07

The study of permutation and combination is concerned with determining the number of different ways of arranging and selecting objects, out of a given number of objects.

In this chapter, we shall learn about some basic counting techniques which will be useful in determining the number of different ways of arranging and selecting objects, without actually listing them.

PERMUTATIONS AND COMBINATIONS

|TOPIC 1|

Fundamental Principles of Counting

There are two fundamental principles of counting which are the base of permutations and combinations.

Fundamental Principle of Multiplication (FPM)

If an event can occur in m different ways, another event can occur in n different ways, then the total number of occurrence of the events in the given order is $m \times n$.

In other words, if there are two operations say E and F such that E can be performed in m ways and associated with each way of performing operation E, operation F can be performed in n ways. Then, the two operations in succession can be performed in $m \times n$ ways.

e.g. In a school, there are 200 boys and 150 girls. The teacher wants to select a boy and a girl to represent the school in inter school competition.

Here, the teacher can select a boy in 200 ways and a girl in 150 ways. So, by principle of multiplication, the teacher can select a boy and a girl in $200 \times 150 = 30000$ ways.

Note

The above principle can be extended to any finite number of events (or operations) as stated below :

If there are n events, say $E_1, E_2, \ldots, E_n$ such that E_1 can occur in m_1 ways, E_2 can occur in m_2 ways, E_3 can occur in m_3 ways and so on, then the total number of ways in which all the events can be occurred in the stated order is $m_1 \times m_2 \times m_3 \times \ldots \times m_n$.

CHAPTER CHECKLIST

- Fundamental Principles of Counting
- Factorial Notation
- Permutations
- Restricted Permutations
- Combinations
- Practical Problems on Combinations

EXAMPLE |1| In a monthly test, the teacher decides that there will be three questions, one from each of the Exercises 7, 8 and 9 of textbook. If there are 11 questions in Exercise 7, 15 in Exercise 8 and 10 in Exercise 9, in how many ways can three questions be selected?

Sol. Clearly, from Exercise 7, one question can be selected in 11 ways.

From Exercise 8, one question can be selected in 15 ways.

and from Exercise 9, one question can be selected in 10 ways.

Since, each operation is performed after performing the previous operation, so we use fundamental principle of multiplication.

$\therefore$ Number of ways of selecting three questions

$$= 11 \times 15 \times 10 = 1650$$

EXAMPLE |2| A room has 7 doors. In how many ways can a man enter the room through one door and come out through a different door?

Sol. Here, we need to perform two operations :

(i) Selecting a door to enter.

(ii) Selecting a door to come out.

Clearly, the man can enter the room through anyone of the seven doors. So, there are seven ways of entering into the room. Note that the man can come out through anyone of the remaining six doors. So, he can come out through a different door in 6 ways.

Hence, by fundamental principle of counting, required number of ways $= 7 \times 6 = 42$

EXAMPLE |3| In how many ways can 6 letters be posted in 5 letter boxes?

Sol. Let L_1, L_2, L_3, L_4, L_5 and L_6 be six letters. Since, each letter can be posted in anyone of the five letter boxes.

So, L_1 can be posted in 5 ways. Similarly, each of L_2, L_3, L_4, L_5 and L_6 can be posted in 5 ways.

Hence, the total number of ways in which all the six letters can be posted is $5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^6 = 15625$ ways.

EXAMPLE |4| In an examination, there are three multiple choice questions and each question has 4 choices. Find the number of ways in which a student can fail to get all answer correct.

Sol. Since, each question can be answered in 4 ways.

So, the total number of ways answering 3 questions is $4 \times 4 \times 4 = 64$.

Out of these possible answers, only one will be correct and hence the number of ways in which a student can fail to get all correct answer is $64 - 1 = 63$.

EXAMPLE |5| The flag of a newly formed forum is in the form of three blocks □□□, each to be coloured differently. If there are six different colours on the whole to choose from, how many such designs are possible?

Sol. Here, we have six colours to colour three blocks.

So, first block can be coloured in 6 ways.

Since, each block should be coloured differently, so second block can be coloured by anyone of the remaining 5 colours.

Thus, second block can be coloured in 5 ways.

Similarly, third block can be coloured in 4 ways.

Since, each block is coloured after colouring the previous block.

$\therefore$ By FPM, required number of designs

$$= 6 \times 5 \times 4 = 120$$

EXAMPLE |6| How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 assuming

(i) repetition of digits allowed? **[NCERT]**

(ii) repetition of digits not allowed?

Sol. (i) There are 5 digits *viz.* 1, 2, 3, 4 and 5.

Since, every digit can be repeated any number of times and we have to form 3-digit numbers.

So, hundred's place can be fill in 5 ways, ten's place can be fill in 5 ways and unit's place can be fill in 5 ways.

Hundred's place	Ten's place	Unit's place
↓ 5 ways	↓ 5 ways	↓ 5 ways

Thus, by the fundamental principle of multiplication, the number of ways in which the 3-digit number can be formed $= 5 \times 5 \times 5 = 125$ ways.

Hence, number of 3-digit numbers is 125.

(ii) When repetition of digit is not allowed.

Hundred's place can be fill in 5 ways.

Ten's place can be fill in 4 ways.

Unit's place can be fill in 3 ways.

Hundred's place	Ten's place	Unit's place
↓ 5 ways	↓ 4 ways	↓ 3 ways

According to FPM, total number of ways

$$= 5 \times 4 \times 3$$
$$= 60 \text{ ways}$$

Hence, number of 3-digit numbers is 60.

EXAMPLE |7| In a certain city, all telephone numbers have six digits, the first two digits always being 41 or 42 or 46 or 62 or 64. How many telephone numbers have all six digits distinct?

Sol. Suppose first two digit is 41. Then we left with 8 digits, namely 0, 2, 3, 5, 6, 7, 8, 9 which can be used in constructing the telephone numbers.

Clearly, third digit can be selected in 8 ways.

Fourth digit can be selected in $8-1=7$ ways.

Fifth digit can be selected in 6 ways.

and sixth digit can be selected in 5 ways.

Now, according to fundamental principle of multiplication, number of telephone starting with 41

$$= 8 \times 7 \times 6 \times 5 = 1680$$

Similarly, if the telephone number start with 42 or 46 or 62 or 64, the remaining 4 digits can be selected in 1680 ways.

Hence, total number of required telephone numbers

$$= 1680 \times 5 = 8400$$

EXAMPLE |8| Find the number of positive integers greater than 6000 and less than 7000 which are divisible by 5, provided that no digit is to be repeated.

Sol. We know a number will be divisible by 5, if either 0 or 5 is at unit place.

$\therefore$ Number of ways to fill the unit's place = 2

Also note that, for number to be greater than 6000 and less than 7000, digit 6 should be fixed at thousand's place.

$\therefore$ Number of ways to fill the thousand's place = 1

Now, we left with 8 digits. So hundred's place can be filled in 8 ways and ten's place can be filled in 7 ways.

Thousand's place	Hundred's place	Ten's place	Unit's place
6			0 or 5
↓	↓	↓	↓
1 way	8 ways	7 ways	2 ways

$\therefore$ Total number of numbers $= 1 \times 8 \times 7 \times 2 = 112$

EXAMPLE |9| A code word is to consist of two distinct English alphabets followed by two distinct numbers between 1 and 9. e.g. CA23 is code word.

(i) How many such code words are there?

(ii) How many of them end with an even integer?

Sol. (i) There are in all 26 English alphabets, we have to choose 2 distinct alphabets.

1st alphabet can be selected in 26 ways.

2nd alphabet can be selected in 25 ways.

Again, out of 9 digits (1 to 9),

1st number can be selected in 9 ways.

2nd number can be selected in 8 ways.

Thus, by the fundamental principle of multiplication, the number of distinct codes

$$= 26 \times 25 \times 9 \times 8 = 46800$$

(ii) As above, two distinct alphabets can be selected in 26×25 ways.

We have, in all 1, 2, 3, 4, 5, 6, 7, 8 and 9 digits, unit place can be filled up in 4 ways (by 2, 4, 6, 8). Ten's place can be filled up in 8 ways.

[since, one of the digits is already used]

Thus, the number of such codes $= 26 \times 25 \times 8 \times 4$

$$= 20800$$

EXAMPLE |10| In how many ways can 3 prizes be distributed among 4 boys, when

(i) no boy gets more than one prize?

(ii) a boy may get any number of prizes?

(iii) no boy gets all the prizes?

Sol. (i) The first prize can be given in 4 ways. The 2nd prize can be given in 3 ways because the boy who got the first prize cannot receive the prize.

The 3rd prize can be given to anyone of the remaining 2 boys in 2 ways.

$\therefore$ Total number of ways in which all the prizes can be given $= 4 \times 3 \times 2 = 24$

(ii) The first prize can be given in 4 ways.

The 2nd prize can also be given in 4 ways, because it may be obtained by the boy who has already received a prize.

Similarly, 3rd prize can be given in 4 ways.

Hence, the number of ways in which all the prizes can be given $= 4 \times 4 \times 4 = 4^3 = 64$

(iii) If any one of the 4 boys may get all the prizes, then, the number of ways in which all the 3 prizes given to a boy = 4

So, the number of ways in which no boy get all the prizes $= 64 - 4 = 60$.

Fundamental Principle of Addition (FPA)

If there are two events such that they can be performed independently in m and n ways respectively, then either of the two events can be occurred in $(m+n)$ ways.

In other words, if an operation can be performed in m different ways and another operation which is independent of the first operation , can be performed in n different ways. Then, either of the two operations can be performed in $(m+n)$ ways.

e.g. In a school, there are 200 boys and 150 girls. The teacher wants to select either a boy or a girl to represent the school in inter school competition.

Here, the teacher can select a boy in 200 ways and a girl in 150 ways.

So, by fundamental principle of addition, the teacher can select a boy or a girl in $(200 + 150) = 350$ ways.

Note

The above principle can be extended to any finite number of operation.

EXAMPLE |11| In how many ways, two books of different languages can be selected from 10 Hindi, 5 English and 7 Sanskrit books?

Sol. Clearly, a book of Hindi and a book of English can be selected in $10 \times 5 = 50$ ways.

A book of English and a book of Sanskrit can be selected in $5 \times 7 = 35$ ways.

A book of Hindi and a book of Sanskrit can be selected in $10 \times 7 = 70$ ways.

Now, by fundamental principle of addition, two books of different languages can be selected in $50 + 35 + 70 = 155$ ways.

EXAMPLE |12| Find the number of different signals that can be generated by arranging atleast 2 flags in order (one below the other) on a vertical staff, if five different flags are available. **[NCERT]**

Sol. Here, we have to find number of different signals that can be generated by arranging atleast 2 flags in order. So a signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags. If a signal consist of 2 flags, then two vacant places are available.

$\therefore$ Number of ways of filling first vacant place = 5
and number of ways of filling second vacant place = 4
Thus, total number of signals consisting 2 flags

$$= 5 \times 4 = 20$$

Now, if a signal consist of 3 flags, then three vacant places are available.

$\therefore$ Number of ways of filling first vacant place = 5
number of ways of filling second vacant place = 4
and number of ways of filling third vacant place = 3
Thus, total number of signals consisting 3 flags

$$= 5 \times 4 \times 3 = 60$$

Similarly, total number of signals consisting 4 flags

$$= 5 \times 4 \times 3 \times 2 = 120$$

and total number of signals consisting 5 flags

$$= 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Since, different signals can be generated by arranging either 2 flags or 3 flags or 4 flags or 5 flags.

$\therefore$ Total number of signals

$$= 20 + 60 + 120 + 120 = 320$$

EXAMPLE |13| How many numbers are there between 100 and 1000 which have exactly one of their digits as 7?

Sol. We have to find numbers between 100 and 1000, i.e. 3-digit numbers which have exactly one of their digits as 7. Now, consider the following cases:

***Case* I When 7 is at unit's place.**

In this case, ten's and hundred's place can be filled by the digits 0, 1, 2, 3, 4, 5, 6, 8 and 9.

Clearly, the number of ways to fill ten's place = 9
[by anyone of the digits 0, 1, 2, 3, 4, 5, 6, 8, 9]
and the number of ways to fill the hundred's place = 8

$\therefore$ Number of such numbers $= 1 \times 9 \times 8 = 72$

***Case* II When 7 is at ten's place.**

In this case, unit's and hundred's place can be filled by the digits 0, 1, 2, 3, 4, 5, 6, 8 and 9.

Clearly, the number of ways to fill unit's place = 9
[by anyone of digits 0, 1, 2, 3, 4, 5, 6, 8, 9]
and the number of ways to fill hundred's place = 8
[$\because$ hundred's place cannot be filled by zero]

$\therefore$ Number of such numbers $= 9 \times 1 \times 8 = 72$

***Case* III When 7 is at hundred's place.**

In this case, unit's and ten's place can be filled by the digits 0, 1, 2, 3, 4, 5, 6, 8 and 9.

Clearly, the number of ways to fill unit's place = 9
[by anyone of the digits 0, 1, 2, 3, 4, 5, 6, 8, 9]
and the number of ways to fill the tens place = 9
[by anyone of the digits 0, 1, 2, 3, 4, 5, 6, 8, 9]

$\therefore$ Number of such numbers

$$= 1 \times 9 \times 9 = 81$$

Hence, total number of required numbers

$$= 72 + 72 + 81 = 225$$

TOPIC PRACTICE 1

OBJECTIVE TYPE QUESTIONS

1 Mohan has 3 pants and 2 shirts. The number of different pairs of a pant and shirt he can wear are

(a) 5 (b) 6
(c) 7 (d) None of these

2 Sabnam has 2 school bags, 3 tiffin boxes and 2 water bottles. In how many ways can she carry these items (choosing one each)?

(a) 11 (b) 12
(c) 13 (d) 14

3 A telegraph has 5 arms and each arm is capable of 4 distincts positions, including the position of rest. The total number of signals that can be made is

(a) 1024 (b) 1023
(c) 1022 (d) 1021

4 A sequence is a ternary sequence, if it contains digits 0, 1 and 2. The total number of ternary sequences of length 9 which either begin with 210 or end with 210, is

(a) 1458 (b) 1431
(c) 729 (d) 707

5 The number of ways in which 3 prize can be distributed to 4 childrens, so that no child gets all the three prizes, are

(a) 64 (b) 62
(c) 60 (d) None of these

VERY SHORT ANSWER Type Questions

6 Pratiksha wants to go abroad by ship and to return by airline. She has a choice of 6 different ships to go and 4 airlines to return. In how many ways can she perform her journey?

7 How many AP's with 10 terms are there whose first terms is in the set {1, 2, 3} and whose common difference is in the set {1, 2, 3, 4, 5}?

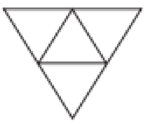
8 How many words (with or without meaning) of three distinct letters of the English alphabets are there?

SHORT ANSWER Type I Questions

9 How many 2-digit even numbers can be formed from the digits 1, 2, 3, 4 and 5, if the digits can be repeated? [NCERT]

10 In how many ways can this diagram be coloured subject to the following two conditions.

(i) Each of the smaller triangle is to be painted with one of three colours : red, black, green.
(ii) No two adjacent regions have the same colour. [NCERT Exemplar]



11 A gentleman has 6 friends to invite. In how many ways can he send invitation cards to them, if he has three servants to carry the cards?

12 Given 5 flags of different colours. How many different signals can be generated, if each signal requires the use of 2 flags, one below the other? [NCERT]

13 How many numbers are there between 100 and 1000 in which all the digits are distinct?

14 How many 4-digit numbers are there, when a digit may be repeated any number of times?

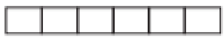
15 How many 3-digit numbers more than 600 can be formed by using the digits 2, 3, 4, 6 and 7.

SHORT ANSWER Type II Questions

16 Five persons entered in the lift cabin on the ground floor of an 8-floor house. Suppose each of them can leave the cabin independently at any floor beginning with the first. Find the total number of ways in which each of the five persons can leave the cabin

(i) at anyone of the 7 floor.
(ii) at different floors.

17 Each of the six squares in the strip, shown in figure, is to be coloured with anyone of 10 different colours so that no two adjacent squares have the same colour. Find the number of ways of colouring the strip.



18 How many natural numbers less than 1000 can be formed with the digits 1, 2, 3, 4 and 5, if repetition of digits is allowed?

19 There are 3 candidates for Classical language, 5 for Mathematics and 4 for Natural Science scholarship.

(i) In how many ways can these scholarship be awarded?
(ii) In how many ways, one of these scholarship be awarded?

20 How many numbers greater than 1000, but not greater than 4000 can be formed with the digits 0, 1, 2, 3, 4, if

(i) repetition of digits is allowed?
(ii) repetition of digits is not allowed?

HINTS & ANSWERS

1. (b) Number of different pairs of a pant and shirt $= 3 \times 2$
   Ans. 6
2. (b) Number of ways she can carry the 3 items $= 2 \times 3 \times 2$
   Ans. 12
3. (b) Since, each arm can be kept in 4 positions and a signal is possible when all the 5 arms
   are simultaneously placed in positions.
   $\therefore$ Total number of ways placing the arms $= 4^5$
   But, this includes one inadmissible case, when all the arms are in the position of rest and then no signal can be made.
   Hence, the required number of signals $= 4^5 - 1$
   Ans. 1023
4. (b) Since, each digit in a ternary sequence of length 9 can be filled in 3 ways.
   Therefore, the number of nine-digit ternary sequence beginning with 210 is 3^6
   and number of nine-digit ternary sequence ending with 210 is also 3^6.
   The number of ternary sequence of 9 digits which begin and end with 210 is 3^3.
   $\therefore$ Required number of such numbers $= 3^6 + 3^6 - 3^3$
   Ans. 1431
5. (c) $\therefore$ Total number of ways of distributing prizes
   $$= 4 \times 4 \times 4 = 64$$
   Number of ways in which one child gets all prizes $= 4$
   $\therefore$ Number of ways in which no child gets all the three prizes $= 64 - 4$
   Ans. 60
6. Number of ways to go by ship $= 6$ and number of ways to return by airline $= 4$.
   Ans. She perform her journey $= 6 \times 4 = 24$
7. There are 3 ways to choose the first term and corresponding to each such way, there are 5 ways of selecting the common difference.
   Ans. 15
8. There are 26 letters of English alphabet and we have to fill three places by distinct letters of English alphabet.
   Ans. Then, number of words $= 26 \times 25 \times 24 = 15600$
9. In unit's place number, only one even number (2, 4) can be placed in 2 ways and in ten's place, anyone of the given number can be placed.
   Ans. 10
10. (ii) The given condition can be satisfied only when we paint the central triangle with any one of the three colours and remaining 3 triangles with anyone of the remaining two colours.
    Ans. (i) 81 (ii) 24
11. Since, the gentleman has 3 servants, so number of ways of sending the invitation card to the first friend = 3.
    Similarly, for each of the remaining friends, there are 3 ways each.
    Ans. Total number of ways $= 3^6 = 729$
12. First, flag can be selected in 5 ways and the second flag can be selected in 4 ways.
    Ans. 20
13. A number between 100 and 1000 has three digits. So, we need to form all possible 3-digit numbers with distinct digits. 0 cannot be placed at hundred's place, so hundred's place can be filled in 9 ways. Since, repetition of digits is not allowed, therefore each of the remaining ten's and unit's place be filled in 9 and 8 ways.
    Ans. 648
14. 0 cannot be placed at thousand's place. So, thousand's place can be filled in 9 ways. Since, repetition of digits is allowed, therefore each of the remaining 3 places can be filled in 10 ways.
    Ans. 9000
15. Digit at hundred's place will be either 6 or 7. So, hundred's place can be filled in 2 ways. Since repetition of digits is allowed, therefore each of the ten's and one's places can be filled in 5 ways.
    Ans. 50
16. (i) Each person can leave the cabin at anyone of the seven floors. So, each person can leave the cabin in 7 ways.
    Ans. 7^5
    (ii) First person can leave the cabin at anyone of the seven floors. Second person can leave the cabin at anyone of the remaining 6 floors. Similarly, we can calculate for others.
    Ans. 2520
17. First square can be coloured in 10 ways (by using anyone of the given colour). Second square can be coloured in 9 ways (by using anyone of the remaining 9 colours). Third square can be coloured in 9 ways (by using anyone of the given colours except the colour used in second square). Similarly, other squares can be coloured.
    Ans. 590490
18. Required numbers will be of 1-digit, 2-digit or 3-digit.
    Ans. 155
19. Number of ways of awarding scholarship for Classical language = 3
    Number of ways of awarding scholarship for Mathematics = 5
    Number of ways of awarding scholarship for Natural Science = 4.
    Ans. (i) 60 (ii) 12
20. Thousand's place can be filled either by 1 or 2 or 3, i.e. thousand's place can be filled in 3 ways.
    Ans. (i) 375 (ii) 72

|TOPIC 2|
Factorial Notation

n Factorial

Many times, we come across the products of the form $1\times 2, 1\times 2\times 3, 1\times 2\times 3\times 4, \ldots$. For our convenience, we use a special notation instead of writing all the factors of such a product. We write

$$1! = 1$$
$$2! = 2\times 1$$
$$3! = 3\times 2\times 1$$
$$\vdots \quad \vdots$$
$$n! = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$$

Thus, the notation $n!$ represent the product of first n natural numbers. We read this notation as 'n factorial' and it is also denoted by n !

