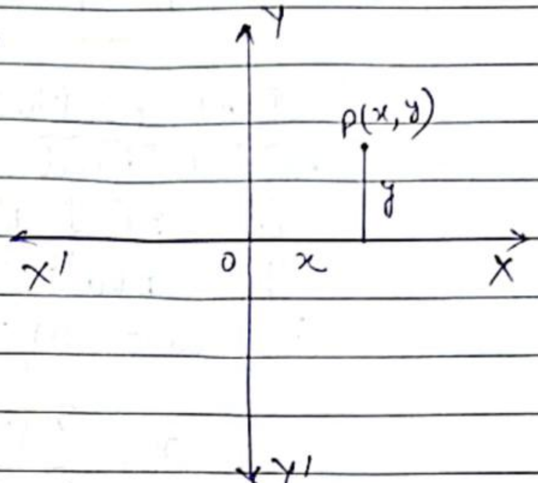
CHAPTER 10

STRAIGHT LINES

Brief Recall :

Co-ordinates :

Let XOX' and YOY' be two fixed straight lines whose fixed directions are $X'OX$ and $Y'OY$ which are mutually perpendicular to each other and meet at 'O'. Then



(i) $X'OX$ is called x-axis

(ii) $Y'OY$ is called y-axis

(iii) The Two together taken in this particular order are called the Rectangular axes or simply axes.

(iv) O is called the origin.

Let P be a point in the plane of rectangular axes. From P draw $PM \perp$ on the x-axis. Then

(i) The length OM is called Abscissa of P, denoted by x, also called x-coordinate.

(ii) The length PM is called Ordinate of P, denoted by y, also called y-coordinate.

(iii) The two together taken in this particular order are called the rectangular co-ordinates or simple Coordinates of P and are denoted by $P(x, y)$

NOTE : (1) The coordinates of Origin are $O(0, 0)$

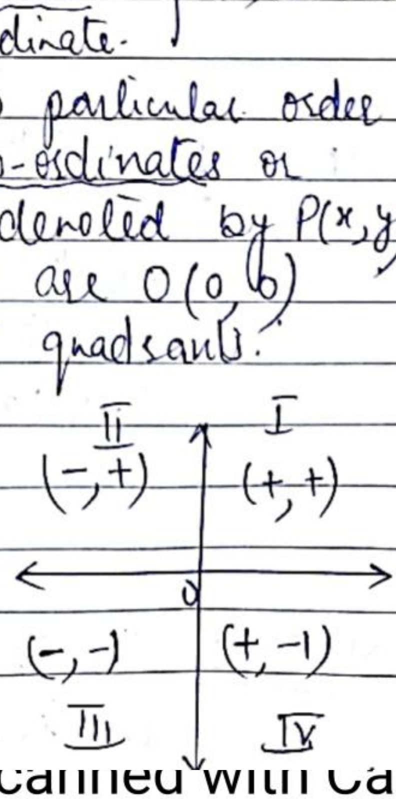
(2) Signs of co-ordinates in different quadrants.

(i) In first quadrant $(+, +)$

(ii) In Second " $(-, +)$

(iii) In Third " $(-, -)$

(iv) In Forth " $(+, -)$



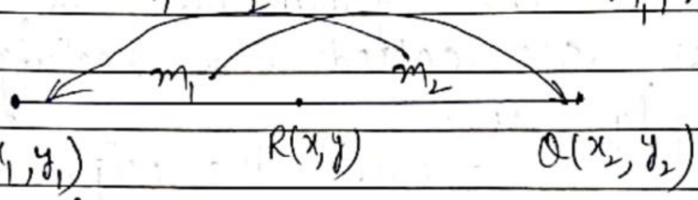
DISTANCE FORMULA: Distance between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$|PQ| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

SECTION FORMULA:

(a) **For Internal Division:** The coordinates $R(x, y)$ of a point R dividing the line segment joining the points $P(x_1, y_1)$ & $Q(x_2, y_2)$ internally in the ratio $m_1 : m_2$ are given by

$$x = \frac{m_1x_2+m_2x_1}{m_1+m_2} \; ; \; y = \frac{m_1y_2+m_2y_1}{m_1+m_2}$$



Mid-Point formula:

Since the midpt divides the line segment in ratio 1:1
∴ from section formula, the coordinates of mid point are given by

$$x = \frac{x_1+x_2}{2} \; ; \; y = \frac{y_1+y_2}{2}$$

(b) **For External Division:**

The Coordinates of a point dividing the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ externally in ratio $m_1 : m_2$ are given by

