Determine order and degree (if defined) of differential equations given in Exercises 1 to 10.

Question 1:

Determine order and degree (if defined) of differential equation

$\frac{d^4y}{dx^4} + \sin(y''') = 0$

Answer 1:

$$\frac{d^4y}{dx^4} + \sin(y''') = 0 \qquad \Rightarrow y'''' + \sin(y''') = 0$$

The highest order derivative present in the differential equation is y''''. Therefore, its order is four.
The given differential equation is not a polynomial equation in its derivatives. Hence, its degree is not defined.

Question 2:

Determine order and degree (if defined) of differential equation $y' + 5y = 0$

Answer 2:

The given differential equation is: $y' + 5y = 0$
The highest order derivative present in the differential equation is y'. Therefore, its order is one.
It is a polynomial equation in y'. The highest power raised to y' is 1. Hence, its degree is one.

Question 3:

Determine order and degree (if defined) of differential equation $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$

Answer 3:

$$\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$$

The highest order derivative present in the given differential equation is $\frac{d^2s}{dt^2}$. Therefore, its order is two.

It is a polynomial equation in $\frac{d^2s}{dt^2}$ and $\frac{ds}{dt}$. The power raised to $\frac{d^2s}{dt^2}$ is 1. Hence, its degree is one.

Question 4:

Determine order and degree (if defined) of differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$

Answer 4:

$$\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$$

The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$. Therefore, its order is 2.

The given differential equation is not a polynomial equation in its derivatives. Hence, its degree is not defined.

Question 5:

Determine order and degree (if defined) of differential equation $\frac{d^2y}{dx^2} = \cos 3x + \sin 3x$

Answer 5:

$$\frac{d^2y}{dx^2} = \cos 3x + \sin 3x \quad so \quad \frac{d^2y}{dx^2} - \cos 3x - \sin 3x = 0$$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$. Therefore, its order is two.

It is a polynomial equation in $\frac{d^2y}{dx^2}$ and the power raised to $\frac{d^2y}{dx^2}$ is 1. Hence, its degree is one.

Question 6:
Determine order and degree (if defined) of differential equation
$(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$

Answer 6:

$$\left(y'''\right)^2 + (y'')^3 + (y')^4 + y^5 = 0$$

The highest order derivative present in the differential equation is y'''.

Therefore, its order is three.

The given differential equation is a polynomial equation in y''', y'' and y'.

The highest power raised to y''' is 2.

Hence, its degree is 2.

Question 7:
Determine order and degree (if defined) of differential equation $y''' + 2y'' + y' = 0$

Answer 7:

$$y''' + 2y'' + y' = 0$$

The highest order derivative present in the differential equation is y'''. Therefore, its order is three.

It is a polynomial equation in $y''', y'', and\ y'$. The highest power raised to y''' is 1.
Hence, its degree is 1.

Question 8:
Determine order and degree (if defined) of differential equation $y' + y = e^x$

Answer 8:

$$y' + y = e^x \Rightarrow y' + y - e^x = 0$$

The highest order derivative present in the differential equation is y'. Therefore, its order is one.
The given differential equation is a polynomial equation in y' and the highest power raised to y' is one.
Hence, its degree is one.

Question 9:
Determine order and degree (if defined) of differential equation $y'' + (y')^2 + 2y = 0$

Answer 9:
$y'' + (y')^2 + 2y = 0$
The highest order derivative present in the differential equation is y''. Therefore, its order is two.
The given differential equation is a polynomial equation in y'' and y' the highest power raised to y' is one.
Hence, its degree is one.

Question 10:
Determine order and degree (if defined) of differential equation $y'' + 2\,y' + \sin y = 0$

Answer 10:

$$y'' + 2\,y' + \sin y = 0$$

The highest order derivative present in the differential equation is y''.
Therefore, its order is two
This is a polynomial equation in y'' and y' and the highest power raised to y'' is one.
Hence, its degree is one.

Question 11:

The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$ is

(A) 3 (B) 2 (C) 1 (D) not defined

Answer11:

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$$

The given differential equation is not a polynomial equation in its derivatives.

Therefore, its degree is not defined.

Hence, the correct answer is (D).

Question 12:

The order of the differential equation $2x^2 \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$ is

(A) 2 (B) 1 (C) 0 (D) not defined

Answer 12:

$$2x^2 \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$$

The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$.

Therefore, its order is two.

Hence, the correct answer is (A).

Mathematics

(Chapter – 9) (Differential Equations) (Exercise 9.2)
(Class – XII)

Question 1:

Verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

$y = e^x + 1 \quad : \quad y'' - y' = 0$

Answer 1:

$y = e^x + 1$

Differentiating both sides of this equation with respect to x, we get:

$\frac{dy}{dx} = \frac{d}{dx}(e^x + 1)$

$y' = e^x \qquad \text{......} (1)$

Now, differentiating equation (1) with respect to x, we get:

$$\frac{d}{dx}(y') = \frac{d}{dx}(e^x) \Rightarrow y'' = e^x$$

Substituting the values of y' *and* y'' in the given differential equation, we get the L.H.S. as:

$y'' - y' = e^x - e^x = 0 = R.H.S$

Thus, the given function is the solution of the corresponding differential equation.

Question 2:

Verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

$y = x^2 + 2x + C \quad : \quad y' - 2x - 2 = 0$

Answer 2:

$$y = x^2 + 2x + C$$

Differentiating both sides of this equation with respect to x, we get:

$$y' = \frac{d}{dx}(x^2 + 2x + C) \Rightarrow y' = 2x + 2$$

Substituting the value of y' in the given differential equation, we get:

$L.H.S = y' - 2x + 2$

$= 2x + 2 - 2x - 2 = 0 = R.H.S$

Hence, the given function is the solution of the corresponding differential equation.

Question 3:

Verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

$y = \cos x + C \quad : \quad y' + \sin x = 0$

Answer 3:

$$y = \cos x + C$$

Differentiating both sides of this equation with respect to x, we get:

$$y' = \frac{d}{dx}(\cos x + C) \Rightarrow y' = -\sin x$$

Substituting the value of y' in the given differential equation, we get:

$\text{L.H.S} = y' + \sin x$

$= -\sin x + \sin x = 0$

= R.H.S.

Hence, the given function is the solution of the corresponding differential equation.

Question 4:

Verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

$y = \sqrt{1+x^2} \quad : y' = \frac{xy}{1+x^2}$

Answer 4:

$$y = \sqrt{1+x^2}$$

Differentiating both sides of the equation with respect to x, we get:

$y' = \frac{d}{dx}\left(\sqrt{1+x^2}\right) \qquad \Rightarrow \; y' = \frac{1}{2\sqrt{1+x^2}}\frac{d}{dx}(1+x^2)$

$\Rightarrow y' = \frac{2x}{2\sqrt{1+x^2}} \qquad \Rightarrow y' = \frac{x}{\sqrt{1+x^2}}$

$\Rightarrow y' = \frac{x}{\sqrt{1+x^2}} \times \sqrt{1+x^2} \qquad \Rightarrow y' = \frac{x}{1+x^2}.y = \frac{xy}{1+x^2}$

$\therefore L.H.S = R.H.S$

Hence, the given function is the solution of the corresponding differential equation.

Question 5:

Verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

$y = Ax \quad : xy' = y (x \neq 0)$

Answer 5:

$$y = Ax$$

Differentiating both sides with respect to x, we get:

$$y' = \frac{d}{dx}(Ax) \Rightarrow \; y' = A$$

Substituting the value of y' in the given differential equation, we get:

L.H.S = $xy' = x.A = Ax = y = \quad R.H.S$

Hence, the given function is the solution of the corresponding differential equation.

Question 6:

Verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

$y = x\sin x \quad : xy' = y + x\sqrt{x^2 - y^2} \; (x \neq 0 \; and \; x > y \; or \; x < -y)$

Answer 6:

$y = x\sin x$

Differentiating both sides of this equation with respect to x, we get:

$$y' = \frac{d}{dx}(x\sin x) \qquad \Rightarrow y' = \sin x.\frac{d}{dx}(x) + x.\frac{d}{dx}(\sin x) \qquad \Rightarrow y' = \sin x + x\cos x$$

Substituting the value of y' in the given differential equation, we get:

$L.H.S \;\; xy' = x(\sin x + x\cos x)$

$= x\sin x + x^2\cos x \; = y + x^2\sqrt{1 - sin^2 x}$

$= y + x^2\sqrt{1 - \left(\frac{y}{x}\right)^2}$

$= y + x\sqrt{y^2 - x^2} \;\; = \;\; R.H.S$

Question 7:

Verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

$xy = \log y + C \quad : y' = \frac{y^2}{1-xy} \, (xy \neq 1)$

Answer 7:

$$xy = \log y + C$$

$$\Rightarrow \frac{d}{dx}(xy) = \frac{d}{dx}(\log y)$$

$$\Rightarrow y\frac{d}{dx}(x) + x.\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

$$\Rightarrow y + xy' = \frac{1}{y}.y'$$

$$\Rightarrow y^2 + xyy' = y'$$

$$\Rightarrow (xy-1)y' = -y^2$$

$$\Rightarrow y' = \frac{y^2}{1-xy}$$

Thus, the given function is the solution of the differential equation.

Question 8:

Verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

$y - \cos y = x \quad : \quad (y \sin y + \cos y + x)y' = y$

Answer 8:

$$y - \cos y = x$$

$$\Rightarrow \frac{dy}{dx} - \frac{d}{dx}(\cos y) = \frac{d}{dx}(x)$$

$$\Rightarrow y' - (-\sin y).y' = 1$$

$$\Rightarrow y'(1+\sin y) = 1$$

$$\Rightarrow y' = \frac{1}{1+\sin y}$$

Therefore,

$$(y\sin y + \cos y + x)y' = (y\sin y + \cos y + y - \cos y) \times \frac{1}{1+\sin y}$$

$$= y(1+\sin y).\frac{1}{1+\sin y}$$

$$= y$$

Thus, the given function is the solution of the differential equation.

Question 9:

Verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

$x + y = tan^{-1}y \quad : y^2y' + y^2 + 1 = 0$

Answer 9:

$x + y = tan^{-1}y$

Differentiating both sides of this equation with respect to x, we get:

$$\frac{d}{dx}(x+y) = \frac{d}{dx}(tan^{-1}y) \Rightarrow 1 + y' = \left[\frac{1}{1+y^2}\right]y'$$

$$\Rightarrow y'\left[\frac{1}{1+y^2} - 1\right] = 1 \quad \Rightarrow y'\left[\frac{1-(1+y^2)}{1+y^2}\right] = 1 \quad \Rightarrow y'\left[\frac{-y^2}{1+y^2}\right] = 1 \quad \Rightarrow y' = \frac{-(1+y^2)}{y^2}$$

Substituting the value of y' in the given differential equation, we get:

$$L.H.S = y^2y' + y^2 + 1 = y^2\left[\frac{-(1+y^2)}{y^2}\right] + y^2 + 1 = -1 - y^2 + y^2 + 1 = 0 = R.H.S$$

Hence, the given function is the solution of the corresponding differential equation.

Question 10:

Verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

$y = \sqrt{a^2 - x^2}\, x \in (-a, a) : x + y\frac{dy}{dx} = 0 (y \neq 0)$

Answer 10:

$$y = \sqrt{a^2 - x^2}$$

Differentiating both sides of this equation with respect to x, we get:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{a^2 - x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{a^2 - x^2}}\frac{d}{dx}(a^2 - x^2) = \frac{1}{2\sqrt{a^2 - x^2}}(-2x) = \frac{-x}{\sqrt{a^2 - x^2}}$$

Substituting the value of $\frac{dy}{dx}$

$$LHS = x + y\frac{dy}{dx} = x + \sqrt{a^2 - x^2} \times \frac{-x}{\sqrt{a^2 - x^2}}$$

$$= x - x = 0 = R.H.S$$

Hence, the given function is the solution of the corresponding differential equation.

Question 11:

The numbers of arbitrary constants in the general solution of a differential equation of fourth order are:

(A) 0 (B) 2 (C) 3 (D) 4

Answer 11:

We know that the number of constants in the general solution of a differential equation of order n is equal to its order.

Therefore, the number of constants in the general equation of fourth order differential equation is four.

Hence, the correct answer is (D).

Question 12:

The numbers of arbitrary constants in the particular solution of a differential equation of third order are:

(A) 3 (B) 2 (C) 1 (D) 0

Answer 12:

In a particular solution of a differential equation, there are no arbitrary constants.

Hence, the correct answer is (D).

(Chapter – 9) (Differential Equations) (Exercise 9.3)
(Class – XII)

For each of the differential equations in Exercises 1 to 10, find the general solution:

Question 1:

Find the general solution: $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$

Answer 1:

The given differential equation is:

$$\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} = \tan^2\frac{x}{2} \qquad \Rightarrow \frac{dy}{dx} = \left(\sec^2\frac{x}{2} - 1\right)$$

Separating the variables, we get

$$dy = \left(\sec^2\frac{x}{2} - 1\right)dx$$

Now, integrating both sides of this equation, we get:

$$\int dy = \int \left(\sec^2\frac{x}{2} - 1\right)dx = \int \sec^2\frac{x}{2}dx - \int dx \Rightarrow y = 2\tan\frac{x}{2} - 1 + C$$

This is the required general solution of the given differential equation.

Question 2:

Find the general solution: $\frac{dy}{dx} = \sqrt{4-y^2}\ (-2 < y < 2)$

Answer 2:

The given differential equation is:

$$\frac{dy}{dx} = \sqrt{4-y^2} \qquad \Rightarrow \int \frac{dy}{\sqrt{4-y^2}} = \int dx$$

Now, integrating both sides of this equation, we get:

$$\int \frac{dy}{\sqrt{4-y^2}} = \int dx \Rightarrow \sin^{-1}\frac{y}{2} = x + C$$

$$\Rightarrow \frac{y}{2} = \sin(x+C) \quad \text{so} \quad y = 2\sin(x+C)$$

This is the required general solution of the given differential equation.

Question 3:

Find the general solution: $\frac{dy}{dx} + y = 1\ (y \neq 1)$

Answer 3:

The given differential equation is: $\frac{dy}{dx} + y = 1$

$$\Rightarrow dy + y\,dx = dx \qquad \Rightarrow dy = (1-y)dx$$

Separating the variables, we get:

$$\Rightarrow \frac{dy}{1-y} = dx$$

Now, integrating both sides, we get:

$$\int \frac{dy}{1-y}dx = \int dx$$

$$\Rightarrow \log(1-y) = x + \log C \quad \Rightarrow -\log C - \log(1-y) = x \quad \Rightarrow \log C\,(1-y) = -x \quad \Rightarrow C(1-y) = e^{-x}$$

$$\Rightarrow 1 - y = \frac{1}{C}e^{-x} \qquad \Rightarrow y = 1 + \frac{1}{C}e^{-x} \qquad \Rightarrow y = 1 + Ae^{-x} \qquad \left(\text{where } A = \frac{1}{C}\right)$$

This is the required general solution of the given differential equation.

