Sample Question Paper - 14
Mathematics (041)
Class- XII, Session: 2021-22
TERM II

Time Allowed: 2 hours **Maximum Marks: 40**

General Instructions:

1. This question paper contains three sections – A, B and C. Each part is compulsory.
2. Section - A has 6 short answer type (SA1) questions of 2 marks each.
3. Section – B has 4 short answer type (SA2) questions of 3 marks each.
4. Section - C has 4 long answer-type questions (LA) of 4 marks each.
5. There is an internal choice in some of the questions.
6. Q 14 is a case-based problem having 2 sub-parts of 2 marks each.

Section - A

[2 Marks each]

1. Find $\int \frac{2^{x+1} - 5^{x-1}}{10^x} dx$.

OR

Evaluate $\int_0^{2\pi} |\sin x| dx$.

2. Find the general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$.

3. Let $\vec{a}$ and $\vec{b}$ be two-unit vectors and θ be the angle between them. Then for what value of θ, $\vec{a} + \vec{b}$ is a unit vector.

4. Find the cartesian equation of the line which passes through the point (– 2, 4, – 5) and is parallel to the line $\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$. AI

5. A refrigerator box contains 2 milk chocolates and 4 dark chocolates. Two chocolates are drawn at random. Find the probability distribution of the number of milk chocolates. What is the most likely outcome?

6. If P(not A) = 0.7, P(B) = 0.7 and $P\left(\frac{B}{A}\right) = 0.5$, then find $P\left(\frac{A}{B}\right)$.

Section - B

[3 Marks each]

7. Find $\int \frac{x}{x^2 + 3x + 2} dx$.

8. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$; given that $y(0) = \sqrt{3}$.

OR

Prove that $x^2 - y^2 = C(x^2 + y^2)^2$ is the general solution of the differential equation $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$, where C is a parameter.

9. If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ respectively are the position vectors of points A, B, C and D, then find the angle between the straight lines AB and CD. Find whether $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear or not.

OR

Find the area of the quadrilateral ABCD, where A(0, 4, 1), B(2, 3, –1), C(4, 5, 0) and D(2, 6, 2).

10. If the lines $\frac{x-1}{-3} = \frac{y-2}{2\lambda} = \frac{z-3}{2}$ and $\frac{x-1}{3\lambda} = \frac{y-1}{2} = \frac{z-6}{-5}$ are perpendicular, find the value of λ. Hence find whether the lines are intersecting or not.

Section - C

[4 Marks each]

11. Show that: $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}\,dx = \frac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right)$.

12. Find the area of the region bounded by the parabola $y^2 = x$ and the line $2y = x$.

OR

Find the area of the region bounded by the parabola $y^2 = 16x$ and the line $x = 4$. AI

13. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c} = 0$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$, then prove that:

$\vec{a} = \pm 2(\vec{b} \times \vec{c})$

Case-Based/Data Based

14. Bag I contains 1 white, 2 black and 3 red balls; Bag II contains 2 white, 1 black and 1 red balls; Bag III contains 4 white, 3 black and 2 red balls. A bag is chosen at random and two balls are drawn from it with replacement. They happen to be one white and one red.

Based on the above information, answer the following questions:

(i) What is the probability that they came from Bag III? **[2]**

(ii) What is the probability that they will come from Bag I? **[2]**

❑❑

Solution

MATHEMATICS 041

Class 12 - Mathematics

Section - A

1. $\int \frac{2^{x+1}-5^{x-1}}{10^x}dx = \int \frac{2^{x+1}-5^{x-1}}{2^x.5^x}dx$

$= \int\left(\frac{2^{x+1}}{2^x.5^x}-\frac{5^{x-1}}{2^x.5^x}\right)dx$

$= \int \frac{2}{5^x}dx - \int \frac{1}{5\cdot 2^x}dx$

$= 2\int 5^{-x}dx - \frac{1}{5}\int 2^{-x}dx$ **1**

$= 2\times\frac{-5^{-x}}{\log 5} - \frac{1}{5}\times\frac{-2^{-x}}{\log 2}+C$

$= \frac{1}{5.2^x \log 2} - \frac{2}{5^x \log 5}+C$ **1**

OR

Let $I = \int_0^{2\pi} |\sin x|\,dx$

$= \int_0^{\pi} |\sin x|\,dx + \int_{\pi}^{2\pi} |\sin x|\,dx$

$= \int_0^{\pi} \sin x dx - \int_{\pi}^{2\pi} \sin x dx$ **1**

$= [-\cos x]_0^{\pi} - [-\cos x]_{\pi}^{2\pi}$

$= [-\cos\pi+\cos 0]-[-\cos 2\pi+\cos\pi]$

$= [1 + 1] - [-1-1] = 2+2=4$ **1**

Commonly Made Error

- Mostly students fail in correctly splitting the modulus function involving trigonometric functions.