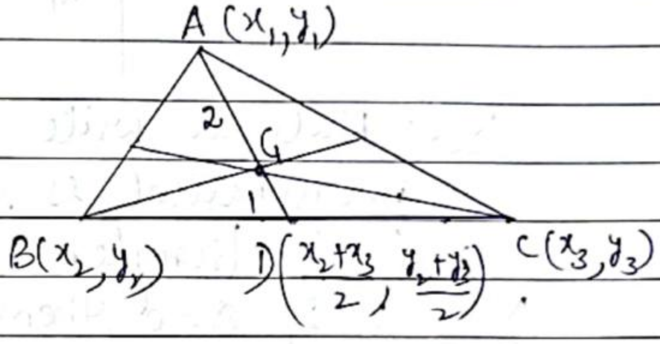
$$x = \frac{m_1x_2-m_2x_1}{m_1-m_2} \; ; \; y = \frac{m_1y_2-m_2y_1}{m_1-m_2}$$

NOTE: In case of external division, change m_2 to $-m_2$ in result of internal division.

CENTROID OF A TRIANGLE :

The point where the medians of a triangle meet i.e. the Point of concurrency of all the three medians of a triangle is called Centroid of Triangle.

Centroid divides each median in ratio 2:1, 2 parts towards vertex and 1 part towards side. By using Mid point formula and section formula, we find that The coordinates of centroid of triangle are given by



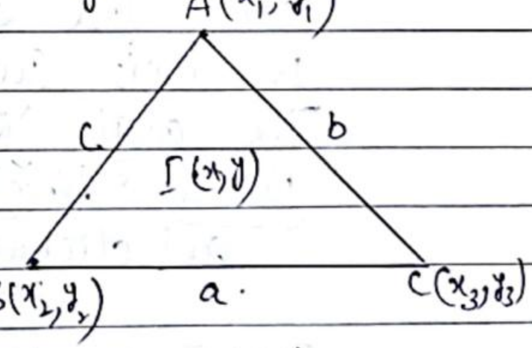
$$x = \frac{x_1+x_2+x_3}{3}$$

$$y = \frac{y_1+y_2+y_3}{3}$$

Incentre of a triangle:

The point, where the internal bisectors of the angles of a triangle meet is called Incentre of triangle.

Let ΔABC be a triangle with vertices $A(x_1, y_1)$; $B(x_2, y_2)$; $C(x_3, y_3)$ and $BC = a$, $CA = b$; $AB = c$.



Then Coordinates of Incentre I are given by

$$x = \frac{ax_1 + bx_2 + cx_3}{a+b+c} \quad ; \quad y = \frac{ay_1 + by_2 + cy_3}{a+b+c}$$

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AREA OF TRIANGLE : Area of ΔABC with vertices $A(x_1, y_1)$; $B(x_2, y_2)$; $C(x_3, y_3)$ is given by

$$\Delta = \frac{1}{2} \left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|$$

Rule to write of Δ

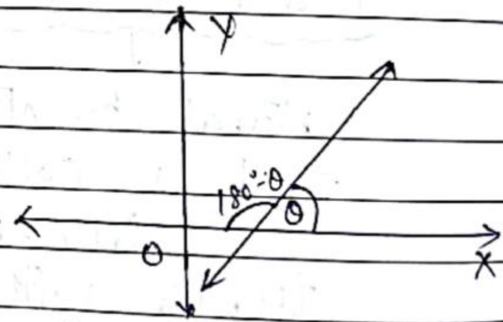
$$\Delta = \frac{1}{2} \left| \begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{array} \right|$$

Note: Rule to write down the area of quadrilateral is same as to write down the area of triangle.

NOTE : If $\Delta = 0$, then the three points will be collinear.

INCLINATION and SLOPE (GRADIENT) of a LINE :

A line in a coordinate plane forms two angles with the x-axis, which are supplementary ie their sum is 180°.



The angle 'θ' made by the line with the positive direction of x-axis and measured anticlockwise is called INCLINATION of the line.
It must be noted that $0° < \theta \leq 180°$

SLOPE OF LINE :

If θ is the inclination of line l, then '$\tan\theta$' is slope or gradient of line. The slope of line is generally denoted by 'm'

ie $\boxed{m = \tan\theta}$; $\theta \neq 90°$

NOTE: Slope of x-axis is Zero and slope of y-axis is not defined.

SLOPE OF LINE WHEN COORDINATES OF ANY TWO POINTS ON LINE ARE GIVEN:

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points on a line l making an angle θ with +ve direction of x-axis ie inclination of line is θ. Obviously $x_1 \neq x_2$ $\because$ otherwise the line will become perpendicular to x-axis and its slope will not be defined. Draw $QR \perp x$-axis and $PM \perp QR$. (As shown in fig)
Now since θ could be acute as well as obtuse So two cases arise.