Question 4:

Find the general solution: $sec^2 x tan\, y\, dx + sec^2 y \tan x\, dy = 0$

Answer 4:

The given differential equation is:

$sec^2 x tan\, y\, dx + sec^2 y \tan x\, dy = 0$

$$\Rightarrow \frac{sec^2 x tan\, y\, dx + sec^2 y \tan x\, dy}{\tan x \tan y} = 0$$

$$\Rightarrow \frac{sec^2 x}{\tan x} dx + \frac{sec^2 y}{\tan y} dy = 0$$

$$\Rightarrow \frac{sec^2 x}{\tan x} dx = -\frac{sec^2 y}{\tan y} dy$$

Integrating both sides of this equation, we get:

$$\int \frac{sec^2 x}{\tan x} dx = -\int \frac{sec^2 y}{\tan y} dy \qquad \dots (1)$$

Let $\tan x = t$

$$\Rightarrow \frac{d}{dx}(\tan x) = \frac{dt}{dx} \qquad \Rightarrow sec^2 x = \frac{dt}{dx} \qquad \Rightarrow \quad sec^2 x\, dx = dt$$

Now, $\int \frac{sec^2 x}{\tan x} dx = \int \frac{1}{t} dt \quad = \log t = \log(\tan x)$

Similarly, $\int \frac{sec^2 x}{\tan x} dy = \log(\tan y)$

Substituting these values in equation (1), we get:

$\Rightarrow \log(\tan x) = -\log(\tan y) + \log C$

$$\Rightarrow \log(\tan x) = \log\left(\frac{C}{\tan y}\right) \qquad \Rightarrow \tan x = \frac{C}{\tan y} \qquad \Rightarrow \tan x \tan y = C$$

This is the required general solution of the given differential equation.

Question 5:

Find the general solution: $(e^x + e^{-x})dy - (e^x - e^{-x})dx = 0$

Answer 5:

The given differential equation is: $(e^x + e^{-x})dy - (e^x - e^{-x})dx = 0$

$$\Rightarrow (e^x + e^{-x})dy = (e^x - e^{-x})dx \qquad \Rightarrow dy = \left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right] dy$$

Integrating both sides of this equation, we get:

$$\int dy = \int \left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right] dx + C$$

$$\Rightarrow y = \int \left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right] dx + C$$

Let $e^x + e^{-x} = t$

Differentiating both sides with respect to x, we get:

$$\frac{d}{dx}(e^x + e^{-x}) = \frac{dt}{dx} \qquad \Rightarrow e^x - e^{-x} = \frac{dt}{dx} \qquad \Rightarrow (e^x - e^{-x})dx = dt$$

Substituting this value in equation (1), we get:

$$\Rightarrow y = \int \frac{1}{t} dt + C \qquad \Rightarrow y = \log(t) + C \qquad \Rightarrow y = \log(e^x + e^{-x}) + C$$

This is the required general solution of the given differential equation.

Question 6:

Find the general solution: $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

Answer 6:

The given differential equation is:

$$\frac{dy}{dx} = (1 + x^2)(1 + y^2) \qquad \Rightarrow \frac{dy}{1 + y^2} = (1 + x^2)dx$$

Integrating both sides of this equation, we get:

$$\int \frac{dy}{1 + y^2} = \int (1 + x^2)dx \quad \Rightarrow \ tan^{-1}y = \int dx + \int x^2 dx \quad \Rightarrow tan^{-1}y = x + \frac{x^3}{3} + C$$

This is the required general solution of the given differential equation.

Question 7:

Find the general solution: $y \log y \, dx - x \, dy = 0$

Answer 7:

The given differential equation is:

$$y \log y \, dx - x \, dy = 0 \qquad \Rightarrow y \log y \, dx = x \, dy \qquad \Rightarrow \frac{dy}{y \log y} = \frac{dx}{x}$$

Integrating both sides, we get:

$$\int \frac{dy}{y \log y} = \int \frac{dx}{x} \qquad \dots(1)$$

Let $\log y = t$

$$\Rightarrow \frac{d}{dy}(\log y) = \frac{dt}{dy} \qquad \Rightarrow \frac{1}{y} = \frac{dt}{dy} \qquad \Rightarrow \ \frac{1}{y}dy = dt$$

Substituting this value in equation (1), we get:

$$\int \frac{dt}{t} = \int \frac{dx}{x} \qquad \Rightarrow \log t = \log x + \log C \qquad \Rightarrow \log(\log y) = \log Cx \qquad \Rightarrow y = e^{Cx}$$

This is the required general solution of the given differential equation.

Question 8:

Find the general solution: $x^5 \frac{dy}{dx} = -y^5$

Answer 8:

The given differential equation is:

$$x^5 \frac{dy}{dx} = -y^5 \qquad \Rightarrow \frac{dy}{y^5} = -\frac{dx}{x^5} \qquad \Rightarrow \frac{dy}{x^5} + \frac{dx}{y^5} = 0$$

Integrating both sides, we get:

$$\int \frac{dy}{x^5} + \int \frac{dx}{y^5} = k \qquad \text{(wher k is any constant)}$$

$$\Rightarrow \int x^{-5} \, dy + \int y^{-5} \, dx = k$$

$$\Rightarrow \frac{x^{-4}}{-4} + \frac{x^{-4}}{-4} = k$$

$$\Rightarrow x^{-4} + y^{-4} = -4k$$

$$\Rightarrow x^{-4} + y^{-4} = C \qquad (C = -4k)$$

This is the required general solution of the given differential equation.

Question 9:

Find the general solution: $\frac{dy}{dx} = sin^{-1}x$

Answer 9:

The given differential equation is:

$$\frac{dy}{dx} = sin^{-1}x \qquad \Rightarrow dy = sin^{-1}x\ dx$$

Integrating both sides, we get:

$$\int dy = \int sin^{-1}x\ dx \qquad \Rightarrow y = \int (sin^{-1}x.1)dx$$

$$\Rightarrow y = sin^{-1}x.\int (1)dx - \int \left[\left(\frac{d}{dx}(sin^{-1}x)\right).\int (1)dx\right]dx$$

$$\Rightarrow y = sin^{-1}x.x - \int \left(\frac{1}{\sqrt{1+x^2}}.x\right)dx \qquad \Rightarrow y = x.sin^{-1}x - \int \frac{-x}{\sqrt{1-x^2}}dx \qquad .....(1)$$

Let $1 - x^2 = t \quad \Rightarrow \frac{d}{dx}(1-x^2) = \frac{dt}{dx} \qquad \Rightarrow -2x = \frac{dt}{dx} \qquad \Rightarrow xdx = -\frac{1}{2}dt$

Substituting this value in equation (1), we get:

$$y = xsin^{-1}x + \int \frac{1}{2\sqrt{t}}dt \qquad \Rightarrow y = xsin^{-1}x + \frac{1}{2}.\int (t)^{-\frac{1}{2}}\ dt \qquad \Rightarrow y = xsin^{-1}x + \frac{1}{2}.\frac{t^{-\frac{1}{2}}}{\frac{1}{2}} + C$$

$$\Rightarrow y = xsin^{-1}x + \frac{1}{2}.\sqrt{t} + C \qquad \Rightarrow y = xsin^{-1}x + \frac{1}{2}.\sqrt{1-x^2} + C$$

This is the required general solution of the given differential equation.

Question 10:

Find the general solution: $e^x \tan y\, dx + (1-e^x)sec^2y\, dy = 0$

Answer 10:

The given differential equation is:

$$e^x \tan y\, dx + (1-e^x)sec^2y\, dy = 0 \qquad \Rightarrow (1-e^x)sec^2y\, dy = -e^x \tan y\, dx$$

Separating the variables, we get:

$$\frac{sec^2y}{\tan y}dy = \frac{-e^x}{1-e^x}dx$$

Integrating both sides, we get:

$$\int \frac{sec^2y}{\tan y}dy = \int \frac{-e^x}{1-e^x}dx \qquad .....(1)$$

Let $\tan y = u$

$$\Rightarrow \frac{d}{dy}(\tan y) = \frac{du}{dy} \qquad \Rightarrow sec^2y = \frac{du}{dy} \qquad \Rightarrow sec^2y\, dy = du$$

$$\int \frac{sec^2y}{\tan y}dy = \int \frac{du}{u} = \log u = \log(\tan y)$$

Now let $1 - e^x = t$

$$\frac{d}{dx}(1-e^x) = \frac{dt}{dx} \qquad \Rightarrow -e^x = \frac{dt}{dx} \qquad \Rightarrow -e^x dx = dt$$

$$\Rightarrow \int \frac{-e^x}{1-e^x}dx = \int \frac{dt}{t} = \log t = \log(1-e^x)$$

Substituting the values of $\int \frac{sec^2y}{\tan y}dy$ and $\int \frac{-e^x}{1-e^x}dx$

$$\Rightarrow \log(\tan y) = \log(1-e^x) + \log C \qquad \Rightarrow \log(\tan y) = \log[C(1-e^x)] \qquad \Rightarrow \tan y = C(1-e^x)$$

This is the required general solution of the given differential equation.

For each of the differential equations in Exercises 11 to 14, find a particular solution satisfying the given condition:

Question 11:

Find a particular solution satisfying the given condition: $(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x;\ y = 1$ when $x = 0$

Answer 11:

The given differential equation is:

$$(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x \qquad \Rightarrow \frac{dy}{dx} = \frac{2x^2 + x}{(x^3 + x^2 + x + 1)} \qquad \Rightarrow dy = \frac{2x^2 + x}{(x^3 + x^2 + x + 1)}dx$$

Integrating both sides, we get:

$$\int dy = \int \frac{2x^2 + x}{(x+1)(x^2+1)}dx \qquad \dots (1)$$

$$\Rightarrow \frac{2x^2 + x}{(x+1)(x^2+1)} = \frac{A}{(x+1)} + \frac{Bx + C}{(x^2+1)} \qquad \dots (2)$$

$$\Rightarrow 2x^2 + x = Ax^2 + A + Bx^2 + Bx + Cx + C \qquad \Rightarrow 2x^2 + x = (A+B)x^2 + (B+C)x + (A+C)$$

Comparing the coefficients of x^2, x and constant, we get:

A + B = 2

B + C = 1

A + C = 0

Solving these equations, we get:

$$A = \frac{1}{2}\ , B = \frac{3}{2} \text{ and } C = \frac{-1}{2}$$

Substituting the values of A, B, and C in equation (2), we get:

$$\frac{2x^2 + x}{(x+1)(x^2+1)} = \frac{1}{2}.\frac{1}{(x+1)} + \frac{1}{2}.\frac{(3x+1)}{(x^2+1)}$$

Therefore, equation (1) becomes:

$$\int dy = \frac{1}{2}\int \frac{1}{(x+1)}dx + \frac{1}{2}\int \frac{(3x+1)}{(x^2+1)}dx$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{2}\int \frac{x}{(x^2+1)}dx - \frac{1}{2}\int \frac{1}{(x^2+1)}dx$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{4}\int \frac{2x}{(x^2+1)}dx - \frac{1}{2}tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{4}\log(x^2+1) - \frac{1}{2}tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{4}[2\log(x+1) + 3\log(x^2+1)] - \frac{1}{2}tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{4}[(x+1)^2(x^2+1)^3] - \frac{1}{2}tan^{-1}x + C \qquad \dots (3)$$

Now y=1 when x = 0

$$\Rightarrow 1 = \frac{1}{4}\log(1) - \frac{1}{2}tan^{-1}(0) + C$$

$$\Rightarrow 1 = \frac{1}{4}\times 0 - \frac{1}{2}\times 0 + C \qquad \Rightarrow C = 1$$

Substituting C = 1 in equation (3), we get:

$$y = \frac{1}{4}[(x+1)^2(x^2+1)^3] - \frac{1}{2}tan^{-1}x + 1$$

Question 12:

Find a particular solution satisfying the given condition: $x(x^2-1)\frac{dy}{dx}=1; y=0$ when $x=2$.

Answer 12:

$x(x^2-1)\dfrac{dy}{dx}=1 \qquad \Rightarrow dy=\dfrac{1}{x(x^2-1)}dx \qquad \Rightarrow dy=\dfrac{1}{x(x-1)(x+1)}dx$

Integrating both sides, we get:

$\int dy=\int\dfrac{1}{x(x-1)(x+1)}dx \qquad \ldots(1)$

Let $\dfrac{1}{x(x-1)(x+1)}=\dfrac{A}{x}+\dfrac{B}{(x-1)}+\dfrac{C}{(x+1)}$

$\Rightarrow 1=\dfrac{A(x-1)(x+1)+Bx(x+1)+Cx(x-1)}{x(x-1)(x+1)} \qquad \Rightarrow 1=\dfrac{(A+B+C)x^2+(B-C)\boldsymbol{x}-A}{x(x-1)(x+1)}$

Comparing the coefficients of x2, x, and constant, we get:

$A=-1,\ B-C=0$ and $A+B+C=0$

Solving these equations, we get

$B=\dfrac{1}{2}$ and $C=\dfrac{1}{2}$

Substituting the values of A, B and C in equation (2), we get:

$$\frac{1}{x(x-1)(x+1)}=\frac{-1}{x}+\frac{1}{2(x-1)}+\frac{1}{2(x+1)}$$

Therefore, equation (1) becomes:

$\int dy=-\int\dfrac{1}{x}dx+\dfrac{1}{2}\int\dfrac{1}{(x-1)}dx+\dfrac{1}{2}\int\dfrac{1}{(x+1)}dx$

$\Rightarrow y=-\log x+\dfrac{1}{2}\log(x-1)+\dfrac{1}{2}\log(x+1)+\log k \ \Rightarrow y=\dfrac{1}{2}\log\left[\dfrac{k^2(x-1)(x+1)}{x^2}\right] \qquad \ldots(3)$

Now, y = 0 when x = 2.

$\Rightarrow 0=\dfrac{1}{2}\log\left[\dfrac{k^2(2-1)(2+1)}{4}\right] \qquad \Rightarrow \log\left(\dfrac{3k^2}{4}\right)=0 \qquad \Rightarrow\dfrac{3k^2}{4}=1 \qquad \Rightarrow k^2=\dfrac{4}{3}$

Substituting the value of k^2 in equation (3), we get:

$y=\dfrac{1}{2}\log\left[\dfrac{4(x-1)(x+1)}{3x^2}\right] \qquad \Rightarrow y=\dfrac{1}{2}\log\left[\dfrac{4(x^2-1)}{3x^2}\right]$

Question 13:

Find a particular solution satisfying the given condition: $\cos\left(\frac{dy}{dx}\right)=a\ (a\in R); y=1$ when $x=0$

Answer 13:

$\cos\left(\dfrac{dy}{dx}\right)=a \qquad \Rightarrow\dfrac{dy}{dx}=cos^{-1}a \qquad \Rightarrow dy=cos^{-1}a\,dx$

Integrating both sides, we get:

$\int dy=cos^{-1}a\int dx \qquad \Rightarrow y=cos^{-1}a.x+C \qquad \Rightarrow y=x.cos^{-1}a+C \qquad \ldots(1)$

Now, y = 1 when x = 0,

$1=0.cos^{-1}a+C \qquad \Rightarrow C=1$

Substituting C = 1 in equation (1), we get:

$y=x.cos^{-1}a+1 \qquad \Rightarrow\dfrac{y-1}{x}=cos^{-1}a \qquad \Rightarrow cos\left(\dfrac{y-1}{x}\right)=a$

Question 14:

Find a particular solution satisfying the given condition: $\frac{dy}{dx} = y\tan x; y = 1$ when $x = 0$

Answer 14:

$\frac{dy}{dx} = ytan\ x \qquad \Rightarrow \frac{dy}{y} = tan\ x\ dx$

Integrating both sides, we get:

$\int \frac{dy}{y} = -\int tan\ x\ dx \quad \Rightarrow \log y = \log(\sec x) + \log C \qquad \Rightarrow \log y = \log(C\sec x)$

$\Rightarrow y = C\sec x \qquad ...(1)$

Now, y = 1 when x = 0.

$\Rightarrow 1 = C\sec 0 \qquad \Rightarrow C = 1$

Substituting C = 1 in equation (1), we get: $y = sec\ x$

Question 15:

Find the equation of a curve passing through the point (0, 0) and whose differential equation is $y' = e^x \sin x$.

