15. Finally

Partial Fractions

Case	Fraction $\frac{N(x)}{D(x)}$	Form of denominator, D(x)	Partial Fraction Form (where A, B and C are unknown constants)
1	$\frac{N(x)}{(ax+b)(cx+d)}$	Linear Factors	$\frac{A}{ax+b}+\frac{B}{cx+d}$
2	$\frac{N(x)}{(ax+b)^2}$	Repeated Linear Factors	$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$
	$\frac{N(x)}{(ax+b)(cx+d)^2}$	Linear and Repeated Linear Factors	$\frac{A}{ax+b}+\frac{B}{cx+d}+\frac{C}{(cx+d)^2}$
3	$\frac{N(x)}{(ax+b)(x^2+c^2)}$	Linear and Quadratic (which cannot be factorised) Factors	$\frac{A}{ax+b}+\frac{Bx+C}{x^2+c^2}$

An expression of the form $\frac{f(x)}{g(x)}$, where f(x) and g(x) are polynomial in x, is called a rational fraction.

1. **Proper rational functions:** Functions of the form $\frac{f(x)}{g(x)}$, where f(x) and g(x) are polynomials and $g(x) \neq 0$, are called rational functions of x.
   If degree of f(x) is less than degree of g(x),then is called a proper rational function.
2. **Improper rational functions:** If degree of f(x) is greater than or equal to degree of g(x), then $\frac{f(x)}{g(x)}$ is called an improper rational function.
3. **Partial fractions:** Any proper rational function can be broken up into a group of different rational fractions, each having a simple factor of the denominator of the original rational function. Each such fraction is called a partial fraction.

If by some process, we can break a given rational function $\frac{f(x)}{g(x)}$ into different fractions, whose denominators are the factors of g(x),then the process of obtaining them is called the resolution or decomposition of $\frac{f(x)}{g(x)}$ into its partial fractions.

Different cases of partial fractions

(1) When the denominator consists of non-repeated linear factors:
To each linear factor (x – a) occurring once in the denominator of a proper fraction, there corresponds a single partial fraction of the form $\frac{A}{x-a}$, where A is a constant to be determined.
If $g(x) = (x - a_1)(x - a_2)(x - a_3) \ldots\ldots (x - a_n)$, then we assume that,

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \ldots\ldots + \frac{A_n}{x-a_n}$$

where $A_1, A_2, A_3, \ldots\ldots A_n$ are constants, can be determined by equating the numerator of L.H.S. to the numerator of R.H.S. (after L.C.M.) and substituting $x = a_1, a_2, \ldots\ldots a_n$.

(2) When the denominator consists of linear factors, some repeated:
To each linear factor (x − a) occurring r times in the denominator of a proper rational function, there corresponds a sum of r partial fractions.
Let $g(x) = (x - a)^k(x - a_1)(x - a_2) \ldots\ldots (x - a_r)$. Then we assume that

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots\ldots + \frac{A_k}{(x-a)^k} + \frac{B_1}{(x-a_1)} + \ldots\ldots + \frac{B_r}{(x-a_r)}$$

Where $A_1, A_2, A_3, \ldots\ldots\ldots A_k$ are constants. To determine the value of constants adopt the procedure as above.

(3) When the denominator consists of non-repeated quadratic factors:
To each irreducible non repeated quadratic factor $ax^2 + bx + c$, there corresponds a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined.
Example :

$$\frac{4x^2+2x+3}{(x^2+4x+9)(x-2)(x+3)} = \frac{Ax+B}{x^2+4x+9} + \frac{C}{x-2} + \frac{D}{x+3}$$

$$(1)\quad \frac{px+q}{x^2(x-a)} = \frac{-q}{ax^2} - \frac{pa+q}{a^2x} + \frac{pa+q}{a^2(x-a)}$$

$$(2)\quad \frac{px+q}{x(x-a)^2} = \frac{q}{a^2x} - \frac{q}{a^2(x-a)} + \frac{pa+q}{a(x-a)^2}$$

$$(3)\quad \frac{px+q}{x(x^2+a^2)} = \frac{q}{a^2x} + \frac{pa^2-qx}{a^2(x^2+a^2)}$$

(4) When the denominator consists of repeated quadratic factors:

To each irreducible quadratic factor $ax^2 + bx + c$ occurring r times in the denominator of a proper rational fraction there corresponds a sum of r partial fractions of the form.