CASE I : When θ is acute angle ($< 90°$)

In fig (i)

$\angle MPQ = \theta$ $\therefore$ Slope of line $l = m = \tan\theta$

In rt $\angle$d ΔPMQ

$$\tan\theta = \frac{MQ}{MP}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

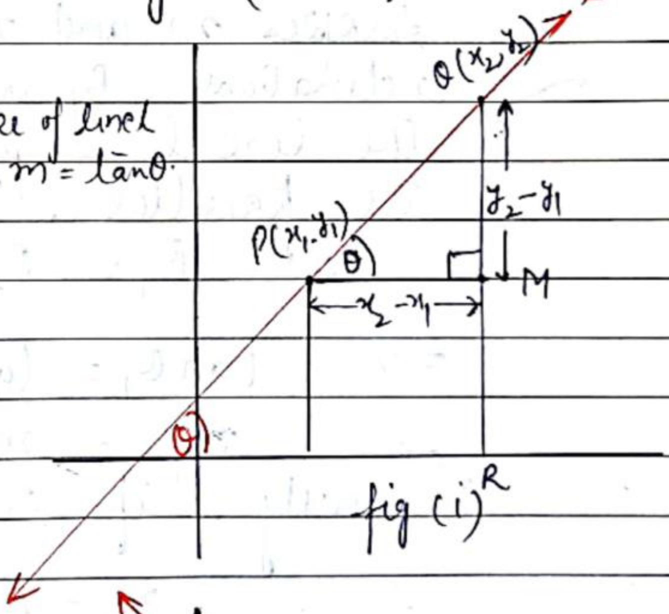


fig (i)

$\Rightarrow$ Slope of line l

$= m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$

CASE II : When θ is obtuse angle ie $90° < \theta < 180°$

In fig (ii) $\angle MPQ = 180° - \theta$

$\therefore$ $\theta = 180° - \angle MPQ$

$\therefore$ $\tan\theta = \tan(180° - \angle MPQ) = -\tan \angle MPQ$

Now In rt $\angle$d Δ MPQ

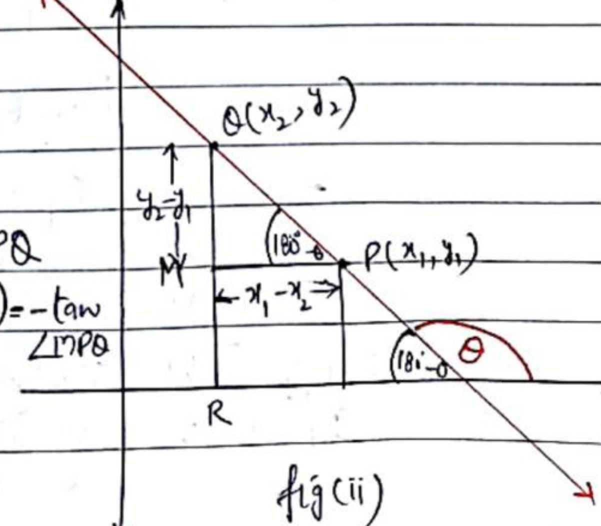


fig (ii)

$$-\tan \angle MPR = -\frac{MQ}{MP} = -\frac{y_2-y_1}{x_1-x_2} = \frac{y_2-y_1}{x_2-x_1}.$$

$\therefore$ In both cases

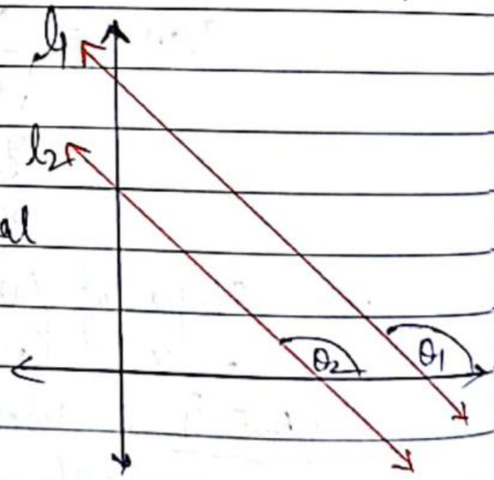
The slope of line l through the points (x_1, y_1) and (x_2, y_2) is given by

$$\boxed{m = \frac{y_2-y_1}{x_2-x_1}}$$

Condition for Parallelism and perpendicularity of lines in terms of their slopes:

Let two non-vertical lines l_1 and l_2 have slopes m_1 and m_2 respectively, having inclinations θ_1 and θ_2.