Clearly, for a natural number n,

$$n! = n(n-1)!$$
$$= n(n-1)(n-2)! \quad [\text{provided } n\geq 2]$$
$$= n(n-1)(n-2)(n-3)! \quad [\text{provided } n\geq 3]$$
$$\vdots \quad \vdots$$
$$= n(n-1)(n-2)(n-3)\ldots 3\cdot 2\cdot 1$$

e.g. $\quad 5! = 5\times 4\times 3\times 2\times 1 = 120$

and $\quad 8! = 8\times (7)! = 8\times 7\times (6!)$

ZERO FACTORIAL

It does not make any sense to define 0! as product of the integers from 1 to 0. So, we define $0! = 1$.

Note Factorial of proper fractions or negative integers are not define. $n!$ is defined only for whole numbers, i.e. for non-negative integers.

EXAMPLE |1| Compute $\frac{8!}{4!}$. Is $\frac{8!}{4!} = 2!$?

Sol. We have, $\frac{8!}{4!} = \frac{8\times 7\times 6\times 5\times 4!}{4!}$

$$[\because n! = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1]$$
$$= 8\times 7\times 6\times 5 = 1680$$

$$= 8\times 7\times 6\times 5 = 1680$$

Also, $\quad 2! = 2\times 1 = 2\neq 1680$

$\therefore \quad \frac{8!}{4!}\neq 2!$

EXAMPLE |2| Evaluate $\frac{n!}{(n-r)!}$ when $r = 3$.

Sol. We have, $\frac{n!}{(n-r)!} = \frac{n!}{(n-3)!} \quad [\because r = 3, \text{ given}]$

$$= \frac{n(n-1)(n-2)(n-3)!}{(n-3)!}$$
$$[\because n! = n(n-1)(n-2)(n-3)!]$$
$$= n(n-1)(n-2)$$

EXAMPLE |3| Convert product into factorials.

(i) $6\cdot 7\cdot 8\cdot 9\cdot 10$ (ii) $2\cdot 4\cdot 6\cdot 8\cdot 10$

Sol. (i) We have, $6\cdot 7\cdot 8\cdot 9\cdot 10 = \frac{(1\cdot 2\cdot 3\cdot 4\cdot 5)\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}{(1\cdot 2\cdot 3\cdot 4\cdot 5)}$

[multiplying numerator and denominator by $1\cdot 2\cdot 3\cdot 4\cdot 5$]

$$= \frac{10!}{5!}$$

(ii) We have, $2\cdot 4\cdot 6\cdot 8\cdot 10$

$$= (2\times 1)\cdot(2\times 2)\cdot(2\times 3)\cdot(2\times 4)\cdot(2\times 5)$$
$$= 2^5\cdot(1\cdot 2\cdot 3\cdot 4\cdot 5) = 2^5\cdot 5!$$

EXAMPLE |4| Compute $\frac{(12!)-(10!)}{9!}$.

Sol. Consider, $\frac{(12!)-(10!)}{9!} = \frac{12\times 11\times 10! - 10!}{9!} = \frac{10!(132-1)}{9!}$

$$= \frac{10\times 9!\times 131}{9!} = 1310$$

EXAMPLE |5| Find the LCM of 6!, 7! and 8!.

Sol. We have, $7! = 7\times 6!$ and $8! = 8\times 7\times 6!$

$\therefore \quad$ LCM of 6!, 7! and 8! = LCM $(6!, 7\times 6!, 8\times 7\times 6!)$

$$= 6!\times 7\times 8 = 8\times 7\times 6! = 8!$$

EXAMPLE |6| If $\frac{1}{7!} + \frac{1}{9!} = \frac{x}{10!}$, then find x.

Sol. Given, $\frac{1}{7!} + \frac{1}{9!} = \frac{x}{10!} \Rightarrow \frac{1}{7!} + \frac{1}{9\times 8\times 7!} = \frac{x}{10\times 9\times 8\times 7!}$

$$\Rightarrow \quad \frac{1}{7!}\left(1+\frac{1}{72}\right) = \frac{1}{7!}\left(\frac{x}{10\times 9\times 8}\right) \Rightarrow \frac{73}{72} = \frac{x}{10\times 9\times 8}$$

$$\therefore \quad x = \frac{73}{72}\times 10\times 9\times 8 = 730$$

EXAMPLE |7| If $(n+3)! = 56\times (n+1)!$, then find n.

Sol. We have, $\quad (n+3)! = 56\times (n+1)!$

$$\Rightarrow \quad (n+3)(n+2)(n+1)! = 56\times (n+1)!$$
$$\Rightarrow \quad (n+3)(n+2) = 56 \quad [\because (n+1)!\neq 0]$$
$$\Rightarrow \quad n^2+5n+6-56 = 0 \Rightarrow n^2+5n-50 = 0$$
$$\Rightarrow \quad n^2+10n-5n-50 = 0 \Rightarrow n(n+10)-5(n+10) = 0$$
$$\Rightarrow \quad (n+10)(n-5) = 0 \Rightarrow n = 5$$

[$\because$ n can't be negative, as n is a natural number]

EXAMPLE |8| Prove that $\frac{(2n)!}{n!} = \{1 \cdot 3 \cdot 5 \ldots (2n-1)\}2^n$.

Sol. LHS $= \frac{(2n)!}{n!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \ldots (2n-2)(2n-1)(2n)}{n!}$

$= \frac{\{1 \cdot 3 \cdot 5 \ldots (2n-1)\}\{2 \cdot 4 \cdot 6 \ldots (2n-2)(2n)\}}{n!}$

$= \frac{\{1 \cdot 3 \cdot 5 \ldots (2n-1)\} \cdot 2^n \{1 \cdot 2 \cdot 3 \ldots (n-1)n\}}{n!}$

$= \frac{\{1 \cdot 3 \cdot 5 \ldots (2n-1)\} \cdot 2^n \cdot n!}{n!}$ $[\because 1 \cdot 2 \cdot 3 \ldots (n-1)\, n = n!]$

$= \{1 \cdot 3 \cdot 5 \ldots (2n-1)\}2^n =$ RHS **Hence proved.**

EXAMPLE |9| Prove that 33! is divisible by 2^{15}. What is the largest integer n such that 33! is divisible by 2^n?

Sol. Clearly, $33! = 1 \times 2 \times 3 \times \ldots \times 30 \times 31 \times 32 \times 33$

$= [2 \cdot 4 \cdot 6 \cdot 8 \ldots 30 \cdot 32](1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \ldots 31 \cdot 33)$

$= [2^1 \cdot 2^2 \cdot (2 \times 3) \cdot 2^3 \cdot (2 \times 5) \cdot (2^2 \times 3) \cdot (2 \times 7) \cdot 2^4$

$(2 \times 9) \cdot (2^2 \times 5) \cdot (2 \times 11) \cdot (2^3 \times 3) \cdot (2 \times 13)$

$(2^2 \times 7) \cdot (2 \times 15) \cdot 2^5] \cdot (1 \cdot 3 \cdot 5 \ldots 31 \cdot 33)$

$= [2^{1+2+1+3+1+2+1+4+1+2+1+3+1+2+1+5}$

$\cdot (3 \cdot 5 \cdot 3 \cdot 7 \cdot 9 \cdot 5 \cdot 11 \cdot 3 \cdot 13 \cdot 7 \cdot 15)](1 \cdot 3 \cdot 5 \ldots 31 \cdot 33)$

$= 2^{31}[3 \cdot 5 \cdot 3 \cdot 7 \cdot 9 \cdot 5 \cdot 11 \cdot 3 \cdot 13 \cdot 7 \cdot 15](1 \cdot 3 \cdot 5 \ldots 31 \cdot 33)$

Thus, 33! is divisible by 2^{15}.

Hence, the maximum power of 2 in 33! is 31.

EXAMPLE |10| If $\frac{n!}{2!(n-2)!}$ and $\frac{n!}{4!(n-4)!}$ are in the ratio 2 : 1, then find the value of n.

Sol. We have, $\frac{n!}{2!(n-2)!} : \frac{n!}{4!(n-4)!} = 2:1$

$\Rightarrow \frac{n!}{2!(n-2)!} \times \frac{4!(n-4)!}{n!} = \frac{2}{1}$

$\Rightarrow \frac{4!(n-4)!}{2!(n-2)!} = 2$

$\Rightarrow \frac{4 \times 3 \times 2! \times (n-4)!}{2! \times (n-2)(n-3)(n-4)!} = 2$

$\Rightarrow \frac{12}{(n-2)(n-3)} = 2$

$\Rightarrow 12 = 2(n-2)(n-3)$

$\Rightarrow 12 = 2(n^2 - 5n + 6)$

$\Rightarrow 6 = n^2 - 5n + 6$

$\Rightarrow n^2 - 5n = 0 \Rightarrow n(n-5) = 0$

$\Rightarrow n = 0$ or 5

When $n = 0$, then $(n-2)!$ and $(n-4)!$ are not defined, so rejecting $n = 0$.

$\therefore$ $n = 5$

TOPIC PRACTICE 2

OBJECTIVE TYPE QUESTIONS

1 Fill in the blanks.

I. When $n = 6, r = 2$, the value of $\frac{n!}{(n-r)!}$ is ...A... .

II. When $n = 9, r = 5$, the value of $\frac{n!}{(n-r)!}$ is ...B... .

III. When $n = 5, r = 2$, the value of $\frac{n!}{r!(n-r)!}$ is ...C... .

IV. 3! + 4! = 7! is ...D... .

Here, A, B, C and D refer to

(a) A→ 10, B→ 15120, C→ 10 , D→ false
(b) A→ 30 , B→ 15120, C→ 20 , D→ true
(c) A→ 30 , B→ 15120, C→ 10 , D→ false
(d) None of the above

2 Match the following columns and choose the correct option from the codes given below.

Column I	Column II
A. 5! equals	1. 5040
B. 7! equals	2. 120
C. 8! equals	3. 4920
D. 7! − 5! equals	4. 40320
E. 4! − 3! equals	5. 18

Codes

	A	B	C	D	E
(a)	1	2	3	4	5
(b)	2	1	3	4	5
(c)	2	1	4	3	5
(d)	5	4	3	2	1

3 If $\frac{1}{9!} + \frac{1}{10!} = \frac{x}{11!}$, then x is equal to

(a) 100 (b) 110
(c) 99 (d) 121

4 The value of $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n!$ is

(a) $(n+1)!$ (b) $(n+1)! + 1$
(c) $(n+1)! - 1$ (d) $n! + 1$

5 If $(n+1)! = 12 \times (n-1)!$ then the value of n is equal to

(a) 4 (b) 3
(c) 16 (d) 11

VERY SHORT ANSWER Type Questions

6 Evaluate the following.
(i) $4! - 3!$ (ii) $2 \times 6! - 3 \times 5!$

7 Compute $\frac{20!}{18!(20-18)!}$.

8 If $\frac{1}{8!} + \frac{1}{9!} = \frac{x}{10!}$, then find x.

9 Prove that

$$n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}.$$

10 Convert the following products into factorials, $(n+1)(n+2)(n+3)\ldots(2n)$.

11 Prove that $(n!)\cdot(n+2) = [n! + (n+1)!]$.

SHORT ANSWER Type I Questions

12 Find n, if $(n+2)! = 2550 \times n!$.

13 Prove that $$\frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!} = \frac{(n+1)!}{r!(n-r+1)!}.$$

SHORT ANSWER Type II Questions

14 Prove that $\frac{(2n+1)!}{n!} = 2^n[1\cdot 3\cdot 5\ldots(2n-1)(2n+1)]$.

15 Prove that 35! is divisible by 2^{12}. What is the largest integer n such that 35! is divisible by 2^n?

16 If $\frac{(2n)!}{3!(2n-3)!}$ and $\frac{n!}{2!(n-2)!}$ are in the ratio 44 : 3, find n.

HINTS & ANSWERS

1. (c) I. $n = 6, r = 2$

$$\frac{n!}{(n-r)!} = \frac{6!}{(6-2)!} = \frac{6\times5\times4!}{4!} = 30$$

II. $n = 9, r = 5$

$$\frac{n!}{(n-r)!} = \frac{9!}{4!} = \frac{9\times8\times7\times6\times5\times4!}{4!} = 15120$$

III. $n = 5, r = 2$

$$\frac{n!}{r!(n-r)!} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!\,3!} = 10$$

IV. $3! + 4! = 6 + 24 = 30$

$$7! = 7\times6\times5\times4\times3\times2\times1 = 5040$$

$\Rightarrow \quad 3! + 4! \neq 7!$

Hence, IV is false.

$A \to 30$, $B \to 15120$, $C \to 10$, $D \to$ false

2. (c) The notation $n!$ represents the product of first n natural numbers *i.e.*, $n! = 1.2.3.\ldots\ldots(n-1).n$

A. $5! = 120$; B. $7! = 5040$; C. $8! = 40320$
D. $7! - 5! = 4920$ E. $4! - 3! = 18$

3. (d) We have, $\frac{1}{9!} + \frac{1}{10!} = \frac{x}{11!}$

$$\Rightarrow \frac{1}{9!} + \frac{1}{10\times9!} = \frac{x}{11\times10\times9!}$$

$$\Rightarrow 1 + \frac{1}{10} = \frac{x}{110} \Rightarrow \frac{11}{10} = \frac{x}{110}$$

$$\Rightarrow x = 121$$

4. (c) We have, $1\cdot1! + 2\cdot2! + 3\cdot3! + \ldots + n\cdot n!$

$$= \sum_{r=1}^{n} r(r!) = \sum_{r=1}^{n}[(r+1)r! - r!]$$
$$= (2! - 1!) + (3! - 2!) + \ldots + [(n+1)! - n!]$$
$$= (n+1)! - 1! = (n+1)! - 1$$

5. (b) We have, $(n+1)! = 12(n-1)!$

$$\Rightarrow (n+1)(n)(n-1)! = 12(n-1)!$$
$$\Rightarrow n^2 + n = 12 \Rightarrow n^2 + n - 12 = 0$$
$$\Rightarrow (n+4)(n-3) = 0$$
$$n = 3 \Rightarrow n \neq -4$$

6. (i) $4! - 3! = 4\times3\times2\times1 - 3\times2\times1 = 18$ (ii) 1080

7. $\frac{20!}{18!(20-18)!} = \frac{20\times19\times18!}{18\times2!} = 190$

8. Solve as Example 6. **Ans.** $x = 100$

9. In LHS, multiply numerator and denominator by $(n-r)!$.

10. Multiply numerator and denominator by $n!$.

Ans. $\frac{(2n)!}{n!}$

11. Consider RHS and write it as $n! + (n+1)n!$.

12. Solve as Example 7. **Ans.** $n = 49$

13. (a) LHS can be written as

$$\frac{n!}{(n-r)!r(r-1)!} + \frac{n!}{(n-r+1)(n-r)!(r-1)!}$$
$$= \frac{n!}{(n-r)!(r-1)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$
$$\left[\begin{array}{ll}\because (n-r+1)(n-r)! = (n-r+1)! \\ \text{and} \quad (n+1)n! = (n+1)!\end{array}\right]$$

14. Solve as Example 8.

15. Solve as Example 9. **Ans.** $n = 32$

16. Solve as Example 10. **Ans.** $n = 6$

|TOPIC 3|
Permutations

A permutation is an arrangement of objects in a definite order. Arrangement can be made by taking some or all objects at a time. e.g. If there are three objects say A, B and C, then the permutations of these three objects taking two at a time are AB, BA, AC, CA BC, CB and the permutations of these three objects taking all at a time are ABC, ACB, BAC, BCA, CAB, CBA.

Here, in each case, number of permutations is 6.

Note that the order of arrangement is important. Because when the order is changed, then different permutation is obtained.

PERMUTATIONS, WHEN ALL THE OBJECTS ARE DISTINCT

When all given objects are distinct, then we can find the number of permutations with the help of following theorem

***Theorem* 1** *The number of permutations of n different objects taken r at a time, where* $0 < r \le n$ *and the objects do not repeat, is* $n(n-1)\ (n-2)\ \dots\ (n-r+1)$, *which is denoted by* ${}^nP_r$ *or* $P(n, r)$.

i.e. $$P(n, r) = {}^nP_r = \frac{n!}{(n-r)!},\ 0 \le r \le n$$

Proof We know that

$$P(n, r) = {}^nP_r = n(n-1)(n-2)(n-3)\dots[n-(r-1)]$$

On multiplying numerator and denominator by $(n-r)(n-r-1)\ \dots\ 3 \times 2 \times 1$, we get

$${}^nP_r = \frac{\begin{bmatrix} n(n-1)(n-2)\dots(n-(r-1)) \\ \times (n-r)(n-(r+1))\dots3\cdot2\cdot1 \end{bmatrix}}{(n-r)(n-(r+1))\dots3\cdot2\cdot1}$$

$$= \frac{n(n-1)(n-2)\dots3\cdot2\cdot1}{(n-r)(n-r-1)\dots3\cdot2\cdot1} = \frac{n!}{(n-r)!}$$

where, $0 \le r \le n$

In Particular,

(i) When $r = 0$, then ${}^nP_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$

(ii) When $r = n$, then ${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$ $[\because 0! = 1]$

Note

Number of permutations of n different things taken all at a time $= n!$

Let us look at different types of problems based on above concept.

Problems based on the formula of $P(n, r)$

EXAMPLE |1| Find the values of the following.

(i) ${}^6P_4$ (ii) $P(15, 2)$

Sol. We know that, ${}^nP_r = P(n, r) = \frac{n!}{(n-r)!}$

(i) $${}^6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!}$$
$$= 6 \times 5 \times 4 \times 3 = 360$$

(ii) $$P(15, 2) = {}^{15}P_2 = \frac{15!}{(15-2)!} = \frac{15!}{13!} = \frac{15 \times 14 \times 13!}{13!}$$
$$= 15 \times 14 = 210$$

EXAMPLE |2| If $P(n, 5) = 20P(n, 3)$, find n.

Sol. We have, $P(n, 5) = 20P(n, 3)$

$$\Rightarrow \frac{n!}{(n-5)!} = 20 \times \frac{n!}{(n-3)!}$$
$$\Rightarrow (n-3)! = 20 \times (n-5)!$$
$$\Rightarrow (n-3)(n-4)(n-5)! = 20 \times (n-5)!$$
$$\Rightarrow (n-3)(n-4) = 20 = 5 \times 4$$
$$\Rightarrow (n-3)(n-4) = (8-3)(8-4)$$

On comparing, we get $n = 8$.

