

$$\iint_R \sin\left(x+\sqrt{y}\right)dA \approx \sin\left(\frac{1}{4}+\sqrt{\frac{1}{4}}\right)\left(\frac{1}{4}\right)+\sin\left(\frac{1}{4}+\sqrt{\frac{3}{4}}\right)\left(\frac{1}{4}\right)$$
$$+\sin\left(\frac{3}{4}+\sqrt{\frac{1}{4}}\right)\left(\frac{1}{4}\right)+\sin\left(\frac{3}{4}+\sqrt{\frac{3}{4}}\right)\left(\frac{1}{4}\right)$$
$$\approx \boxed{.8819906}$$

Apply the Midpoint Rule with the region R divided into 16 squares, the side [0, 1] is divided into 4 equal parts as [0, ¼], [1/4, ½], [1/2, ¾], [3/4, 1] and similarly along y – axis

So, the respective midpoints of the subintervals are $\overline{x_1} = \frac{1}{8}, \overline{x_2} = \frac{3}{8}, \overline{x_3} = \frac{5}{8}, \overline{x_2} = \frac{7}{8}$, and

$\overline{y_1} = \frac{1}{8}, \overline{y_2} = \frac{3}{8}, \overline{y_3} = \frac{5}{8}, \overline{y_2} = \frac{7}{8}$

Further, the area of the smaller square with the side $\frac{1}{4}$ is $\Delta A = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$

Using these in the function and the formula

$$\iint_R \sin\left(x+\sqrt{y}\right) dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta A = \sum_{i=1}^{4}\sum_{j=1}^{4} f(\bar{x}_i, \bar{y}_j)\Delta A$$

We get

$$\begin{aligned}\iint_R \sin\left(x+\sqrt{y}\right) dA \approx\ & f(\bar{x}_1,\bar{y}_1)\Delta A + f(\bar{x}_1,\bar{y}_2)\Delta A + f(\bar{x}_1,\bar{y}_3)\Delta A + f(\bar{x}_1,\bar{y}_4)\Delta A \\ & + f(\bar{x}_2,\bar{y}_1)\Delta A + f(\bar{x}_2,\bar{y}_2)\Delta A + f(\bar{x}_2,\bar{y}_3)\Delta A + f(\bar{x}_2,\bar{y}_4)\Delta A \\ & + f(\bar{x}_3,\bar{y}_1)\Delta A + f(\bar{x}_3,\bar{y}_2)\Delta A + f(\bar{x}_3,\bar{y}_3)\Delta A + f(\bar{x}_3,\bar{y}_4)\Delta A \\ & + f(\bar{x}_4,\bar{y}_1)\Delta A + f(\bar{x}_4,\bar{y}_2)\Delta A + f(\bar{x}_4,\bar{y}_3)\Delta A + f(\bar{x}_4,\bar{y}_4)\Delta A\end{aligned}$$

$$\begin{aligned}\approx & \sin\left(\frac{1}{8}+\sqrt{\frac{1}{8}}\right)\left(\frac{1}{16}\right) + \sin\left(\frac{1}{8}+\sqrt{\frac{3}{8}}\right)\left(\frac{1}{16}\right) + \sin\left(\frac{1}{8}+\sqrt{\frac{5}{8}}\right)\left(\frac{1}{16}\right) + \sin\left(\frac{1}{8}+\sqrt{\frac{7}{8}}\right)\left(\frac{1}{16}\right) \\ & + \sin\left(\frac{3}{8}+\sqrt{\frac{1}{8}}\right)\left(\frac{1}{16}\right) + \sin\left(\frac{3}{8}+\sqrt{\frac{3}{8}}\right)\left(\frac{1}{16}\right) + \sin\left(\frac{3}{8}+\sqrt{\frac{5}{8}}\right)\left(\frac{1}{16}\right) + \sin\left(\frac{3}{8}+\sqrt{\frac{7}{8}}\right)\left(\frac{1}{16}\right) \\ & + \sin\left(\frac{5}{8}+\sqrt{\frac{1}{8}}\right)\left(\frac{1}{16}\right) + \sin\left(\frac{5}{8}+\sqrt{\frac{3}{8}}\right)\left(\frac{1}{16}\right) + \sin\left(\frac{5}{8}+\sqrt{\frac{5}{8}}\right)\left(\frac{1}{16}\right) + \sin\left(\frac{5}{8}+\sqrt{\frac{7}{8}}\right)\left(\frac{1}{16}\right) \\ & + \sin\left(\frac{7}{8}+\sqrt{\frac{1}{8}}\right)\left(\frac{1}{16}\right) + \sin\left(\frac{7}{8}+\sqrt{\frac{3}{8}}\right)\left(\frac{1}{16}\right) + \sin\left(\frac{7}{8}+\sqrt{\frac{5}{8}}\right)\left(\frac{1}{16}\right) + \sin\left(\frac{7}{8}+\sqrt{\frac{7}{8}}\right)\left(\frac{1}{16}\right)\end{aligned}$$