Answering Tip

- Practice problems in modulus functions and trigonometric functions.

2. $\frac{dy}{dx} = e^{x+y}$

$= e^x \cdot e^y$

$\Rightarrow \frac{dy}{e^y} = e^x dx$

$\Rightarrow e^{-y}dy = e^x dx$ **1**

Integrating both sides, we get:

$\int e^{-y}dy = \int e^x dx$

$\Rightarrow -e^{-y} = e^x + k$

$\Rightarrow e^x + e^{-y} = -k$

$\Rightarrow e^x + e^{-y} = C$ (where, $C = -k$) **1**

3. Let $\vec{a}$ and $\vec{b}$ be two-unit vectors and θ be the angle between them.

Then, $|\vec{a}| = |\vec{a}+\vec{b}| = |\vec{b}| = 1.$

Now, $\vec{a}+\vec{b}$ is a unit vector then

$|\vec{a}+\vec{b}| = 1$

$\Rightarrow (\vec{a}+\vec{b})^2 = 1$

$\Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 1$

$\Rightarrow \vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b} = 1$

$\Rightarrow |\vec{a}|^2+2\vec{a}.\vec{b}+|\vec{b}|^2 = 1$ **1**

$\Rightarrow 1^2+2|\vec{a}||\vec{b}|\cos\theta+1^2 = 1$

$\Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = \frac{2\pi}{3}$

So that, $|\vec{a}+\vec{b}|$ is a unit vector if $\theta = \frac{2\pi}{3}$. **1**

4. Given line $\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$

[If two lines are parallel, then they both have proportional direction ratio] or $$\frac{x-(-3)}{3} = = \frac{y-4}{-5} = \frac{z-(-8)}{6}$$

Here, given point is (– 2, 4, – 5) with D.R's. 3, – 5, 6 **1**

Therefore, cartesian equation of parallel line will be :

$$\frac{x+2}{a} = \frac{y-4}{b} = \frac{z+5}{c}$$

or $$\frac{x+2}{3} = \frac{y-4}{-5} = \frac{z+5}{6}$$ **1**

5. Let X denote the number of milk chocolates drawn

X	P(X)
0	$\frac{4}{6}\times\frac{3}{5}=\frac{12}{30}$
1	$\left(\frac{2}{6}\times\frac{4}{5}\right)\times 2=\frac{16}{30}$
2	$\frac{2}{6}\times\frac{1}{5}=\frac{2}{30}$

1½

Most likely outcome is getting one chocolate of each type. ½

Commonly Made Error

- Students get confused whether the items are drawn with replacement or without replacement.

Answering Tip

- At random means without replacement.

6. $$P(\bar{A}) = 0.7 \Rightarrow 1-P(A)=0.7 \Rightarrow P(A)=0.3$$

$$P(A\cap B) = P(A)\cdot P\left(\frac{B}{A}\right)=0.3\times 0.5=0.15$$ **1**

$$P\left(\frac{A}{B}\right) = \frac{P(A\cap B)}{P(B)}=\frac{0.15}{0.7}=\frac{15}{70} \text{ or } \frac{3}{14}$$ **1**

Section - B

7. $$\int\frac{x}{x^2+3x+2}dx = \frac{1}{2}\int\frac{2x+3}{x^2+3x+2} - \frac{3}{2}\int\frac{dx}{x^2+3x+2}$$ ½

$$= \frac{1}{2}\log|x^2+3x+2| - \frac{3}{2}\int\frac{dx}{\left(x+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}$$ **1**

$$= \frac{1}{2}\log|x^2+3x+2| - \frac{3}{2} \times\frac{1}{2\times\frac{1}{2}}\log\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right|+C$$ **1**

$$\Rightarrow \int\frac{x}{x^2+3x+2}dx = \frac{1}{2}\log|x^2+3x+2| - \frac{3}{2}\log\left|\frac{x+1}{x+2}\right|+C$$ ½