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots\ldots\ldots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

where, A’s and B’s are constants to be determined.

Partial fractions of improper rational functions

If degree of is greater than or equal to degree of g(x), then $\frac{f(x)}{g(x)}$ is called an **improper rational function** and every rational function can be transformed to a proper rational function by dividing the numerator by the denominator.
We divide the numerator by denominator until a remainder is obtained which is of lower degree than the denominator.

General method of finding out the constants

1. Express the given fraction into its partial fractions in accordance with the rules written above.
2. Then multiply both sides by the denominator of the given fraction and you will get an identity which will hold for all values of x.
3. Equate the coefficients of like powers of x in the resulting identity and solve the equations so obtained simultaneously to find the various constant is short method. Sometimes, we substitute particular values of the variable x in the identity obtained after clearing of fractions to find some or all the constants. For non-repeated linear factors, the values of x used as those for which the denominator of the corresponding partial fractions become zero.

What is Mathematical Induction in Discrete Mathematics?

First principle of Mathematical induction

The proof of proposition by mathematical induction consists of the following three steps :

Step I : (Verification step) : Actual verification of the proposition for the starting value "i".

Step II : (Induction step) : Assuming the proposition to be true for "k", $k \geq i$ and proving that it is true for the value $(k + 1)$ which is next higher integer.

Step III : (Generalization step) : To combine the above two steps. Let p(n) be a statement involving the natural number n such that

1. p(1) is true i.e. p(n) is true for $n = 1$.
2. $p(m + 1)$ is true, whenever p(m) is true i.e. p(m) is true $\Rightarrow p(m + 1)$ is true.

Then p(n) is true for all natural numbers n.

Second principle of Mathematical induction

The proof of proposition by mathematical induction consists of following steps :

Step I : (Verification step) : Actual verification of the proposition for the starting value i and $(i + 1)$.

Step II : (Induction step) : Assuming the proposition to be true for $k - 1$ and k and then proving that it is true for the value $k + 1$; $k \geq i + 1$.

Step III : (Generalization step) : Combining the above two steps. Let p(n) be a statement involving the natural number n such that

1. p(1) is true i.e. p(n) is true for $n = 1$ and
2. $p(m + 1)$ is true, whenever p(n) is true for all n, where $i \leq n \leq m$.

Then p(n) is true for all natural numbers.
For $a \neq b$, The expression is divisible by
(a) $a + b$, if n is even.
(b) $a - b$, if n is odd or even.

Divisibility problems

To show that an expression is divisible by an integer

1. If *a*, *p*, *n*, *r* are positive integers, then first of all we write $a^{pn+r} = a^{pn} . a^r = (a^p)^n . a^r$.
2. If we have to show that the given expression is divisible by *c*.

Then express, $a^p = [1 + (a^p - 1)]$, if some power of $(a^p - 1)$ has *c* as a factor. $a^p = [2 + (a^p - 2)]$, if some power of $(a^p - 2)$ has *c* as a factor.
$a^p = [k + (a^p - k)]$, if some power of $(a^p - k)$ has *c* as a factor.

Mathematical Induction Problems with Solutions

1. For all positive integral values of *n*, $3^{2n} - 2n + 1$ is divisible by
(a) 2
(b) 4
(c) 8
(d) 12

Solution:
Putting $n = 2$ in $3^{2n} - 2n + 1$ then, $3^{2(2)} - 2\times2 + 1 = 81 - 4 + 1 = 78$, which is divisible by 2.

2. If $n \in N$, then $x^{2n-1} + y^{2n-1}$ is divisible by
(a) $x + y$
(b) $x - y$

(c) $x^2 + y^2$
(d) $x^2 + xy$

Solution:
$x^{2n-1} + y^{2n-1}$ is always contain equal odd power. So it is always divisible by x + y.

3. If n ∈ N, then $7^{2n} + 2^{3n-3} . 3^{n-1}$ is always divisible by
(a) 25
(b) 35
(c) 45
(d) None of these

Solution:

Putting $n = 1$ in $7^{2n} + 2^{3n-3}.3^{n-1}$

then, $7^{2\times1} + 2^{3\times1-3}.3^{1-1}$

$= 7^2 + 2^0.3^0 = 49 + 1 = 50$ ......(i)

Also, $n = 2$

$7^{2\times2} + 2^{3\times2-3}.3^{2-1} = 2401 + 24 = 2425$ ......(ii)

From (i) and (ii) it is always divisible by 25.