$\approx \boxed{.8657496}$

Apply the Midpoint Rule with the region R divided into 64 squares, we follow that each side is divided into 8 equal subintervals and so along the $x-$axis, the mid points are $\frac{1}{16}, \frac{3}{16}, \ldots, \frac{15}{16}$ and similarly along y - axis.

$$\iint_R \sin\left(x+\sqrt{y}\right) dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta A$$

$$= \sum_{i=1}^{8}\sum_{j=1}^{8} f(\bar{x}_i, \bar{y}_j)\Delta A$$

$$= f(\bar{x}_1,\bar{y}_1)\Delta A + f(\bar{x}_1,\bar{y}_2)\Delta A + f(\bar{x}_1,\bar{y}_3)\Delta A + f(\bar{x}_1,\bar{y}_4)\Delta A$$
$$+ f(\bar{x}_1,\bar{y}_5)\Delta A + f(\bar{x}_1,\bar{y}_6)\Delta A + f(\bar{x}_1,\bar{y}_7)\Delta A + f(\bar{x}_1,\bar{y}_8)\Delta A$$
$$+ f(\bar{x}_2,\bar{y}_1)\Delta A + f(\bar{x}_2,\bar{y}_2)\Delta A + f(\bar{x}_2,\bar{y}_3)\Delta A + f(\bar{x}_2,\bar{y}_4)\Delta A$$
$$+ f(\bar{x}_2,\bar{y}_5)\Delta A + f(\bar{x}_2,\bar{y}_6)\Delta A + f(\bar{x}_2,\bar{y}_7)\Delta A + f(\bar{x}_2,\bar{y}_8)\Delta A$$
$$+ \ldots + f(\bar{x}_8,\bar{y}_8)\Delta A$$

In this case, we see that $\Delta A = \frac{1}{8}\times\frac{1}{8} = \frac{1}{64}$

So, the respective integral is

$$\iint_R \sin\left(x+\sqrt{y}\right) dA \approx \sin\left(\frac{1}{16}+\sqrt{\frac{1}{16}}\right)\left(\frac{1}{64}\right) + \sin\left(\frac{1}{16}+\sqrt{\frac{3}{16}}\right)\left(\frac{1}{64}\right)$$

$$+\sin\left(\frac{1}{16}+\sqrt{\frac{5}{16}}\right)\left(\frac{1}{64}\right)+\sin\left(\frac{1}{16}+\sqrt{\frac{7}{16}}\right)\left(\frac{1}{64}\right)+\sin\left(\frac{1}{16}+\sqrt{\frac{9}{16}}\right)\left(\frac{1}{64}\right)$$

$$+\sin\left(\frac{1}{16}+\sqrt{\frac{11}{16}}\right)\left(\frac{1}{64}\right)+\sin\left(\frac{1}{16}+\sqrt{\frac{13}{16}}\right)\left(\frac{1}{64}\right)+\sin\left(\frac{1}{16}+\sqrt{\frac{15}{16}}\right)\left(\frac{1}{64}\right)$$

$$+\sin\left(\frac{3}{16}+\sqrt{\frac{1}{16}}\right)\left(\frac{1}{64}\right)+\sin\left(\frac{3}{16}+\sqrt{\frac{3}{16}}\right)\left(\frac{1}{64}\right)+\ldots+\sin\left(\frac{15}{16}+\sqrt{\frac{15}{16}}\right)\left(\frac{1}{64}\right)$$

$$\approx \boxed{.8604900}$$

We use CAS(Maple) for the application of the Midpoint Rule for the region R divided into 256 equal squares.

So, each side is divided into 16 equal subintervals of length $\frac{1}{16}$

$$\Delta A = \frac{1}{16} \times \frac{1}{16} = \frac{1}{256}$$

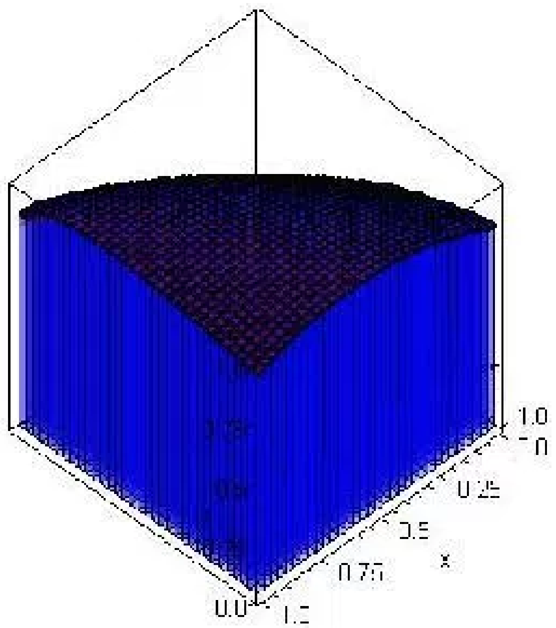
$$\iint_R \sin\left(x+\sqrt{y}\right) dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\Delta A$$
$$= \sum_{i=1}^{16} \sum_{j=1}^{16} f(\overline{x}_i, \overline{y}_j)\Delta A$$