8. From the given, equation,

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

Integrating, we get

$$\tan^{-1}y = \tan^{-1}x+C$$ **1**

As $x = 0, y = \sqrt{3}$ so $\tan^{-1}\sqrt{3} = C$ or $C = \frac{\pi}{3}$

Now, the required solution

$$\tan^{-1}y = \tan^{-1}x+\frac{\pi}{3}$$ **1**

$$\tan^{-1}\frac{y-x}{1+xy} = \frac{\pi}{3}$$

$$\frac{y-x}{1+xy} = \tan\frac{\pi}{3}=\sqrt{3}$$

$$y-x = \sqrt{3}(1+xy)$$ **1**

OR

$$x^2-y^2 = C(x^2+y^2)^2 \quad \text{...(i)}$$

Differentiating above

$$2x-2yy' = 2C(x^2+y^2)(2x+2yy')$$

or $$(x-yy') = \frac{x^2-y^2}{y^2+x^2}(2x+2yy')$$

using eq. (i)

or $$(y^2+x^2)(x-yy') = (x^2-y^2)(2x+2yy')$$ **1**

$$-(y^2+x^2)yy^1+x(y^2+x^2) = (x^2-y^2)2x +(x^2-y^2)2yy^1$$

$$[2y(x^2-y^2)+y(y^2+x^2)]\frac{dy}{dx}=2x(x^2-y^2)- x(y^2+x^2)$$

or $$(y^3-3x^2y)\frac{dy}{dx} = (x^3-3xy^2)$$

or $$(y^3-3x^2y)dy = (x^3-3xy^2)dx$$ **1**

Hence $x^2 - y^2 = C(x^2 + y^2)^2$ is the solution of given differential equation.

9. $$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA}=\hat{i}+4\hat{j}-\hat{k}$$

$$\overrightarrow{CD} = \overrightarrow{OD}-\overrightarrow{OC}=-2\hat{i}-8\hat{j}+2\hat{k}$$ **1**

Let required angle be θ.

Then $\cos\theta = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}||\overrightarrow{CD}|} = \dfrac{-2-32-2}{\sqrt{18}\sqrt{72}} = -1$ **1**

$\Rightarrow \quad \theta = 180°$ or π

Since $\theta = \pi$ so $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear. **1**

OR

We have, A(0, 4, 1), B(2, 3, –1), C(4, 5, 0) and D(2, 6, 2).

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2-0)\hat{i} + (3-4)\hat{j} + (-1-1)\hat{k}$

$= 2\hat{i} - \hat{j} - 2\hat{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (4-2)\hat{i} + (5-3)\hat{j} + (0+1)\hat{k}$

$= 2\hat{i} + 2\hat{j} + \hat{k}$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = (2-4)\hat{i} + (6-5)\hat{j} + (2-0)\hat{k}$

$= -2\hat{i} + \hat{j} + 2\hat{k}$

$\overrightarrow{DA} = \overrightarrow{OA} - \overrightarrow{OD} = (0-2)\hat{i} + (4-6)\hat{j} + (1-2)\hat{k}$

$= -2\hat{i} - 2\hat{j} - \hat{k}$ **1½**

Since,

$\overrightarrow{AB} = -\overrightarrow{CD}$ and

$\overrightarrow{BC} = -\overrightarrow{DA}$

Thus quadrilateral formed is parallelogram.

∴ Area of quadrilateral *ABCD*

$= |\overrightarrow{AB} \times \overrightarrow{BC}|$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{vmatrix}$

$= |3\hat{i} - 6\hat{j} + 6\hat{k}|$

$= \sqrt{9+36+36}$

= 9 sq. units **1½**

10. The equations of the given lines are:

$$\frac{x-1}{-3} = \frac{y-2}{2\lambda} = \frac{z-3}{2}$$

and $$\frac{x-1}{3\lambda} = \frac{y-1}{2} = \frac{z-6}{-5}$$

On comparing these lines with $\dfrac{x-x_1}{a_1} = \dfrac{y-y_1}{b_1} = \dfrac{z-z_1}{c_1}$ and $\dfrac{x-x_2}{a_2} = \dfrac{y-y_2}{b_2} = \dfrac{z-z_2}{c_2}$, we get

$a_1 = -3$, $b_1 = 2\lambda$, $c_1 = 2$ and $a_2 = 3\lambda$, $b_2 = 2$, $c_2 = -5$

Since, lines are perpendicular,

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

So, $(-3)(3\lambda) + (2\lambda)(2) + (2)(-5) = 0$

$\Rightarrow \quad -9\lambda + 4\lambda - 10 = 0$

$\Rightarrow \quad -5\lambda - 10 = 0$

$\Rightarrow \quad -5\lambda = 10$

$\Rightarrow \quad \lambda = -2$ **1**

Hence, for $\lambda = -2$ the given lines are perpendicular.