4. If n ∈ N, then $11^{n+2} + 12^{2n+1}$ is divisible by
(a) 113
(b) 123
(c) 133
(d) None of these

Solution:

Putting $n = 1$ in $11^{n+2} + 12^{2n+1}$

We get, $11^{1+2} + 12^{2\times1+1} = 11^3 + 12^3 = 3059$, which is divisible by 133.

5. The remainder when 5^{99} is divided by 13 is
(a) 6
(b) 8
(c) 9
(d) 10

Solution:

$5^{99} = (5)(5^2)^{49} = 5(25)^{49} = 5(26-1)^{49}$

$= 5 \times (26) \times$ (Positive terms) $- 5$, So when it is divided by 13 it gives the remainder -5 or $(13 - 5)$ *i.e.*, 8.

6. When 2^{301} is divided by 5, the least positive remainder is
(a) 4
(b) 8
(c) 2
(d) 6

Solution:

$$2^4 \equiv 1 \pmod 5; \Rightarrow (2^4)^{75} \equiv (1)^{75} \pmod 5$$

$$i.e.\ 2^{300} \equiv 1 \pmod 5 \Rightarrow 2^{300} \times 2 \equiv (1.2) \pmod 5$$

$$\Rightarrow 2^{301} \equiv 2 \pmod 5,\ \therefore \text{ Least positive remainder is } 2.$$

7. For a positive integer *n*,

Let $a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{(2^n) - 1}$. Then

(a) $a(100) \le 100$ (b) $a(100) > 100$

(c) $a(200) \le 100$ (d) $a(200) > 100$

Solution:

It can be proved with the help of mathematical induction

that $\frac{n}{2} > a(n) \le n$.

$\therefore \frac{200}{2} < a(200) \Rightarrow a(200) > 100$ and $a(100) \le 100$.

8. $10^n + 3(4^{n+2}) + 5$ is divisible by ($n \in N$)

(a) 7
(b) 5
(c) 9
(d) 17
(e) 13

Solution:

$10^n + 3(4^{n+2}) + 5$

Taking $n = 2$; $10^2 + 3 \times 4^4 + 5$

$= 100 + 768 + 5 = 873$

Therefore this is divisible by 9.

The Binomial Theorem

You are faced with the problem of expanding $(x + y)^{10}$. What to do??? Do you really have to multiply this expression times itself 10 times?? That could take forever.

The Binomial Theorem

- Gives us the coefficients for a binomial expansion
- The values in a row of Pascal's triangle are the coefficients in a binomial expansion of the same degree as the row.
- A binomial expansion of degree n is $(a + b)^n$.
- The variables are

$$a^nb^0 + a^{n-1}b^1 + \ldots + a^1b^{n-1} + a^nb^0 + a^0b^n$$

Let's investigate:

When binomial expressions are expanded, is there any type of pattern developing which might help us expand more quickly? Consider the following expansions:

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

What observations can we make in general about the expansion of $(a+b)^n$

1. The expansion is a series (an adding of terms).
2. The number of terms in each expansion is one more than n. (terms = n + 1)
3. The power of a starts with an and decreases by one in each successive term ending with a0. The power of b starts with b0 and increases by one in each successive term ending with bn.
4. The power of b is always one less than the "number" of the term. The power of a is always n minus the power of b.
5. The sum of the exponents in each term adds up to n.
6. The coefficients of the first and last terms are each one.
7. The coefficients of the middle terms form an interesting (but perhaps not easily recognized) pattern where each coefficient can be determined from the previous term. The coefficient is the product of the previous term's coefficient and a's index, divided by the number of that previous term.

Check it out: $(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

The second term's coefficient is determined by a^4: $\frac{(1)(4)}{1} = 4$

The third term's coefficient is determined by $4a^3b$: $\frac{(4)(3)}{2} = 6$

To Get Coefficient
From the Previous Term: $\frac{(coefficient)(index\ of\ a)}{number\ of\ term}$

(This pattern will eventually be expressed as a combination of the form ${}_nC_k$.)