$$\iint_R \sin\left(x+\sqrt{y}\right) dA \approx \boxed{0.85875}$$

Similarly, using Maple for 32 subintervals on [0, 1] both along x – axis and along y – axis, we get $\iint_R \sin\left(x+\sqrt{y}\right) dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\Delta A$

$$= \sum_{i=1}^{32} \sum_{j=1}^{32} f(\overline{x}_i, \overline{y}_j)\Delta A$$
$$\approx \boxed{0.85816}$$

The graph of the region using 32 subintervals along both axis is



Chapter 15 Multiple Integrals 15.1 17E

Let f be a constant function. Now, based on this, write the following:

$f(x,y)=k$ (Constant) and $R=[a,b]\times[c,d]$

Show that $\iint_R k\ dA=k(b-a)(d-c)$

The first step is to divide the rectangle R into subrectangles.

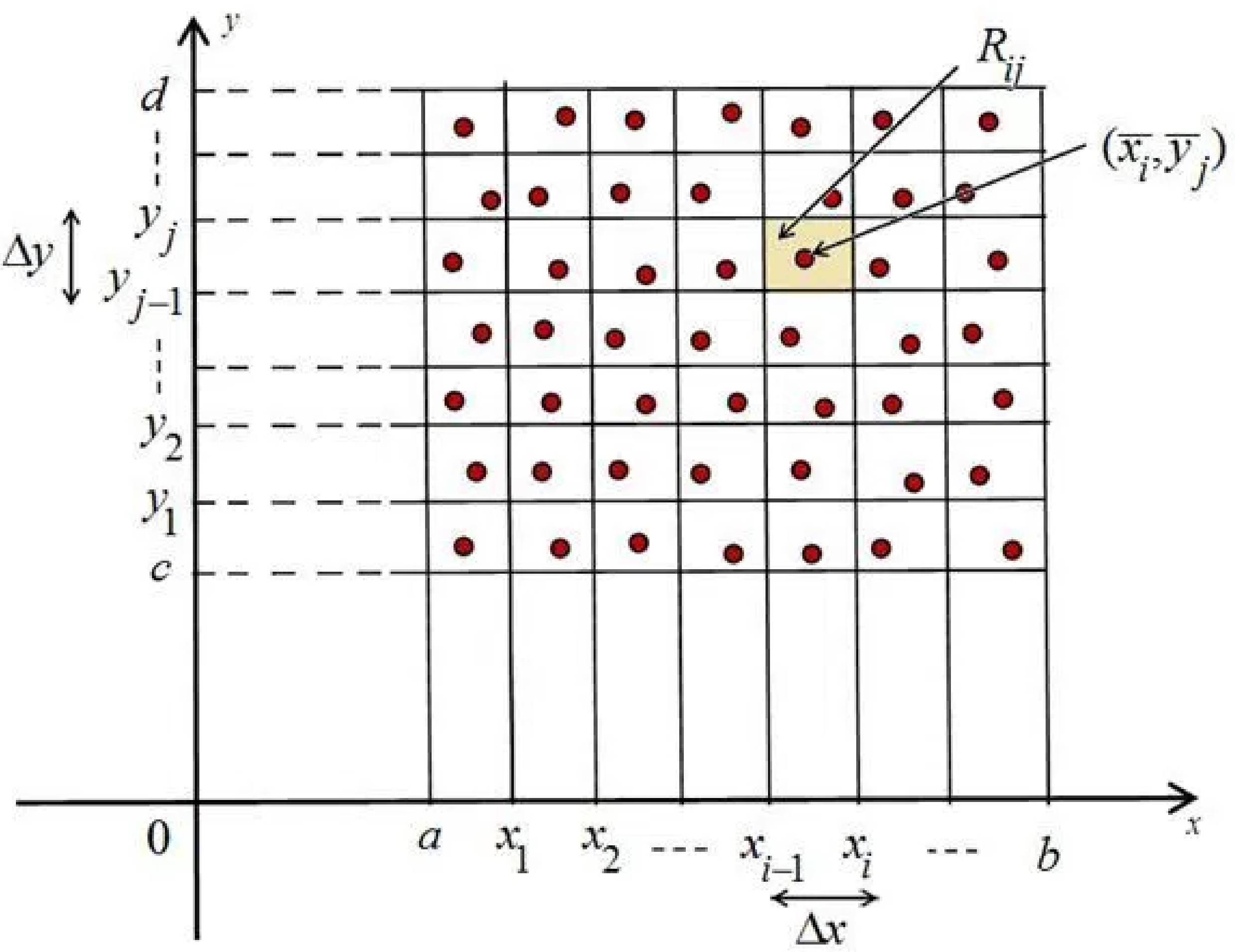
Accomplish this by dividing the interval $[a,b]$ into m subintervals $[x_{i-1},x_i]$ of equal width $\Delta x=(b-a)/m$ and dividing the interval $[c,d]$ into n subintervals $[y_{j-1},y_j]$ of equal width $\Delta y=(d-c)/n$.