EXAMPLE |3| Find the value of r, if ${}^5P_r = 2 \times {}^6P_{r-1}$.

Sol. Given, ${}^5P_r = 2 \times {}^6P_{r-1}$

$$\therefore \frac{5!}{(5-r)!} = 2 \times \frac{6!}{(6-r+1)!} \quad \left[\because {}^nP_r = \frac{n!}{(n-r)!}\right]$$
$$\Rightarrow \frac{5!}{(5-r)!} = 2 \times \frac{6 \times 5!}{(7-r)!}$$
$$\Rightarrow \frac{1}{(5-r)!} = \frac{12}{(7-r)(6-r)(5-r)!}$$
$$\Rightarrow \frac{1}{1} = \frac{12}{(7-r)(6-r)}$$
$$\Rightarrow (7-r)(6-r) = 12$$
$$\Rightarrow 42 - 7r - 6r + r^2 = 12$$
$$\Rightarrow r^2 - 13r + 30 = 0$$
$$\Rightarrow r^2 - 10r - 3r + 30 = 0$$
$$\Rightarrow r(r-10) - 3(r-10) = 0$$
$$\Rightarrow (r-3)(r-10) = 0$$
$$\Rightarrow r = 3, 10$$
$$\therefore r = 3$$

EXAMPLE |4| If $P(9, r) = 3024$, find r.

Sol. We have, $P(9, r) = 3024 \Rightarrow {}^9P_r = 3024$

$$\Rightarrow \frac{9!}{(9-r)!} = 3024 \Rightarrow \frac{9!}{(9-r)!} = 9 \times 336$$

$$\Rightarrow \frac{9!}{(9-r)!} = 9 \times 8 \times 7 \times 6$$

$$\Rightarrow \frac{9!}{(9-r)!} = \frac{9 \times 8 \times 7 \times 6 \times 5!}{5!}$$

[multiplying numerator and denominator by 5!]

$$\Rightarrow \frac{9!}{(9-r)!} = \frac{9!}{5!} \Rightarrow (9-r)! = 5!$$

On comparing, we get

$$9 - r = 5 \Rightarrow r = 4$$

EXAMPLE |5| If ${}^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_r$, find r.

Sol. We have, ${}^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_r$

$$\Rightarrow \frac{9!}{(9-5)!} + 5 \cdot \frac{9!}{(9-4)!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \frac{9!}{4!} + 5 \cdot \frac{9!}{5!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \frac{9!}{4!} + 5 \cdot \frac{9!}{5 \times 4!} = \frac{10 \times 9!}{(10-r)!}$$

[dividing both sides by 9!]

$$\Rightarrow \frac{1}{4!} + \frac{1}{4!} = \frac{10}{(10-r)!} \Rightarrow \frac{2}{4!} = \frac{10}{(10-r)!}$$

$$\Rightarrow (10-r)! = 5 \times 4! = 5! \Rightarrow (10-r)! = 5!$$

On comparing, we get $10 - r = 5 \Rightarrow r = 5$

EXAMPLE |6| If $P(2n-1, n) : P(2n+1, n-1) = 22 : 7$, find n.

Sol. We have, ${}^{2n-1}P_n : {}^{2n+1}P_{n-1} = 22 : 7$

$$\therefore \frac{\dfrac{(2n-1)!}{(2n-1-n)!}}{\dfrac{(2n+1)!}{(2n+1-n+1)!}} = \frac{22}{7}$$

$$\frac{(2n-1)!}{(n-1)!} \times \frac{(n+2)!}{(2n+1)!} = \frac{22}{7}$$

$$\Rightarrow \frac{(2n-1)!}{(n-1)!} \times \frac{(n+2)(n+1)n(n-1)!}{(2n+1)(2n)(2n-1)!} = \frac{22}{7}$$

$$\Rightarrow \frac{(n+2)(n+1)n}{(2n+1)(2n)} = \frac{22}{7} \Rightarrow \frac{(n+1)(n+2)}{2(2n+1)} = \frac{22}{7}$$

$$\Rightarrow 7(n^2 + 3n + 2) = 44(2n+1)$$

$$\Rightarrow 7n^2 + 21n + 14 = 88n + 44$$

$$\Rightarrow 7n^2 - 67n - 30 = 0$$

$$\Rightarrow 7n^2 - 70n + 3n - 30 = 0$$

$$\Rightarrow 7n(n-10) + 3(n-10) = 0$$

$$\Rightarrow (n-10)(7n+3) = 0$$

$$\Rightarrow n = 10 \text{ or } n = \frac{-3}{7}$$

$$\therefore n = 10 \quad [\because n \text{ can't be negative}]$$

EXAMPLE |7| Prove that

$$P(n, r) = P(n-1, r) + r \cdot P(n-1, r-1).$$

Sol. Consider RHS $= P(n-1, r) + r \cdot P(n-1, r-1)$

$$= \frac{(n-1)!}{(n-1-r)!} + r \cdot \frac{(n-1)!}{(n-1-r+1)!}$$

$$= \frac{(n-1)!}{(n-r-1)!} + r \cdot \frac{(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!}{(n-r-1)!} + \frac{r \cdot (n-1)!}{(n-r)(n-r-1)!}$$

$$= \frac{(n-1)!}{(n-r-1)!}\left[1 + \frac{r}{n-r}\right]$$

$$= \frac{(n-1)!}{(n-r-1)!}\left[\frac{n-r+r}{(n-r)}\right]$$

$$= \frac{n(n-1)!}{(n-r)(n-r-1)!}$$

$$= \frac{n!}{(n-r)!}$$

$= P(n, r) = \text{LHS}$ **Hence proved.**

Simple Applications on Permutation

Here we will discuss some simple practical problems, where the formula for ${}^nP_r$ and its meaning can be applicable.

EXAMPLE |8| In how many ways two different rings can be worn in four fingers with atmost one in each finger?

Sol. Clearly, required number of ways is same as number of permutations (or arrangements) of 4 different things taken 2 at a time.

So, required number of ways $= {}^4P_2$

$$= \frac{4!}{2!} = \frac{4 \times 3 \times 2!}{2!} = 12$$

EXAMPLE |9| In how many ways can 6 persons occupy 4 vacants seats?

Sol. Required number of ways is same as number of permutations of 6 different things taken 4 at a time.

Hence, required number of ways

$$= {}^6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!}$$

$$= 6 \times 5 \times 4 \times 3 = 360$$

EXAMPLE |10| How many 3-digit numbers can be formed by using the digits 1 to 9, if no digit is repeated?

Sol. Required number of 3-digit numbers is same as number of permutations of 9 different things taken 3 at a time.

Hence, total number of 3-digit numbers $= {}^9P_3$

$$= \frac{9!}{(9-3)!} = \frac{9!}{6!} = \frac{9 \times 8 \times 7 \times 6!}{6!} = 504$$

EXAMPLE |11| From a committee of 8 persons, in how many ways can we choose a chairman and a vice-chairman assuming one person can not hold more than one position? **[NCERT]**

Sol. Since, a person cannot hold more than one position, therefore required number of ways is same as number of permutations of 8 different things taken 2 at a time.

Hence, required number of ways $= {}^8P_2$

$$= \frac{8!}{(8-2)!}$$

$$= \frac{8!}{6!} = \frac{8 \times 7 \times 6!}{6!} = 56$$

EXAMPLE |12| How many words, with or without meaning can be made from the letters of the word 'MONDAY,' assuming that no letter is repeated, if

(i) 4 letters are used at a time?

(ii) all letters are usedz at a time? **[NCERT]**

Sol. Here, the word 'MONDAY' contains 6 distinct letters.

(i) Required number of words = Number of permutations of 6 different things taken 4 at a time

$$= {}^6P_4 = \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!} = 360$$

(ii) Required number of words = Number of permutations of 6 different things taken all at a time

$$= {}^6P_6 = 6! = 720$$

EXAMPLE |13| How many different signals can be made by using any number of flags from 6 flags of different colours? **[NCERT Exemplar]**

Sol. Clearly, the signals can be made by using one or two or three or four or five or six flags.

Now, number of signals when one flag is used at a time $= {}^6P_1$.

Number of signals when two flags are used at a time $= {}^6P_2$.

Similarly, number of signals when three, four, five, six flags are used at a time are ${}^6P_3$, ${}^6P_4$, ${}^6P_5$, ${}^6P_6$, respectively.

Hence, by fundamental principle of addition, total number of signals formed from 6 flags of different colours $= {}^6P_1 + {}^6P_2 + {}^6P_3 + {}^6P_4 + {}^6P_5 + {}^6P_6$

$$= \frac{6!}{(6-1)!} + \frac{6!}{(6-2)!} + \frac{6!}{(6-3)!} + \frac{6!}{(6-4)!} + \frac{6!}{(6-5)!} + \frac{6!}{(6-6)!}$$

$$= \frac{6!}{5!} + \frac{6!}{4!} + \frac{6!}{3!} + \frac{6!}{2!} + \frac{6!}{1!} + \frac{6!}{0!}$$

$$= 6 + 6 \times 5 + 6 \times 5 \times 4 + 6 \times 5 \times 4 \times 3 + 6 \times 5 \times 4 \times 3 \times 2 + 6!$$

$$= 6 + 30 + 120 + 360 + 720 + 720 = 1956$$

EXAMPLE |14| Three men have 4 coats, 5 waistcoats and 6 caps. In how many ways, can they wear them?

Sol. Here, all the 4 coats are distinct, all 5 waistcoats are distinct and all 6 caps are also distinct.

So, number of ways in which three men can wear 4 coats will be equal to the number of arrangements of 4 different things taken 3 at a time.

$\therefore$ Number of ways in which three men can wear 4 coats

$$= {}^4P_3 = \frac{4!}{(4-3)!} = 4 \times 3 \times 2 = 24$$

Similarly, number of ways in which three men can wear 5 waistcoats $= {}^5P_3 = \frac{5!}{(5-3)!} = \frac{5 \times 4 \times 3 \times 2!}{2!} = 5 \times 4 \times 3 = 60$

and number of ways in which three men can wear 6 caps

$$= {}^6P_3 = \frac{6!}{(6-3)!} = \frac{6 \times 5 \times 4 \times 3!}{3!} = 6 \times 5 \times 4 = 120$$

Since, all three men wear coat, waistcoat and cap, so by the fundamental principal of multiplication,

Required number of ways $= {}^4P_3 \times {}^5P_3 \times {}^6P_3$

$$= 24 \times 60 \times 120 = 172800$$

Permutations with Repetitions

When repetition of objects is allowed, then number of permutations can be obtained with the help of following theorem.

Theorem 2 *The number of permutation of n different objects taken r at a time, when each may be repeated any number of times in each arrangement, is n^r (permutation with repetitions).*

Note The number of permutations of n different objects, all at a time, when each may be repeated any number of times in each arrangement, is n^n.

EXAMPLE |15| In how many ways, 3 prizes can be given to 7 boys when each boy is eligible for any of the prizes?

Sol. Here, three prizes can be given to the same boy but two boys cannot get the same prize. Thus, boys can be repeated. So, here we have $n = 7$ and $r = 3$.

Hence, required number of ways $= 7^3 = 343$.

EXAMPLE |16| How many 4-digit numbers can be formed with the digits 1, 2, 3, 4, 5, 6 when a digit may be repeated any number of times in any arrangement?

Sol. Required number of numbers is same as number of permutation of 6 different things taken 4 at a time, when a digit may be repeated any number of times.

Hence, required number of numbers $= 6^4 = 1296$.

EXAMPLE |17| Find the number of 5-digit telephone numbers having atleast one of their digits repeated. **[NCERT Exemplar]**

Sol. Using the digits 0, 1, 2, ..., 9, the number of 5-digits telephone numbers which can be formed is 10^5 (since, repetition is allowed).

The number of 5-digits telephone numbers, which have none of the digits repeated $= {}^{10}P_5 = \frac{10!}{5!} = 30240$

Hence, the required number of telephone numbers having atleast one of their digits repeated

$$= 10^5 - {}^{10}P_5$$
$$= 100000 - 30240$$
$$= 69760$$

Permutations When all the Objects are not Distinct Objects

When all the objects are not distinct, i.e. some objects are of same kind, then we can find the number of permutations with the help of following theorems (without proof).

Theorem 3 *The number of permutations of n objects, where p objects are of the same kind or identical and other are distinct, is given by* $\frac{n!}{p!}$.

EXAMPLE |18| How many permutations of the letters of the words 'INDIA' are there?

Sol. Here, we have 5 letters, two of which are of the same kind and others are different.

$\therefore$ Required number of permutations (or arrangements)

$$= \frac{5!}{2!}$$
$$= 5 \times 4 \times 3 = 60$$

Theorem 4 *The number of permutations of n objects, where p_1 objects are of one kind, p_2 are of second kind, ... p_k are of kth kind and the rest if any, are of different kind is* $\frac{n!}{p_1!\,p_2!\dots p_k!}$.

EXAMPLE |19| Find the number of permutations of the letters of the word 'INDEPENDENCE'.

Sol. Here, we have 12 letters of which 3 are N's, 4 are E's, 2 are D's and rest are different.

$\therefore$ Required number of permutations $= \frac{12!}{3!4!2!}$

[here, $n = 12$, $p_1 = 3$, $p_2 = 4$ and $p_3 = 2$]

$$= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{6 \times 2 \times 4!}$$
$$= 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5$$
$$= 1663200$$

EXAMPLE |20| In how many ways, 5 flags in which 3 are red, 1 is white and 1 is blue, be arranged on a staff, one below the other, if flags of one colour are not distinguishable?

Sol. Since, out of 5 flags, 3 are of the same kind (red) and others are different.

$\therefore$ Required number of ways

$$= \frac{5!}{3!1!1!}$$
$$= 5 \times 4 = 20$$

TOPIC PRACTICE 3

OBJECTIVE TYPE QUESTIONS

1 $P(n-1, r) + r \cdot P(n-1, r-1)$ equals

(a) $P(n-1, r+1)$ (b) $P(n, r-1)$
(c) $P(n, r)$ (d) $P(n+1, r+1)$

2 The number of permutations of n different things taken r at a time, when repetition is allowed, is

(a) r^n (b) n^r
(c) $\frac{n!}{(n-r)!}$ (d) None of these

3 Consider the following statements

Statement I ${}^nP_r = \frac{n!}{(n-r)!}, 0 \le r \le n$.

Statement II ${}^nP_r = n(n-1)(n-2) \dots\dots (n-r+1)$, $0 \le r \le n$.

Which of the above statement (s) is/are true?

(a) Only I (b) Only II
(c) Both I and II (d) Neither I nor II

4 In a class of 10 students there are 3 girls A, B, C. The number of different ways they can be arranged in a row such that no two of three girls are consecutive are

(a) $4! \times 300$ (b) $7! \times 336$
(c) $6! \times 300$ (d) $6! \times 336$

5 The total number of 9-digits numbers which have all different digits is **[NCERT Exemplar]**

(a) $10!$ (b) $9!$
(c) $9 \times 9!$ (d) $10 \times 10!$

VERT SHORT ANSWER Type Questions

6 Evaluate (i) ${}^{75}P_2$ (ii) ${}^8P_8$.

7 If ${}^nP_4 = 360$, find the value of n.

8 If ${}^nP_4 : {}^nP_5 = 1:2$, find n.

SHORT ANSWER Type I Questions

9 If ${}^{10}P_r = 5040$, find the value of r.

10 Find the value of n such that
$$P(n, 5) = 42\, P(n, 3),\, n > 4.$$

11 Find r, if $5 \cdot {}^4P_r = 6 \cdot {}^5P_{r-1}$.

12 If ${}^{n-1}P_3 : {}^{n+1}P_3 = 5 : 12$, find n.

13 If ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5$, find n.

14 If ${}^{22}P_{r+1} : {}^{20}P_{r+2} = 11 : 52$, find r.

15 Prove that following.

(i) ${}^nP_n = 2 \cdot {}^nP_{n-2}$ (ii) ${}^nP_n = {}^nP_{n-1}$

(iii) ${}^nP_r = n \cdot {}^{n-1}P_{r-1}$

16 How many numbers lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4 and 5, if the repetition of digits is not allowed?

17 How many 3-digit numbers are there with distinct digits with each digit odd?

18 Find the number of 4-digit numbers that can be formed with the digits 2, 3, 4, 7 and using each digit only once.

19 In how many ways can 18 people line up at a ticket window of a cinema hall?

20 How many words, with or without meaning can be formed using all the letters of the word EQUATION, using each letter exactly once? **[NCERT]**

21 Find the number of different 4-letter words, with or without meaning that can be formed from the letters of the word 'NUMBER'.

22 How many different signals can be made by 5 flags from 8 flags of different colours?

23 From a committee of 12 persons, in how many ways can we choose a chairman and a vice-chairman assuming one person can not hold more than one position?

24 Sunil wants to arrange 3 Economics, 2 History and 4 Language books on a shelf. If the books of the same subject are different. Determine the number of all possible arrangements.

25 Eight students are contesting the election for the presidentship of the student's union. In how many ways can their names be listed on the ballot papers?

26 In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if the discs of the same colour are indistinguishable?

SHORT ANSWER Type II Questions

27 If x denotes the number of permutations of $(n+2)$ things taken all at a time, y denotes the number of permutations of n things taken 12 at a time and z denotes the number of permutation of $n-12$ things taken all at a time such that $x = 132yz$, find the value of n.

28 Ten different letters of English alphabet are given. Words with five letters are formed from these given letters. Find the number of words which have atleast one letter repeated.

HINTS & ANSWERS

1. (c) We have, $P(n-1, r) + r \cdot P(n-1, r-1)$

$$= \frac{(n-1)!}{(n-1-r)!} + r \cdot \frac{(n-1)!}{[(n-1)-(r-1)]!}$$
$$= \frac{(n-1)!}{(n-r-1)!} + r \cdot \frac{(n-1)!}{(n-r)!}$$
$$= \frac{(n-1)!}{(n-r-1)!}\left[1 + \frac{r}{n-r}\right]$$
$$= \frac{(n-1)!}{(n-r-1)!} \cdot \frac{n}{n-r}$$
$$= \frac{n!}{(n-r)!} \quad = P(n, r)$$

2. (b) The number of permutations = The number of ways of filling r vacant places by n different things

= □ □------□ (r places)

∴ Number of permutations
$$= n \times n \times \ldots\ldots \times n \,(r \text{ times}) = n^r$$

3. (c) We know, ${}^nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1)$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)(n-r)\ldots 3 \times 2 \times 1}{(n-r)(n-r-1)\ldots 3 \times 2 \times 1}$$
$$= \frac{n!}{(n-r)!}$$

Thus, both I and II are true.

4. (b) $X\, B_1\, X\, B_2\, X\, B_3\, X\, B_4\, X\, B_5\, X\, B_6\, X\, B_7\, X$

First, we arrange 7 boys in 7! ways.

Now for girls, we have eight places and 3 girls need to be arranged. This can be done in ${}^8P_3$ ways.