8. Another famous pattern is also developing regarding the coefficients. If the coefficients are "pulled off" of the terms and arranged, they form a triangle known as Pascal's triangle. (The use of Pascal's triangle to determine coefficients can become tedious when the expansion is to a large power.)

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

(The two outside edges of the triangle are comprised of ones. The other terms are each the sum of the two terms immediately above them in the triangle.)

By pulling these observations together with some mathematical syntax, a theorem is formed relating to the expansion of binomial terms:

Binomial Theorem

(or Binomial Expansion Theorem)

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Most of the syntax used in this theorem should look familiar. The $\binom{n}{k}$ notation is just another way of writing a combination such as ${}_nC_k$ (read "*n choose k*").

$$\binom{n}{k} = {}_nC_k = \frac{n!}{(n-k)!k!}$$

Our pattern to obtain the coefficient using the previous term (in observation #6), $\frac{(coefficient)(index\ of\ a)}{number\ of\ term}$, actually leads to the ${}_nC_k$ used in the binomial theorem.

Here is the connection. Using our coefficient pattern in a general setting, we get:

$$(a+b)^n = 1\cdot a^n + \frac{n}{1}\cdot a^{n-1}\cdot b + \frac{n(n-1)}{1\cdot 2}a^{n-2}\cdot b^2 + \boxed{\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}\cdot b^3} + \ldots + b^n$$

Let's examine the coefficient of the fourth term, the one in the box.

If we write a combination ${}_nC_k$ using $k = 3$, (for the previous term), we see the connection:

$${}_nC_3 = \frac{n!}{(n-3!)3!} = \frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}3!} = \frac{n(n-1)(n-2)}{3!} = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$$

The Binomial Theorem can also be written in its expanded form as:

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n$$

Remember that $\binom{n}{k} = {}_nC_k = \frac{n!}{(n-k)!k!}$ and that $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$.

Examples using the Binomial Theorem:

1. Expand $(x+2)^5$.

Let $a = x$, $b = 2$, $n = 5$ and substitute. (Do not substitute a value for k.)

$$(x+2)^5 = \sum_{k=0}^{5}\binom{5}{k}x^{5-k}2^k$$

$$(x+2)^5 = \binom{5}{0}x^5 2^0 + \binom{5}{1}x^{5-1}2^1 + \binom{5}{2}x^{5-2}2^2 + \binom{5}{3}x^{5-3}2^3 + \binom{5}{4}x^{5-4}2^4 + \binom{5}{5}x^{5-5}2^5$$

The fastest and most accurate way to calculate combinations is to use your graphing calculator.
For calculator help, see the link at the end of the page.

$$(x+2)^5 = x^5 + 5x^4\cdot 2 + 10x^3\cdot 2^2 + 10x^2\cdot 2^3 + 5x^1\cdot 2^4 + 2^5$$
$$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$$

2. Expand $(2x^4 - y)^3$.

Look carefully!!
This can be a tricky one!

The "tricks" involve the use of an expression with an exponent as the "a" value, and also the change of sign between the terms. For the Binomial Theorem, this problem is actually $((2x^4)+(-y))^3$.

(Let $a = (2x^4)$, $b = (-y)$, $n = 3$ and substitute. The parentheses are a "must have"!!!)

$$(2x^4 - y)^3 = \sum_{k=0}^{3} \binom{3}{k} (2x^4)^{3-k}(-y)^k$$

$$(2x^4 - y)^3 = \binom{3}{0}(2x^4)^3(-y)^0 + \binom{3}{1}(2x^4)^2(-y)^1 + \binom{3}{2}(2x^4)^1(-y)^2 + \binom{3}{3}(2x^4)^0(-y)^3$$

Grab your calculator. Be sure to raise the entire parentheses to the indicated power and watch out for signs.

$$\begin{aligned}(2x^4 - y)^3 &= (8x^{12}) + 3\cdot(4x^8)(-y) + 3\cdot(2x^4)(y^2) + (-y^3)\\ &= 8x^{12} + (-12x^8y) + 6x^4y^2 + (-y^3)\\ &= 8x^{12} - 12x^8y + 6x^4y^2 - y^3\end{aligned}$$