If the line l_1 is parallel to l_2: If the lines are parallel, then their inclications are equal, i.e. $\theta_1 = \theta_2$



$\Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow m_1 = m_2.$

Conversely, if slopes are equal

$\Rightarrow m_1 = m_2$

$\Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow \theta_1 = \theta_2$ (By property of tangent fn between 0° and 180°)

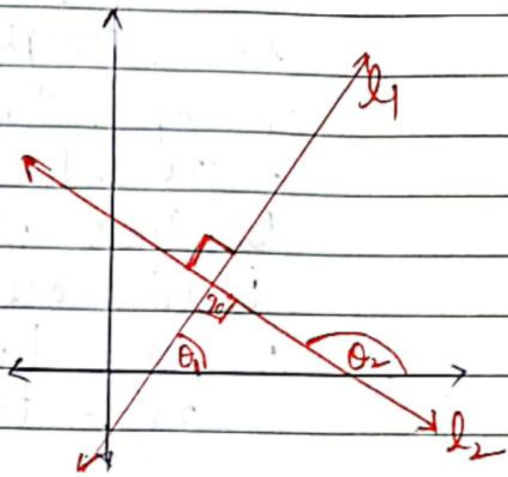
$\Rightarrow$ lines are parallel.

Hence Two non-vertical lines l_1 and l_2 are parallel if and only if their slopes are equal.

If the lines l_1 and l_2 are perpendicular:

Since $l_1 \perp l_2$

$\therefore$ from fig



$$\theta_2 = 90° + \theta_1$$

$$\Rightarrow \tan\theta_2 = \tan(90° + \theta_1)$$

$$\Rightarrow \tan\theta_2 = -\cot\theta_1$$

$$\Rightarrow \tan\theta_2 = -\frac{1}{\tan\theta_1}$$

$$\Rightarrow m_2 = -\frac{1}{m_1}$$

ie $m_1 m_2 = -1$

Conversely: If $m_1 m_2 = -1$

$$\Rightarrow \tan\theta_1 \cdot \tan\theta_2 = -1$$

$$\Rightarrow \tan\theta_1 = -1/\tan\theta_2 = -\cot\theta_2$$

$$\Rightarrow \tan\theta_1 = \tan(90 + \theta_2) \text{ or } \tan(\theta_2 - 90°)$$

$\Rightarrow$ θ_1 and θ_2 differ by 90°

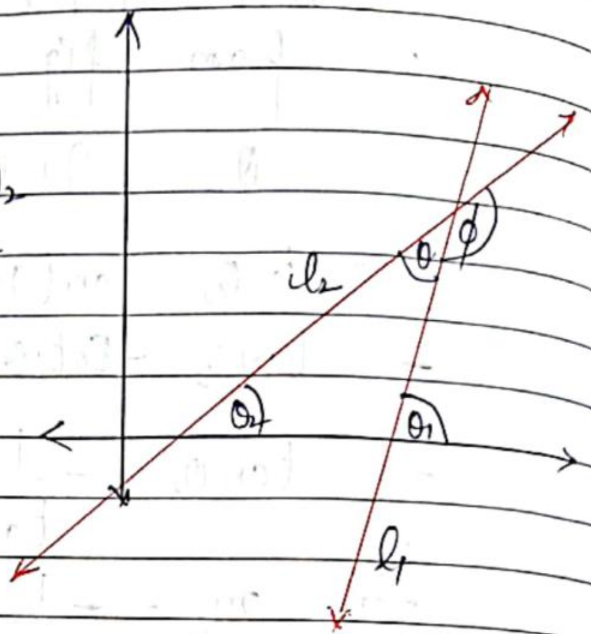
$\Rightarrow$ lines l_1 and l_2 are perpendicular.

Hence Two non-vertical lines are perpendicular to each other if and only iff their slopes are negative reciprocals of each other

ie $m_2 = -\frac{1}{m_1}$ or $m_1 m_2 = -1$.

ANGLE BETWEEN TWO STRAIGHT LINES:

Let l_1 and l_2 be two straight lines having slopes m_1 and m_2 resp. and let θ_1 and θ_2 be their respective inclinations.