Draw the parallel lines to the coordinate axes through the end points of these subintervals, as in the following figure.

Form the sub-rectangles as follows:

$$R_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j]$$
$$=\{(x,y):x_{i-1}\le x\le x_i,\ y_{j-1}\le y\le y_j\}$$

Each, with an area $\Delta A=\Delta x\,\Delta y$.



Recall the midpoint rule for double integrals.

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta A$$

Here, $\bar{x}_i$ is the mid – point of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the mid – point of $[y_{j-1}, y_j]$.

Since $f(x,y) = k$, then by using the midpoint rule, write the following:

$$\iint_R k\;dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta A$$

$$= [k + k + k + ...+ k(mn \text{ times})]\Delta A$$

(Since *f* is a constant function, and $f(\bar{x}_i, \bar{y}_j) = k\ \forall\ i,\ j$, the total numbers of triangles are $m \times n$.)

Thus, $\iint_R k\,dA \approx mn \cdot k \cdot \Delta A$

$= mn \cdot k \cdot \Delta x\ \Delta y$ Since: $\Delta A = \Delta x\,\Delta y$

$= \frac{mn}{1} \cdot \frac{k}{1} \cdot \frac{(b-a)}{m}\frac{(d-c)}{n}$ Since: $\Delta x = (b-a)/m$ and $\Delta y = (d-c)/n$

$= \frac{mn \cdot k(b-a)(d-c)}{mn}$ Multiply the numerators and denominators

$= \frac{\cancel{mn} \cdot k(b-a)(d-c)}{\cancel{mn}}$ Cancel out the common term, mn

$= k(b-a)(d-c)$

Therefore, $\boxed{\iint_R k\,dA = k(b-a)(d-c)}$

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The objective is to show that

$0 \le \iint_R \sin \pi x \cos \pi y dA \le \frac{1}{32}$ where $R = \left[0, \frac{1}{4}\right] \times \left[\frac{1}{4}, \frac{1}{2}\right]$

Observe that, $\sin \pi x \ge 0$ for all $x \in \left[0, \frac{1}{4}\right]$ and $\cos \pi y \ge 0$ for all $y \in \left[\frac{1}{4}, \frac{1}{2}\right]$

Thus, $\iint_R \sin \pi x \cos \pi y dA \ge 0$ where $R = \left[0, \frac{1}{4}\right] \times \left[\frac{1}{4}, \frac{1}{2}\right]$ …… (1)

Also, notice that

$$\sin \pi x \le \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad \forall x \in \left[0, \frac{1}{4}\right]$$

And, $\cos \pi y \le \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad \forall y \in \left[\frac{1}{4}, \frac{1}{2}\right]$

Therefore, $\iint_R \sin \pi x \cos \pi y \, dA \le \iint_R \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} dA$

$$= \frac{1}{2}\iint_R 1 \cdot dA$$

Recall the fact that, if f is a constant function, $f(x, y) = k$ and $R = [a,b] \times [c,d]$ then

$$\iint_R k\,dA = k(b-a)(d-c)$$

Using the above fact,

$\frac{1}{2}\iint_R 1 \cdot dA = \frac{1}{2} \cdot 1 \cdot \left(\frac{1}{4} - 0\right) \cdot \left(\frac{1}{2} - \frac{1}{4}\right)$ Here $R = \left[0, \frac{1}{4}\right] \times \left[\frac{1}{4}, \frac{1}{2}\right]$

$$= \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4}$$

$$= \frac{1}{32}$$

Thus $\iint_R \sin \pi x \cos \pi y \, dA \le \frac{1}{32}$ ...... (2)

From the inequalities (1) and (2), we have that

$0 \le \iint_R \sin \pi x \cos \pi y \, dA \le \frac{1}{32}$ where $R = \left[0, \frac{1}{4}\right] \times \left[\frac{1}{4}, \frac{1}{2}\right]$