Now, given lines can be written as after substituting value of $\lambda = -2$

$$\frac{x-1}{-3} = \frac{y-2}{-4} = \frac{z-3}{2}$$

and $$\frac{x-1}{-6} = \frac{y-1}{2} = \frac{z-6}{-5}$$

The coordinate of any point on first line are given by

$$\frac{x-1}{-3} = \frac{y-2}{-4} = \frac{z-3}{2} = r \quad \text{(say)}$$

or

$$x = -3r+1,\ y = -4r+2,\ z = 2r+3$$

So, the coordinates of a general point on first line are $(-3r + 1, -4r + 2, 2r + 3)$.

The coordinates of any point on second line are given by

$$\frac{x-1}{-6} = \frac{y-1}{2} = \frac{z-6}{-5} = s \quad \text{(say)}$$

or $\quad x = -6s+1, y = 2s+1, z = -5s+6$

So, the coordinates of a general point on second line are $(-6s + 1, 2s + 1, -5s + 6)$

If the lines intersect, then they have a common point. So, for some value of λ and μ, we must have

$$-3r + 1 = -6s + 1, -4r + 2 = 2s + 1, 2r + 3 = -5s + 6$$

or $\quad r = 2s, -4r - 2s = -1, 2r + 5s = 3$

Solving first two of these two equations, we get $r = \dfrac{1}{5}$ and $s = \dfrac{1}{10}$. These values of r and s do not satisfy the third equation.

Hence, the given lines do not intersect. **2**

Commonly Made Error

- Some students compute the shortest distance to show that the lines are intersecting.

Answering Tip

- Learn the concepts of parallel, perpendicular, skew and intersection lines.

Section - C

11. $$I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx \qquad \text{...(i)}$$

By applying property

$$\int_0^a f(x) = \int_0^a f(a-x)$$

$$I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos x + \sin x} dx \qquad \text{...(ii)} \quad \mathbf{1}$$

Adding eqn. (i) & (ii)

$$\therefore \quad 2I = \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{1}{\cos\left(x - \frac{\pi}{4}\right)} dx \qquad \mathbf{1}$$

$$\Rightarrow \quad 2I = \frac{1}{\sqrt{2}} \left[\log\left|\sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right)\right|\right]_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}} \left[\log\left|\sqrt{2}+1\right| - \log\left|\sqrt{2}-1\right|\right]$$

$$= \frac{1}{\sqrt{2}} \log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

$$I = \frac{2}{2\sqrt{2}} \log\left|\sqrt{2}+1\right|$$

$$I = \frac{1}{\sqrt{2}} \log\left|\sqrt{2}+1\right| \qquad \mathbf{2}$$

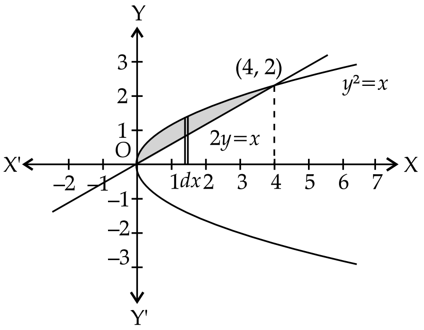
12. When $y^2 = x$ and $2y = x$

Solving we get $y^2 = 2y$

$\Rightarrow y = 0, 2$ and when $y = 2, x = 4$ and $y = 0 \Rightarrow x = 0$

So, points of intersection are (0, 0) and (4, 2). **1**

Graphs of parabola $y^2 = x$ and $2y = x$ are as shown in the adjoining figure :



1

From the figure, area of the shaded region,

$$A = \int_0^4 \left[\sqrt{x} - \frac{x}{2}\right] dx$$

$$= \left[\frac{2}{3} x^{3/2} - \frac{1}{2} \cdot \frac{x^2}{2}\right]_0^4$$

$$= \frac{2}{3} \cdot (4)^{3/2} - \frac{16}{4} - 0 = \frac{16}{3} - 4$$

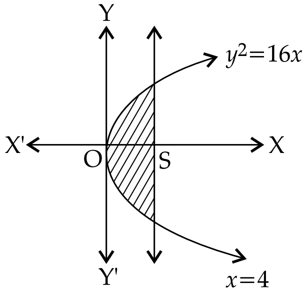
$$= \frac{4}{3} \text{ sq. units} \qquad \mathbf{2}$$