$\therefore\ m_1 = \tan\theta_1$

$m_2 = \tan\theta_2$

Now from fig

$$\theta_1 = \theta + \theta_2 \quad \left[\text{Exterior angle of } \Delta \text{ is equal to sum of opposite interior angles}\right]$$

$$\Rightarrow \theta = \theta_1 - \theta_2$$

$$\Rightarrow \tan\theta = \tan(\theta_1 - \theta_2)$$

$$= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

i.e. $$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)}$$

NOTE 1. If $\tan\theta$ is +ve then θ is acute angle and if $\tan\theta$ is −ve, the θ is obtuse angle. But to find the angle between the lines means to find acute angle between the lines. Therefore if $\tan\theta$ is −ve

We dissegard the -ve sign and find the angle between the lines.

NOTE 2: If ϕ is the other angle between the lines, then $\phi = \pi - \theta$

$$\Rightarrow \tan\phi = \tan(\pi - \theta)$$

$$= -\tan\theta$$

$$= -\frac{m_1 - m_2}{1 + m_1 m_2}.$$

Combining, we get

$$\tan\theta = \pm \frac{m_1 - m_2}{1 + m_1 m_2}.$$

Which is called Complete angle formula.

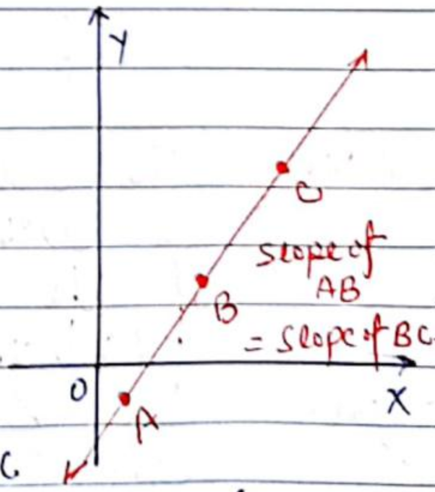
Also we can write $\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$.

COLLINEARITY OF THREE POINTS:

Three points A, B, C in xy-plane will be collinear iff they lie on same line.

ie line segments AB and BC are segments of same line.

$\Rightarrow$ slope of AB = slope of BC

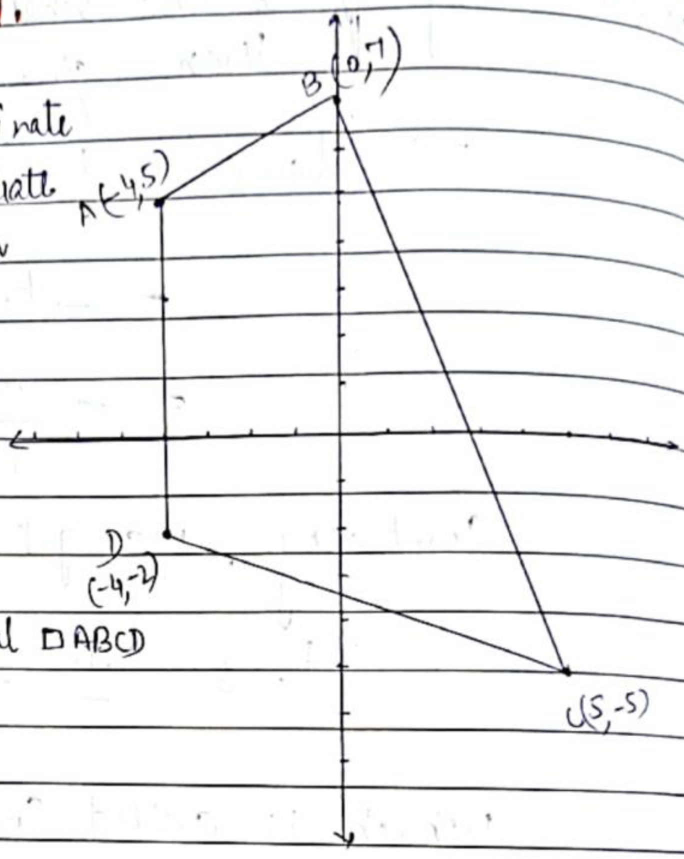


So Three points A, B, C will be collinear iff slope of AB = slope of BC.

EXERCISE 10.1

Q.No 1: Draw a quadrilateral in the Cartesian plane whose vertices are $(-4,5)$, $(0,7)$, $(5,-5)$ and $(-4,-2)$. Also find its area.

Soln: Take retangular coordinate axes and mark appropriate scales. and let them $A(-4,5)$; $B(0,7)$ $C(5,-5)$; $D(-4,-2)$. So $\square ABCD$ is the required Quadrilateral.