Answer 15:

The differential equation of the curve is:

$y' = e^x \sin x \qquad \Rightarrow \frac{dy}{dx} = e^x \sin x \qquad \Rightarrow dy = e^x \sin x\ dx$

Integrating both sides, we get:

$\int dy = \int e^x \sin x\ dx \qquad ...(1)$

Let $I = \int e^x \sin x\ dx$

$\Rightarrow I = \sin x \int e^x\ dx - \int \left(\frac{d}{dx}(\sin x). \int e^x dx\right) dx$

$\Rightarrow I = \sin x . e^x - \int \cos x. e^x\ dx$

$\Rightarrow I = \sin x . e^x - \left[\cos x. \int e^x dx - \int \left(\frac{d}{dx}(\cos x). \int e^x dx\right) dx\right]$

$\Rightarrow I = \sin x . e^x - \left[\cos x. e^x - \int (-sin\ ). e^x dx\right]$

$\Rightarrow I = e^x \sin x - e^x \cos x - I$

$\Rightarrow 2I = e^x(\sin x - \cos x)$

$\Rightarrow I = e^x \frac{(\sin x - \cos x)}{2} + C$

Substituting this value in equation (1), we get:

$y = e^x \frac{(\sin x - \cos x)}{2} + C \qquad ...(2)$

Now, the curve passes through point (0, 0).

$\Rightarrow 0 = e^0 \frac{(\sin 0 - \cos 0)}{2} + C \qquad \Rightarrow 0 = 1.\frac{(0-1)}{2} + C \qquad \Rightarrow C = \frac{1}{2}$

Substituting $C = \frac{1}{2}$ in equation (2), we get;

$y = e^x \frac{(\sin x - \cos x)}{2} + \frac{1}{2} \qquad \Rightarrow 2y = e^x(\sin x - \cos x) + 1$

$\Rightarrow 2y - 1 = e^x(\sin x - \cos x)$

Hence, the required equation of the curve is $2y - 1 = e^x(\sin x - \cos x)$

Question 16:

For the differential equation $xy\frac{dy}{dx} = (x+2)(y+2)$, find the solution curve passing through the point (1, –1).

Answer 16:

The differential equation of the given curve is:

$$xy\frac{dy}{dx} = (x+2)(y+2)$$

$$\Rightarrow \left(\frac{y}{y+2}\right)dy = \left(\frac{x+2}{x}\right)dx \quad \Rightarrow \left(1-\frac{y}{y+2}\right)dy = \left(1+\frac{2}{x}\right)dx$$

Integrating both sides, we get:

$$\int\left(1-\frac{y}{y+2}\right)dy = \int\left(1+\frac{2}{x}\right)dx$$

$$\Rightarrow \int dy - 2\int\frac{y}{y+2}dy = \int dx + 2\int\frac{1}{x}dx$$

$$\Rightarrow y - 2\log(y+2) = x + 2\log x + C$$

$$\Rightarrow y - x - C = \log x^2 + \log(y+2)^2$$

$$\Rightarrow y - x - C = \log[x^2(y+2)^2] \quad \dots(1)$$

Now, the curve passes through point (1, –1).

$$\Rightarrow -1-1-C = \log[(-1)^2(-1+2)^2] \quad \Rightarrow -2-C = \log 1 = 0 \quad \Rightarrow C = -2$$

Substituting C = –2 in equation (1), we get:

$$y - x + 2 = \log[x^2(y+2)^2]$$

This is the required solution of the given curve.

Question 17:

Find the equation of a curve passing through the point (0, –2) given that at any point (x, y) on the curve, the product of the slope of its tangent and y-coordinate of the point is equal to the x-coordinate of the point.

Answer 17:

Let x and y be the x-coordinate and y-coordinate of the curve respectively.

We know that the slope of a tangent to the curve in the coordinate axis is given by $\frac{dy}{dx}$,

According to the given information, we get:

$$y.\frac{dy}{dx} = x \quad \Rightarrow y.dy = x.dx$$

Integrating both sides, we get:

$$\int y.dy = \int x.dx \quad \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$$

$$\Rightarrow y^2 - x^2 = 2C \quad \dots(1)$$

Now, the curve passes through point (0, –2).

$$(-2)^2 - 0^2 = 2C \quad \Rightarrow 2C = 4$$

Substituting 2C = 4 in equation (1), we get: $y^2 - x^2 = 4$

This is the required equation of the curve.

Question 18:

At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (–4, –3). Find the equation of the curve given that it passes through (–2, 1).

Answer 18:

It is given that (x, y) is the point of contact of the curve and its tangent.

The slope (m_1) of the line segment joining (x, y) and (–4, –3) is $\frac{y+3}{x+4}$

We know that the slope of the tangent to the curve is given by the relation, slope (m_2) of the given information = $\frac{dy}{dx}$

According to the given information:

$m_2 = 2m_1$ $\Rightarrow \frac{dy}{dx} = \frac{2(y+3)}{x+4}$ $\Rightarrow \frac{dy}{y+3} = \frac{2dx}{x+4}$

Integrating both sides, we get:

$\int \frac{dy}{y+3} = 2\int \frac{dx}{x+4}$ $\Rightarrow \log(y+3) = 2\log(x+4) + \log C$ $\Rightarrow \log(y+3) = \log C.(x+4)^2$

$\Rightarrow y + 3 = C(x+4)^2$ $\Rightarrow 4 = 4C$ $\Rightarrow C = 1$

Substituting C = 1 in equation (1), we get: $y + 3 = C(x+4)^2$

This is the required equation of the curve.

Question 19:

The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after t seconds.

Answer 19:

Let the rate of change of the volume of the balloon be k (where k is a constant).

$\Rightarrow \frac{dv}{dt} = k$ $\Rightarrow \frac{d}{dt}\left(\frac{4}{2}\pi r^3\right) = k$ $\left[\text{Volume of sphere} = \frac{4}{2}\pi r^3\right]$

$\Rightarrow \frac{4}{2}\pi 3r^2.\frac{dr}{dt} = k$ $\Rightarrow 4\pi r^2\, dr = kdt$

Integrating both sides, we get:

$4\pi \int r^2\, dr = k\int dt$ $\Rightarrow 4\pi \frac{r^3}{3} = kt + C$

$\Rightarrow 4\pi r^3 = 3(kt + C)$ ... (1)

Now, at t = 0, r = 3;

$\Rightarrow 4\pi(3)^3 = 3(k \times 0 + C)$ $\Rightarrow 108\pi = 3C$ $\Rightarrow C = 36\pi$

At t = 3, r = 6:

$\Rightarrow 4\pi \times 6^3 = 3(k \times 3 + C)$ $\Rightarrow 864\pi = 3(3k + 36\pi)$ $\Rightarrow 3k = -288\pi - 36\pi = 252\pi$

$\Rightarrow k = 84\pi$

Substituting the values of k and C in equation (1), we get:

$4\pi r^3 = 3(84\pi t + 36\pi)$ $\Rightarrow 4\pi r^3 = 4\pi(63t + 27)$ $\Rightarrow r^3 = 63t + 27$ $\Rightarrow r = (63t + 27)^{\frac{1}{3}}$

Thus, the radius of the balloon after t seconds is $(63t + 27)^{\frac{1}{3}}$.

Question 20:

In a bank, principal increases continuously at the rate of r% per year. Find the value of r if ₹100 doubles itself in 10 years (loge 2 = 0.6931).

Answer 20:

Let p, t, and r represent the principal, time, and rate of interest respectively.

It is given that the principal increases continuously at the rate of r% per year.

$\Rightarrow \frac{dp}{dt} = \left(\frac{r}{100}\right)p$ $\Rightarrow \frac{dp}{p} = \left(\frac{r}{100}\right)dt$

Integrating both sides, we get:

$\int \frac{dp}{p} = \frac{r}{100}\int dt$ $\Rightarrow \log p = \frac{rt}{100} + k$ $\Rightarrow p = e^{\frac{rt}{100}+k}$ .... (1)

It is given that when $t = 0$, $p = 100$.

$\Rightarrow 100 = e^k$ ... (2)

Now, if $t = 10$, then $p = 2 \times 100 = 200$.

Therefore, equation (1) becomes:

$\Rightarrow 200 = e^{\frac{r}{10}+k}$

$\Rightarrow 200 = e^{\frac{r}{10}}.e^{k}$

$\Rightarrow 200 = e^{\frac{r}{10}}.100$ (from (2))

$e^{\frac{r}{10}} = 2$

$\Rightarrow \frac{r}{10} = log_e 2 = 0.6931$

$\Rightarrow r = 6.931$

Hence, the value of r is 6.93%

Question 21:

In a bank, principal increases continuously at the rate of 5% per year. An amount of ₹ 1000 is deposited with this bank, how much will it worth after 10 years $(e^{0.5}) = 1.648)$.

Answer 21:

Let p and t be the principal and time respectively.

It is given that the principal increases continuously at the rate of 5% per year.

$\Rightarrow \frac{dp}{dt} = \left(\frac{5}{100}\right)p \qquad \Rightarrow \frac{dp}{dt} = \left(\frac{p}{20}\right) \qquad \Rightarrow \frac{dp}{p} = \frac{dt}{20}$

Integrating both sides, we get:

$\int \frac{dp}{p} = \frac{1}{20}\int dt \qquad \Rightarrow log\, p = \frac{1}{20} + C \qquad \Rightarrow p = e^{\frac{1}{20}+C} \qquad ....(1)$

Now, when $t = 0$, $p = 1000$.

$\Rightarrow 1000 = e^{c}$ ... (2)

At t = 10, equation (1) becomes:

$\Rightarrow p = e^{\frac{1}{2}+C} = e^{0.5} \times e^{C} \qquad \Rightarrow p = 1.648 \times 1000$

$\Rightarrow p = 1648$, Hence, after 10 years the amount will worth ₹ 1648.

Question 22:

In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

Answer 22:

Let y be the number of bacteria at any instant t.

It is given that the rate of growth of the bacteria is proportional to the number present.

$\frac{dy}{dt} \propto y \qquad \Rightarrow \frac{dy}{dt} = ky$ (where k is a constant)

$\Rightarrow \frac{dy}{y} = kdt$

Integrating both sides, we get:

$\int \frac{dy}{y} = k\int dt \qquad \Rightarrow log\, y = kt + C \qquad ....(1)$

Let y0 be the number of bacteria at t = 0.

$\Rightarrow log\, y_0 = C$

Substituting the value of C in equation (1), we get:

$log\, y = kt + log\, y_0 \qquad \Rightarrow log\left(\frac{y}{y_0}\right) = kt \qquad \Rightarrow kt = log\left(\frac{y}{y_0}\right) \qquad ...(2)$

Also, it is given that the number of bacteria increases by 10% in 2 hours.

$$\Rightarrow y = \frac{110}{100} y_0$$

$$\Rightarrow \frac{y}{y_0} = \frac{110}{100} \qquad \ldots (3)$$

Substituting this value in equation (2), we get:

$$\Rightarrow k.2 = log\left(\frac{11}{10}\right)$$

Therefore, equation (2) becomes:

$$\frac{1}{2} log\left(\frac{11}{10}\right).t = log\left(\frac{y}{y_0}\right)$$

$$\Rightarrow t = \frac{2\,log\left(\frac{y}{y_0}\right)}{log\left(\frac{11}{10}\right)} \qquad \ldots (4)$$

Now, let the time when the number of bacteria increases from 100000 to 200000 be t_1.
$\Rightarrow y = 2y_0$ at $t = t_1$
From equation (4), we get:

$$t_1 = \frac{2\,log\left(\frac{y}{y_0}\right)}{log\left(\frac{11}{10}\right)} = \frac{2\,log\,2}{log\left(\frac{11}{10}\right)}$$

Hence, in $\frac{2\log 2}{\log\left(\frac{11}{10}\right)}$ hours the number of bacteria increases from 100000 to 200000.

Question 23:

The general solution of the differential *equation* $\frac{dy}{dx} = e^{x+y} is$

(A) $e^x + e^{-y} = C$ (B) $e^x + e^y = C$
(C) $e^{-x} + e^y = C$ (D) $e^{-x} + e^{-y} = C$

Answer 23:

$$\frac{dy}{dx} = e^{x+y} = e^x.e^y$$

$$\Rightarrow \frac{dy}{e^y} = e^x dx$$

$$\Rightarrow e^{-y} = e^x + k$$

$$\Rightarrow e^x + e^{-y} = -k$$

$$\Rightarrow e^x + e^{-y} = c \qquad (c = -k)$$

Hence, the correct answer is (A).

Mathematics

(Chapter – 9) (Differential Equations) (Exercise 9.4)
(Class – XII)

Question 1:
Show that the given differential equation is homogeneous and solve each of them.

$$(x^2 + xy)dy = (x^2 + y^2)dx$$

Answer 1:
The given differential equation i.e., $(x^2 + xy)dy = (x^2 + y^2)dx$ can be written as:

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy} \qquad \dots(1)$$

Let $F(x, y) = \frac{x^2 + y^2}{x^2 + xy}$

Now $F(\lambda x, \lambda y) = \frac{(\lambda x)^2 + (\lambda y)^2}{(\lambda x)^2 + (\lambda x)(\lambda y)} = \frac{x^2 + y^2}{x^2 + xy} = \lambda^0 . F(x, y)$

This shows that equation (1) is a homogeneous equation.
To solve it, we make the substitution as:

$$y = vx$$

Differentiating both sides with respect to x, we get: $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting the values of v and $\frac{dy}{dx}$ in equation (1), we get:

$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2 + (vx)^2}{x^2 + x(vx)} \qquad \Rightarrow v + x\frac{dv}{dx} = \frac{1 + v^2}{1 + v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1 + v^2}{1 + v} - v = \frac{(1 + v^2) - v(1 + v)}{1 + v} \qquad \Rightarrow x\frac{dv}{dx} = \frac{1 - v}{1 + v}$

$\Rightarrow \left(\frac{1 + v}{1 - v}\right)dv = \frac{dx}{x} \qquad \Rightarrow \left(\frac{2 - 1 + v}{1 - v}\right)dv = \frac{dx}{x} \qquad \Rightarrow \left(\frac{2}{1 - v} - 1\right)dv = \frac{dx}{x}$

Integrating both sides, we get:

$-2\log(1 - v) - v = \log x - \log k \qquad \Rightarrow v = -2\log(1 - v) - \log x + \log k$

$\Rightarrow v = log\left[\frac{k}{(1 - v)^2}\right] \qquad \Rightarrow \frac{y}{x} = \log\left[\frac{k}{\left(1 - \frac{y}{x}\right)^2}\right] \qquad \Rightarrow \frac{y}{x} = \log\left[\frac{kx}{(x - y)^2}\right]$

$\Rightarrow \frac{kx}{(x - y)^2} = e^{\frac{y}{x}} \qquad \Rightarrow (x - y)^2 = kxe^{\frac{y}{x}}$

This is the required solution of the given differential equation.

Question 2:
Show that the given differential equation is homogeneous and solve each of them.

$$y' = \frac{x + y}{x}$$

Answer 2:
The given differential equation: $y' = \frac{x+y}{x}$

$$\Rightarrow \frac{dy}{dx} = \frac{x + y}{x} \qquad \dots(1)$$

Let $F(x, y) = \frac{x + y}{x}$

Now $F(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x} = \frac{x + y}{x} = \lambda^0 F(x, y)$

Thus, the given equation is a homogeneous equation.
To solve it, we make the substitution as: $y = vx$

Differentiating both sides with respect to x, we get: $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{x + vx}{x} \qquad \Rightarrow v + x\frac{dv}{dx} = 1 + v \qquad \Rightarrow x\frac{dv}{dx} = 1 \quad \Rightarrow dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$v = \log x + C \qquad \Rightarrow \frac{y}{x} = \log x + C \qquad \Rightarrow y = x\log x + Cx$$

This is the required solution of the given differential equation.