Binomial Theorem for any Index

Binomial theorem for positive integral index

The rule by which any power of binomial can be expanded is called the binomial theorem.
If n is a positive integer and x, y ∈ C then

$$(x+y)^n = {}^nC_0 x^{n-0} y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \ldots\ldots\ldots + {}^nC_r x^{n-r} y^r + \ldots\ldots + {}^nC_{n-1} x y^{n-1} + {}^nC_n x^0 y^n$$

$$\text{i.e.,}\quad (x+y)^n = \sum_{r=0}^{n} {}^nC_r . x^{n-r} . y^r \qquad \ldots\ldots\text{(i)}$$

Here ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots {}^nC_n$ are called binomial coefficients and

$${}^nC_r = \frac{n!}{r!(n-r)!} \text{ for } 0 \le r \le n.$$

Binomial theorem for any Index

Statement :

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)\ldots\ldots(n-r+1)}{r!}x^r + \ldots \text{terms up to } \infty$$

when n is a negative integer or a fraction, where , otherwise expansion will not be possible.
If first term is not 1, then make first term unity in the following way,

$$(x+y)^n = x^n\left[1 + \frac{y}{x}\right]^n, \text{ if } \left|\frac{y}{x}\right| < 1.$$

General term :

$$T_{r+1} = \frac{n(n-1)(n-2)......(n-r+1)}{r!} x^r$$

Some important expansions

(i) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + ... + \frac{n(n-1)(n-2)......(n-r+1)}{r!} x^r + ...$

(ii) $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!} x^2 - .... + \frac{n(n-1)(n-2).....(n-r+1)}{r!} (-x)^r + ...$

(iii) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + ... + \frac{n(n+1)......(n+r-1)}{r!} x^r + ...$

(iv) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + ... + \frac{n(n+1)......(n+r-1)}{r!} (-x)^r + ...$

(v) $(1+x)^{-1} = 1 - x + x^2 - x^3 + ... \infty$

(vi) $(1-x)^{-1} = 1 + x + x^2 + x^3 + ... \infty$

(vii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... \infty$

(viii) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... \infty$

(ix) $(1+x)^{-3} = 1 - 3x + 6x^2 - .... \infty$

(x) $(1-x)^{-3} = 1 + 3x + 6x^2 + ... \infty$

Problems on approximation by the binomial theorem :

We have,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + ...$$

If x is small compared with 1, we find that the values of x^2, x^3, x^4, ….. become smaller and smaller.
∴ The terms in the above expansion become smaller and smaller. If x is very small compared with 1, we might take 1 as a first approximation to the value of $(1 + x)^n$ or $(1 + nx)$ as a second approximation.

Three / Four consecutive terms or Coefficients

(1) **If consecutive coefficients are given:** In this case divide consecutive coefficients pair wise. We get equations and then solve them.

(2) **If consecutive terms are given :** In this case divide consecutive terms pair wise *i.e.* if four consecutive terms be $T_r, T_{r+1}, T_{r+2}, T_{r+3}$ then find $\frac{T_r}{T_{r+1}}, \frac{T_{r+1}}{T_{r+2}}, \frac{T_{r+2}}{T_{r+3}} \Rightarrow \lambda_1, \lambda_2, \lambda_3$ (say) then divide λ_1 by λ_2 and λ_2 by λ_3 and solve.

Some important points

(1) Pascal's Triangle

$$
\begin{array}{ccccccccccccl}
 & & & & & 1 & & & & & & & (x+y)^0 \\
 & & & & 1 & & 1 & & & & & & (x+y)^1 \\
 & & & 1 & & 2 & & 1 & & & & & (x+y)^2 \\
 & & 1 & & 3 & & 3 & & 1 & & & & (x+y)^3 \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & & (x+y)^4 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & & (x+y)^5
\end{array}
$$

Pascal's triangle gives the direct binomial coefficients.
Example : $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

(2) Method for finding terms free from radicals or rational terms in the expansion of $(a^{1/p} + b^{1/q})^N$ $\forall$ a, b $\in$ prime numbers:

Find the general term

$$T_{r+1} = {}^N C_r (a^{1/p})^{N-r} (b^{1/q})^r = {}^N C_r \, a^{\frac{N-r}{p}} . b^{\frac{r}{q}}$$

Putting the values of $0 \le r \le N$, when indices of a and b are integers.
Number of irrational terms = Total terms – Number of rational terms.