Area of quadrilateral $\square ABCD$

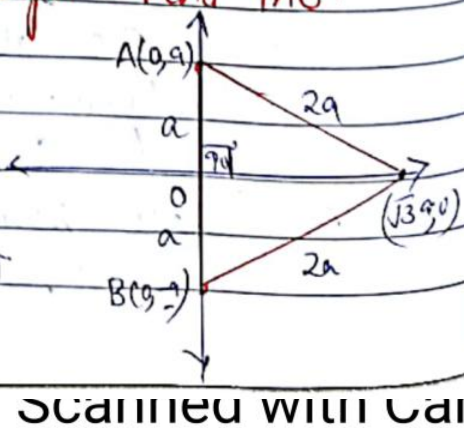
$$= \frac{1}{2}\begin{vmatrix} -4 & 5 \\ 0 & 7 \\ 5 & -5 \\ -4 & -2 \\ -4 & -5 \end{vmatrix}$$

$$= \frac{1}{2}\left|-28-0+0-35-10-20-20-8\right|$$

$$= \frac{1}{2}\left|-121\right| = \frac{1}{2}\times 121 = 60.5 \text{ square units.}$$

Q.No 2: The base of an equilateral triangle with side $2a$ lies along y-axis such that the mid point of the base is at the origin. Find the vertices of the triangle.

Sol.: Let ΔABC be the equilateral triangle with base $AB = 2a$ along y-axis. Now O is mid point



of AB

$\therefore \quad OA = a = OB$

$\therefore$ A is $A(0, a)$ and B is $B(0, -a)$

Also $AC = 2a = BC$

Since ΔABC is equilateral triange (समभुजी त्रिभुज)

$\therefore \angle COA = 90°$

$\therefore$ In rt $\angle d$ ΔAOC

$OC^2 = AC^2 - OA^2$

$= (2a)^2 - (a)^2 = 4a^2 - a^2 = 3a^2$

$\Rightarrow OC = \sqrt{3a^2} = \sqrt{3}\, a.$

$\therefore$ C is $C(\sqrt{3}a, 0)$

QNo 3: Find the distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ when (i) PQ is parallel to y-axis (ii) PQ is parallel to x-axis

Sol: By distance formula distance between P and Q is

$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

(i) When PQ is parallel to y-axis

$\Rightarrow x_1 = x_2$

$\therefore |PQ| = \sqrt{0 + (y_2 - y_1)^2} = \sqrt{(y_2 - y_1)^2} = |y_2 - y_1|$

(ii) When PQ is parallel to x-axis

$y_1 = y_2$

$\therefore |PQ| = \sqrt{(x_2 - x_1)^2 + 0^2} = \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|$.

QNo 4: Find a point on x-axis which is equidistant from the points (7, 6) and (3, 4)

Soln: Let $P(x, 0)$ be any point on x-axis which is equidistant from points $A(7, 6)$ and $B(3, 4)$

(ATQ means According to question)

ATQ. $PA = PB$

$\Rightarrow PA^2 = PB^2$

$\Rightarrow (x-7)^2+(0-6)^2 = (x-3)^2+(0-4)^2$

$\Rightarrow x^2+49-14x+36 = x^2+9-6x+16$

$\Rightarrow -8x = -60$

$\Rightarrow x = \frac{-60}{-8} = \frac{15}{2}$

$\therefore$ Required point is $\left(\frac{15}{2}, 0\right)$

Q No. 5: Find the slope of line which passes through the origin and mid point of the line segment joining the points $P(0,-4)$ and $B(8,0)$

Sol.: Let A be the mid point of $P(0,-4)$ and $B(8,0)$

$\therefore$ By, Mid point formula

A is $A\left(\frac{0+8}{2}, \frac{-4+0}{2}\right)$ ie $A(4,-2)$

$\therefore$ Slope m of line passing through origin $O(0,0)$ and $A(4,-2)$

$$m = \frac{-2-0}{4-0} = \frac{-2}{4} = -\frac{1}{2}$$

Q No 6: Without using the Pythagoras theorem show that the points $(4,4)$, $(3,5)$ and $(-1,-1)$ are vertices of a rt $\angle$d Δ.

Sol.: Let $A(4,4)$; $B(3,5)$ and $C(-1,-1)$ be the given points.

Let m_1, m_2, m_3 be the slopes of AB, BC, CA respectively.