Total ways $= 7! \times {}^8P_3$

Ans. $7! \times 336$

5. (c) The total number of 9-digits numbers, having all digit are different $= 9\ {}^9P_8$

Ans. $9 \times 9!$

6. Solve as Example 1. **Ans.** (i) 5550 (ii) 40320

7. Write the given equation as
$n(n-1)(n-2)(n-3) = 6 \times 5 \times 4 \times 3 = 6(6-1)(6-2)(6-3)$
and then compare it. **Ans.** $n = 6$

8. Solve as Example 2. **Ans.** $n = 6$

9. Solve as Example 4. **Ans.** $r = 4$

10. Solve as Example 2. **Ans.** $n = 10$

11. Solve as Example 3. **Ans.** $r = 3$

12. Solve as Example 6. **Ans.** $n = 8$

13. Solve as Example 6. **Ans.** $n = 4$

14. Solve as Example 6. **Ans.** $r = 7$

15. Consider RHS and use the formula ${}^nP_r = \dfrac{n!}{(n-r)!}$ to simplify RHS.

16. Required numbers are 3-digit numbers, formed with the given digits. Solve as Example 10. **Ans.** 60

17. Required numbers are 3-digit numbers formed with the digits 1, 3, 5, 7 and 9. **Ans.** 60

18. Required number of numbers is same as number of permutations of 4 distinct objects taken all at a time.
Ans. 24

19. Solve as Example 14. **Ans.** 18!

20. Their are 8 distinct letters in a given word. So, total number of words can be formed is 8. **Ans.** 40320

21. Their are 6 distinct letters in a given word. So, number of different 4-letter words can be formed is ${}^6P_4$.
Ans. 360

22. Required number of signals is same as number of permutations of 8 different things taken 5 at a time.
Ans. 6720

23. Solve as Example 11. **Ans.** 132

24. Number of possible arrangements is same as number of permutations of 9 different things taking all at a time.
Ans. 362880

25. Required number of ways is same as number of permutations of 8 different things taken all at a time.
Ans. 40320

26. (i) Solve as Example 20.
(ii) Here, $n = 9$, $p_1 = 4$, $p_2 = 3$ and $p_3 = 2$
Ans. Required number of ways $= \dfrac{9!}{4!\,3!\,2!} = 1260$

27. We have, $(n+2)! = 132\ {}^nP_{12}(n-12)!$
On solving, we get $n = 10$, which is not satisfied ${}^nP_{12}$.
Hence, no value of n exists.

28. Solve as Example 17. **Ans.** 69760

|TOPIC 4|
Restricted Permutations

Here, we shall discuss permutations of objects under certain conditions, e.g. permutation when certain objects occur together, when position of particular objects are fixed, when a particular object occurs in every arrangement etc.

SOME IMPORTANT RESULTS

(i) If r particular things out of n different things are to be together, then we count these r particular things as one things and remaining $(n-r)$ things as separate things.

Then, total number of things $= (n-r)+1$
$= n-r+1.$

Since, these are different things, therefore number of permutations of these things $= (n-r+1)!$.

But, r particular things can also be arrange among themselves in $r!$ ways.

$\therefore$ Required number of permutations $= (n-r+1)!\,r!$

Note

If r particular things are identical, then required number of permutations $= (n-r+1)!$

(ii) The number of permutations of n objects taken r at a time, when a particular object is taken in each arrangement, is $r \cdot {}^{n-1}P_{r-1}$.

(iii) The number of permutations of n objects taken r at a time, when a particular object is never taken in each arrangement, is ${}^{n-1}P_r$.

(iv) The number of permutations of n different objects taken r at a time in which two specific objects always occur together, is $2!\,(r-1) \cdot {}^{n-2}P_{r-2}$.

Restricted Permutation when all the objects are distinct

EXAMPLE |1| Find the number of different words that can be formed from the letters of the word 'TRIANGLE', so that

(i) all vowels occur together.

(ii) all vowels do not occur together.

Sol. There are 8 distinct letters in the word TRIANGLE, out of which 3 are vowels, namely A, E, I and 5 are consonants, namely T, R, N, G, L.

(i) Since, the vowels have to occur together, so let us assume them as a single object (AEI).

$$\boxed{A\ E\ I}\ T, R, N, G, L$$

Now, this single object together with 5 remaining letters will be counted as 6 objects and these can be arrange in ${}^6P_6 = 6!$ ways.

Corresponding to each of these permutations, we have $3! = 6$ permutations of the three vowels A, E and I taken all at a time.

Hence, by fundamental principle of multiplication, the required number of words $= 6! \times 3! = 4320$

(ii) Clearly, required number of words = Number of all possible arrangements of 8 letters taken all at a time − Number of permutation in which the vowels are always together

$= {}^8P_8 - 6! \times 3! = 8! - 6! \times 3!$

$= 8 \times 7 \times 6! - 6! \times 3!$

$= 6!(56 - 6)$

$= 720 \times 50 = 36000$

EXAMPLE |2| In how many ways can 5 children be arranged in a line such that

(i) two particular children of them are always together?

(ii) two particular children of them are never together? **[NCERT Exemplar]**

Sol. (i) Let us take 2 particular children together as one.

Now, the remaining 4 (particular childrens and other three childrens) can be arranged in $4! = 24$ ways.

Again two particular children taken together can be arranged in $2! = 2$ ways.

Hence, there are $24 \times 2 = 48$ ways of arrangement.

(ii) Clearly, required number of arrangements = Number of permutation of 5 children taken all at a time − Number of permutation of children in which two particular children are together

$= 5! - 4! \times 2$

$= 5 \times 4! - 4! \times 2$

$= 4!(5 - 2)$

$= 24 \times 3 = 72.$

Hence, required number of arrangements = 72

EXAMPLE |3| How many words, with or without meaning, can be formed using all the letters of the word EQUATION at a time so that the vowels and consonant occurs together? **[NCERT]**

Sol. There 8 distinct letters in the word EQUATION, out of which 5 are vowels, namely A, E, I, O, U and 3 are consonants namely Q, T, N.

Now, let us take all vowels together as one object and all consonants together as another object.

(A, E, I, O, U) (Q, T, N)

Clearly, these objects can be arranged in $2! = 2$ ways.

Again, all vowels taken together can be arranged in $5! = 120$ ways and all consonants taken together can be arranged in $3! = 6$ ways.

Hence, by fundamental principle of multiplication required number of words $= 2 \times 5! \times 3!$

$= 2 \times 120 \times 6 = 1440$

EXAMPLE |4| In how many ways, can 8 Indians, 4 Americans and 4 Englishmen be seated in a row so that all persons of the same nationality sit together?

Sol. Let us take persons of same nationality as one unit. Then, there are 3 units which can be arranged in $3! = 6$ ways. Now, in each of arrangements, 8 Indians can be arranged among themselves in 8! ways, 4 Americans can be arranged among themselves in 4! ways and 4 Englishmen can be arranged among themselves in 4! ways.

Hence, by fundamental principle of multiplication, required number of ways $= 3! \times 8! \times 4! \times 4!$.

EXAMPLE |5| All the letters of the word 'EAMCOT' are arranged in different possible ways. Find the number of such arrangements in which no two vowels are adjacent to each other. **[NCERT Exemplar]**

Sol. There are 3 consonants and 3 vowels.

Since, no two vowels can occur together, therefore the possible choices for vowels are the places marked as '×'.

$$\times M \times C \times T \times$$

Clearly, the vowels can be arranged in ${}^4P_3$ ways and the consonants can be arranged in 3! ways. Hence, by fundamental principle of multiplication, required number of ways $= {}^4P_3 \times 3! = \frac{4!}{1!} \times 3! = 24 \times 6 = 144$

EXAMPLE |6| It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible? **[NCERT]**

Sol. There are 9 seats, out of which 4 are at even places and rest are at odd places. Thus, there are 4 even places. Clearly, 4 women can be placed in 4 even places in 4! ways. In rest of the five places, five men can be placed in 5! ways.

Hence, required number of ways $= 4! \times 5! = 2880.$

EXAMPLE |7| The letters of the word 'TUESDAY' are arranged in a line, each arrangement ending with letter S. How many different arrangements are possible? How many of them start with letter D?

Sol. Here, the given word contains 7 letters. Since, all words must end with S, so we fix S at last place. Now, remaining 6 letters can be arranged in $P(6, 6) = 6!$ ways.

$\therefore$ Total number of words $= 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

If the words begin with D and end with S, we fix D at the first place and S at the last place. Now, remaining 5 letters can be arranged in $P(5, 5) = 5!$

$$= 5 \times 4 \times 3 \times 2 \times 1 = 120 \text{ ways}$$

$\therefore$ Required number of words $= 120$

EXAMPLE |8| Find the number of ways in which 5 boys and 5 girls be seated in a row, so that

(i) no two girls sit together.

(ii) boys and girls sit alternatively. **[NCERT Exemplar]**

Sol. We have 5 boys and 5 girls.

(i) Since, no two girls sit together, therefore the possible choices for girls are the places marked as '×'.

$$\times B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times$$

Clearly, the girls can be arranged in ${}^6P_5$ ways and the boys can be arranged in 5! ways.

Hence, by fundamental principle of multiplication, required number of ways = ${}^6P_5 \times 5!$

$$= \frac{6!}{(6-5)!} \times 5! = 6! \times 5! = 86400$$

(ii) Let us first seat the 5 girls (or 5 boys). This can be done in 5! ways.

Now, for each such arrangement, the 5 boys can be seated only at the cross marked place as shown below:

(I) $\times G_1 \times G_2 \times G_3 \times G_4 \times G_5$

(II) $G_1 \times G_2 \times G_3 \times G_4 \times G_5 \times$

In case I, 5 boys can be seated in ${}^5P_5 = 5!$ ways

Thus, number of ways of seating $= 5! \times 5! = (5!)^2$

Similarly, in case II, number of ways of seating $= (5!)^2$

Hence, required number of ways $= (5!)^2 + (5!)^2$

$$= 2 \cdot (5!)^2 = 28800$$

FINDING THE NUMBER OF NUMBERS HAVING DISTINCT DIGITS WHICH ARE FORMED UNDER CERTAIN CONDITIONS

For solving this type of problem, we used the following steps

Step I Firstly, decide that from how many digits the required number will be formed.

Step II Fill up the places on which restrictions are present and let the number of ways of filling up these places be k.

Step III Find the number of ways of filling the remaining places with remaining digits by using the formula ${}^nP_r$.

Step IV Required number of numbers is $k \cdot {}^nP_r$.

EXAMPLE |9| How many numbers between 400 and 1000 can be made with the digits 2, 3, 4, 5, 6 and 0, when repetition of digits is not allowed?

Sol. Given digits are, 2, 3, 4, 5, 6 and 0.

Any number between 400 and 1000 will be of three digits.

Since, the number should be greater than 400.

Therefore, hundred's place can be filled up by anyone of the digits 4, 5 and 6 in 3 ways.

$\therefore \quad k = 3$

Now, remaining two places can be filled up by the remaining five digits in ${}^5P_2$ ways.

Hence, required numbers $= 3 \times {}^5P_2 = 3 \times \frac{5!}{3!} = 60$

EXAMPLE |10| Determine the number of natural numbers smaller than 10^4, in the decimal notation of which all the digits are distinct.

Sol. The required natural numbers consist of number of 4-digit, 3-digit, 2-digit and 1-digit.

Clearly, total number of 4-digit natural numbers

$$= 9 \times {}^9P_3$$

[$\because$ in thousand place, 0 does not exist]

Total number of 3-digit natural numbers $= 9 \times {}^9P_2$

Total number of 2-digit natural numbers $= 9 \times {}^9P_1$

and total number of 1-digit natural numbers = 9

Hence, required number of natural numbers

$$= 9 \times {}^9P_3 + 9 \times {}^9P_2 + 9 \times {}^9P_1 + 9$$

$$= 9({}^9P_3 + {}^9P_2 + {}^9P_1 + 1)$$

$$= 9\left(\frac{9!}{6!} + \frac{9!}{7!} + \frac{9!}{8!} + 1\right)$$

$$= 9(9 \times 8 \times 7 + 9 \times 8 + 9 + 1)$$

$$= 9(504 + 72 + 10) = 9 \times 586 = 5274$$

EXAMPLE |11| How many different 4-digit numbers can be formed from the digits 2, 3, 4 and 6, if each digit is used only once in a number? Further, how many of these numbers

(i) end at 4? (ii) end at 3? (iii) end at 3 or 4?

Sol. The number of 4-digit numbers formed by the digits 2, 3, 4 and 6

$$= P(4, 4) = \frac{4!}{(4-4)!} = \frac{4!}{0!} \quad \left[\because {}^nP_r = \frac{n!}{(n-r)!}\right]$$

$$= \frac{4!}{1} = 4 \times 3 \times 2 \times 1 = 24 \quad [\because 0! = 1]$$

(i) In this case, '4' is fixed at the unit's place. Therefore, remaining 3-digits can be 2, 3 and 6. Hence, the required number of 4-digits number = $P(3, 3)$

$$= \frac{3!}{(3-3)!} \qquad \left[\because {}^nP_r = \frac{n!}{(n-r)!}\right]$$

$$= \frac{3!}{0!} = 3 \times 2 \times 1 = 6 \qquad \text{...(i)}$$

(ii) In this case, '3' is fixed at unit's place. Therefore, remaining 3-digits can be 2, 4 and 6. Hence, the required number of 4-digits numbers = $P(3, 3)$

$$= \frac{3!}{0!} = 6 \qquad \text{...(ii)}$$

(iii) The number of 4-digit numbers ending with either 3 or 4

$$= 6 + 6 = 12 \qquad \text{[from Eqs. (i) and (ii)]}$$

Restricted Permutation When all the Objects are not distinct

EXAMPLE |12| In how many different ways, the letters of the word 'ALGEBRA' can be arranged in a row, if

(i) the two A's are together?

(ii) the two A's are not together?

Sol. There are 7 letters in the word ALGEBRA. In these letters, 'A' occurs twice and rest all are different.

(i) Since, two A's are always together, so let us consider two A's as one letter.

$\boxed{\text{AA}}$, L, G, E, B, R

Now, we have 6 letters, which can be arranged in ${}^6P_6 = 6!$ ways

Hence, required number of ways = $6! = 720$

[$\because$ two A's can be arranged among themselves in one way only]

(ii) Required number of ways = Number of all possible arrangements of the letters of the word ALGEBRA − Number of arrangements in which two A's are together

$$= \frac{7!}{2!} - 6! = 6!\left(\frac{7}{2} - 1\right) = 720 \times \frac{5}{2} = 360 \times 5 = 1800$$

EXAMPLE |13| Find the number of different words that can be formed from the letters of the word 'INTERMEDIATE' such that two vowels never come together.

[NCERT Exemplar]

Sol. There are 12 letters in the word INTERMEDIATE.

In these letters, I occurs twice, T occurs twice, E occurs thrice and rest all are distinct. Also the word INTERMEDIATE contains 6 vowels I, I, E, E, E and A.

Since, two vowels never come together, therefore possible choices for vowels are the places marked with'×'.

× N× T× R× M× D× T×

Clearly, number of ways of arranging vowels

$$= {}^7P_6 \times \frac{1}{3!} \times \frac{1}{2!}$$

[$\because$ E occurs three times and I occurs two times]

and number of ways of arranging 6 consonants of which two are alike $= \frac{6!}{2!}$

Hence, required number of words

$$= {}^7P_6 \times \frac{1}{3!} \times \frac{1}{2!} \times \frac{6!}{2!}$$

$$= \frac{7! \times 6!}{3! \times 2! \times 2!} = 151200$$

EXAMPLE |14| How many different words can be formed by using all the letters of the word 'ALLAHABAD'? In how many of them, vowels occupy the even position?

Sol. There are 9 letters in the word ALLAHABAD. In these letters, A occurs 4 times, L occurs twice and rest all are different.

$\therefore$ Total number of words

$$= \frac{9!}{4!\,2!} = 7560$$

Now, consider the case, where vowels occupy the even position.

Since, the word ALLAHABAD contains 4 vowels, A, A, A and A, and there are 4 even places, therefore 4 vowels can be arranged in 4 even places.

This can be done in

$${}^4P_4 \times \frac{1}{4!} = 4! \times \frac{1}{4!} = 1 \text{ way.}$$

Now, we left with 5 places and 5 letters, which can be arranged in ${}^5P_5 \times \frac{1}{2!} = \frac{5!}{2!}$ ways.

[$\because$ out of 5 letters, 2 are alike]

Hence, the number of words in which vowels occupy the even position $= \frac{5!}{2!} = 60.$

EXAMPLE |15| How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2 and 1, so that odd digits always occupy the odd places?

Sol. The given digits are 1, 2, 3, 4, 3, 2, 1. Out of these, 1, 3, 3, 1 are odd digits.

Since, there are 4 odd places, therefore odd digits can occupy the odd places in ${}^4P_4 \times \frac{1}{2!} \times \frac{1}{2!} = \frac{4!}{2!\,2!} = 6$ ways.

Now, we left with 3 places in which remaining 3 digits (2, 2 and 1) can be arranged in ${}^3P_3 \times \frac{1}{2!} = \frac{3!}{2!} = 3$ ways

Hence, required number of numbers = $6 \times 3 = 18$

EXAMPLE |16| In how many ways can the letters of the word 'PERMUTATIONS' be arranged, if

(i) word start with P and end with S.

(ii) vowels are all together. **[NCERT]**

Sol. There are 12 letters in the word PERMUTATIONS.

In these letter T occurs twice and rest all are different.

Also, the word PERMUTATIONS contains all the five vowels, namely, A, E, I, O and U.

(i) Since, the word start with P and end with S, so let us fix P at first place and S at last place.

P	I	II	III	IV	V	VI	VII	VIII	IX	X	S

Now, we left with 10 letters (E, R, M, U, T, A, T, I, O, N) which can be arranged in $\frac{10!}{2!}$ ways.

[$\because$ T occurs twice]

Hence, required number of ways $= \frac{10!}{2!}$

$= 1814400$

(ii) Since, the vowels are occur together, so let us take all vowels together and consider it as one letter.

[A E I O U], P, R, M, T, T, N, S

Now, we have 8 letters, which can be arranged in $\frac{8!}{2!}$ ways.

Corresponding to each such arrangement, all the five vowels can be arranged among themselves in ${}^5P_5 = 5!$ ways.

Hence, required number of ways $= \frac{8!}{2!} \times 5!$

$= 2419200$

TOPIC PRACTICE 4

OBJECTIVE TYPE QUESTIONS

1 The number of 5-digit telephone numbers having atleast one of their digits repeated is

(a) 90000 (b) 10000
(c) 30240 (d) 69760

2 Fill in the blanks.

I. The number of permutations of n objects, where P objects are of the same kind and rest are all different equals ...A... .

II. The number or permutations of n objects, where P_1 objects are of one kind, P_2 are of second kind, P_k are of k^{th} kind and the rest, if any, are of different kind is ...B... .

Here, A and B refer to

(a) $\frac{n!}{P!\,2!}$, $\frac{n!}{(P_1 + P_2 + \ldots + P_k)!}$

(b) $\frac{n!}{P!}$, $\frac{n!}{P_1!\,P_2!\ldots P_k!}$

(c) $\frac{P!}{n!}$, $\frac{(P_1 + P_2 \ldots + P_k)}{n!}$

(d) None of the above

3 The sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time, is

(a) 93321 (b) 93322
(c) 93323 (d) 93324

4 The number of arrangements can be made with the letters of the word 'MATHEMATICS' in which all vowels are together, is

(a) 120920 (b) 120930
(c) 120940 (d) 120960

5 How many number lying between 999 and 10000 can be formed with the help of the digits 0, 2, 3, 6, 7, 8, when the digits are not be repeated?

(a) 100 (b) 200 (c) 300 (d) 400

VERY SHORT ANSWER Type Questions

6 In how many ways can the letters of the word 'GOLDEN' be arranged so that N and E are always together.

7 In how many ways can the letters of the word 'PENCIL' be arranged so that N is always next to E?

8 Find the number of different words that can be formed with the letters of the word ENGLISH so that the word begin with E and end with I?

SHORT ANSWER Type I Questions

9 How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 5, 6, 7, if no digits is repeated?

10 In how many ways 3 Mathematics books, 4 History books, 3 Chemistry books and 2 Biology books can be arranged on a shelf so that all books of the same subjects are together? **[NCERT Exemplar]**

11 Find the number of different words that can be formed from the letters of the word 'TRIANGLE' so that no vowels are together.

12 In how many ways can 5 girls and 3 boys be seated in a row so that no two boys are together? **[NCERT]**

13 Find the number of ways in which the letters of the word 'MACHINE' can be arranged so that the vowels occupy the odd places.

14 How many words can be formed out of the letters of the word 'ORIENTAL' so that the vowels always occupy the odd places?

15 How many different words can be formed with the letters of the word SUNDAY? How many of these begin with N? How many begin N and end in Y?