Question 3:

Show that the given differential equation is homogeneous and solve each of them.

$$(x - y)dy - (x + y)dx = 0$$

Answer 3:

The given differential equation: $(x - y)dy - (x + y)dx = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{x + y}{x - y} \qquad \dots (1)$$

Let $F(x, y) = \dfrac{x + y}{x - y}$

Now $F(\lambda x, \lambda y) = \dfrac{\lambda x + \lambda y}{\lambda x - \lambda y} = \dfrac{x + y}{x - y} = \lambda^0 F(x, y)$

Thus, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as: $y = vx$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(vx) \qquad \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substitute the value of y and $\frac{dy}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x + vx}{x - vx} = \frac{1 + v}{1 - v} \qquad \Rightarrow x\frac{dv}{dx} = \frac{1 + v}{1 - v} - v = \frac{1 + v - v(1 - v)}{1 - v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 + v^2}{1 - v} \qquad \Rightarrow \frac{1 - v}{(1 + v^2)}dv = \frac{dx}{x} \qquad \Rightarrow \left(\frac{1}{(1 + v^2)} - \frac{v}{(1 + v^2)}\right)dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$tan^{-1}v - \frac{1}{2}\log(1 + v^2) = \log x + C$$

$$\Rightarrow tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left(1 + \left(\frac{y}{x}\right)^2\right) = \log x + C$$

$$\Rightarrow tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left(\frac{x^2 + y^2}{x^2}\right) = \log x + C$$

$$\Rightarrow tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}[\log(x^2 + y^2) - \log x^2] = \log x + C \ \Rightarrow tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\log(x^2 + y^2) + C$$

This is the required solution of the given differential equation.

Question 4:

Show that the given differential equation is homogeneous and solve each of them.

$$(x^2 - y^2)dx + 2xy\,dy = 0$$

Answer 4:

The given differential equation: $(x^2 - y^2)dx + 2xy\,dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{-(x^2 - y^2)}{2xy} \quad \ldots (1)$$

Let $F(x, y) = \dfrac{-(x^2 - y^2)}{2xy}$

Now $F(\lambda x, \lambda y) = \left[\dfrac{(\lambda x)^2 - (\lambda y)^2}{2(\lambda x)(\lambda y)}\right] = \dfrac{-(x^2 - y^2)}{2xy} = \lambda^0 F(x, y)$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as: $y = vx$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx) \qquad \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the value of y and $\frac{dy}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = -\left[\frac{x^2 - (vx)^2}{2x(vx)}\right] \qquad \Rightarrow v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v} \Rightarrow x\frac{dv}{dx} = -\frac{(1 - v^2)}{2v} \quad \Rightarrow \frac{2v}{1 - v^2}dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$\Rightarrow \log(1 - v^2) = -\log x + \log C = \log\frac{C}{x} \qquad \Rightarrow 1 - v^2 = \frac{C}{x}$$

$$\Rightarrow \left[1 - \left(\frac{y}{x}\right)^2\right] = \frac{C}{x} \qquad \Rightarrow x^2 + y^2 = Cx$$

This is the required solution of the given differential equation.

Question 5:

Show that the given differential equation is homogeneous and solve each of them.

$$x^2\frac{dy}{dx} = x^2 - 2y^2 + xy$$

Answer 5:

The given differential equation: $x^2\frac{dy}{dx} = x^2 - 2y^2 + xy$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 - 2y^2 + xy}{x^2} \quad \ldots (1)$$

Let $F(x, y) = \dfrac{x^2 - 2y^2 + xy}{x^2}$

Now $F(\lambda x, \lambda y) = \dfrac{(\lambda x)^2 - 2(\lambda y)^2 + (\lambda x)(\lambda y)}{(\lambda x)^2} = \dfrac{x^2 - 2y^2 + xy}{x^2} = \lambda^0 F(x, y)$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as: $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$

$$v + x\frac{dv}{dx} = 1 - 2v^2 + v \qquad \Rightarrow x\frac{dv}{dx} = 1 - 2v^2 \qquad \Rightarrow \frac{dv}{1 - 2v^2} = \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}.\frac{dv}{\frac{1}{2} - v^2} = \frac{dx}{x} \qquad \Rightarrow \frac{1}{2}\left[\frac{dv}{\left(\frac{1}{\sqrt{2}}\right)^2 - v^2}\right] = \frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}.\frac{1}{2.\frac{1}{\sqrt{2}}}\log\left|\frac{\frac{1}{\sqrt{2}} + \frac{y}{x}}{\frac{1}{\sqrt{2}} - \frac{y}{x}}\right| = \log|x| + C \qquad \Rightarrow \frac{1}{2\sqrt{2}}\log\left|\frac{\frac{1}{\sqrt{2}} + \frac{y}{x}}{\frac{1}{\sqrt{2}} - \frac{y}{x}}\right| = \log|x| + C$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\log\left|\frac{x + \sqrt{2}y}{x - \sqrt{2}y}\right| = \log|x| + C$$

This is the required solution for the given differential equation.

Question 6:

Show that the given differential equation is homogeneous and solve each of them.

$$x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$$

Answer 6:

The given differential equation: $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$

$$\Rightarrow x\,dy = \left[y + \sqrt{x^2 + y^2}\right]dx \qquad \Rightarrow \frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} \qquad \ldots(1)$$

Let $F(x, y) = \dfrac{y + \sqrt{x^2 + y^2}}{x}$

Now $F(\lambda x, \lambda y) = \dfrac{\lambda x + \sqrt{(\lambda x)^2 + (\lambda y)^2}}{\lambda x} = \dfrac{y + \sqrt{x^2 + y^2}}{x} = \lambda^0 F(x, y)$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as: $y = vx$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(vx) \qquad \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of v and $\frac{dy}{dx}$ in equation (1), we get:

$$\Rightarrow v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + (vx)^2}}{x} \qquad \Rightarrow v + x\frac{dv}{dx} = v + \sqrt{1 + v^2} \qquad \Rightarrow \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

Integrating both sides, we get:

$$\log\left|v + \sqrt{1 + v^2}\right| = \log|x| + C \qquad \Rightarrow \log\left|\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right| = \log|x| + C$$

$$\Rightarrow \log\left|\frac{y + \sqrt{x^2 + y^2}}{x}\right| = \log|x| + C \qquad \Rightarrow y + \sqrt{x^2 + y^2} = Cx^2$$

This is the required solution of the given differential equation.

Question 7:

Show that the given differential equation is homogeneous and solve each of them.

$$\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y\,dx=\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x\,dy$$

Answer 7:

The given differential equation: $\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y\,dx=\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x\,dy$

$$\Rightarrow \frac{dy}{dx}=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x} \quad \ldots(1)$$

Let $F(x,y)=\dfrac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$

Now $F(\lambda x,\lambda y)=\dfrac{\left\{\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right)+\lambda y\sin\left(\frac{\lambda y}{\lambda x}\right)\right\}y}{\left\{\lambda y\sin\left(\frac{\lambda y}{\lambda x}\right)-\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right)\right\}x}=\lambda^0F(x,y)$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as: $y = vx$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$\Rightarrow v+x\frac{dv}{dx}=\frac{(x\cos v+vx\sin v).vx}{(vx\sin v-x\cos v).x} \qquad \Rightarrow v+x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v}{v\sin v-\cos v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v-v^2\sin v+v\cos v}{v\sin v-\cos v} \qquad \Rightarrow x\frac{dv}{dx}=\frac{2v\cos v}{v\sin v-\cos v}$$

$$\Rightarrow \left[\frac{v\sin v-\cos v}{2v\cos v}\right]dv=\frac{2\,dx}{x} \qquad \Rightarrow \left(\tan v-\frac{1}{v}\right)dv=\frac{2\,dx}{x}$$

Integrating both sides, we get:

$$\Rightarrow \log\left(\frac{\sec v}{v}\right)=\log(Cx^2)$$

$$\Rightarrow \left(\frac{\sec v}{v}\right)=Cx^2 \qquad \Rightarrow \sec v=Cx^2v$$

$$\Rightarrow \sec\left(\frac{y}{x}\right)=Cx^2.\frac{y}{x}= Cxy$$

$$\Rightarrow \cos\left(\frac{y}{x}\right)=\frac{1}{Cxy}=\frac{1}{C}.\frac{1}{xy}$$

$$\Rightarrow\ xy\cos\left(\frac{y}{x}\right)=k \qquad \left(\text{Where } k=\frac{1}{C}\right)$$

This is the required solution of the given differential equation.

Question 8:
Show that the given differential equation is homogeneous and solve each of them.

$$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$$

Answer 8:

The given differential equation: $x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$

$$\Rightarrow x\frac{dy}{dx} = y - x\sin\left(\frac{y}{x}\right) \qquad \Rightarrow \frac{dy}{dx} = \frac{y - x\sin\left(\frac{y}{x}\right)}{x} \qquad \dots(1)$$

$Let\ F(x,y) = \frac{y - x\sin\left(\frac{y}{x}\right)}{x}$

Now $F(\lambda x, \lambda y)\frac{\lambda y - \lambda x\sin\left(\frac{\lambda y}{\lambda x}\right)}{\lambda x} = \frac{y - x\sin\left(\frac{y}{x}\right)}{x} = \lambda^0 F(x,y)$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as: $y = vx$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(vx) \qquad \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{vx - x\sin v}{x} \qquad \Rightarrow v + x\frac{dv}{dx} = v - \sin v \qquad \Rightarrow -\frac{dv}{\sin v} = \frac{dx}{x} \qquad \Rightarrow cosec\ v\ dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$\log|cosec\ v - \cot v| = -\log x + \log C = \log\frac{C}{x}$$

$$\Rightarrow cosec\left(\frac{y}{x}\right) - \cot\left(\frac{y}{x}\right) = \frac{C}{x} \qquad \Rightarrow \frac{1}{\sin\left(\frac{y}{x}\right)} - \frac{\cos\left(\frac{y}{x}\right)}{\sin\left(\frac{y}{x}\right)} = \frac{C}{x} \qquad \Rightarrow x\left[1 - \cos\left(\frac{y}{x}\right)\right] = C\sin\left(\frac{y}{x}\right)$$

This is the required solution of the given differential equation.

Question 9:
Show that the given differential equation is homogeneous and solve each of them.

$$y\,dx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$$

Answer 9:

The given differential equation: $y\,dx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$

$$\Rightarrow y\,dx = \left[2x - x\log\left(\frac{y}{x}\right)\right]dy \qquad \Rightarrow \frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} \qquad \dots(1)$$

Let $F(x,y) = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$

Now $F(\lambda x, \lambda y) = \frac{\lambda y}{2(\lambda x) - (\lambda x\_\log\left(\frac{\lambda y}{\lambda x}\right)} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} = \lambda^0 F(x,y)$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as: $y = vx$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(vx) \qquad \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substitute this value of y and $\frac{dy}{dx}$

$v + x\frac{dv}{dx} = \frac{vx}{2x - x\log v}$ $\Rightarrow v + x\frac{dv}{dx} = \frac{v}{2 - \log v}$ $\Rightarrow x\frac{dv}{dx} = \frac{v}{2 - \log v} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{v - 2v + v\log v}{2 - \log v}$ $\Rightarrow \frac{2 - \log v}{v(\log v - 1)}dv = \frac{dx}{x}$ $\Rightarrow \left[\frac{1 + (1 - \log v)}{v(\log v - 1)}\right]dv = \frac{dx}{x}$

$\Rightarrow \left[\frac{1}{v(\log v - 1)} - \frac{1}{v}\right]dv = \frac{dx}{x}$

Integrating both sides, we get:

$\int \frac{1}{v(\log v - 1)}dv - \int \frac{1}{v}dv = \int \frac{1}{x}dx$ $\Rightarrow \int \frac{1}{v(\log v - 1)}dv - \log v = \log x + \log C$ ... (2)

Let $\log(v - 1) = t$ $\Rightarrow \frac{d}{dv}(\log v - 1) = \frac{dt}{dv}$ $\Rightarrow \frac{1}{v} = \frac{dt}{dv}$ $\Rightarrow \frac{dv}{v} = dt$

Therefore, equation (2) becomes:

$\int \frac{dt}{t} - \log v = \log x + \log C$ $\Rightarrow \log t - \log\left(\frac{y}{x}\right) = \log(Cx)$

$\Rightarrow \log\left[\log\left(\frac{y}{x}\right) - 1\right] - \log\left(\frac{y}{x}\right) = \log(Cx)$ $\Rightarrow \log\left[\frac{\log\left(\frac{y}{x}\right) - 1}{\frac{y}{x}}\right] = \log(Cx)$

$\Rightarrow \frac{y}{x}\left[\log\left(\frac{y}{x}\right) - 1\right] = Cx$ $\Rightarrow \log\left(\frac{y}{x}\right) - 1 = Cy$

This is the required solution of the given differential equation.

Question 10:

Show that the given differential equation is homogeneous and solve each of them.

$$\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$$

Answer 10:

The given differential equation: $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

$\Rightarrow \left(1 + e^{\frac{x}{y}}\right)dx = -e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy$ $\Rightarrow \frac{dy}{dx} = \frac{-e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)}{\left(1 + e^{\frac{x}{y}}\right)}$ ... (1)

Let $F(x, y) = \frac{-e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)}{\left(1 + e^{\frac{x}{y}}\right)}$

Now $F(\lambda x, \lambda y) = \frac{-e^{\frac{\lambda x}{\lambda y}}\left(1 - \frac{\lambda x}{\lambda y}\right)}{\left(1 + e^{\frac{\lambda x}{\lambda y}}\right)} = \frac{-e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)}{\left(1 + e^{\frac{x}{y}}\right)} = \lambda^0 F(x, y)$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as: $x = vy$

$\frac{d}{dy}(x) = \frac{d}{dy}(vy)$ $\Rightarrow \frac{dy}{dx} = v + y\frac{dv}{dy}$

Substitute the value of x and $\frac{dy}{dx}$

$$v + y\frac{dv}{dy} = \frac{-e^v(1-v)}{(1+e^v)} \quad \Rightarrow y\frac{dv}{dy} = \frac{-e^v + ve^v}{1+e^v} - v \quad \Rightarrow y\frac{dv}{dy} = \frac{-e^v + ve^v - v - ve^v}{1+e^v}$$

$$\Rightarrow y\frac{dv}{dy} = -\left[\frac{v - e^v}{1+e^v}\right] \quad \Rightarrow \left[\frac{v-e^v}{1+e^v}\right]dv = -\frac{dy}{y}$$

Integrating both sides, we get:

$$log(v + e^v) = -\log y + \log C = \log\left(\frac{C}{y}\right) \quad \Rightarrow \left[\frac{x}{y} + e^{\frac{x}{y}}\right] = \frac{C}{y} \quad \Rightarrow x + ye^{\frac{x}{y}} = C$$

This is the required solution of the given differential equation.