$\therefore m_1 = \frac{5-4}{3-4} = \frac{1}{-1} = -1$

$m_2 = \frac{-1-5}{-1-3} = \frac{-6}{-4} = \frac{3}{2}$

$m_3 = \frac{4+1}{4+1} = \frac{5}{5} = 1$

Now since $m_1 m_3 = -1 \times (1) = -1$

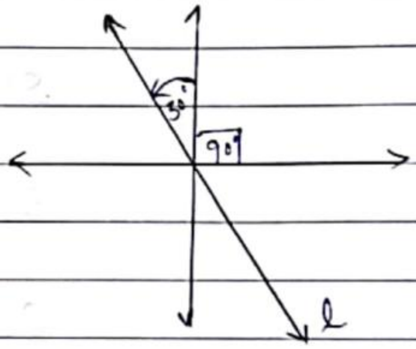
$\Rightarrow AB \perp CA$

Also $m_1 \neq m_2 \Rightarrow AB$ is not parallel to BC

$\therefore \triangle ABC$ is rt $\angle$d $\triangle$.

Q.No.7: Find the slope of line which makes an angle of 30° with the positive direction of y-axis measured anticlockwise.

Sol: Since the line makes an angle of 30° with y-axis
$\therefore$ line will make an angle of $\theta = 90^\circ + 30^\circ$ with +ve direction of x-axis



$\therefore$ slope of line m

$= \tan(120^\circ)$

$= \tan(90^\circ + 30^\circ)$

$= -\cot 30^\circ$

$= -\sqrt{3}$

OR $\tan(180^\circ - 60^\circ)$

$= -\tan 60^\circ$

$= -\sqrt{3}$.

Q.No.8: Find the value of x for which the points $(x, -1)$ $(2, 1)$ and $(4, 5)$ are collinear.

Soln: Let $A(x, -1)$; $B(2, 1)$; $C(4, 5)$ be given pts.

Since A, B and C are collinear.

$\therefore$ slope of AB = slope of BC

$\Rightarrow \frac{1-(-1)}{2-x} = \frac{5-1}{4-2}$

$\Rightarrow \frac{1+1}{2-x} = \frac{4}{2}$

$\Rightarrow \frac{2}{2-x} = \frac{2}{1}$

$\Rightarrow 2 = 2(2-x)$

$\Rightarrow 2 = 4-2x$

$\Rightarrow 2x = 2$

$\Rightarrow \boxed{x = 1.}$

QNo.9: Without using distance formula, show that the points $(-2,-1)$ $(4,0)$, $(3,3)$ and $(-3,2)$ are the vertices of a parallelogram.

Soln. Let $A(-2,-1)$, $B(4,0)$, $C(3,3)$, $D(-3,2)$ be given points.

Slope of $AB = \frac{0+1}{4+2} = \frac{1}{6}$

Slope of $BC = \frac{3-0}{3-4} = \frac{3}{-1} = -3$

Slope of $CD = \frac{2-3}{-3-3} = \frac{-1}{-6} = \frac{1}{6}$

Slope of $DA = \frac{-1-2}{-2+3} = \frac{-3}{1} = -3$

Since Slope of AB $\neq$ Slope of BC

$\Rightarrow$ A, B, C are not collinear.

Also slope of CD $\neq$ Slope of DA

$\Rightarrow$ A, C, D are not collinear.

$\therefore$ A, B, C, D form a quadrilateral.

Again slope of AB = Slope of CD

$\Rightarrow AB \parallel CD$

$\parallel$ly Slope of BC = slope of DA

$\Rightarrow BC \parallel DA$

$\Rightarrow$ Opposite sides of quadrilateral are parallel.

$\Rightarrow$ ABCD is a parallelogram.

Q No 10: Find the angle between x-axis and line joining the points ~~x~~ (3, -1) and (4, -2)

Sol: Let θ be the angle which the line joining (3, -1) and (4, -2) makes with x-axis

$$\therefore \tan\theta = \text{slope of line}$$

$$\Rightarrow \tan\theta = \frac{-2-(-1)}{4-3}$$

$$\Rightarrow \tan\theta = \frac{-2+1}{1}$$

$$\Rightarrow \tan\theta = -1$$

$$\Rightarrow \tan\theta = -\tan 45^\circ$$

$$\Rightarrow \tan\theta = \tan(180-45^\circ)$$

$$\Rightarrow \tan\theta = \tan 135^\circ$$

$$\Rightarrow \theta = 135^\circ$$

Q No 11: The slope of a line is double of slope of another line. If the tangent of angle between them is $\frac{1}{3}$ find the slopes of lines.

Sol: Let slope of one line be m.

slope of other line $= 2m$.