16 How many 4-digit numbers are there with no digit repeated? **[NCERT]**

17 In how many ways can the letters of the word 'MATHEMATICS' be arranged so that vowels are always together?

18 In how many ways can 7 plus (+) signs and 5 minus (–) signs be arranged in a row so that no two minus signs are together?

SHORT ANSWER Type II Questions

19 Find the number of different 8 letters arrangements that can be made from the letters of the word 'DAUGHTER' so that

(i) all vowels occur together.

(ii) all vowels do not occur together.

20 In how many ways can 10 examination papers be arranged so that the best and the worst papers are never together?

21 How many numbers lying between 100 and 1000 can be formed with the digits 0, 1, 2, 3, 4, 5, if the repetition of the digits is not allowed? **[NCERT]**

22 In how many of the distinct permutations of the letters in 'MISSISSIPPI' do the four I's not come together?

23 How many different words can be formed with the letters of the word 'HARYANA'? How many of these

(i) have H and N together?

(ii) begin with H and end with N?

(iii) have three vowels together?

LONG ANSWER Type Questions

24 Three married couples are to be seated in a row having six seats in a cinema halls. If spouses are to be seated next to each other, in how many ways can they be seated? Also, find the number of ways of their seating, if all the ladies sit together. **[NCERT Exemplar]**

25 Find the number of ways in which 5 boys and 5 girls be seated in a row, so that

(i) all the girls sit together.

(ii) all the girls and all the boys sit together.

(iii) all the girls are never sit together.

26 How many numbers greater than 1000000 can be formed by using the digits 2, 3, 0, 3, 4, 2, 3?

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1. (d) If all the digits repeated, then number of 5 digit telephone numbers can be formed in 10^5 ways and if no digit repeated, then 5-digit telephone numbers can be formed in ${}^{10}P_5$ ways.

$\therefore$ Required number of ways $= 10^5 - {}^{10}P_5 = 69760$

2. (b) I. The number of permutations of n objects, where P objects are of the same kind and rest are all different $= \dfrac{n!}{P!}$

II. The number of permutations of n objects, where P_1 objects are of one kind, P_2 are of second kind,...., P_k are of k^{th} kind and the rest if any, are of different kind, is $\dfrac{n!}{P_1!P_2!...P_k!}$

3. (d) Total number of numbers formed with the digits 2, 3, 4, 5 taken all at a time

= Number of arrangement of 4 digits, taken all at a time $= {}^4P_4 = 24$

Consider the digits in the unit's places in all these numbers. Each of the digit 2, 3, 4, 5 occurs in $3!(= 6)$ times in the unit's place.

So, total for the digits in the unit's place in all the numbers $= (2+3+4+5) \times 3! = 84$.

Since, each of the digits 2, 3, 4, 5 occurs 3! times in any one of the remaining places.

So, the sum of the digits in the ten's, hundred's, thousand's places in all the numbers

$$= (2+3+4+5) \times 3! = 84$$

$\therefore$ Sum of all the numbers $= 84\,(10^0 + 10^1 + 10^2 + 10^3)$

Ans. 93324

4. (d) Considering these four vowels as one letter we have 8 letters (M, T, H, M, T, C, S and one letter obtained by combining all vowels), out of which M occurs twice, T occurs twice and the rest all different. These 8 letters can be arranged in $\dfrac{8!}{2! \times 2!}$ ways. But the four vowel, (A, E, A, I) can be put together in $\dfrac{4!}{2!}$ ways.

Hence, total number of arrangements in which vowels are always together $= \dfrac{8!}{2! \times 2!} \times \dfrac{4!}{2!}$

Ans. 120960

5. (c) The number of 4-digit numbers formed by digits 0, 2, 3, 6, 7, 8 is ${}^6P_4 = 360$.

But here those numbers are also involved which begin from 0. So, we take those numbers as three-digit numbers.

Taking initial digit 0, the number of ways to fill remaining 3 places from five digits 2, 3, 6, 7, 8 are ${}^5P_3 = 60$.

So, the required numbers = 360 − 60

Ans. 300

6. Let us take E and N together and consider it as one letter. Further, solve as Example 2, part (i). **Ans.** 240

7. Here we consider E and N together as one unit.

$\therefore$ Required number of ways $= 5! = 120$ **Ans.** 120

8. Solve as Example 16 (i). **Ans.** 120

9. For even numbers, even number should be placed on unit (2, 4, 6)

$\therefore$ 3-digit distinct even number of ways $= 6 \times 5 \times 3$.

Ans. 90

10. Solve as Example 4. **Ans.** 41472

11. Firstly, fix the alternate position of consonant on C's position is $5! \times C_1 \times C_2 \times C_3 \times C_4 \times C_5 \times$

Now, in any six '×' position, 3 vowels can be arrange in ${}^6P_3$ ways. **Ans.** 14400

12. Solve as Example 4. **Ans.** 14400

13. Firstly, fix the odd position of vowels (A, I, E) 1, 2, 3, 4, 5, 6, 7 i.e. ${}^4P_3$ ways and sent of the 5 places can be arranged in 5! ways. **Ans.** 576

14. Solve as Example 6. **Ans.** 576

15. Solve as Example 16. part (i) **Ans.** 720, 120, 24

16. Solve as Example 11. **Ans.** 4536

17. Solve as Example 16, part (ii). **Ans.** 120960

18. Solve as Example 8. **Ans.** 56

19. Solve as Example 1. **Ans.** (i) 4320 (ii) 36000

20. Required number of ways = Number of all possible arrangement of 10 papers − Number of arrangement of paper in which best and worst occur together **Ans.** $8 \times 9!$

21. **Ans.** 100

22. Solve as Example 12 (ii).

Firstly find the number of permutations in which four I's come together

Required number of ways = Number of permutation of letters of the word MISSISSIPPI

− Number of permutation in which four I's come together

Ans. 33810

23. Solve as Example 16.

Ans. 840; (i) 240 (ii) 20 (iii) 120

24. Consider each married couple as one unit, then the three units can be arranged in $3! = 6$ ways.

Again each couple can be arranged in $2! = 2$ ways.

If the ladies sit together, then consider all ladies together as one unit. Now this single unit together with 3 remaining mens will be counted as 4 objects.

Ans. 48; 144

25. (i) Take all the girls together as one unit.

Ans. $6! \times 5!$

(ii) Take all the girls together as one unit and all the boys together as another unit. **Ans.** $2! \times 5! \times 5!$

(iii) Required number of ways = Number of all possible arrangements of 5 boys and 5 girls − Number of arrangement in which all girls sit together.

Ans. $10! - 5! \times 6!$

26. Since, 1000000 is a 7-digit number and the number of digits to be used is also 7. Therefore, the number to be counted will be 7-digit only.

Now, let us arrange the given seven digits. This can be done in $\frac{7!}{2!3!} = 420$ ways.

But these arrangements will include those also where 0 is at the million's place.

So, keeping 0 fixed at million's place, the remaining 6 digits can be arranged in $\frac{6!}{2!3!} = 60$ ways.

Hence, required number of numbers $= 420 - 60 = 360$.

|TOPIC 5|
Combinations

The word **combination** means selection.

Each of the different selection, which is made by taking some or all of a number of different objects at a time, irrespective of their arrangements is called a combination.

e.g. The different combinations formed from three letters A, B, C taking two at a time are AB, AC, BC.

Difference between Permutations and Combinations

The process of selecting objects is called combination and that of arranging objects is called permutation.

If we have 4 objects A, B, C and D, the possible selection (or combination) and arrangement (or permutation) of 3 objects out of 4 are given below. This will help you to understand clearly the difference between permutations and combinations.

Selection ↓ Combination	Arrangement ↓ Permutation
ABC	*ABC*, *ACB*, *BAC*, *BCA*, *CAB*, *CBA*
ABD	*ABD*, *ADB*, *BAD*, *BDA*, *DAB*, *DBA*
ACD	*ACD*, *ADC*, *CAD*, *CDA*, *DAC*, *DCA*
BCD	*BCD*, *BDC*, *CBD*, *CDB*, *DBC*, *DCB*
Total 4 combinations	24 permutations

MEANING OF ${}^nC_r$

The number of combinations of n different things taken r at a time is meant the number of groups of r things which can be formed from n things and generally it is denoted by symbol ${}^nC_r$ or $C(n, r)$.

COMBINATIONS OF n DIFFERENT THINGS TAKEN r AT A TIME

The number of combinations of n distinct objects taken r at a time is given by

$$^nC_r = \frac{n!}{r!(n-r)!}$$

Proof Let $\quad {}^nC_r = x \qquad \text{...(i)}$

Each one of these x combinations contains r things and these r things can be arranged among themselves in $r!$ ways. Hence, one combination gives $r!$ permutations.

So, x combinations will give rise to $(x \cdot r!)$ permutations. But the number of permutations of n things taken r at a time is $\frac{n!}{(n-r)!}$.

$$\therefore \quad x \cdot r! = \frac{n!}{(n-r)!} \Rightarrow x = \frac{n!}{r!(n-r)!}$$

$$\Rightarrow \quad {}^nC_r = \frac{n!}{r!(n-r)!} \qquad \text{[from Eq. (i)]}$$

$$\therefore \quad {}^nC_r = \frac{n!}{r!(n-r)!};\ 1 \le r \le n$$

In particular,

(*i*) When $r = 0$, then ${}^nC_0 = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \times n!} = 1$

${}^nC_0 = 1$, can be interpreted as the number of combinations of n different things taken nothing at all is considered to be 1.

Selecting nothing at all is the same as leaving behind all the objects and there is only one way of doing so. In this way, we define

$${}^nC_0 = 1$$

(*ii*) When $r = n$, then

$${}^nC_n = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = \frac{1}{0!} = 1$$

$$\therefore \quad {}^nC_0 = {}^nC_n = 1$$

It means number of combination taking all the objects at a time is always 1.

EXAMPLE |1| Evaluate the following.

(i) ${}^{14}C_3$ (ii) ${}^{100}C_{99}$ (iii) ${}^{35}C_{35}$ (iv) ${}^{30}C_0$

Sol. (i) ${}^{14}C_3 = \frac{14!}{3!(14-3)!} \qquad \left[\because {}^nC_r = \frac{n!}{r!(n-r)!}\right]$

$$= \frac{14!}{3! \times 11!} = \frac{14 \times 13 \times 12 \times 11!}{(3 \times 2 \times 1) \times 11!}$$

$$= \frac{14 \times 13 \times 12}{6} = 14 \times 13 \times 2 = 364$$

(ii) ${}^{100}C_{99} = \frac{100!}{99!(100-99)!} = \frac{100!}{99! \times 1!} = \frac{100 \times 99!}{99! \times 1} = 100$

(iii) ${}^{35}C_{35} = 1 \qquad [\because {}^nC_n = 1]$

(iv) ${}^{30}C_0 = 1 \qquad [\because {}^nC_0 = 1]$

EXAMPLE |2| If ${}^nP_r = 840$ and ${}^nC_r = 35$, find r. **[NCERT Exemplar]**

Sol. We know that, $\quad {}^nP_r = r! \times {}^nC_r$

$$\therefore \quad 840 = r! \times 35$$

$$\Rightarrow \quad r! = \frac{840}{35}$$

$$\Rightarrow \quad r! = 24 = 4!$$

$$\Rightarrow \quad r = 4$$

EXAMPLE |3| If the ratio ${}^{2n}C_3 : {}^nC_3$ is equal to $11 : 1$, find the value of n. **[NCERT]**

Sol. We have, $\quad {}^{2n}C_3 : {}^nC_3 = 11 : 1$

$$\Rightarrow \quad \frac{{}^{2n}C_3}{{}^nC_3} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{\frac{2n(2n-1)(2n-2)}{3\cdot 2\cdot 1}}{\frac{n(n-1)(n-2)}{3\cdot 2\cdot 1}} = \frac{11}{1}$$

$$\Rightarrow \frac{2n(2n-1)2(n-1)}{3\cdot 2\cdot 1} \times \frac{3\cdot 2\cdot 1}{n(n-1)(n-2)} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{4(2n-1)}{n-2} = \frac{11}{1}$$

$$\Rightarrow \quad 8n - 4 = 11n - 22$$

$$\Rightarrow \quad 18 = 3n$$

$$\Rightarrow \quad n = 6$$

Hence, the value of n is 6.

EXAMPLE |4| If ${}^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$, then find the value of n.

Sol. We have, ${}^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$

$$\Rightarrow \frac{{}^{n+2}C_8}{{}^{n-2}P_4} = \frac{57}{16} \Rightarrow \frac{\frac{(n+2)!}{8!(n+2-8)!}}{\frac{(n-2)!}{(n-2-4)!}} = \frac{57}{16}$$

$$\Rightarrow \quad \frac{(n+2)!}{8!(n-6)!} \times \frac{(n-6)!}{(n-2)!} = \frac{57}{16}$$

$$\Rightarrow \frac{(n+2)(n+1)n(n-1)(n-2)!}{8!} \times \frac{1}{(n-2)!} = \frac{57}{16}$$

$$\Rightarrow (n+2)(n+1)n(n-1) = \frac{57}{16} \times 8!$$

$$= \frac{19 \times 3}{16} \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$\Rightarrow (n-1)n(n+1)(n+2) = 19 \times 3 \times 7 \times 6 \times 5 \times 4 \times 3$$

$$\Rightarrow (n-1)n(n+1)(n+2) = 19 \times (3 \times 7) \times (6 \times 3) \times (5 \times 4)$$

$$\Rightarrow (n-1)n(n+1)(n+2) = 18 \times 19 \times 20 \times 21$$

On comparing both sides, we get

$$n - 1 = 18 \Rightarrow n = 19$$

EXAMPLE |5| Prove that $r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}$.

Sol. LHS $= r \cdot {}^nC_r = r \cdot \dfrac{n!}{r!(n-r)!}$ $\qquad \left[\because {}^nC_r = \dfrac{n!}{r!(n-r)!}\right]$

$$= \frac{r \cdot n!}{r(r-1)!(n-r)!} \qquad [\because r! = r(r-1)!]$$

$$= \frac{n!}{(r-1)!(n-r)!} \qquad \text{...(i)}$$

RHS $= n \cdot {}^{n-1}C_{r-1} = n \cdot \dfrac{(n-1)!}{(r-1)!\{(n-1)-(r-1)\}!}$

$$= \frac{n\cdot(n-1)!}{(r-1)!(n-r)!} = \frac{n!}{(r-1)!(n-r)!}$$

$$[\because n(n-1)! = n!] \text{ ...(ii)}$$

From Eqs. (i) and (ii), we get

LHS = RHS

$$\Rightarrow \quad r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}$$ **Hence proved.**

EXAMPLE |6| If ${}^{n-1}C_r : {}^nC_r : {}^{n+1}C_r = 6 : 9 : 13$, then find the values of n and r.

Sol. We have, $\quad \dfrac{{}^{n-1}C_r}{{}^nC_r} = \dfrac{6}{9}$

$$\Rightarrow \quad \frac{\frac{(n-1)!}{r!(n-1-r)!}}{\frac{n!}{r!(n-r)!}} = \frac{2}{3}$$

$$\Rightarrow \quad \frac{(n-1)!}{(n-(r+1))!} \times \frac{(n-r)!}{n!} = \frac{2}{3}$$

$$\Rightarrow \quad \frac{(n-1)!}{(n-(r+1))!} \times \frac{(n-r)(n-(r+1))!}{n(n-1)!} = \frac{2}{3}$$

$$\Rightarrow \quad \frac{n-r}{n} = \frac{2}{3}$$

$$\Rightarrow \quad 3n - 3r = 2n$$

$$\Rightarrow \quad n - 3r = 0 \qquad \text{...(i)}$$

Also, we have $\quad \dfrac{{}^nC_r}{{}^{n+1}C_r} = \dfrac{9}{13}$

$$\Rightarrow \quad \frac{n!}{r!(n-r)!} \times \frac{r!(n+1-r)!}{(n+1)!} = \frac{9}{13}$$

$$\Rightarrow \quad \frac{n!}{(n-r)!} \times \frac{(n-(r-1))!}{(n+1)n!} = \frac{9}{13}$$

$$\Rightarrow \quad \frac{(n-(r-1))(n-r)!}{(n-r)!(n+1)} = \frac{9}{13}$$

$$\Rightarrow \quad \frac{n-r+1}{n+1} = \frac{9}{13}$$

$$\Rightarrow \quad 13n - 13r + 13 = 9n + 9$$

$$\Rightarrow \quad 4n - 13r = -4 \qquad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$n = 12 \text{ and } r = 4$$

Some Important Theorems

Theorem 1 $\quad {}^nP_r = {}^nC_r \cdot r!,\ 0 < r \le n$

Proof The number of combinations of n distinct objects taken r at a time is ${}^nC_r$. In these combinations, r things can be arranged among themselves in $r!$ ways. So, we have $r!$ permutations.

Therefore, the total number of permutations of n different things taken r at a time is ${}^nC_r \times r!$, which is equal to ${}^nP_r$.

$\therefore \quad {}^nP_r = r! \times {}^nC_r,\ 0 < r \le n$ **Hence proved.**

Theorem 2 $\quad {}^nC_r = {}^nC_{n-r},\ 0 \le r \le n$

Note This theorem is used to simplify the calculation when r is large.

Theorem 3 $\quad {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Proof

$$\text{LHS} = {}^nC_r + {}^nC_{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!(n-r+1)}{r!(n-r+1)(n-r)!} + \frac{n!r}{r(r-1)!(n-r+1)!}$$

$$= \frac{n!(n-r+1)}{r!(n-r+1)!} + \frac{n!r}{r!(n-r+1)!} = n!\left[\frac{n-r+1+r}{r!(n-r+1)!}\right]$$

$$= \frac{(n+1)n!}{r!(n+1-r)!} = \frac{(n+1)!}{r!(n+1-r)!} = {}^{n+1}C_r = \text{RHS}$$

Hence proved.

Theorem 4 *If* ${}^nC_x = {}^nC_y$, *then either* $x = y$ *or* $x + y = n$.

EXAMPLE |7| Evaluate ${}^{13}C_6 + {}^{13}C_5$.

Use the formula, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ and simplify it.

Sol. We have, ${}^{13}C_6 + {}^{13}C_5 = {}^{14}C_6 \quad [\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

$$= \frac{14!}{6!(14-6)!} \qquad \left[\because {}^nC_r = \frac{n!}{r!(n-r)!}\right]$$

$$= \frac{14!}{6! \times 8!} = \frac{14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8!}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 8!}$$

$$= 7 \times 13 \times 11 \times 3 = 3003$$

EXAMPLE |8| Evaluate $\sum_{r=1}^{5} {}^5C_r$.

Sol. Consider, $\sum_{r=1}^{5} {}^5C_r = {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$

$$= {}^5C_1 + {}^5C_2 + {}^5C_2 + {}^5C_1 + {}^5C_0 \quad [\because {}^nC_r = {}^nC_{n-r}]$$

$$= 5 + 2 \times {}^5C_2 + 5 + 1 \quad [\because {}^nC_1 = n \text{ and } {}^nC_0 = 1]$$

$$= 11 + 2 \times \frac{5 \times 4}{2 \times 1} = 11 + 20 = 31$$

EXAMPLE |9| If ${}^nC_{10} = {}^nC_{12}$, then find the value of ${}^{23}C_n$.

Sol. We know that, ${}^nC_x = {}^nC_y$

$\Rightarrow \quad x + y = n$ or $x = y$

Here $x \ne y$, so $x + y = n$.

$\Rightarrow \quad n = 10 + 12 = 22$

Now, ${}^{23}C_n = {}^{23}C_{22} = {}^{23}C_{23-22} = {}^{23}C_1 = 23$

EXAMPLE |10| If ${}^{16}C_r = {}^{16}C_{r+2}$, find ${}^rC_4$.