Question 11:

For the differential equation, find the particular solution satisfying the given condition:

$$(x+y)dy + (x-y)dx = 0; y = 1 \text{ when } x = 1$$

Answer 11:

The given differential equation: $(x+y)dy + (x-y)dx = 0$

$$\Rightarrow (x+y)dy = -(x-y)dx \quad \Rightarrow \frac{dy}{dx} = \frac{-(x-y)}{x+y} \quad \dots(1)$$

Let $F(x,y) = \frac{-(x-y)}{x+y}$

Now $F(\lambda x, \lambda y) = \frac{-(\lambda x - \lambda y)}{\lambda x + \lambda y} = \frac{-(x-y)}{x+y} = \lambda^0 F(x,y)$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as: $y = vx$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx) \quad \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{-(x - vx)}{x + vx} \quad \Rightarrow v + x\frac{dv}{dx} = \frac{v-1}{v+1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{v - 1 - v(v+1)}{v+1} \Rightarrow x\frac{dv}{dx} = \frac{v - 1 - v^2 - v}{v+1} = \frac{-(1+v^2)}{v+1}$$

$$\Rightarrow \frac{(v+1)}{1+v^2}dv = -\frac{dx}{x} \quad \Rightarrow \left[\frac{v}{1+v^2} + \frac{1}{1+v^2}\right]dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}\log(1+v^2) + tan^{-1}v = -\log x + k$$

$$\Rightarrow \log(1+v^2) + 2tan^{-1}v = -2\log x + 2k \Rightarrow \log[(1+v^2).x^2] + 2tan^{-1}v = 2k$$

$$\Rightarrow \log\left[\left(1 + \frac{y^2}{x^2}\right).x^2\right] + 2tan^{-1}\frac{y}{x} = 2k$$

$$\Rightarrow \log(x^2 + y^2) + 2tan^{-1}\frac{y}{x} = 2k \quad \dots(2)$$

Now, y = 1 at $x = 1$.

$$\log 2 + 2tan^{-1}1 = 2k \quad \Rightarrow \log 2 + 2 \times \frac{\pi}{4} = 2k \quad \Rightarrow \frac{\pi}{2} + \log 2 = 2k$$

Substituting the value of 2k in equation (2), we get:

$$\log(x^2 + y^2) + 2tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2} + \log 2$$

This is the required solution of the given differential equation.

Question 12:

For the differential equation, find the particular solution satisfying the given condition:

$$x^2dy + (xy + y^2)dx = 0;\ y = 1 \text{ when } x = 1$$

Answer 12:

The given differential equation: $x^2dy + (xy + y^2)dx = 0$

$\Rightarrow x^2dy = -(xy + y^2)dx \qquad \Rightarrow \frac{dy}{dx} = \frac{-(xy + y^2)}{x^2} \quad \ldots(1)$

Let $F(x, y) = \frac{-(xy + y^2)}{x^2}$

Now $F(\lambda x, \lambda y) = \frac{[(\lambda x)(\lambda y) + (\lambda y)^2]}{(\lambda x)^2} = \frac{-(xy + y^2)}{x^2} = \lambda^0 F(x, y)$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as: $y = vx$

$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx) \qquad \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substitute the value of y and $\frac{dy}{dx}$

$v + x\frac{dv}{dx} = \frac{[x.vx + (vx)^2]}{x^2} = -v - v^2 \qquad \Rightarrow x\frac{dv}{dx} = -v^2 - 2v = -v(v + 2)$

$\Rightarrow \frac{dv}{v(v + 2)} = -\frac{dx}{x} \qquad \Rightarrow \frac{1}{2}\left[\frac{(v - 2) - v}{v(v + 2)}\right]dv = -\frac{dx}{x} \qquad \Rightarrow \frac{1}{2}\left[\frac{1}{v} - \frac{1}{v + 2}\right]dv = -\frac{dx}{x}$

Integrating both sides, we get:

$\frac{1}{2}[\log v - \log(v + 2)] = -\log x + \log C \qquad \Rightarrow \frac{1}{2}[\log v - \log(v + 2)] = -\log x + \log C$

$\Rightarrow \frac{1}{2}\log\left(\frac{v}{v + 2}\right) = \log\frac{C}{x} \qquad \Rightarrow \frac{v}{v + 2} = \left(\frac{C}{x}\right)^2 \qquad \Rightarrow \frac{\frac{y}{x}}{\frac{y}{x} + 2x} = \frac{C^2}{x^2} \qquad \Rightarrow \frac{x^2y}{y + 2x} = C^2 \qquad \ldots(2)$

Now, y = 1 at $x = 1$. Therefore $\frac{1}{1+2} = C^2 = \frac{1}{3}$

Substituting $C^2 = \frac{1}{3}$, we have

$$\frac{x^2y}{y + 2x} = \frac{1}{3} \qquad \Rightarrow y + 2x = 3x^2y$$

This is the required solution of the given differential equation.

Question 13:

For the differential equation, find the particular solution satisfying the given condition:

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0;\ y = \frac{\pi}{4} \text{ when } x = 1$$

Answer 13:

The given differential equation: $\left[xsin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{\left[x sin^2\left(\frac{y}{x}\right) - y\right]}{x} \qquad \dots (1)$$

Let $F(x, y) = \dfrac{-(x-y)}{x+y}$

Now $F(\lambda x, \lambda y) = \dfrac{\left[\lambda x sin^2\left(\lambda \frac{\lambda y}{x}\right) - \lambda y\right]}{\lambda x} = \dfrac{\left[x sin^2\left(\frac{y}{x}\right) - y\right]}{x} = \lambda^0 F(x, y)$

Therefore, the given differential equation is a homogeneous equation.

To solve this differential equation, we make the substitution as: $y = vx$

$\Rightarrow \dfrac{d}{dx}(y) = \dfrac{d}{dx}(vx) \qquad \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1), we get:

$\Rightarrow v + x\dfrac{dv}{dx} = \dfrac{-\left[x sin^2 v - vx\right]}{x} \qquad \Rightarrow v + x\dfrac{dv}{dx} = -[sin^2 v - v] = v - sin^2 v$

$\Rightarrow x\dfrac{dv}{dx} = -sin^2 v \qquad \Rightarrow \dfrac{dv}{sin^2 v} = -\dfrac{dx}{x} \qquad \Rightarrow cosec^2 v dv = -\dfrac{dx}{x}$

Integrating both sides, we get:

$-\cot v = -\log|x| - C \qquad \Rightarrow \cot v = \log|x| + C \qquad \Rightarrow \cot\left(\dfrac{y}{x}\right) = \log|x| + C \qquad \Rightarrow \cot\left(\dfrac{y}{x}\right) = \log|Cx|$

Now, $\quad y = \dfrac{\pi}{4}$ $at\ x = 1 \quad \Rightarrow \cot\left(\dfrac{\pi}{4}\right) = \log|C| \qquad \Rightarrow 1 = \log C \quad \Rightarrow \quad C = e^1 = e$

Substituting C = e in equation (2), we get: $\cot\left(\frac{y}{x}\right) = \log|ex|$

This is the required solution of the given differential equation.

Question 14:

For the differential equation, find the particular solution satisfying the given condition:

$$\frac{dy}{dx} - \frac{y}{x} + cosec\left(\frac{y}{x}\right) = 0; y = 0 \text{ when } x = 1$$

Answer 14:

The given differential equation: $\frac{dy}{dx} - \frac{y}{x} + cosec\left(\frac{y}{x}\right) = 0 \quad \Rightarrow \frac{dy}{dx} = \frac{y}{x} - cosec\left(\frac{y}{x}\right)$

Let $F(x, y) = \dfrac{y}{x} - cosec\left(\dfrac{y}{x}\right)$

Now $F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} - cosec\left(\frac{\lambda y}{\lambda x}\right) = \frac{y}{x} - cosec\left(\frac{y}{x}\right) = \lambda^0 F(x, y)$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as: $y = vx$

$\Rightarrow \dfrac{d}{dx}(y) = \dfrac{d}{dx}(vx) \qquad \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$v + x\dfrac{dv}{dx} = v - cosec\ v \qquad \Rightarrow \dfrac{dv}{cosec\ v} = \dfrac{dx}{x} \qquad \Rightarrow -\sin v\ dv = \dfrac{dx}{x}$

Integrating both sides, we get: $\cos v = \log x + \log C = \log|Cx| \qquad \dots (2)$

This is the required solution of the given differential equation. Now, y = 0 at x = 1.

$\cos(0) = \log C \qquad \Rightarrow 1 = \log C \qquad \Rightarrow C = e^1 = e$

Substituting C = e in equation (2), we get: $\cos\left(\frac{y}{x}\right) = \log|ex|$

This is the required solution of the given differential equation.

Question 15:

For the differential equation, find the particular solution satisfying the given condition:

$$2xy + y^2 - 2x^2\frac{dy}{dx} = 0; y = 2 \text{ when } x = 1$$

Answer 15:

The given differential equation: $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$

$\Rightarrow 2x^2\frac{dy}{dx} = 2xy + y^2 \qquad \Rightarrow \frac{dy}{dx} = \frac{2xy + y^2}{2x^2} \qquad \ldots (1)$

Let $F(x, y) = \frac{2xy + y^2}{2x^2}$

Now $F(\lambda x, \lambda y) = \frac{2(\lambda x)(\lambda y) + (\lambda y)^2}{2(\lambda x)^2} = \frac{2xy + y^2}{2x^2} = \lambda^0 F(x, y)$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as: $y = vx$

$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx) \qquad \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1), we get:

$v + x\frac{dv}{dx} = \frac{2x(vx) + (vx)^2}{2x^2}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{2v + v^2}{2}$

$\Rightarrow v + x\frac{dv}{dx} = v + \frac{v^2}{2} \qquad \Rightarrow \frac{2}{v^2}dv = \frac{dx}{x}$

Integrating both sides, we get:

$$\frac{v^{-2+1}}{-2+1} = \log|x| + C$$

$\Rightarrow -\frac{2}{v} = \log|x| + C$

$\Rightarrow -\frac{2}{\frac{y}{x}} = \log|x| + C$

$\Rightarrow -\frac{2x}{y} = \log|x| + C \qquad \ldots (2)$

Now, y = 2 at x = 1. We have $-1 = \log(0) + C \qquad \Rightarrow C = -1$

Substituting C = –1 in equation (2), we get:

$$-\frac{2x}{y} = \log|x| - 1 \qquad \Rightarrow \frac{2x}{y} = 1 - \log|x|$$

This is the required solution of the given differential equation.

Question16:

A homogeneous differential equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution

(A) $y = vx$ (B) $v = yx$ (C) $x = vy$ (D) $x = v$

Answer 16:

For solving homogeneous equation of form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$, we need to make substitution as $x = vy$

.

Thus, the correct option is C.

Question17:

Which of the following is a homogeneous differential equation?

(A) $(4x + 6y + 5)dy - (3y + 2x + 4)dx = 0$ (B) $(xy)dx - (x^3 + y^3)dy = 0$

(C) $(x^3 + 2y^2)dx + 2xy\,dy = 0$ (D) $y^2dx + (x^2 - xy - y^2)\,dy = 0$

Answer 17:

Function F(x, y) is said to be the homogenous function of degree n, if

$F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any non − zero constant (λ).

Consider the equation given in alternative (D):

$$y^2dx + (x^2 - xy - y^2)\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y^2}{(x^2 - xy - y^2)}$$

Let $F(x, y) = \frac{-y^2}{(x^2 - xy - y^2)}$

Now $F(\lambda x, \lambda y) = \frac{-(\lambda y)^2}{(\lambda y)^2 - (\lambda x)(\lambda y) - (\lambda x)^2}$

$$\Rightarrow \frac{\lambda^2 y^2}{\lambda^2(y^2 + xy - x^2\,)}$$

$$\Rightarrow \lambda^0\left(\frac{y^2}{y^2 + xy - x^2}\right)$$

$$\Rightarrow \lambda^0 F(x, y)$$

Hence, the differential equation given in alternative (D) is a homogenous equation.

Mathematics

(Chapter – 9) (Differential Equations) (Exercise 9.5)
(Class – XII)

Question 1:

For the given differential equation, find the general solution:

$$\frac{dy}{dx}+2y=\sin x$$

Answer 1:

The given differential equation: $\frac{dy}{dx}+2y=\sin x$

This is in the form of $\frac{dy}{dx}+py=Q(where\quad p=2and\ Q=\sin x)$

Now I.F = $e^{\int p\,dx}=e^{\int 2\,dx}=e^{2x}$

The solution of the given differential equation is given by the relation,

$y(I.F.)=\int(Q\times I.F.)\,dx+C$

$\Rightarrow ye^{2x}=\int \sin x\,.e^{2x}\,dx+C\quad ...(1)$

Let $I=\int \sin x\,.e^{2x}$

$\Rightarrow I=\sin x\int e^{2x}dx-\int\left(\frac{d}{dx}(\sin x).\int e^{2x}dx\right)dx\qquad \Rightarrow I=\sin x\,.\frac{e^{2x}}{2}-\int\left(\cos x\,.\frac{e^{2x}}{2}\right)dx$

$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\left[\cos x\int e^{2x}dx-\int\left(\frac{d}{dx}(\cos x).\int e^{2x}dx\right)dx\right]$

$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\left[\cos x.\frac{e^{2x}}{2}-\int\left[(-\sin x).\frac{e^{2x}}{2}\right]dx\right]$

$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{e^{2x}\cos x}{4}-\frac{1}{4}\int(\sin x.e^{2x})dx\qquad \Rightarrow I=\frac{e^{2x}}{4}(2\sin x-\cos x)-\frac{1}{4}I$

$\Rightarrow \frac{5}{4}I=\frac{e^{2x}}{4}(2\sin x-\cos x)\quad \Rightarrow I=\frac{e^{2x}}{5}(2\sin x-\cos x)$

Therefore, equation (1) becomes:

$ye^{2x}=\frac{e^{2x}}{5}(2\sin x-\cos x)+C\qquad \Rightarrow y=\frac{1}{5}(2\sin x-\cos x)+Ce^{-2x}$

This is the required general solution of the given differential equation.

Question 2:

For the given differential equation, find the general solution:

$$\frac{dy}{dx}+3y=e^{-2x}$$

Answer 2:

The given differential equation: $\frac{dy}{dx}+py=Q$ (where $p=2$ and $Q=e^{-2x}$)

Now I.F $\quad ye^{2x}=e^{\int p\,dx}=e^{\int 3\,dx}=e^{3x}$

The solution of the given differential equation is given by the relation,

$y(I.F)=\int(Q\times I.F)\,dx+C\qquad \Rightarrow ye^{3x}=\int(e^{-2x}\times e^{3x})+C$

$\Rightarrow\ ye^{3x}=\int e^{x}+C$

$\Rightarrow y=e^{-2x}+Ce^{-3x}$

This is the required general solution of the given differential equation.

Question 3:

For the given differential equation, find the general solution:

$$\frac{dy}{dx}+\frac{y}{x}=x^2$$

Answer 3:

The given differential equation: $\frac{dy}{dx}+py=Q$(where $p=\frac{1}{x}$ and $Q=x^2$)

The solution of the given differential equation is given by the relation,

$$y(I.F)=\int(Q\times I.F)\,dx+C$$

$$\Rightarrow y(x)=\int(x^2.x)dx+C \quad \Rightarrow yx=\int(x^3)dx+C \quad \Rightarrow xy=\frac{x^4}{4}+C$$

This is the required general solution of the given differential equation.

Question 4:

For the given differential equation, find the general solution:

$$\frac{dy}{dx}+(\sec x)y=\tan x \quad \left(0\le x<\frac{\pi}{2}\right)$$

Answer 4:

The given differential equation: $\frac{dy}{dx}+py=Q$ (where $p=\sec x$ and $Q=\tan x$)

Now $I.F.=e^{\int p\,dx}=e^{\int \sec x\,dx}=e^{\log(\sec x+tan x)}$

The general solution of the given differential equation is given by the relation,

$$y(I.F)=\int(Q\times I.F)\,dx+C$$

$$\Rightarrow y(\sec x+\tan x)=\int \tan x\,(\sec x+\tan x)\,dx+C$$

$$\Rightarrow y(\sec x+\tan x)=\int \sec x\tan x\,dx+\int tan^2 x dx+C$$

$$\Rightarrow y(\sec x+\tan x)=\sec x+\int(sec^2x-1)dx+C \Rightarrow y(\sec x+\tan x)=\sec x+\tan x+C$$

This is the required general solution of the given differential equation.