Parabola $y^2 = 16x$ and line $x = 4$

at $x = 4$, $y^2 = 64 \Rightarrow y = \pm 8$

Hence, the point of intersection (4, 8) and (4, –8)

OR



1

$$\text{Area} = 2\int_0^4 \sqrt{16x}\, dx \qquad \mathbf{1}$$

$$= 2 \times 4 \int_0^4 x^{1/2} dx$$

$$= 8\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^4 \qquad \mathbf{1}$$

$$= \frac{16}{3}\left[(4)^{3/2} - 0\right]$$

$$= \frac{16}{3} \times 8$$

$$= \frac{128}{3} \text{ sq. units} \qquad \mathbf{1}$$

13. As given

$\vec{a} \cdot \vec{b} = 0$, $\vec{a} \cdot \vec{c} = 0 = \vec{a} \perp$ both $\vec{b}$ and $\vec{c}$

$|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$ as $\vec{a}, \vec{b}, \vec{c}$ are unit vectors

or $\quad \vec{a} \parallel (\vec{b} \times \vec{c})$

Let $\quad \vec{a} = \lambda(\vec{b} \times \vec{c})$, **1**

then $\quad |\vec{a}| = |\lambda|\, |(\vec{b} \times \vec{c})|$

or $\quad \dfrac{|\vec{a}|}{|(\vec{b} \times \vec{c})|} = |\lambda|$ **1** or $|\lambda| = \dfrac{1}{\sin\frac{\pi}{6}} = 2$

$\therefore \quad \lambda = \pm 2$ **1**

$\therefore \quad \vec{a} = \pm 2(\vec{b} \times \vec{c})$ **1**

Case-Based/Data Based

14. Let E_1 = Bag I is chosen, E_2 = Bag II is chosen, E_3 = Bag III is chosen, A = The two balls drawn from the chosen bag are one white and one red.

$$P(E_1) = \frac{1}{3} = P(E_2) = P(E_3),$$

$$P\left(\frac{A}{E_1}\right) = \frac{1}{6}\times\frac{3}{6}\times 2,\ P\left(\frac{A}{E_2}\right) = \frac{2}{4}\times\frac{1}{4}\times 2,$$

$$P\left(\frac{A}{E_3}\right) = \frac{4}{9}\times\frac{2}{9}\times 2.$$

(i) By Bayes' Theorem,

Required probability

$$= P\left(\frac{E_3}{A}\right) = \frac{P(E_3)\times P(A/E_3)}{\sum_{i=1}^{3} P(E_i)\times P(A/E_i)}$$

$$= \frac{\frac{1}{3}\times\frac{4}{9}\times\frac{2}{9}\times 2}{\frac{1}{3}\times\frac{1}{6}\times\frac{3}{6}\times 2+\frac{1}{3}\times\frac{2}{4}\times\frac{1}{4}\times 2+\frac{1}{3}\times\frac{4}{9}\times\frac{2}{9}\times 2}$$

$$= \frac{\frac{16}{243}}{\frac{1}{18}+\frac{1}{12}+\frac{16}{243}}$$ **2**

$$= \frac{\frac{16}{243}}{\frac{54+81+64}{972}} = \frac{64}{199}$$

(ii) By Bayes' theorem

Required probability

$$= P\left(\frac{E_1}{A}\right) = \frac{P(E_1)\times P\left(\frac{A}{E_1}\right)}{\sum_{i=1}^{3} P(E_i)\times P\left(\frac{A}{E_i}\right)}$$

$$= \frac{\frac{1}{3}\times\frac{1}{6}\times\frac{3}{6}\times 2}{\frac{1}{3}\times\frac{1}{6}\times\frac{3}{6}\times 2+\frac{1}{3}\times\frac{2}{4}\times\frac{1}{4}\times 2+\frac{1}{3}\times\frac{4}{9}\times\frac{2}{9}\times 2}$$

$$= \frac{\frac{1}{18}}{\frac{1}{18}+\frac{1}{12}+\frac{16}{243}}$$ **2**

$$= \frac{\frac{1}{18}}{\frac{54+81+64}{972}} = \frac{54}{199}$$