Sol. We know that, ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

$\therefore \quad r + (r+2) = 16 \quad [\because r \ne r+2]$

$\Rightarrow \quad 2r + 2 = 16$

$\Rightarrow \quad 2r = 14 \Rightarrow r = 7$

Hence, ${}^7C_4 = {}^7C_3 = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$

EXAMPLE |11| If ${}^8C_r - {}^7C_3 = {}^7C_2$, find r.

Sol. We have, ${}^8C_r - {}^7C_3 = {}^7C_2$

$\Rightarrow \quad {}^8C_r = {}^7C_3 + {}^7C_2$

$\Rightarrow \quad {}^8C_r = {}^8C_3 \quad [\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

$\Rightarrow \quad r = 3$ or $r + 3 = 8$

$[\because {}^nC_x = {}^nC_y \Rightarrow x = y \text{ or } x + y = n]$

$\Rightarrow \quad r = 3$ or $r = 5$

EXAMPLE |12| Evaluate ${}^nC_r + 2 \cdot {}^nC_{r-1} + {}^nC_{r-2}$.

Sol. Consider, ${}^nC_r + 2 \cdot {}^nC_{r-1} + {}^nC_{r-2}$

$$= {}^nC_r + {}^nC_{r-1} + {}^nC_{r-1} + {}^nC_{r-2}$$

$$= ({}^nC_r + {}^nC_{r-1}) + ({}^nC_{r-1} + {}^nC_{r-2})$$

$$= {}^{n+1}C_r + {}^{n+1}C_{r-1} = {}^{n+2}C_r$$

$[\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

EXAMPLE |13| Find the value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$.

Sol. We have, ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$

$$= {}^{47}C_4 + {}^{52-1}C_3 + {}^{52-2}C_3 + {}^{52-3}C_3 + {}^{52-4}C_3 + {}^{52-5}C_3$$

$$= {}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3$$

$$= ({}^{47}C_3 + {}^{47}C_4) + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$$

$$= {}^{48}C_4 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$$

$[\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]$

$$= {}^{49}C_4 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 = {}^{50}C_4 + {}^{50}C_3 + {}^{51}C_3$$

$$= {}^{51}C_4 + {}^{51}C_3 = {}^{52}C_4$$

$$= \frac{52 \times 51 \times 50 \times 49}{4 \times 3 \times 2 \times 1} = 13 \times 17 \times 25 \times 49 = 270725$$

TOPIC PRACTICE 5

OBJECTIVE TYPE QUESTIONS

1 The number of ways a team of 2 players can be formed out of a group of 3 lawn tennis players X, Y, Z is

(a) 7 (b) 6 (c) 3 (d) 4

2 If there are 5 different objects A, B, C, D, E. The number of combinations of 3 different objects is ...A... . Here, A refers to

(a) 9 (b) 12 (c) 11 (d) 10

3 If ${}^nC_{12} = {}^nC_8$, then n is equal to

(a) 20 (b) 12 (c) 6 (d) 30

4 If $n = {}^mC_2$, then the value of ${}^nC_2$ is

(a) ${}^{m+1}C_4$ (b) ${}^{m-1}C_4$ (c) ${}^{m+2}C_4$ (d) $3 \times {}^{m+1}C_4$

5 ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7$ is equal to ....

(a) 0 (b) 2 (c) 1 (d) 3

VERY SHORT ANSWER Type Questions

6 Evaluate the following.

(i) ${}^{10}C_8$ (ii) ${}^{50}C_{50}$

7 If ${}^nP_r = 720$ and ${}^nC_r = 120$, find r.

8 Verify that $2 \times {}^7C_4 = {}^8C_4$.

9 If ${}^nC_9 = {}^nC_8$, find ${}^nC_{17}$.

10 If ${}^{24}C_x = {}^{24}C_{2x+3}$, find x.

11 Evaluate ${}^{10}C_4 + {}^{10}C_5$.

SHORT ANSWER Type I Questions

12 If $\alpha = {}^mC_2$, then find the value of ${}^{\alpha}C_2$.

13 Prove that ${}^nC_r \times {}^rC_s = {}^nC_s \times {}^{n-s}C_{r-s}$

14 Evaluate ${}^{31}C_{26} - {}^{30}C_{26}$.

15 Evaluate ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7$.

SHORT ANSWER Type II Questions

16 Determine n, if ${}^{2n}C_3 : {}^nC_3 = 12:1$. **[NCERT]**

17 If ${}^{28}C_{2r} : {}^{24}C_{2r-4} = 225:11$, find r.

18 Prove that ${}^{2n}C_n = \dfrac{2^n\{1 \cdot 3 \cdot 5 \ldots (2n-1)\}}{n!}$.

LONG ANSWER Type Question

19 Evaluate ${}^{20}C_5 + \sum_{r=1}^{5} {}^{25-r}C_4$.

20 If ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$, then find ${}^rC_2$. **[NCERT Exemplar]**

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1. (c) There are only 3 possible ways in which the team could be constructed.
These are XY, YZ and ZX.

2. (d) We have 5 different objects A, B, C, D and E. Taking 3 at a time, the combinations will be

ABC, ABD, ABE, BCD
BCE, CDE, ACE, ACD
ADE and BDE.

3. (a) Given that, ${}^nC_{12} = {}^nC_8$

$\Rightarrow {}^nC_{n-12} = {}^nC_8$ $[\because {}^nC_r = {}^nC_{n-r}]$

Ans. 20

4. (d) We have, $n = {}^mC_2 = \dfrac{m(m-1)}{2}$

$$\therefore \quad {}^nC_2 = \frac{n(n-1)}{2} = \frac{m(m-1)}{4}\left\{\frac{m(m-1)}{2} - 1\right\}$$
$$= 3\left\{\frac{1}{24}(m+1)\,m\,(m-1)(m-2)\right\}$$

Ans. $3 \times {}^{m+1}C_4$

5. (a) ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7$

$= {}^{15}C_{15-8} + {}^{15}C_{15-9} - {}^{15}C_6 - {}^{15}C_7$ $[\because {}^nC_r = {}^nC_{n-r}]$

$= {}^{15}C_7 + {}^{15}C_6 - {}^{15}C_6 - {}^{15}C_7$

Ans. 0

6. Solve as Example 1. **Ans.** (i) 45 (ii) 1

7. Solve as Example 2. **Ans.** $r = 3$

8. Evaluate LHS and RHS separately and verify the result.

9. Solve as Example 9. **Ans.** 1

10. Solve as Example 9. **Ans.** 7

11. Solve as Example 7. **Ans.** 462

12. $\alpha = \dfrac{m(m-1)}{2}$ and ${}^{\alpha}C_2 = \dfrac{\alpha(\alpha-1)}{2}$

$$\Rightarrow \quad {}^{\alpha}C_2 = \frac{\frac{m(m-1)}{2}\left(\frac{m(m-1)}{2} - 1\right)}{2}$$

Ans. $\dfrac{(m+1)m(m-1)(m-2)}{8}$

13. Simplify LHS and RHS separately and get the result.

14. $^{31}C_{26} - {}^{30}C_{26} = {}^{31}C_5 - {}^{30}C_4$

$$= \left[\left(\frac{31\times 30\times 29\times 28\times 27}{5\times 4\times 3\times 2\times 1}\right) - \left(\frac{30\times 29\times 28\times 27}{4\times 3\times 2\times 1}\right)\right]$$

Ans. 142506

15. Given expression = $^{16}C_9 - {}^{16}C_7 = {}^{16}C_7 - {}^{16}C_7$ **Ans.** 0

16. Solve as Example 3. **Ans.** 5

17. Use $^nC_r = \dfrac{n!}{r!(n-r)!}$ and simplify. **Ans.** 7

18. $^{2n}C_n = \dfrac{(2n)!}{n!\,n!} = \dfrac{(2n)(2n-1)(2n-2)\ldots 4\cdot 3\cdot 2\cdot 1}{n!\,n!}$

$$= \frac{\{(2n-1)(2n-3)\ldots 3\cdot 1\}\{(2n)(2n-2)\ldots 4\cdot 2\}}{n!\cdot n!}$$

$$= \frac{\{(2n-1)(2n-3)\ldots 3\cdot 1\}2^n\{n(n-1)\ldots 2\cdot 1\}}{n!\,n!}$$

$$= \frac{2^n\{(2n-1)(2n-3)\ldots 3\cdot 1\}\times n!}{n!\,n!}$$

19. Solve as Example 13. **Ans.** 42504

20. Now solving $\dfrac{^nC_{r-1}}{^nC_r} = \dfrac{36}{84}$

and $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{84}{126}$, we get

$$r = 3$$

$\therefore$ $^rC_2 = {}^3C_2 = 3$ **Ans.** 3

|TOPIC 6|

Practical Problems on Combinations

Here, we will discuss some practical problems, where the formula for ^nC_r and its meaning can be applied.

|TYPE I|

SIMPLY BASED ON THE FORMULA OF ^nC_r

EXAMPLE |1| In how many ways, can 5 sportsmen be selected from a group of 10?

Here, order of selection is immaterial. So, we use the concept of combination.

Sol. Required number of ways = $^{10}C_5$

$$= \frac{10\times 9\times 8\times 7\times 6}{5\times 4\times 3\times 2\times 1} = 2\times 3\times 7\times 6 = 252$$

EXAMPLE |2| If there are 15 persons in a party and if each two of them shake hands with each other. How many hand-shakes happen in the party?

Note that when two person shake hands, then it is counted as one hand-shake, not two, i.e. order of hand-shake is immaterial. So, we use the concept of combination.

Sol. The total number of hand-shakes is same as the number of ways of selecting 2 persons among 15 persons = $^{15}C_2$.

$$= \frac{15\times 14}{2\times 1} = 15\times 7 = 105$$

EXAMPLE |3| Everybody in a room shakes hands with everybody else. The total number of hand-shakes is 66. Find the total number of persons in the room. **[NCERT Exemplar]**

Sol. Let total number of persons be n.

Since, total number of hand-shakes = 66

$\therefore$ $^nC_2 = 66 \Rightarrow \dfrac{n(n-1)}{2} = 66$

$\Rightarrow$ $n^2 - n - 132 = 0 \Rightarrow (n-12)(n+11) = 0$

$\therefore$ $n = 12$ [$\because$ n cannot be negative]

|TYPE II|

FINDING NUMBER OF COMBINATIONS WHEN SOME PARTICULAR THINGS ARE INCLUDED OR EXCLUDED

EXAMPLE |4| In how many ways, can a cricket team of 11 players be selected out of 16 players,

(i) if two particular players are always be included?

(ii) if two particular players are always be excluded?

Sol. (i) Since, two particular players are always included, so we have to select 9 players out of 14 players. This can be done in $^{14}C_9$ ways.

Hence, required number of ways = $^{14}C_9 = {}^{14}C_5$ [$\because {}^nC_r = {}^nC_{n-r}$]

$$= \frac{14\times 13\times 12\times 11\times 10}{5\times 4\times 3\times 2\times 1}$$

$$= 14\times 13\times 11 = 2002$$

(ii) Since, two particular players are always excluded, so we have to select 11 players out of 14 players. This can be done in $^{14}C_{11}$ ways.

Hence, required number of ways = $^{14}C_{11} = {}^{14}C_3$ [$\because {}^nC_r = {}^nC_{n-r}$]

$$= \frac{14\times 13\times 12}{3\times 2\times 1} = 14\times 13\times 2 = 364$$

EXAMPLE |5| From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can the excursion party be chosen? **[NCERT]**

Sol. Here, we have to choose 10 students out of a class of 25 students, such that the given condition satisfied.

Let us make the following cases and find the number of possible choices in each case.

***Case* I When 3 particular students join the party**

In this case, we have to choose 7 students out of 22 students. This can be done in $^{22}C_7$ ways.

***Case* II When 3 particular students does not join the party** In this case, we have to choose 10 students out of 22 students. This can be done in $^{22}C_{10}$ ways.

Hence, total number of ways in which excursion party can be chosen $= {}^{22}C_7 + {}^{22}C_{10}$.

EXAMPLE |6| We wish to select 6 persons from 8 but if the person A is chosen, then B must be chosen. In how many ways, can the selection be made?

Sol. Let us make the following cases and find the number of possible selection in each case.

***Case* I When A is chosen**

In this case B must be chosen also. So we have to choose 4 persons out of 6 persons. This can be done in

$$^6C_4 = {}^6C_2 = \frac{6\times 5}{2\times 1} = 15 \text{ ways.}$$

***Case* II When A is not chosen**

In this case, we have to choose 6 persons out of 7 persons. This can be done in $^7C_6 = {}^7C_1 = 7$ ways.

Hence, total number of ways of selecting 6 persons

$$= 15 + 7 = 22$$

|TYPE III|

FINDING NUMBER OF COMBINATION WHEN GIVEN NUMBER OF THINGS ARE TO BE SELECTED FROM DIFFERENT GROUPS OF THINGS

EXAMPLE |7| In how many ways, can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls? **[NCERT]**

Sol. There are 5 boys and 4 girls. We have to select 3 boys out of 5 boys and 3 girls out of 4 girls.

Clearly, 3 boys out of 5 boys can be selected in $^5C_3$ ways, and 3 girls out of 4 girls can be selected in $^4C_3$ ways.

Hence, by fundamental principle of multiplication, number of ways of selecting 3 boys and 3 girls

$$= {}^5C_3 \times {}^4C_3$$

$$= {}^5C_2 \times {}^4C_1 \qquad [\because {}^nC_r = {}^nC_{n-r}]$$

$$= \frac{5\times 4}{2\times 1} \times 4 \qquad [\because {}^nC_1 = n]$$

$$= 10 \times 4 = 40$$

EXAMPLE |8| In an examination, Yamini has to select 4 questions from each part. There are 6, 7 and 8 questions in part I, part II and part III, respectively. What is the number of possible combinations in which she can choose the questions?

Sol.

Name of parts and number of questions in the parts	Number of questions selected by Yamini
Part I = 6	4
Part II = 7	4
Part III = 8	4

Clearly, total number of ways of selection

= Number of ways of selecting 4 questions from 6 questions × Number of ways of selecting 4 questions from 7 questions × Number of ways of selecting 4 questions from 8 questions

$$= {}^6C_4 \times {}^7C_4 \times {}^8C_4 \qquad [\because {}^nC_r = {}^nC_{n-r}]$$

$$= {}^6C_2 \times {}^7C_3 \times {}^8C_4$$

$$= \frac{6\times 5}{2\times 1} \times \frac{7\times 6\times 5}{3\times 2\times 1} \times \frac{8\times 7\times 6\times 5}{4\times 3\times 2\times 1}$$

$$= 15 \times 35 \times 70 = 36750 \text{ ways}$$

Hence, the total number of possible combination in which she can choose the questions is 36750 ways.

EXAMPLE |9| Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls, if each selection consists of 3 balls of each colour. **[NCERT]**

Sol. The required number of ways

= Number of ways of selecting 3 balls from 6 red balls × Number of ways of selecting 3 balls from 5 white balls × Number of ways of selecting 3 balls from 5 blue balls

$$= {}^6C_3 \times {}^5C_3 \times {}^5C_3 = {}^6C_3 \times {}^5C_2 \times {}^5C_2$$

$$= \frac{6\times 5\times 4}{3\times 2\times 1} \times \frac{5\times 4}{2\times 1} \times \frac{5\times 4}{2\times 1}$$

$$= 20 \times 10 \times 10 = 2000$$

Hence, the number of ways of selecting 3 balls of each colour is 2000.

EXAMPLE |10| There are 10 professors and 20 students out of whom a committee of 2 professors and 3 students is to be formed. Find the number of ways in which this can be done. Further find in how many of these committees

(i) a particular professors is included?

(ii) a particular student is included?

(iii) a particular student is excluded?

Sol. Clearly, for forming a committee of 2 professors and 3 students, we have to select 2 professors out of 10 professors and 3 students out of 20 students.

This can be done in $^{10}C_2 \times {}^{20}C_3$ ways.

Thus, required number of ways

$$= \frac{10 \times 9}{2 \times 1} \times \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 45 \times 60 \times 19 = 51300$$

(i) When a particular professor is included, then we have to select 1 professor out of 9 professors and 3 students out of 20 students.

This can be done in $^9C_1 \times {}^{20}C_3$ ways.

Thus, required number of ways

$$= 9 \times \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 9 \times 60 \times 19 = 10260$$

(ii) When a particular students is included, then we have to select 2 professors out of 10 professors and 2 students out of 19 students.

This can be done in $^{10}C_2 \times {}^{19}C_2$.

Thus, required number of ways

$$= \frac{10 \times 9}{2 \times 1} \times \frac{19 \times 18}{2 \times 1} = 45 \times 171 = 7695$$

(iii) When a particular is excluded, then we have to select 2 professors out of 10 professors and 3 students out of 19 students. This can be done in $^{10}C_2 \times {}^{19}C_3$ ways.

Thus, required number of ways

$$= \frac{10 \times 9}{2 \times 1} \times \frac{19 \times 18 \times 17}{3 \times 2 \times 1}$$

$$= 45 \times 969 = 43605$$

EXAMPLE |11| A bag contains six white marbles and five red marbles. Find the number of ways in which four marbles can be drawn from the bag, if

(i) they can be of any colour.

(ii) two must be white and two red.

(iii) they must all be of the same colour. **[NCERT Exemplar]**

Sol. Total number of marbles

$$= 6 \text{ white} + 5 \text{ red } = 11 \text{ marbles}$$

(i) Since, the 4 marbles can be of any colour, therefore we have to select 4 marbles out of 11 marbles.

This can be done in $^{11}C_4$ ways.

Hence, required number of ways = 330 ways.

(ii) Since out of four marbles two are white and two are red, therefore we have to select 2 white marbles out of 6 white marbles and 2 red marbles out of 5 red marbles.

This can be done in $^6C_2 \times {}^5C_2$ ways.

Hence, required number of ways $= 15 \times 10 = 150$

(iii) Since, all the 4 marbles must be of same colour, therefore we have to select 4 marbles of white colour or red colour.

Clearly, number of ways of selecting 4 white marbles out of 6 white marbles $= {}^6C_4 = {}^6C_2 = 15$

and number of ways of selecting 4 red marbles out of 5 red marbles $= {}^5C_4 = {}^5C_1 = 5$

Hence, required number of ways = 15 + 5 = 20 ways.

| TYPE IV |

FINDING NUMBER OF COMBINATIONS WHEN SELECTION OF THINGS INVOLVES GIVEN UNDER CONDITIONS

EXAMPLE |12| A committee of 7 members has to be formed from 9 boys and 4 girls. In how many ways, can this be done when the committee consists of

(i) exactly 3 girls?

(ii) atleast 3 girls?

(iii) atmost 3 girls? **[NCERT]**

Sol. There are 9 boys and 4 girls. We have to form a committee of 7 members.

(i) Since, the committee has exactly 3 girls. Therefore, we have to choose 3 girls out of 4 girls and the rest 4 members out of 9 boys.

This can be done in $^4C_3 \times {}^9C_4$ ways.

Hence, number of ways of forming a committee

$$= {}^4C_3 \times {}^9C_4 = {}^4C_1 \times \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1}$$

$$= 4 \times 9 \times 7 \times 2 = 504 \text{ ways}$$

(ii) Since the committee has at least 3 girls. Therefore, we can form the committee in the following ways

(a) Selecting 3 girls and 4 boys

(b) Selecting 4 girls and 3 boys

Hence, the number of ways of forming a committee

$$= {}^4C_3 \times {}^9C_4 + {}^4C_4 \times {}^9C_3$$

$$= 504 + 1 \times \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 504 + 84 = 588 \text{ ways}$$

(iii) Since, the committee has atmost 3 girls, therefore we can form committee in the following ways

(a) Selecting 7 boys and no girl

(b) Selecting 6 boys and 1 girl

(c) Selecting 5 boys and 2 girls

(d) Selecting 4 boys and 3 girls

Hence, the number of ways of forming a committee

$$= {}^9C_7 \times {}^4C_0 + {}^9C_6 \times {}^4C_1 + {}^9C_5 \times {}^4C_2 + {}^9C_4 \times {}^4C_3$$

$$= {}^9C_2 + {}^9C_3 \times 4 + {}^9C_4 \times {}^4C_2 + {}^9C_4 \times {}^4C_1$$

$$[\because {}^nC_r = {}^nC_{n-r}, {}^nC_1 = n \text{ and } {}^nC_0 = 1]$$

$$= 36 + 84 \times 4 + 126 \times 6 + 126 \times 4$$

$$= 36 + 336 + 756 + 504 = 1632$$

EXAMPLE |13| A committee of 10 is to be formed from 8 gentlemen and 9 ladies. In how many ways this can be done if atleast five ladies have to be included in a committee? In how many of these committees

(i) the ladies are in majority?