Question 5:

For the given differential equation, find the general solution:

$$\cos^2 x\frac{dy}{dx}+y=\tan x \quad \left(0\le x<\frac{\pi}{2}\right)$$

Answer 5:

The given differential equation: $\cos^2 x\frac{dy}{dx}+y=\tan x \Rightarrow \frac{dy}{dx}+y\,sec^2x=\tan x\sec^2 x$

The given equation is in the form $\frac{dy}{dx}+py=Q$ (where $p=\sec^2 x$ and $Q=\tan x\sec^2 x$)

Now $I.F.=e^{\int p\,dx}=e^{\int \sec^2 x\,dx}=e^{\tan x}$

The general solution of the given differential equation is given by the relation,

$$y(I.F)=\int(Q\times I.F)\,dx+C \quad \Rightarrow ye^{\tan x}=\int \tan x\sec^2 x\,e^{\tan x}\,dx+C$$

Let $\tan x=t \quad \Rightarrow \sec^2 x\,dx=dt$

Therefore, the solution of differential become

$$ye^t = \int te^t\, dt + C$$

$\Rightarrow ye^t = t.e^t - \int e^t\, dt + C$ [Using Integration by part]

$\Rightarrow ye^t = t.e^t + e^t + C$

$\Rightarrow ye^{\tan x} = \tan x\,.e^{\tan x} - e^{\tan x} + C$

$\Rightarrow y = \tan x + 1 + Ce^{-\tan x}$

This is the required general solution of the given differential equation.

Question 6:

For the given differential equation, find the general solution:

$$x\frac{dy}{dx} + 2y = x^2 \log x$$

Answer 6:

The given differential equation: $\frac{dy}{dx} + \frac{2}{x}y = x\log x$

This equation is in the form of a linear differential equation as:

$$\frac{dy}{dx} + py = Q \text{ (where } p = \frac{2}{x} \text{ and } Q = x\log x)$$

Now I.F = $e^{\int p\,dx} = e^{\int \frac{2}{x}dx} = e^{\log x^2} = x^2$

The general solution of the given differential equation is given by the relation,

$$y(I.F) = \int (Q \times I.F)\, dx + C$$

$\Rightarrow y(x^2) = \int (x\log x.x^2)\, dx + C \quad \Rightarrow x^2y = \int (x^3 \log x)dx + C$

$\Rightarrow x^2y = \log x \int x^3 dx - \int \left(\frac{d}{dx}(\log x).\int x^3 dx\right) dx + C$

$\Rightarrow x^2y = \log x.\frac{x^4}{4} - \int \left(\frac{1}{x}.\frac{x^4}{4}\right) dx + C \quad \Rightarrow x^2y = \frac{x^4 \log x}{4} - \frac{1}{4}\int x^3 dx + C$

$\Rightarrow x^2y = \frac{x^4 \log x}{4} - \frac{1}{4}.\frac{x^4}{4} + C \quad \Rightarrow x^2y = \frac{x^4}{16}(4\log x - 1) + C \quad \Rightarrow y = \frac{x^2}{16}(4\log x - 1) + Cx^{-2}$

This is the required general solution of the given differential equation.

Question 7:

For the given differential equation, find the general solution:

$$x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$$

Answer 7:

The given differential equation: $x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$

$\Rightarrow \frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x^2}$

This equation is the form of a linear differential equation as:

$$\frac{dy}{dx} + py = Q \left(\text{where } p = \frac{1}{x\log x} \text{ and } Q = \frac{2}{x^2}\right)$$

Now I.F = $e^{\int p\,dx} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$

The general solution of the given differential equation is given by the relation,

$y(I.F) = \int (Q \times I.F)\,dx + C \quad \Rightarrow y\log x = \int \left(\frac{2}{x^2}\log x\right)dx + C \quad \ldots(1)$

Now, $\int \left(\frac{2}{x^2}\log x\right)dx = 2\int \left(\log x . \frac{1}{x^2}\right)dx$

$= 2\left[\log x . \int \frac{1}{x^2}dx - \int \left\{\frac{d}{dx}(\log x)\int \frac{1}{x^2}dx\right\}dx\right]$

$= 2\left[\log x . \left(-\frac{1}{x}\right) - \int \left\{\frac{1}{x}\left(-\frac{1}{x}\right)\right\}dx\right] = 2\left[-\frac{\log x}{x} + \int \frac{1}{x^2}dx\right] = 2\left[-\frac{\log x}{x} - \frac{1}{x}\right] = -\frac{2}{x}(1 + \log x)$

Substituting the value of $\int \left(\frac{2}{x^2}\log x\right)dx$ in equation (1), we get

$$y\log x = -\frac{2}{x}(1 + \log x) + C$$

This is the required general solution of the given differential equation.

Question 8:

For the given differential equation, find the general solution:

$$(1 + x^2)dy + 2xy\,dx = \cot x\,dx \qquad (x \neq 0)$$

Answer 8:

The given differential equation: $(1 + x^2)dy + 2xy\,dx = \cot x\,dx$

$\Rightarrow \frac{dy}{dx} + \frac{2xy}{1 + x^2} = \frac{\cot x}{1 + x^2}$

This equation is a linear differential equation of the form:

$$\frac{dy}{dx} + py = Q \left(where \quad p = \frac{2x}{1 + x^2} \quad and\ Q = \frac{\cot x}{1 + x^2}\right)$$

Now I.F = $e^{\int p\,dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$

The general solution of the given differential equation is given by the relation,

$$y(I.F) = \int (Q \times I.F)\,dx + C$$

$\Rightarrow y(1 + x^2) = \int \left[\frac{\cot x}{1 + x^2} \times (1 + x^2)\right]dx + C \Rightarrow y(1 + x^2) = \int \cot x\,dx + C$

$\Rightarrow y(1 + x^2) = \log|\sin x| + C$

This is the required general solution of the given differential equation.

Question 9:

For the given differential equation, find the general solution:

$$x\frac{dy}{dx} + y - x + xy\cot x = 0 \quad (x \neq 0)$$

Answer 9:

The given differential equation: $x\frac{dy}{dx} + y - x + xy\cot x = 0$

$\Rightarrow x\frac{dy}{dx} + y(1 + x\cot x) = x \qquad \Rightarrow \frac{dy}{dx} + \left(\frac{1}{x} + \cot x\right)y = 1$

This equation is a linear differential equation of the form:

$$\frac{dy}{dx}+py=Q\ \left(where\quad p=\frac{1}{x}+\cot x\quad and\ Q=1\right)$$

Now I.F = $e^{\int p\,dx}=e^{\int\left(\frac{1}{x}+\cot x\right)dx}=e^{\log x+\log(\sin x)}=e^{\log(x\sin x)}=x\sin x$

The general solution of the given differential equation is given by the relation,

$$y(I.F)=\int(Q\times I.F)\,dx+C$$

$\Rightarrow y(xsin\ x)=\int(1\times x\sin x)dx+C$ $\Rightarrow y(xsin\ x)=\int x\sin x\,dx+C$

$\Rightarrow y(xsin\ x)=\mathrm{x}\int\sin x\,dx-\int\left\{\frac{d}{dx}(\mathrm{x})\int\sin x\,dx\right\}dx+C$

$\Rightarrow y(xsin\ x)=x(-\cos x)-\int 1.(-\cos x)dx+C$ $\Rightarrow y(xsin\ x)=-xcos\ x+\sin x+C$

$\Rightarrow y=\frac{-x\cos x}{x\sin x}+\frac{\sin x}{x\sin x}+\frac{C}{x\sin x}$

$\Rightarrow y=-\cot x+\frac{1}{x}+\frac{C}{x\sin x}$

This is the required general solution of the given differential equation.

Question 10:

For the given differential equation, find the general solution:

$$(x+y)\frac{dy}{dx}=1$$

Answer 10:

The given differential equation: $(x+y)\frac{dy}{dx}=1$

$\Rightarrow\frac{dy}{dx}=\frac{1}{x+y}$ $\Rightarrow\frac{dx}{dy}=x+y$ $\Rightarrow\frac{dx}{dy}-x=y$

This is a linear differential equation of the form:

Now I.F = $e^{\int p\,dy}=e^{\int -dy}=e^{-y}$

The general solution of the given differential equation is given by the relation,

$$x(I.F)=\int(Q\times I.F)\,dy+C$$

$\Rightarrow xe^{-y}=\int(y.e^{-y})\,dy+C$

$\Rightarrow xe^{-y}=\mathrm{y}\int e^{-y}\,dy-\int\left\{\frac{d}{dy}(\mathrm{y})\int e^{-y}\,dy\right\}dy+C$

$\Rightarrow xe^{-y}=y(e^{-y})-\int(-e^{-y})dy+C$

$\Rightarrow xe^{-y}=-ye^{-y}-e^{-y}+C$

$\Rightarrow x=-y-1+Ce^{y}$

$\Rightarrow x+y+1=Ce^{y}$

This is the required general solution of the given differential equation.

Question 11:

For the given differential equation, find the general solution:

$$ydx + (x - y^2)dy = 0$$

Answer 11:

The given differential equation: $ydx + (x - y^2)dy = 0$

$\Rightarrow ydx = (y^2 - x)dy \qquad \Rightarrow \frac{dy}{dx} = \frac{y^2 - x}{y} = y - \frac{x}{y} \qquad \Rightarrow \frac{dy}{dx} + \frac{x}{y} = y$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + py = Q \left(where \quad p = \frac{1}{y} \quad and\ Q = y\right)$$

Now I.F = $e^{\int p\,dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$

The general solution of the given differential equation is given by the relation,

$$x(I.F) = \int (Q \times I.F)\,dy + C$$

$\Rightarrow xy = \int (y.y)\,dy + C$

$\Rightarrow xy = \int (y^2)\,dy + C$

$\Rightarrow xy = \frac{y^3}{3} + C$

$\Rightarrow x = \frac{y^3}{3} + \frac{C}{x}$

This is the required general solution of the given differential equation.

Question 12:

For the given differential equation, find the general solution:

$$(x + 3y^2)\frac{dy}{dx} = y \quad (y > 0).$$

Answer 12:

The given differential equation: $(x + 3y^2)\frac{dy}{dx} = y$

$\Rightarrow \frac{dy}{dx} = \frac{y}{(x + 3y^2)} \qquad \Rightarrow \frac{dx}{dy} = \frac{x + 3y^2}{y} = \frac{x}{y} + 3y \qquad \Rightarrow \frac{dx}{dy} - \frac{x}{y} = 3y$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + py = Q \left(\text{where} \quad p = -\frac{1}{y} \quad \text{and } Q = 3y\right)$$

Now I.F = $e^{\int p\,dy} = e^{-\int \frac{1}{y} dy} = e^{\log y} = e^{\log\left(\frac{1}{y}\right)} = \frac{1}{y}$

The general solution of the given differential equation is given by the relation,

$$x(I.F) = \int (Q \times I.F)\,dy + C$$

$\Rightarrow x\left(\frac{1}{y}\right) = \int \left(3y \times \frac{1}{y}\right) dy + C \qquad \Rightarrow \frac{x}{y} = 3y + C \qquad \Rightarrow x = 3y^2 + C$

This is the required general solution of the given differential equation.

Question 13:

For the differential equation, find a particular solution satisfying the given condition:

$$\frac{dy}{dx} + 2y\tan x = \sin x; \quad y = 0 \text{ when } x = \frac{\pi}{3}$$

Answer 13:

The given differential equation: $\frac{dx}{dy} + 2y\tan x = \sin x$

This is a linear equation of the form: $\frac{dy}{dx} + py = Q$ (where $p = 2\tan x$ and $Q = \sin x$)

Now I.F = $e^{\int p\,dy} = e^{\int 2\tan x\,dy} = e^{2\log|\sec x|} = e^{\log(sec^2 x)} = sec^2 x$

The general solution of the given differential equation is given by the relation,

$$y(I.F) = \int (Q \times I.F)\,dx + C$$

$\Rightarrow y(sec^2 x) = \int (\sin x\,.\,sec^2 x)\,dx + C \quad \Rightarrow ysec^2 x = \int (\sec x\,.\tan x)dx + C$

$\Rightarrow ysec^2 x = \sec x + C \quad \dots(1)$

Now, $y = 0$ at $x = \frac{\pi}{3}$, Therefore,

$0 \times sec^2\frac{\pi}{3} = \sec\frac{\pi}{3} + C \;\Rightarrow 0 = 2 + C \;\Rightarrow C = -2$

Substituting C = −2 in equation (1), we get:

$y\,sec^2 x = \sec x - 2 \quad \Rightarrow y = \cos x - 2\,sec^2 x$

Hence, the required solution of the given differential equation is $y = \cos x - 2\,sec^2 x$.

Question 14:

For the differential equation, find a particular solution satisfying the given condition:

$$(1+x^2)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2}; \quad y = 0 \text{ when } x = 1$$

Answer 14:

The given differential equation: $(1+x^2)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2}$

$\Rightarrow \frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{(1+x^2)^2}$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + py = Q \left(\text{where } p = \frac{2x}{1+x^2} \text{ and } Q = \frac{1}{(1+x^2)^2}\right)$$

The general solution of the given differential equation is given by the relation,

$y(I.F) = \int (Q \times I.F)\,dx + C$

$\Rightarrow y(1+x^2) = \int \left[\frac{1}{(1+x^2)^2}.(1+x^2)\right]dx + C \quad \Rightarrow y(1+x^2) = \int \frac{1}{(1+x^2)}dx + C$

$\Rightarrow y(1+x^2) = tan^{-1}1 + C \quad \dots(1)$

Now, y = 0 at x = 1. Therefore,

$$0 = tan^{-1}1 + C \quad \Rightarrow C = -\frac{\pi}{4}$$

Substituting $C = \frac{\pi}{4}$ in equation (1), we get: $y(1+x^2) = tan^{-1}1 - \frac{\pi}{4}$

This is the required particular solution of the given differential equation.

Question 15:

For the differential equation, find a particular solution satisfying the given condition:

$$\frac{dy}{dx} - 3y\cot x = \sin 2x; \quad y = 2 \text{ when } x = \frac{\pi}{2}$$

Answer 15:

The given differential equation: $\frac{dy}{dx} - 3y\cot x = \sin 2x$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + py = Q \text{ (where } p = -3\cot x \text{ and } Q = \sin 2x)$$

Now I.F = $e^{\int p\,dy} = e^{-3\int \cot x\,dy} = e^{-3\log|\sin x|} = e^{\log\left(\frac{1}{sin^3 x}\right)} = \frac{1}{sin^3 x}$

The general solution of the given differential equation is given by the relation,

$$y(I.F) = \int (Q \times I.F)\,dx + C$$

$y.\frac{1}{sin^3 x} = \int \left[\sin 2x.\frac{1}{sin^3 x}\right] dx + C \quad \Rightarrow y.cosec^3 x = 2\int (\cot x.cosec\ x)\,dx + C$

$\Rightarrow y.cosec^3 x = 2cosec\ x + C \quad \Rightarrow y = -\frac{2}{cosec^2 x} + \frac{3}{cosec^3 x}$

$y = -2sin^2 x + Csin^3 x \qquad ...(1)$

Now, $y = 2\ at\ x = \frac{\pi}{2}$

Therefore, we get: $2 = -2 + C \quad \Rightarrow C = 4$

Substituting C = 4 in equation (1), we get:

$y = -2sin^2 x + 4sin^3 x \qquad \Rightarrow y = 4sin^3 x - 2sin^2 x$

This is the required particular solution of the given differential equation.

Question 16:

Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

Answer 16:

Let F (x, y) be the curve passing through the origin.

At point (x, y), the slope of the curve will be $\frac{dy}{dx}$.