Let θ be the angle between the lines

$$\therefore \tan\theta = \left|\frac{2m-m}{1+2m\times m}\right| \quad \left[\because \tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|\right]$$

$$\Rightarrow \frac{1}{3} = \left|\frac{m}{1+2m^2}\right|$$

$$\Rightarrow \frac{m}{1+2m^2} = \pm\frac{1}{3}$$

$$\Rightarrow \text{Either } \frac{m}{1+2m^2} = \frac{1}{3} \quad \text{or} \quad \frac{m}{1+2m^2} = -\frac{1}{3}$$

$\Rightarrow$ Either $3m = 1+2m^2$ or $-3m = 1+2m^2$

$\Rightarrow$ Either $2m^2 - 3m + 1 = 0$ or $2m^2 + 1 + 3m = 0$

$\Rightarrow$ Either $m = \frac{-(-3) \pm \sqrt{(-3)^2 - 4\times2\times1}}{2\times2}$ or $m = \frac{-3 \pm \sqrt{9-8}}{4}$

$= \frac{3 \pm \sqrt{9-8}}{4}$ $\quad$ $= \frac{-3 \pm 1}{4}$

$= \frac{3 \pm 1}{4}$ $\quad$ $= -1, \frac{-1}{2}$

$= 1, \frac{1}{2}$

$\therefore$ Slopes are either $1, 2$ or $\frac{1}{2}, 1$ or $-1, -2$ or $-\frac{1}{2}, -1$.

Q No. 12 A line passes through (x_1, y_1) and (h, k). If slope of line is m show that $k - y_1 = m(h - x_1)$

Sol: Since the line passes through (x_1, y_1) and (h, k)

$\therefore$ Slope $m = \frac{k - y_1}{h - x_1}$

$\Rightarrow m(h - x_1) = k - y_1$

or $k - y_1 = m(h - x_1)$

Q No 13 If three points $(h, 0)$, (a, b), $(0, k)$ lie on a line show that $\frac{a}{h} + \frac{b}{k} = 1$.

Sol: Let the given points be $A(h, 0)$ $B(a, b)$, $C(0, k)$

Since given points lie on a line.

$\therefore$ A, B, C are collinear points.

$\therefore$ Slope of AB = slope of AC

$\Rightarrow \frac{b-0}{a-h} = \frac{k-0}{0-h}$

or $\dfrac{b}{a-h} = \dfrac{k}{-h}$

$\Rightarrow \quad -bh = k(a-h)$

$\Rightarrow \quad -bh = ka - kh$

$\Rightarrow \quad ka + bh = kh$

Dividing both sides by kh.

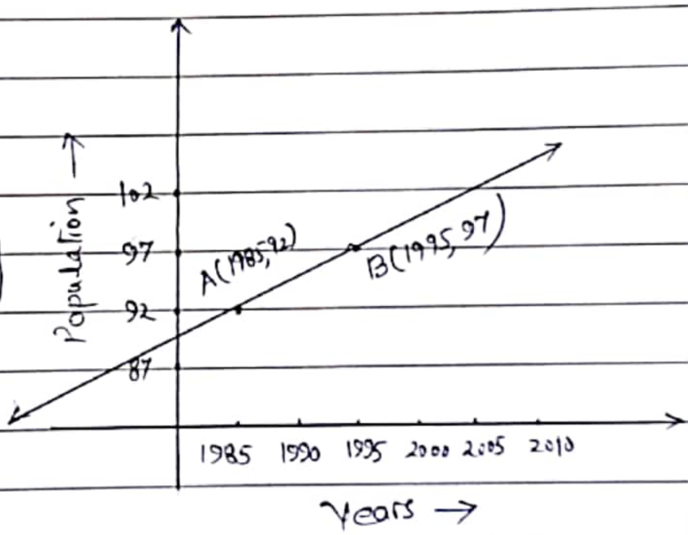
$$\frac{ka}{kh} + \frac{bh}{kh} = \frac{kh}{kh}$$

$$\Rightarrow \quad \frac{a}{h} + \frac{b}{k} = 1.$$

QNo 14: Consider the following population and year graph. Find the slope of line AB using it. Find what will be population in 2010.

Sol. From graph.

Slope of line AB



$$m = \frac{97-92}{1995-1985} \quad \left[\because m = \frac{y_2 - y_1}{x_2 - x_1}\right]$$

$$= \frac{5}{10}$$

$$m = \frac{1}{2}$$

Now Let y be population in year 2010

$\therefore$ C(2010, y) lies on AB

$\therefore$ Slope of AB = Slope of BC

$\Rightarrow \quad \dfrac{1}{2} =$ slope of BC

$$\Rightarrow \quad \frac{1}{2} = \frac{y-97}{2010-1995}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{y-97}{15} \quad \Rightarrow \quad 15 = 2y - 194$$

$$\Rightarrow \quad 2y = 209 \quad \Rightarrow \quad y = \frac{209}{2} = 104.5.$$