(ii) the gentlemen are in majority?

Sol. There are 9 ladies and 8 gentlemen. We have to form a committee of 10, consisting of atleast 5 ladies.

This can be formed by selecting

(a) 5 ladies and 5 gentlemen (b) 6 ladies and 4 gentlemen

(c) 7 ladies and 3 gentlemen (d) 8 ladies and 2 gentlemen

(e) 9 ladies and 1 gentleman

Now, the number of ways of forming a committee

$$= {}^9C_5 \times {}^8C_5 + {}^9C_6 \times {}^8C_4 + {}^9C_7 \times {}^8C_3 + {}^9C_8 \times {}^8C_2 + {}^9C_9 \times {}^8C_1$$

$$= {}^9C_4 \times {}^8C_3 + {}^9C_3 \times {}^8C_4 + {}^9C_2 \times {}^8C_3 + {}^9C_1 \times {}^8C_2 + {}^9C_0 \times {}^8C_1 \quad [\because {}^nC_r = {}^nC_{n-r}]$$

$$= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \times \frac{8 \times 7 \times 6}{3 \times 2 \times 1} + \frac{9 \times 8 \times 7}{3 \times 2 \times 1} \times \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} + \frac{9 \times 8}{2 \times 1} \times \frac{8 \times 7 \times 6}{3 \times 2 \times 1} + 9 \times \frac{8 \times 7}{2 \times 1} + 1 \times 8$$

$$= 126 \times 56 + 84 \times 70 + 36 \times 56 + 9 \times 28 + 8$$

$$= 7056 + 5880 + 2016 + 252 + 8 = 15212$$

(i) Clearly, ladies are in majority in (b), (c), (d) and (e) cases as discussed above.

$\therefore$ Number of committees in which ladies are in majority $= 15212 - {}^9C_5 \times {}^8C_5$

$$= 15212 - 7056 = 8156$$

(ii) Clearly, gentlemen are not in majority in the cases discussed above.

Thus, there is no committee in which gentlemen are in majority. Hence, number of such committee = 0.

EXAMPLE |14| Determine the number of 5 cards combinations out of a deck of 52 cards, if atleast one of the 5 cards has to be a king.

Sol. Selection of 5 cards can be done in the following ways

Case	I	II	III	IV
King	1	2	3	4
Other cards	4	3	2	1

The number of combinations in case I, II, III and IV are respectively

${}^4C_1 \times {}^{48}C_4$; ${}^4C_2 \times {}^{48}C_3$; ${}^4C_3 \times {}^{48}C_2$ and ${}^4C_4 \times {}^{48}C_1$

Hence, required number of combinations

$$= {}^4C_1 \times {}^{48}C_4 + {}^4C_2 \times {}^{48}C_3 + {}^4C_3 \times {}^{48}C_2 + {}^4C_4 \times {}^{48}C_1$$

$$= 4 \times \frac{48 \times 47 \times 46 \times 45}{4 \times 3 \times 2 \times 1} + \frac{4 \times 3}{2 \times 1} \times \frac{48 \times 47 \times 46}{3 \times 2 \times 1} + 4 \times \frac{48 \times 47}{2 \times 1} + 1 \times 48$$

$$= 8 \times 47 \times 46 \times 45 + 48 \times 47 \times 46 + 2 \times 48 \times 47 + 48$$

$$= 778320 + 103776 + 4512 + 48 = 886656$$

EXAMPLE |15| A sports team of 11 students is to be constituted, choosing atleast 5 from class XI and atleast 5 from class XII. If there are 20 students in each of these classes, then how many ways can the team be constituted? **[NCERT Exemplar]**

Sol. To constitute a sports team, there are two cases

***Case* I** Choose 5 students from class XI and 6 students from class XII.

$\therefore$ In this case, number of ways $= {}^{20}C_5 \times {}^{20}C_6$

***Case* II** Choose 6 students from class XI and 5 students from class XII.

In this case, number of ways $= {}^{20}C_6 \times {}^{20}C_5$

$\therefore$ Required number of ways

$$= ({}^{20}C_5 \times {}^{20}C_6) + ({}^{20}C_6 \times {}^{20}C_5)$$

$$= 2({}^{20}C_5 \times {}^{20}C_6)$$

Hence, the total number of ways of constituting a sports team of 11 students is $2({}^{20}C_5 \times {}^{20}C_6)$ ways.

EXAMPLE |16| In an examination, a question paper consist of 12 questions divided into two parts I and II containing 5 and 7 questions, respectively. A students is required to attempt 8 questions in all, selecting atleast 3 from each part. In how many ways, can a student select the questions? **[NCERT]**

Sol. Student may select 8 questions, according to following scheme.

	I (5 Questions)	II (7 Questions)
(i)	3	5
(ii)	4	4
(iii)	5	3

Now, the required number of ways

$$= {}^5C_3 \times {}^7C_5 + {}^5C_4 \times {}^7C_4 + {}^5C_5 \times {}^7C_3$$

$$= {}^5C_2 \times {}^7C_2 + {}^5C_1 \times {}^7C_3 + {}^5C_0 \times {}^7C_3$$

$$= 10 \times 21 + 5 \times 35 + 1 \times 35$$

$$= 210 + 175 + 35 = 420.$$

Hence, the total number of ways of selecting the questions by the student is 420 ways.

EXAMPLE |17| A man has 7 friends. In how many ways, can he invite one or more of them to party?

Sol. A man may invite one of them, two of them, three of them, ... or all of them and this can be done in ${}^7C_1, {}^7C_2, {}^7C_3, \ldots, {}^7C_7$ ways, respectively.

$\therefore$ Total number of ways

$$= {}^7C_1 + {}^7C_2 + {}^7C_3 + {}^7C_4 + {}^7C_5 + {}^7C_6 + {}^7C_7$$

$$= 7 + 21 + 35 + 35 + +21 + 7 + 1 = 127$$

| TYPE V |

PROBLEMS BASED ON GEOMETRY

EXAMPLE |18| How many chords can be drawn through 21 points on a circle? **[NCERT]**

We can get a chord by joining any two points. So, here for finding the number of chords, we make combination of 2 points from 21 points.

Sol. Clearly, number of chords that can be drawn from 21 points $= {}^{21}C_2 = \frac{21 \times 20}{2 \times 1} = 210$

Hence, total number of chords that can be drawn through 21 points on a circle is 210.

EXAMPLE |19| If 20 lines are drawn in a plane such that no two of them are parallel and no three are concurrent, in how many points will they intersect each other? **[NCERT Exemplar]**

Sol. There are 20 lines in a plane. Since, it is given that no two lines are parallel, therefore all lines are intersecting lines. Also, as no three lines are concurrent, i.e. no three lines intersect at a point, therefore only pair of lines intersect at a point.

We know that a pair of lines intersect at a unique point. Therefore, corresponding to every pair of lines, we have a unique point of intersection.

Thus, number of point of intersection = Number of pair of lines $= {}^{20}C_2 = \frac{20 \times 19}{2 \times 1} = 190.$

EXAMPLE |20| If a polygon has 44 diagonals, then find the number of its sides. **[NCERT Exemplar]**

Sol. Let n be the number of sides of a polygon.

Then, number of vertices of polygon $= n$

Now, number of diagonals $= {}^{n}C_2 - n = 44$

[$\because$ diagonal will be formed by joining any two non-adjacent vertices polygon, therefore total number of diagonal is ${}^{n}C_2 - n$]

$$\Rightarrow \frac{n(n-1)}{2} - n = 44 \Rightarrow n^2 - 3n - 88 = 0$$

$$\Rightarrow \quad n = -8 \text{ or } 11 \Rightarrow n = 11$$

[$\because n$ cannot be negative]

EXAMPLE |21| Find the number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines. **[NCERT Exemplar]**

Sol. Clearly, a parallelogram is formed by choosing two parallel lines from the set of 4 parallel lines and two parallel lines from the set of 3 parallel lines.

$\therefore$ Total number of parallelogram formed

= Number of ways of selecting two lines from set I × Number of ways of selecting two lines from set II

$= {}^{4}C_2 \times {}^{3}C_2 = 6 \times 3 = 18$

EXAMPLE |22| Out of 18 points in a plane, no three are in the same line except five points which are collinear. Find the number of lines that can be formed by joining these points in pairs. **[NCERT Exemplar]**

Sol. There are 18 points in a plane, out of which five are collinear.

Note that, required number of lines

= (Number of lines formed by joining 18 points, taken 2 at a time)

− (Number of lines formed by joining 5 collinear points, taken 2 at a time)

+1 (line joined by collinear points) $= {}^{18}C_2 - {}^{5}C_2 + 1$

$= 153 - 10 + 1$

$= 144$

EXAMPLE |23| Find the number of triangles that are formed by choosing the vertices from a set of 12 points, seven of which lie on the same line. **[NCERT Exemplar]**

Sol. We know that, a triangle will be formed by taking three non-collinear points at a time.

$\therefore$ Required number of triangles

$$= {}^{12}C_3 - {}^{7}C_3$$

$$= \frac{12 \times 11 \times 10}{3 \times 2 \times 1} - \frac{7 \times 6 \times 5}{3 \times 2}$$

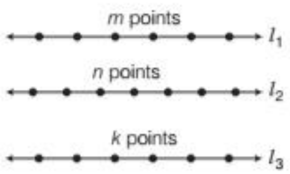
$$= 220 - 35$$

$$= 185$$

Hence, total number of such triangle is 185.

EXAMPLE |24| The straight lines l_1, l_2 and l_3 are parallel and lie in the same plane. A total numbers of m points are taken on l_1; n points on l_2 ; k points on l_3. Find the maximum number of triangles formed with vertices at these points. **[NCERT Exemplar]**

Sol. Total number of points $= m + n + k$



This can give ${}^{m+n+k}C_3$ number of triangles. But m points on l_1, taking 3 at a time, gives ${}^{m}C_3$ combinations which produce no triangle.

Similarly, ${}^{n}C_3$ and ${}^{k}C_3$ number of triangles cannot formed.

$\therefore$ Required number of triangles

$$= {}^{m+n+k}C_3 - {}^{m}C_3 - {}^{n}C_3 - {}^{k}C_3$$

|TYPE VI|

PRACTICAL PROBLEMS INVOLVING BOTH PERMUTATIONS AND COMBINATIONS

EXAMPLE |25| The English alphabet has 5 vowels and 21 consonants. How many words with 2 different vowels and 2 different consonants can be formed from the alphabet?

Sol. Here, we have 5 vowels and 21 consonants. We have to select 2 vowels out of 5 vowels and 2 consonants out of 21 consonants. Also, we have to arrange these 4 letters which can be done in 4! ways.

$\therefore$ Required number of words = ${}^5C_2 \times {}^{21}C_2 \times 4!$

$$= \frac{5 \times 4}{2} \times \frac{21 \times 20}{2} \times 24$$

$$= 10 \times 210 \times 24 = 50400$$

Hence, the total number of words having 2 consonants and 2 vowels is 50400.

EXAMPLE |26| How many words with or without meaning, each of 2 vowels and 3 consonants can be formed from the letters of the word 'DAUGHTER'? **[NCERT]**

Sol. There are 8 letters in the word 'DAUGHTER' including 3 vowels and 5 consonants. We have to select 2 vowels out of 3 vowels and 3 consonants out of 5 consonants.

$\therefore$ Number of ways of selection

$$= {}^3C_2 \times {}^5C_3 = 3 \times 10 = 30$$

Since, each word containing 5 letters can be arranged among themselves in 5! ways.

So, total number of words

$$= 5! \times 30 = 120 \times 30 = 3600$$

HOW TO CHECK WHETHER A GIVEN PROBLEM WILL BE SOLVED BY PERMUTATION OR COMBINATION

It is not easy to check that given problem will be solved by permutation or combination but it can be make easy, if we remember some points related to permutation and combination which are given below

S.No.	In permutation	In combination
1.	Arrangement of objects have to find.	Selection of objects have to find.
2.	Order of objects is important.	Order of objects is not important.
3.	Generally, problems based on numbers, on letters of a word and on arrangements of people in a row, are given.	Generally, problems based on forming committee of people, on selecting group of objects, on cards and on geometry ,are given.

|TOPIC PRACTICE 6|

OBJECTIVE TYPE QUESTIONS

1 The number of ways in which 2 black and 3 red balls can be selected from a bag containing 5 black and 6 red balls is

(a) 170 (b) 190
(c) 180 (d) 200

2 The number of ways in which a mixed double game can be arranged from amongst 9 married couples, if no husband and wife play in the same game, is

(a) 756 (b) 1512
(c) 3024 (d) None of these

3 In a polygon no three diagonals are concurrent. If the total number of points of intersection of diagonals interior to the polygon is 70, then the number of diagonals of polygon is

(a) 8 (b) 20 (c) 28 (d) 32

4 Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then n equals

(a) 5 (b) 7 (c) 6 (d) 4

5 Eighteen guests have to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three others on the other side. The number of ways in which the seating arrangement can be made, is

(a) $\frac{11!}{6!5!} \times 9! \times 9!$ (b) $9! \times 9!$
(c) $\frac{11!}{6!5!} \times 5!6!$ (d) None of these

VERY SHORT ANSWER Type Questions

6 From a class of 32 students, 4 are to be chosen for a competition. In how many ways can this be done?

7 If there are 10 persons in a party and each two of them shake hands with each other, how many hand-shakes happen in the party?

8 In how many ways can a student choose a programme of 5 courses, if 9 courses are available and 2 specific courses are compulsory for every student?

9 In an examination, a student has to answer 4 questions out of 5 questions, questions 1 and 2 are however compulsory. Find the number of ways in which the student can make the choice. **[NCERT Exemplar]**

10 A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected.

11 Find the number of chords that can be drawn through 16 points on a circle.

SHORT ANSWER Type I Questions

12 A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done?

13 In a football championship, 153 matches were played. Every two teams played one match with each other. Find the number of teams, partici- pating in the championship. **[NCERT Exemplar]**

14 Find the number of ways in which a team of eleven players can be selected from 22 players, if two particular players are always including and four are always excluding. **[NCERT Exemplar]**

15 A question paper has two parts, Part A and Part B, each containing 10 questions. If the student has to choose 8 from Part A and 5 from Part B, in how many ways can he choose the questions?

16 Determine the number of 5 cards combination out of a deck of 52 cards, if there is exactly one ace in each combination.

17 Out of 6 gentlemen and 4 ladies, a committee of 5 is to be formed. In how many ways, can this be done so as to include atleast one lady in each committee?

18 If m parallel lines in plane are intersected by a family of n parallel lines, then find the number of parallelograms formed.

19 A polygon has 65 diagonals, the find the number of its sides.

20 Find the number of triangles, that are formed by choosing the vertices from a set of 10 points, 6 of which lie on the same line.

SHORT ANSWER Type II Questions

21 A boy has 3 library tickets and 8 books of his interest in the library. Of these 8, he does not want to borrow Mathematics Part II, unless Mathematics Part I is also borrowed. In how many ways can he choose the three books to be borrowed? **[NCERT Exemplar]**

22 In how many ways can a cricket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicket keepers, assuming that the team of 11 players requires 5 batsmen, 3 all-rounders, 2 bowlers and 1 wicket keeper?

23 For the post of 5 teachers, there are 23 applicants, 2 posts are reserved for SC candidates and there are 7 SC candidates among the applicants. In how many ways can the selection be made?

24 A committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways this can be done, when

(i) atleast two ladies are included?

(ii) atmost two ladies are included?

25 A box contains two white, three black and four red balls. In how many ways can three balls be drawn from the box, if atleast one black ball is to be included in the draw. **[NCERT Exemplar]**

26 Find the number of ways in which we can choose a committee from 4 men and 6 women, so that the committee includes atleast 2 men and exactly twice as many women as men. **[NCERT Exemplar]**

27 A candidate is required to answer 7 questions out of 12 questions, which are divided into two groups each containing 6 questions. He is not permitted to attempt more than 5 questions from either group. Find the number of different ways of doing questions. **[NCERT Exemplar]**

28 There are 10 points in a plane, no three of which are in the same straight line, excepting 4 points, which are collinear.

(i) Find the number of straight lines obtained from the pairs of these points.

(ii) Find the number of triangles that can be formed with the vertices as these points.

29 Find the total number of words formed by 2 vowels and 3 consonants taken from 4 vowels and 5 consonants. **[NCERT Exemplar]**

30 How many words each of 3 vowels and 2 consonants can be formed from the letters of the word 'INVOLUTE'? **[NCERT]**

31 A committee of 8 is to be formed from 7 gentleman and 8 ladies. In how many ways is atleart 4 ladies have to be included in a committee?

LONG ANSWER Type Question

32 A man has 8 friends. In how many ways, can be invite one or more them to party?

33 What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these

(i) four cards are of the same suit?

(ii) four cards belong to four different suits?

(iii) are face cards?

(iv) two are red and two are black cards?

(v) cards are of the same colour? **[NCERT]**

HINTS & ANSWERS

1. (d) Out of 5 black balls, 2 black balls can be selected in $^5C_2$ ways.

Out of 6 red balls, 3 red balls can be selected in $^6C_3$ ways.

FPC, total number of ways

$$= {}^5C_2 \times {}^6C_3 \qquad \left[\because \; {}^nC_2 = \frac{n(n-1)}{2} \text{ and } {}^nC_3 = \frac{n(n-1)(n-2)}{6}\right]$$

Ans. 200 ways

2. (c) Out of 9 men two men can be chosen in $^9C_2$ ways.

Since, no husband and wife are to play in the same game, so we have to select two women from the remaining 7 women. If M_1, M_2 and W_1, W_2 are chosen, then a team can be constituted in 4 ways *viz*.

M_1W_2; M_1W_1 ; M_2W_1; M_2W_2. Thus, the number of ways of arranging the game $= {}^9C_2 \times {}^7C_2 \times 4$

Ans. 3024

3. (b) Let there be n-sides of the polygon. Then, number of vertices is also n.

We know that in a convex polygon every group of four vertices determines two diagonals and hence, one point of intersection.

$\therefore \quad {}^nC_4 = 70$

$\Rightarrow \quad n(n-1)(n-2)(n-3) = 70 \times 24$

$\Rightarrow \quad n = 8$

$\therefore$ Number of diagonals $= {}^nC_2 - n$

$= {}^8C_2 - 8$

Ans. 20

4. (b) We have,

$T_n = {}^nC_3$

$\therefore \quad T_{n+1} - T_n = 21$

$\Rightarrow \quad {}^{n+1}C_3 - {}^nC_3 = 21$

$\Rightarrow \quad {}^nC_2 + {}^nC_3 - {}^nC_3 = 21 \; [\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

$\Rightarrow \quad {}^nC_2 = 21$

Ans. 7

5. (a) Since, four particular guests wants to sit on a particular side A (say) and three others on the other side B (say).

So, we are left with 11 guests. Out of which we choose 5 for side A in $^{11}C_5$ ways and the remaining 6 for side B in $^6C_6$ ways. Hence, the number of selections for the two sides is $^{11}C_5 \times {}^6C_6$.

Now, 9 persons on each side of the table can be arranged among themselves in 9! ways.