According to the given information: $\frac{dy}{dx} = x + y$

$\Rightarrow \frac{dy}{dx} - y = x$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + py = Q \text{ (where } p = -1 \text{ and } Q = x)$$

Now I.F = $e^{\int p\,dy} = e^{\int(-1)\,dx} = e^{-x}$

The general solution of the given differential equation is given by the relation,

$$y(I.F) = \int (Q \times I.F)\,dx + C$$

$\Rightarrow ye^{-x} = \int xe^{-x}\,dx + C$

Now, $\int xe^{-x}\,dx = \text{x}\int e^{-x}\,dx - \int\left\{\frac{d}{dx}(\text{x})\int e^{-x}\,dx\right\}dx$

$= -xe^{-x} - \int -e^{-x}\,dx \quad = -xe^{-x} + (-e^{-x}) \quad = -e^{-x}(x+1)$

Substituting in equation (1), we get: $ye^{-x} = -e^{-x}(x+1) + C$

$\Rightarrow y = -(x+1) + Ce^{x} \quad \Rightarrow x + y + 1 = Ce^{x} \quad \ldots(1)$

The curve passes through the origin.

Therefore, equation (1) becomes: $0 + 0 + 1 = Ce^{0} \quad \Rightarrow C = 1$

Substituting C = 1 in equation (1), we get: $x + y + 1 = e^{x}$

Hence, the required equation of curve passing through the origin is $x + y + 1 = e^{x}$.

Question 17:

Find the equation of a curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

Answer 17:

Let F (x, y) be the curve and let (x, y) be a point on the curve.

The slope of the tangent to the curve at (x, y) is $\frac{dy}{dx}$.

According to the given information: $\frac{dy}{dx} + 5 = x + y \quad \Rightarrow \frac{dy}{dx} - y = x - 5$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + py = Q \text{ (where } p = -1 \text{ and } Q = x - 5)$$

Now I.F = $e^{\int p\,dy} = e^{\int(-1)\,dx} = e^{-x}$

The general equation of the curve is given by the relation,

$$y(I.F) = \int (Q \times I.F)\,dx + C$$

$\Rightarrow y(e^{-x}) = \int (x-5)e^{-x}\,dx + C \quad \ldots(1)$

Now $\int (x-5)e^{-x}\,dx = (x-5)\int e^{-x}dx - \int\left\{\frac{d}{dx}(\text{x}-5).\int e^{-x}\,dx\right\}dx$

$= (x-5)(e^{-x}) - \int(-e^{-x})dx \quad = (5-x)e^{-x} + (e^{-x}) \quad = (4-x)e^{-x}$

Therefore, equation (1) becomes: $ye^{-x} = (4-x)e^{-x} + C$

$\Rightarrow y = (4-x) + Ce^{x} \quad \Rightarrow x + y - 4 = Ce^{x} \quad \ldots(2)$

The curve passes through point (0, 2). Therefore, equation (2) becomes:

$0 + 2 - 4 = Ce^{0} \quad \Rightarrow -2 = C \quad \Rightarrow C = -2$

Substituting C = −2 in equation (2), we get:

$x + y - 4 = -2e^{x} \quad \Rightarrow y = 4 - x - 2e^{x}$

This is the required equation of the curve.

Question 18:

The Integrating Factor of the differential equation $x\frac{dy}{dx} - y = 2x^2$ is

(A) e^{-x} (B) e^{-y} (C) $\frac{1}{x}$ (D) x

Answer 18:

The given differential equation is: $x\frac{dy}{dx} - y = 2x^2 \Rightarrow \frac{dy}{dx} - \frac{y}{x} = 2x$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + py = Q \left(\text{where } p = -\frac{1}{x} \text{ and } Q = 2x\right)$$

Now I.F = $e^{\int p\, dy}$

$= e^{\int \left(-\frac{1}{x}\right) dx}$

$= e^{-\log x}$

$= e^{\log(x^{-1})}$

$= x^{-1} = \frac{1}{x}$

Hence, the correct answer is (C).

Question 19:

The Integrating Factor of the differential equation $(1-y^2)\frac{dx}{dy} + yx = ay\ (-1 < y < 1)$ is

(A) $\frac{1}{y^2-1}$ (B) $\frac{1}{\sqrt{y^2-1}}$ (C) $\frac{1}{1-y^2}$ (D) $\frac{1}{\sqrt{1-y^2}}$

Answer 19:

$$\left(1-y^2\right)\frac{dx}{dy} + yx = ay$$

$$\Rightarrow \frac{dx}{dy} + \frac{yx}{1-y^2} = \frac{ay}{1-y^2}$$

$$\frac{dx}{dy} + p_1 x = Q_1$$

$$\left(\text{where, } p_1 = \frac{y}{1-y^2} \text{ and } Q_1 = \frac{ay}{1-y^2}\right)$$

$$\therefore I.F. = e^{\int p_1 dy} = e^{\int \frac{y}{1-y^2} dy} = e^{-\frac{1}{2}\log\left(1-y^2\right)} = e^{\log\left[\frac{1}{\sqrt{1-y^2}}\right]} = \frac{1}{\sqrt{1-y^2}}$$

Thus, the correct option is D.

Mathematics

(Chapter – 9) (Differential Equations) (Miscellaneous Exercise)
(Class – XII)

Question 1:

For each of the differential equations given below, indicate its order and degree (if defined).

(i) $\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$ **(ii)** $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$ **(iii)** $\frac{d^4y}{dx^4}-\sin\left(\frac{d^3y}{dx^3}\right)=0$

Answer 1:

(i) The differential equation is given as: $\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$

$\Rightarrow \frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y-\log x=0$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$.

Thus, its order is two. The highest power raised to $\frac{d^2y}{dx^2}$ is one.

Hence, its degree is one.

(ii) The differential equation is given as: $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$

$\Rightarrow \left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y-\sin x=0$

The highest order derivative present in the differential equation is $\frac{dy}{dx}$.

Thus, its order is one.

The highest power raised to $\frac{dy}{dx}$ is three. Hence, its degree is three.

(iii) The differential equation is given as: $\frac{d^4y}{dx^4}-\sin\left(\frac{d^3y}{dx^3}\right)=0$

The highest order derivative present in the differential equation is $\frac{d^4y}{dx^4}$.

Thus, its order is four.

However, the given differential equation is not a polynomial equation. Hence, its degree is not defined.

Question 2:

For each of the exercises given below, verify that the given function (implicit or explicit) is a solution of the corresponding differential equation.

(i) $xy = a\,e^x + b\,e^{-x} + x^2$: $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}-xy+x^2-2=0$

(ii) $y=e^x(a\cos x+b\sin x)$: $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$

(iii) $y=x\sin 3x$: $\frac{d^2y}{dx^2}+9y-6\cos 3x=0$

(iv) $x^2=2y^2\log y$: $(x^2+y^2)\frac{dy}{dx}-xy=0$

Answer 2:

(i) $xy = ae^x + be^{-x} + x^2$: $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$

$$xy = ae^x + be^{-x} + x^2 \qquad \ldots(1)$$

Differentiating both sides with respect to x, we get:

$$x\frac{dy}{dx} + y.1 = a\frac{d}{dx}\left(e^x\right) + b\frac{d}{dx}\left(e^{-x}\right) + \frac{d}{dx}\left(x^2\right)$$

$$\Rightarrow x\frac{dy}{dx} + y = ae^x + be^{-x} + 2x$$

Again, differentiating both sides with respect to x, we get:

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = ae^x + be^{-x} + 2$$

$$\Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = ae^x + be^{-x} + 2 \qquad \ldots(2)$$

Now, we have $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$

$$\begin{aligned} LHS &= x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 \\ &= ae^x + be^{-x} + 2 - \left(ae^x + be^{-x} + x^2\right) + x^2 - 2 \qquad \left[\text{using (1) and (2)}\right] \\ &= ae^x + be^{-x} + 2 - ae^x - be^{-x} - x^2 + x^2 - 2 \\ &= 0 \\ &= RHS \end{aligned}$$

Thus, the given function is a solution of the corresponding differential equation.

(ii) $y = e^x(a\cos x + b\sin x) = ae^x\cos x + be^x\sin x$

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = a.\frac{d}{dx}(e^x\cos x) + b.\frac{d}{dx}(e^x\sin x)$$

$\Rightarrow \frac{dy}{dx} = a(e^x\cos x - e^x\sin x) + b.(e^x\sin x - e^x\cos x)$

$\Rightarrow \frac{dy}{dx} = (a+b)e^x\cos x + (b-a)e^x\sin x$

Again, differentiating both sides with respect to x, we get:

$$\frac{d^2y}{dx^2} = (a+b)\frac{d}{dx}(e^x\cos x) + (b-a)\frac{d}{dx}(e^x\sin x)$$

$\Rightarrow \frac{d^2y}{dx^2} = e^x[a\cos x - a\sin x + b\cos x - b\sin x + b\sin x + b\cos x - a\sin x - a\cos x]$

$\Rightarrow \frac{d^2y}{dx^2} = [2e^x(b\cos x - a\sin x)]$

Now, on substituting the values of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ in the L.H.S. of the given differential equation, we get:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y$$

$= 2e^x(b\cos x - a\sin x) - 2e^x[(a+b)\cos x + (b-a)\sin x] + 2e^x(a\cos x + b\sin x)$

$= e^x[(2b\cos x - 2a\sin x) - (2a\cos x + 2b\cos x) - (2b\sin x - 2a\sin x) + (2a\cos x + 2b\sin x)]$

$= e^x[(2b - 2a - 2b + 2a)\cos x] + e^x[(-2a - 2b + 2a + 2b)\sin x] = 0$

L.H.S.= R.H.S.

Hence, the given function is a solution of the corresponding differential equation.

(iii) $y = x\sin 3x$

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = \frac{d}{dx}(x\sin 3x) = \sin 3x + x.\cos 3x.3$$

$\Rightarrow \frac{dy}{dx} = \sin 3x + 3x.\cos 3x$

Again, differentiating both sides with respect to x, we get:

$\frac{d^2y}{dx^2} = \frac{d}{dx}(\sin 3x) + 3\frac{d}{dx}(x.\cos 3x) \quad \Rightarrow \frac{d^2y}{dx^2} = 6\cos 3x - 9x\sin 3x$

Substituting the value of $\frac{d^2y}{dx^2}$ in the L.H.S. of the given differential equation, we get:

$$\frac{d^2y}{dx^2} + 9y - 6\cos 3x$$

$\Rightarrow (6.\cos 3x - 9x\sin 3x) + 9x\sin 3x - 6\cos 3x \;\; = 0$
L.H.S.= R.H.S.
Hence, the given function is a solution of the corresponding differential equation.

(iv) $x^2 = 2y^2 \log y$
Differentiating both sides with respect to x, we get:

$$2x = 2.\frac{d}{dx}(y^2 \log y)$$

$\Rightarrow x = \left[2y.\log y.\frac{dy}{dx} + y^2.\frac{1}{y}.\frac{dy}{dx}\right] \qquad \Rightarrow x = \frac{dy}{dx}(2y\log y + y) \qquad \Rightarrow \frac{dy}{dx} = \frac{x}{y(1+2\log y)}$

Substituting the value of $\frac{dy}{dx}$ in the L.H.S. of the given differential equation, we get:

$$(x^2+y^2)\frac{dy}{dx} - xy$$

$\Rightarrow (2y^2\log y + y^2).\frac{x}{y(1+2\log y)} - xy \quad \Rightarrow y^2(1+2\log y).\frac{x}{y(1+2\log y)} - xy \quad \Rightarrow xy - xy = 0$

L.H.S.= R.H.S.
Hence, the given function is a solution of the corresponding differential equation.

Question 3:

Prove that $x^2 - y^2 = c(x^2+y^2)^2$ is the general solution of differential equation, $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$, where c is parameter.

Answer 3:

Given equation: $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$

$\Rightarrow \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} \qquad \ldots(1)$

This is a homogeneous equation.
To simplify it, we need to make the substitution as: $y = vx$

$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx) \qquad \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting the values of y and $\frac{dv}{dx}$, we have

$v + x\frac{dv}{dx} = \frac{x^3 - 3x(vx)^2}{(vx)^3 - 3x^2(vx)} \qquad \Rightarrow v + x\frac{dv}{dx} = \frac{1-3v^2}{v^3-3v} \qquad \Rightarrow x\frac{dv}{dx} = \frac{1-3v^2}{v^3-3v} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{1-3v^2 - v(v^3-3v)}{v^3-3v} \qquad \Rightarrow x\frac{dv}{dx} = \frac{1-v^4}{v^3-3v} \qquad \Rightarrow \left(\frac{v^3-3v}{1-v^4}\right)dv = \frac{dx}{x}$

Integrating both sides, we get:

$\int\left(\frac{v^3-3v}{1-v^4}\right)dv = \log x + \log C' \qquad \ldots(2)$

Now, $\int\left(\frac{v^3-3v}{1-v^4}\right)dv = \int \frac{v^3}{1-v^4}dv - 3\int \frac{v}{1-v^4}dv$

$\Rightarrow \int\left(\frac{v^3-3v}{1-v^4}\right)dv = I_1 - 3I_2$ where $I_1 = \frac{v^3}{1-v^4}dv$ and $I_2 = \frac{v}{1-v^4}dv \qquad \ldots(3)$

Let $1 - v^4 = t$

$\Rightarrow \frac{d}{dv}(1-v^4) = \frac{dt}{dv} \qquad \Rightarrow -4v^3 = \frac{dt}{dv} \qquad \Rightarrow v^3 dv = -\frac{dt}{4}$

Now, $I_1 = \int \frac{-dt}{4t} = -\frac{1}{4}\log t = -\frac{1}{4}\log(1-v^4)$ and $I_2 = \int \frac{vdv}{1-v^4} = \int \frac{vdv}{1-(v^2)^2}$

let $v^2 = p$

$\Rightarrow \frac{d}{dv}(v^2) = \frac{dp}{dv}$ $\Rightarrow 2v = \frac{dp}{dv}$ $\Rightarrow vdv = \frac{dp}{2}$

$\Rightarrow I_2 = \frac{1}{2}\int \frac{dp}{1-p^2} \Rightarrow \frac{1}{2\times 2}\log\left|\frac{1+p}{1-p}\right| \Rightarrow \frac{1}{4}\log\left|\frac{1+v^2}{1-v^2}\right|$

Substituting the values of I_1 and I_2 in equation (3), we get:

$$\int\left(\frac{v^3-3v}{1-v^4}\right)dv = -\frac{1}{4}\log(1-v^4) - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right|$$

Therefore, equation (2) becomes:

$$\frac{1}{4}\log(1-v^4) - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right| = \log x + \log C'$$

$\Rightarrow -\frac{1}{4}\log\left[(1-v^4)\left(\frac{1+v^2}{1-v^2}\right)^3\right] = \log C'x$ $\Rightarrow \frac{(1+v^2)^4}{(1-v^2)^2} = (C'x)^{-4}$ $\Rightarrow \frac{\left(1+\frac{y^2}{x^2}\right)^4}{\left(1-\frac{y^2}{x^2}\right)^2} = \frac{1}{C'^4x^4}$

$\Rightarrow \frac{(x^2+y^2)^4}{x^4(x^2-y^2)^2} = \frac{1}{C'^4x^4}$ $\Rightarrow (x^2-y^2)^2 = C'^4(x^2+y^2)^4$ $\Rightarrow (x^2-y^2) = C'^2(x^2+y^2)^2$

$\Rightarrow x^2 - y^2 = C(x^2+y^2)^2$, where $C = C'^2$

Hence, the given result is proved.

Question 4:

Find the general solution of the differential equation $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

Answer 4:

The given differential equation: $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

$\Rightarrow \frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$ $\Rightarrow \frac{dy}{\sqrt{1-y^2}} = \frac{-dx}{\sqrt{1-x^2}}$

Integrating both sides, we get: $sin^{-1}y = -sin^{-1}x + C$ $\Rightarrow sin^{-1}y + sin^{-1}x = C.$

Question 5:

Show that the general solution of the differential equation $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$ is given by (x + y + 1) = A (1 – x – y – 2xy), where A is parameter.

Answer 5:

The given differential equation: $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{(y^2+y+1)}{x^2+x+1}$ $\Rightarrow \frac{dy}{(y^2+y+1)} = \frac{-dx}{x^2+x+1}$ $\Rightarrow \frac{dy}{(y^2+y+1)} + \frac{dx}{x^2+x+1} = 0$

Integrating both sides, we get:

$$\frac{dy}{(y^2+y+1)}+\frac{dx}{x^2+x+1}=C$$

$$\Rightarrow \int \frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=C \quad \Rightarrow \frac{2}{\sqrt{3}}tan^{-1}\left[\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]+tan^{-1}\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]=C$$

$$\Rightarrow tan^{-1}\left[\frac{2y+1}{\sqrt{3}}\right]+tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right]=\frac{\sqrt{3}C}{2} \qquad \Rightarrow tan^{-1}\left[\frac{\frac{2y+1}{\sqrt{3}}+\frac{2x+1}{\sqrt{3}}}{1-\frac{(2y+1)}{\sqrt{3}}.\frac{(2x+1)}{\sqrt{3}}}\right]=\frac{\sqrt{3}C}{2}$$

$$\Rightarrow tan^{-1}\left[\frac{\frac{2x+2y+2}{\sqrt{3}}}{1-\left(\frac{4xy+2x+2y+1}{3}\right)}\right]=\frac{\sqrt{3}C}{2} \qquad \Rightarrow tan^{-1}\left[\frac{2\sqrt{3}(x+y+1)}{3-4xy-2x-2y-1}\right]=\frac{\sqrt{3}C}{2}$$

$$\Rightarrow tan^{-1}\left[\frac{\sqrt{3}(x+y+1)}{2(1-x-y-2xy)}\right]=\frac{\sqrt{3}C}{2}$$

$$\Rightarrow \frac{\sqrt{3}(x+y+1)}{2(1-x-y-2xy)}=\tan\left(\frac{\sqrt{3}C}{2}\right)=B \ \ where\ B=\tan\left(\frac{\sqrt{3}C}{2}\right)$$

$$\Rightarrow x+y+1=A\frac{2B}{\sqrt{3}}(1-xy-2xy)$$

$$\Rightarrow x+y+1=A(1-x-y-2xy), where\ A=\frac{2B}{\sqrt{3}}$$

Hence, the given result is proved.

Question 6:

Find the equation of the curve passing through the point $\left(0,\frac{\pi}{4}\right)$ whose differential equation is $\sin x\, cos\ y\, dx+\cos x \sin y\, dy=0$

Answer 6:

The differential equation of the given curve: $\sin x cos\ y\, dx+\cos x \sin y\, dy=0$

$$\Rightarrow \frac{\sin x cos\ y\, dx+\cos x\sin y\, dy}{\cos x\cos y}=0 \qquad \Rightarrow \tan x\, dx+\tan y\, dy=0$$

Integrating both sides, we get: $\log(\sec x)+\log(\sec y)=\log C$

$\Rightarrow \log(\sec x.\sec y)=\log C \qquad \Rightarrow \sec x.\sec y=C \qquad ...(1)$

The curve passes through point $\left(0,\frac{\pi}{4}\right)$

$\Rightarrow 1\times\sqrt{2}=C \quad i.e. C=\sqrt{2}$

On substituting $C=\sqrt{2}$, we have

$$\sec x.\sec y=\sqrt{2} \qquad \Rightarrow \sec x.\frac{1}{\cos y}=\sqrt{2} \qquad \Rightarrow \cos y=\frac{\sec\ x}{\sqrt{2}}$$

Hence, the required equation of the curve is $\cos y=\frac{\sec\ x}{\sqrt{2}}$.

Question 7:

Find the particular solution of the differential equation

$(1+e^{2x})dy+(1+y^2)e^x dx=0,$ given that $y = 1$ when $x = 0$.

Answer 7:

The given differential equation: $(1+e^{2x})dy+(1+y^2)e^x dx=0$

$\Rightarrow \frac{dy}{1+y^2}+\frac{e^x dx}{1+e^{2x}}=0$

Integrating both sides, we get:

$$tan^{-1}y+\int\frac{e^x dx}{1+e^{2x}}=C \quad ...(1)$$

Let $e^x=t\ so\ e^{2x}=t^2$

$$\Rightarrow \frac{d}{dx}(e^x)=\frac{dt}{dx} \qquad \Rightarrow e^x=\frac{dt}{dx}\ \ so\ e^x\ dx=dt$$

Substituting these values in equation (1), we get:

$tan^{-1}y+\int\frac{dt}{1+t^2}=C$

$\Rightarrow tan^{-1}y+tan^{-1}t=C \qquad \Rightarrow tan^{-1}y+tan^{-1}(e^x)=C$

Now, y = 1 at x = 0, therefore

$\Rightarrow tan^{-1}1+tan^{-1}1=C \qquad \Rightarrow \frac{\pi}{4}+\frac{\pi}{4}=C \qquad \Rightarrow C=\frac{\pi}{2}$

Substituting $C=\frac{\pi}{2}$ in equation, we get: $tan^{-1}y+tan^{-1}(e^x)=\frac{\pi}{2}$

This is the required particular solution of the given differential equation.

Question 8:

Solve the differential equation $ye^{\frac{x}{y}}dx=\left(xe^{\frac{x}{y}}+y^2\right)dy \quad (y\neq 0)$.

Answer 8:

The given differential equation: $ye^{\frac{x}{y}}dx=\left(xe^{\frac{x}{y}}+y^2\right)dy$

$\Rightarrow ye^{\frac{x}{y}}\frac{dx}{dy}=xe^{\frac{x}{y}}+y^2$

$\Rightarrow e^{\frac{x}{y}}\left[y.\frac{dx}{dy}-x\right]=y^2 \qquad \Rightarrow e^{\frac{x}{y}}\frac{\left[y.\frac{dx}{dy}-x\right]}{y^2}=1 \quad ...(1)$

Let $e^{\frac{x}{y}}=z$ Differentiating it with respect to y, we get:

$\frac{d}{dy}\left(e^{\frac{x}{y}}\right)=\frac{dz}{dy} \qquad \Rightarrow e^{\frac{x}{y}}\left[\frac{y.\frac{dx}{dy}-x}{y^2}\right]=\frac{dz}{dy} \qquad ...(2)$

From equation (1) and equation (2), we get: $\frac{dz}{dy}=1$

$\Rightarrow dz=dy$

Integrating both sides, we get: $z=y+C$

$\Rightarrow e^{\frac{x}{y}}=y+C$

This is the required solution of the given differential equation.

Question 9:

Find a particular solution of the differential equation $(x-y)(dx+dy)=dx-dy$, given that y = – 1, when x = 0. (Hint: put $x-y\ =\ t$)

Answer 9:

The given differential equation: $(x-y)(dx-dy)=dx-dy$

$\Rightarrow (x-y+1)dy=(1-x+y)dx \quad \Rightarrow \dfrac{dy}{dx}=\dfrac{1-(x-y)}{x-(x-y)} \qquad \dots (1)$

Let $(x-y)=t$

$\Rightarrow \dfrac{d}{dx}(x-y)=\dfrac{dt}{dx} \qquad \Rightarrow 1-\dfrac{dy}{dx}=\dfrac{dt}{dx}$ so $1-\dfrac{dt}{dx}=\dfrac{dy}{dx}$

Substituting the value of $x-y$ and $\frac{dy}{dx}$ in equation (1), we get:

$1-\dfrac{dt}{dx}=\dfrac{1-t}{1+t} \qquad \Rightarrow \dfrac{dt}{dx}=1-\left(\dfrac{1-t}{1+t}\right) \qquad \Rightarrow \dfrac{dt}{dx}=\dfrac{(1+t)-(1-t)}{1+t}$

$\Rightarrow \dfrac{dt}{dx}=\dfrac{2t}{1+t} \qquad \Rightarrow \left(\dfrac{1+t}{t}\right)dt=2dx \qquad \Rightarrow \left(1+\dfrac{1}{t}\right)dt=2dx$

Integrating both sides, we get:

$t+\log|t|=2x+C$

$\Rightarrow (x-y)+\log|x-y|=x+y+C \quad \dots (2)$

Now, y = −1 at x = 0.

Therefore, equation (2) becomes: log 1 = 0 – 1 + C ⇒ C = 1

Substituting C = 1 in equation (2) we get: $\log|x-y|=x+y+1$

This is the required particular solution of the given differential equation.

Question 10:

Solve the differential equation $\left[\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}-\dfrac{y}{\sqrt{x}}\right]\dfrac{dx}{dy}=1 \ (x\neq 0)$.

Answer 10:

The given differential equation: $\left[\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}-\dfrac{y}{\sqrt{x}}\right]\dfrac{dx}{dy}=1$

$\Rightarrow \dfrac{dx}{dy}=\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}-\dfrac{y}{\sqrt{x}} \qquad \Rightarrow \dfrac{dx}{dy}+\dfrac{y}{\sqrt{x}}=\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}$

This equation is a linear differential equation of the form

$\Rightarrow \dfrac{dx}{dy}+Py=Q$, where $P=\dfrac{1}{\sqrt{x}}$ and $Q=\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}$

Now, $\text{I.F}=e^{\int P dx}=e^{\int \frac{1}{\sqrt{x}}dx}=e^{2\sqrt{x}}$

Therefore, the solutions is given by: $y(I.F)=\int(Q\times I.F)dx+C$

$\Rightarrow ye^{2\sqrt{x}}=\displaystyle\int \frac{1}{\sqrt{x}}dx+C$

$\Rightarrow ye^{2\sqrt{x}}=2\sqrt{x}+C$

This is the required solution of the given differential equation.

Question 11:

Find a particular solution of the differential equation $\frac{dy}{dx} + y\,cotx = 4\,x\,cosec\,x\ (x \neq 0)$, given that y = 0 when $x = \frac{\pi}{2}$.

Answer 11:

The given differential equation: $\frac{dy}{dx} + y\,cotx = 4\,x\,cosec\,x$

This equation is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \qquad \text{where } P = \cot x \text{ and } Q = 4x\,cosec\,x$$

Now, I.F $= e^{\int P dx} = e^{\int \cot x dx} = e^{\log|\sin x|} = \sin x$

The general solution of the given differential equation is given by,

$$y(I.F) = \int (Q \times I.F)dx + C$$

$$\Rightarrow y\sin x = \int (4x\,cosec\,x.\sin x)dx + C \qquad \Rightarrow y\sin x = 4.\frac{x^2}{2} + C \qquad \Rightarrow y\sin x = 2x^2 + C \qquad ...(1)$$

Now, y = 0 at $x = \frac{\pi}{2}$, therefore the equation (1) becomes:

$$0 = 2 \times \frac{\pi^2}{4} + C \quad \Rightarrow C = \frac{\pi^2}{2}$$

Substituting $C = -\frac{\pi^2}{2}$, we get $y\sin x = 2x^2 - \frac{\pi^2}{2}$

This is the required particular solution of the given differential equation.

Question 12:

Find a particular solution of the differential equation $(x + 1)\frac{dy}{dx} = 2e^{-y} - 1$, given that y = 0 when $x = 0$.

Answer 12:

The given differential equation: $(x+1)\frac{dy}{dx} = 2e^{-y} - 1$

$$\Rightarrow \frac{dy}{2e^{-y} - 1} = \frac{dx}{x+1} \qquad \Rightarrow \frac{e^y dy}{2 - e^y} = \frac{dx}{x+1} \qquad \Rightarrow \int \frac{e^y dy}{2 - e^y} = \log|x+1| + \log C \qquad ...(1)$$

Let $2 - e^y = t$

$$\Rightarrow \frac{d}{dy}(2 - e^y) = \frac{dt}{dy} \qquad \Rightarrow -e^y = \frac{dt}{dy} \qquad \Rightarrow e^y dy = -dt$$

Substituting this value in equation (1), we get:

$$\int \frac{-dt}{t} = \log|x+1| + C \qquad \Rightarrow -\log|t| = \log|C(x+1)| \qquad \Rightarrow -\log|2 - e^y| = \log|C(x+1)|$$

$$\Rightarrow \frac{1}{2 - e^y} = C(x+1) \qquad \Rightarrow \ 2 - e^y = \frac{1}{C(x+1)} \qquad ...(2)$$

Now, y = 0 at $x = 0$, therefore the equation (2) becomes:

$$2 - 1 = \frac{1}{C} \quad \Rightarrow \quad C = 1$$

Substituting C = 1 in equation (2), we get:

$$2 - e^y = \frac{1}{x+1} \qquad \Rightarrow e^y = 2 - \frac{1}{x+1} = \frac{2x+2-1}{x+1} \qquad \Rightarrow e^y = \frac{2x+1}{x+1} \Rightarrow y = \log\left|\frac{2x+1}{x+1}\right|, (x \neq -1)$$

This is the required particular solution of the given differential equation.

Question 13:

The general solution of the differential equation $\frac{y\,dx - x\,dy}{y} = 0$ is

(A) $xy = C$ (B) $x = Cy^2$ (C) $y = Cx$ (D) $y = Cx^2$

Answer 13:

The given differential equation: $\frac{y\,dx - x\,dy}{y} = 0$

$\Rightarrow \frac{y\,dx - x\,dy}{xy} = 0 \qquad \Rightarrow \frac{1}{x}dx - \frac{1}{y}dy = 0$

Integrating both sides, we get:

$\Rightarrow \log|x| - \log|y| = \log k \;\Rightarrow \log\left|\frac{x}{y}\right| = \log k \;\Rightarrow \frac{x}{y} = k \quad \Rightarrow y = \frac{1}{k}x \quad \Rightarrow y = Cx$, where $C = \frac{1}{k}$

Hence, the correct answer is (C).

Question 14:

The general solution of a differential equation of the type $\frac{dx}{dy} + P_1 x = Q_1$ is

(A) $ye^{\int P_1 dy} = \int\left(Q_1 e^{\int P_1 dy}\right)dy + C$ (B) $ye^{\int P_1 dx} = \int\left(Q_1 e^{\int P_1 dx}\right)dx + C$

(C) $xe^{\int P_1 dy} = \int\left(Q_1 e^{\int P_1 dy}\right)dy + C$ (D) $xe^{\int P_1 dx} = \int\left(Q_1 e^{\int P_1 dx}\right)dx + C$

Answer 14:

The integrating factor of the given differential equation $\frac{dy}{dx} + P_1 x = Q_1$ is $e^{\int P_1 dy}$

The general solution of the differential equation is given by,

$x(I.F.) = \int (Q \times I.F.)dy + C \qquad \Rightarrow xe^{\int P_1 dy} = \int\left(Q_1 e^{\int P_1 dy}\right)dy + C$

Hence, the correct answer is (C).

Question 15:

The general solution of the differential equation $e^x dy + (ye^x + 2x)dx = 0$ is

(A) $xe^y + x^2 = C$ (B) $xe^y + y^2 = C$

(C) $ye^x + x^2 = C$ (D) $ye^y + x^2 = C$

Answer 15:

The given differential equation: $e^x dy + (ye^x + 2x)dx = 0$

$\Rightarrow e^x\frac{dy}{dx} + ye^x + 2x = 0 \qquad \Rightarrow \frac{dy}{dx} + y = -2xe^{-x}$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \qquad \text{where } P = 1 \text{ and } Q = -2xe^{-x}$$

Now, I.F $= e^{\int P dx} = e^{\int dx} = e^x$

The general solution of the given differential equation is given by,

$$y(I.F) = \int (Q \times I.F)dx + C$$

$\Rightarrow ye^x = \int (-2xe^{-x}.e^x)dx + C \qquad \Rightarrow ye^x = -x^2 + C \qquad \Rightarrow ye^x + x^2 = C$

Hence, the correct answer is (C).