Hence, the total number of arrangements

$= {}^{11}C_5 \times {}^6C_6 \times 9! \times 9!$

Ans. $\frac{11!}{6!\,5!} \times 9! \times 9!$

6. Solve as Example 1.
Ans. 35960

7. Solve as Example 3.
Ans. 45

8. Since, 2 specific courses are compulsory, therefore we have to select 3 courses out of rest 7 courses i.e. $^7C_3$ ways.
Ans. 35

9. Since, questions 1 and 2 are compulsory, so students has to select two questions out of rest three questions.
Ans. 3

10. Here, we have to select 2 black balls out of 5 black balls and 3 red balls out of 6 red balls. i.e. $^5C_2 \times {}^6C_3$.
Ans. 200

11. Solve as Example 18. **Ans.** 120

12. Required number of ways $= {}^5C_3$
Ans. 10

13. Let the number of team participating in championship be n.

Since, it is given that every two teams played one match with each other.

$\therefore$ Total number of match played $= {}^nC_2$.

According to the question, $^nC_2 = 153$

Ans. $n = 18$

14. We have to select 9 players out of 16 players **Ans.** 11440

15. Solve as Example 8. **Ans.** 11340

16. **Ans.** 778320

17. Solve as Example 13. **Ans.** 246

18. Solve as Example 21.

Ans. $\frac{mn(m-1)(n-1)}{4}$

19. Solve as Example 20. **Ans.** $n = 65$

20. Solve as Example 23. **Ans.** 100

21. Let us make the following cases and find the number of possible choices in each case.

***Case* I** Boy borrows Mathematics Part II

In this case, boy borrows Mathematics Part I also.

So, number of possible choices are ${}^6C_1$.

***Case* II** Boy does not borrow Mathematics Part II.

In this case, possible choices are ${}^7C_3$.

Ans. 41

22. Required number of ways

$$= {}^{10}C_5 \times {}^5C_3 \times {}^8C_2 \times {}^2C_1$$

Ans. 141120

23. **Ans.** 11760

24. Solve as Example 12.

Ans. (i) 186 (ii) 186

25. Selection of three balls, consisting of atleast one black ball can be done in following ways

(i) Selecting 1 black ball and 2 non-black balls

(ii) Selecting 2 black balls and 1 non-black ball

(iii) Selecting 3 black balls and 0 non-black ball

Required number of ways

$$= {}^3C_1 \times {}^6C_2 + {}^3C_2 \times {}^6C_1 + {}^3C_3$$

Ans. 64

26. Committee can be choose in following ways

(i) Selecting 2 men and 4 women

(ii) Selecting 3 men and 6 women

Ans. 94

27. Total number of ways of attempting 7 questions out of 12 questions = Number of ways of

(Attempting 2 from group I and 5 from group II)

+ (Attempting 3 from group I and 4 from group II)

+ (Attempting 4 from group I and 3 from group II)

+ (Attempting 5 from group I and 2 from group II)

Solve as Example 16. **Ans.** 780

28. (i) Solve as Example 22. **Ans.** 40

(ii) Solve as Example 23. **Ans.** 116

29. Solve as Example 25. **Ans.** 7200

30. Solve as Example 26. **Ans.** 2880

31. Solve as Example 13. **Ans.** 4516

32. Solve as Example 17. **Ans.** 255

33. Number of ways of choosing 4 cards = ${}^{52}C_4$

Ans. 270725

(i) There are 4 suits and each suit have 13 cards. So, from each suit, 4 cards can be choosed in ${}^{13}C_4$ ways separately.

$\therefore$ Required number of ways

$$= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4$$

Ans. 2860

(ii) We have to select one card from each suit.

Required number of ways = ${}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1$

Ans. 13^4

(iii) There are 12 face cards and we have to select 4 cards out of these.

Required number of ways = ${}^{12}C_4$. **Ans.** 495

(iv) There are 26 red cards and 26 black cards.

$\therefore$ Required number of ways = ${}^{26}C_2 \times {}^{26}C_2$

Ans. 105625

(v) Required number of ways = ${}^{26}C_2 \times {}^{26}C_2$

Ans. 29900

SUMMARY

- **Fundamental Principle of Counting** If there are two events such that they can be performed independently in m and n ways respectively, then
  (i) either of the two events can be performed in $(m + n)$ ways. (Addition Rule)
  (ii) the total number of occurrence of the events in the given order is $m \times n$. (Multiplication Rule)
- The continued product of first n natural numbers, is called the 'n factorial' and it is denoted by $n!$.
$$n! = n(n-1)(n-2)..3.2.1$$
- A **permutation** is an arrangement in a definite order of a number of objects taken some or all at a time.
- If n and r are positive integers such that $1 \leq r \leq n$, then the number of permutations of n distinct things taken r at a time is given by $P(n, r) = {}^nP_r = \frac{n!}{(n-r)!}, 0 < r \leq n$.
- The number of permutations of n different objects, taken r at a time when each may be repeated any number of times in each arrangement is n^r.
- The number of permutations of n objects, where p objects are of the same kind or identical and other are distinct is given by $\frac{n!}{p!}$.
- The number of permutations of n objects, where p_1 objects are of one kind, p_2 are of second kind, ...p_k are of kth kind and the rest if any, are of different kind is $\frac{n!}{p_1!p_2!..p_k!}$
- The number of permutations of n different objects taken π object together is $(n - \pi + 1)!$.
- The number of permutations of n different objects taken r at a time, when a particular object is taken in each arrangement, is $r \cdot {}^{n-1}P_{r-1}$.
- The number of permutations of n different objects taken r at a time, when a particular object is never taken in each arrangement, is ${}^{n-1}P_r$.
- The number of permutations of n different objects taken r at a time in which two specific objects always occur together, is $2!(r-1) \cdot {}^{n-2}P_{r-2}$.
- A combination is the selection of an object taken some or all at a time.
- The number of combinations of n distinct objects taken r at a time is given by
$${}^nC_r = \frac{n!}{r!(n-r)!}$$
- Some of the Important Results are
  (i) ${}^nP_r = {}^nC_r \,.\, r!, 0 < r \leq n.$
  (ii) ${}^nC_r = {}^nC_{n-r}, 0 < r \leq n.$
  (iii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C.$
  (iv) If ${}^nC_x = {}^nC_y$, then either $x = y$ or $x + y = n$.

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

1. Suppose you have a suitcase with a number lock. The number lock has 4 wheels each labelled with 10 digits from 0 to 9. The lock can be opened, if 4 specific digits are arranged in a particular sequence with no repetition. Some how, you have forgotten this specific sequence of digits. You remember only the first digit which is 7. In order to open the lock, how many sequences of 3 digits you may have to check with?

(a) 720 (b) 504
(c) 500 (d) None of these

2. The sum of the digits in unit place of all the numbers formed with the help of 3, 4, 5 and 6 taken all at a time is **[NCERT Exemplar]**

(a) 432 (b) 108
(c) 36 (d) 18

3. Match the following columns and choose the correct option from the codes given below.

Column I		Column II	
A.	$\frac{7!}{5!}$ equals	1.	28
B.	$\frac{12!}{(10!)(2!)}$ equals	2.	42
C.	$\frac{8!}{6! \times 2!}$ equals	3.	66

Codes

	A	B	C		A	B	C
(a)	1	2	3	(b)	1	3	2
(c)	3	2	1	(d)	2	3	1

4. Consider the following statements

Statement I If $5\,{}^4P_r = 6\,{}^5P_{r-1}$, then $r = 8, 3$

Statement II If ${}^5P_r = {}^6P_{r-1}$, then $r = 5$

Which of the above statements (s) is/are true?

(a) Only I (b) Only II
(c) Both I and II (d) Neither I nor II

5. Identify the correct combination of true (T) and false (F) of the given two statements.

Statement I The number of permutations of letters of word 'ROOT' are 12.

Statement II The number of permutation of letters of word 'INSTITUTE' are $\frac{9!}{2!\,3!}$.

(a) F T (b) TF
(c) FF (d) T T

6. The number of natural numbers smaller than 10^4, in the decimal notation of which all the digits are distinct is

(a) 5274 (b) 5275
(c) 5276 (d) 5277

7. The number of ways in which three girls and nine boys can be seated in two vans, each having numbered seats 3 in the front and 4 at the back, such that 3 girls are always sit together in a back row on adjacent seats, is

(a) ${}^{11}P_9 \times 4! \times 3$ (b) ${}^{11}P_9 \times 4 \times 3!$
(c) ${}^{14}P_{12} \times 4! \times 3$ (d) ${}^{14}P_{12} \times 4 \times 3!$

8. The number of combinations of 4 different objects A, B, C, D taken 2 at a time is

(a) 4 (b) 6
(c) 7 (d) 8

9. Person wishes to make up as many different parties as he can out of his 20 friends such that each party consists of the same number of persons. The number of friends he should invite are ...A.... Here, A refers to

(a) 8 (b) 9
(c) 10 (d) None of these

10. A five digit number divisible by 3 is to be formed using the numbers 0, 1, 2, 3, 4 and 5, without repetition. The total number of ways this can be done, is **[NCERT Exemplar]**

(a) 216 (b) 240
(c) 600 (d) 3125

VERY SHORT ANSWER Type Questions

11. Prove that $\frac{1}{9!}+\frac{1}{10!}+\frac{1}{11!}=\frac{122}{11!}$.

12. If $\frac{1}{9!}+\frac{1}{10!}=\frac{x}{11!}$, then find the value of x.

13. Evaluate $\frac{n!}{(n-r)!}$, when $n=9$ and $r=5$.

14. Evaluate the following.
(i) ${}^5P_2$ (ii) ${}^{10}P_4$ (iii) ${}^6P_6$

15. If $P(n, 4)=12 \cdot P(n, 2)$, find n.

16. If $P(11, r)=P(12, r-1)$, find r.

17. Find the value of n such that
$P(n, 4): P(n-1, 4)=5: 3, n>4$

18. If ${}^{56}P_{r+6}: {}^{54}P_{r+3}=30800: 1$, then find r.

19. If ${}^nC_{12}={}^nC_5$, find the value of n.

20. If ${}^nC_8={}^nC_6$, then find ${}^nC_2$.

21. If ${}^{18}C_x={}^{18}C_{x+2}$, then find x.

SHORT ANSWER Type I Questions

22. Find n, if $(n+1)!=12 \times(n-1)!$.

23. Find the value of n, if $\frac{n!}{(n-2)!}=930, n \geq 2$.

24. If ${}^{15}C_r: {}^{15}C_{r-1}=11: 5$, find r.

25. If ${}^{2n}C_3: {}^nC_2=44: 3$, find n.

26. In a cinema hall, there are three entrance doors and two exit doors. In how many ways can a person enter the hall and then come out?

27. 25 buses are running between Delhi and Noida. In how many ways, can a person go from Delhi to Noida and return by a different bus?

28. There are 6 multiple choice questions in an examination. How many sequences of answers are possible, if the first three questions have 4 choices each and remaining three have 3 each?

29. In how many ways, can 50 voters vote for the 4 candidates contesting the election for the post of secretary of their association?

30. How many numbers are there between 100 and 1000 such that every digit is either 2 or 9?

SHORT ANSWER Type II Questions

31. How many 4-digit numbers can be formed by using the digits 1 to 9, if repetition of digits is not allowed? **[NECRT]**

32. How many 3 letter words can be made using the letters of the word 'ORIENTAL'?

33. Find how many arrangements can be made with the letters of the word 'MATHEMATICS'?

34. How many words can be formed using the letter A thrice, the letter B twice and the letter C thrice?

35. In how many ways can the letters of the word 'PENCIL' be arranged so that N and E are always together?

36. How many different words can be formed with the letters of the word 'TRIANGLE' so that the word begin with T and end with E.

37. From a group of 15 cricket players, a team of 11 players is to be chosen. In how many ways can this be done?

38. In how many ways a committee of 3 men and 2 women can be chosen from 7 men and 5 women?

39. 7 points lie on a circle. How many chords can be drawn by joining these points?

40. A flag is in the form of three blocks, each to be coloured differently. If there are 8 different colours to choose from, then how many flags are possible?

41. How many 4 letters code can be formed using the first 10 letters of the English alphabet, if no letters can be repeated? **[NCERT]**

42. How many 5-digit telephone numbers can be constructed using the digits 0 to 9, if each number starts with 67 (e.g. 67125 etc.) and no digit appear more than once? **[NCERT]**

43. How many 3-digit numbers can be formed using the digits 0, 2, 3, 6, 8, when the digits may be repeated any number of times?

44. How many 3-digit numbers can be formed without using the digits 0, 2, 3, 4, 5 and 6?

45. A coin is tossed 6 times and the outcomes are recorded. How many possible outcomes are there?

46. In how many ways 5 rings of different types can be worn in 4 fingers?

47. Find the number of ways in which 8 distinct toys can be distributed among 5 children.

48. In how many ways can three jobs I, II and III be assigned to three persons A, B and C, if one person is assigned only one job and all are capable of doing each job?

49. From among the 35 teachers in a school, one principal and one vice-principal are to be appointed. In how many ways can this be done?

LONG ANSWER Type Questions

50. How many 6-digit numbers can be formed from the digits 0, 1, 3, 5, 7 and 9, which are divisible by 10 and no digit is repeated?

51. 18 mice were placed in two experimental groups and one control group, with all group equally large. In how many ways, can the mice be placed into three groups? **[NCERT Exemplar]**

52. There are 6 English, 4 Sanskrit and 5 Hindi books. In how many ways can they be arranged on a shelf so as to keep all the books of the same language together?

53. In a class of 10 students, there are 4 girls A, B, C and D. In how many different ways can they be arranged in a row such that no two of the four girls are consecutive.

54. In how many ways can the letters of the word 'STRANGE' be arranged so that the vowels occupy only the odd places?

55. How many words can be formed with the letters of the word 'UNIVERSITY', the vowels remain together?

56. In how many ways can the letters of the word 'ARRANGE' be arranged so that the two R's are never together?

57. In how many ways can 21 identical books on English and 19 identical books on Hindi be placed in a row on a shelf so that two books on Hindi may not be together?

58. If the different permutations of the word 'EXAMINATION' are listed as in a dictionary, then how many words are there in the list before the first word starting with E? **[NCERT]**

59. Out of 5 men and 2 women, a committee of 3 is to be formed. In how many ways can it be formed, if atleast one women is to be included?

60. There are 12 points in a plane of which 5 points are collinear. Find the number of lines obtained by joining these points in pairs.

61. Three balls are drawn from a bag containing 5 red, 4 white and 3 black balls. Find the number of ways in which this can be done, if atleast 2 balls are red. **[NCERT Exemplar]**

62. Find the number of combinations, when some or all of n objects are taken at a time. **[NCERT Exemplar]**

63. How many automobile licence plates can be made, if each plate contains two different letters followed by three different digits? **[NCERT Exemplar]**

64. How many odd numbers less than 1000 can be formed by using the digits 0, 2, 5, 7 when the repetition of digits is allowed?

65. In how many ways can 10 books be arranged on a shelf, so that a particular pair of books shall be
    (i) always together (ii) never together.

66. In how many ways can 6 balls of different colours, namely white, black, blue, red, green and yellow be arranged in a row in such a way that the white and black balls are never together?

67. How many numbers divisible by 5 and lying between 3000 and 4000 can be formed by using the digits 3, 4, 5, 6, 7, 8, when no digit is repeated in any such number?

68. How many different numbers, greater than 50000 can be formed by using the digits 0, 1, 1, 5, 9?

69. The letters of the word 'SURITI' are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word 'SURITI'.

70. How many different selections of 4 books can be made from 10 different books, if
    (i) there is no restriction?
    (ii) two particular books are always selected?
    (iii) two particular books are never selected?

71. A group consist of 4 girls and 7 boys. In how many ways, a team of 5 members be selected, if the team has
    (i) no girl?
    (ii) atleast one boy and one girl?
    (iii) atleast 3 girls? **[NCERT]**

72. A student has to answer 10 questions, choosing atleast 4 from each of Parts A and B. If there are 6 questions in Part A and 7 in Part B, in how many ways can the student choose 10 questions? **[NCERT Exemplar]**

73. A tea party is arranged for 16 persons along two sides of a long table with 8 chairs on each side. Four persons wish to sit on one particular side and two on the other side. In how many ways can they be seated?

74. There are 4 routes between Delhi and Patna. In how many different ways can a man go from Delhi to Patna and return, if for returning

(i) any of the routes is taken?

(ii) the same route is taken?

(iii) the same route is not taken?

75. Find the number of arrangements of the letters of the word 'INDEPENDENCE'. In how many of these arrangements

(i) do the words start with P?

(ii) do all the vowels always occur together?

(iii) do all the vowels never occur together?

(iv) do the words begin with I and end in P? **[NCERT]**

76. Using the digits 1, 2, 3, 4, 5, 6, 7, a number of 4 different digits is formed. Find how many numbers are formed? Further, find how many of these are

(i) exactly divisible by 2?

(ii) exactly divisible by 25?

(iii) exactly divisible by 4? **[NCERT Exemplar]**

CASE BASED Questions

77. Five students Ajay, Shyam, Yojana, Rahul and Akansha are sitting in a play ground in a line.

Based on the above information answer the following questions.

(i) Total number of ways of sitting arrangement of five students is

(a) 120 (b) 60

(c) 24 (d) None of these

(ii) Total number of arrangement of sitting if Ajay and Yojana sit together is

(a) 60 (b) 48 (c) 72 (d) 120

(iii) Total number of arrangement Yojana and Rahul sitting at extreme position is

(a) 24 (b) 36 (c) 48 (d) 12

(iv) Total number of arrangement if shyam is sitting in the middle is

(a) 24 (b) 12 (c) 6 (d) 36

(v) Total number of arrangement sitting Yojana and Rahul not sit together, is

(a) 72 (b) 120 (c) 60 (d) 144

78. Republic day is a national holiday of India. It honours the date on which the constitution of India came into effect on 26 January 1950 replacing the Government of India Act (1935) as the gavering document of India and thus, turning the nation into a newly formed republic.

Answer the following question which are based on the word "REPUBLIC".

(i) Find the number of arrangements of the letters of the word 'REPUBLIC'.

(a) 40300 (b) 30420 (c) 40320 (d) 40400

(ii) How many arrangements start with a vowel?

(a) 12015 (b) 15120 (a) 12018 (d) 15100

(iii) Which concept is used for 'finding the arrangements start with a vowel?

(a) Permutation (b) F.P.M

(c) Combination (d) F.P.A

(iv) If the number of arrangements of the letters of the word 'REPUBLIC' is *abcde*, the $(a + b + c + d + e)$ is

(a) 10 (b) 9 (c) 8 (d) 15

(v) If the number of arrangements start with a vowel is *abcde*, then $(a + b) - (d + e)$ is

(a) 2 (b) 3 (c) 4 (d) 5

79. In the BCCI board, Jagmohan Dalmia is a Cricket team selector. He selected a cricket team from 17 players in which only 5 players can bowl. Then, answer the following question which are based on above it.

(i) In how many ways exactly 4 bowlers must include out of 11 players.

(a) 2960 (b) 3960 (c) 4960 (d) 5960

(ii) Which concept is used for finding the number of required players?

(a) Permutation (b) Combination

(c) F.P.M. (d) F.P.A.

(iii) In how many ways exactly 3 bowlers must include out of 11 players.

(a) 3950 (b) 4950 (c) 5950 (d) 6950

(iv) If the number of ways of selecting exactly 3 bowlers must include out of 11 players is *abcd*, then value of $(a + b + c + d)$ is

(a) 10 (b) 18 (c) 20 (d) 25

(v) If the number of ways of selecting exactly 4 bowlers must include out of players is *abcd* then value of $(a + b + c + d)$ is

(a) 18 (b) 20 (c) 22 (d) 24

HINTS & ANSWERS

1. (b) If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is $m \times n$. (multiplication principle)

It is given that 1st digit on number lock is 7.

$\therefore$ Number of sequences of 3 digits $= 9 \times 8 \times 7$

Ans. 504

2. (b) Number of numbers ending with $6 = \square\ \square\ \square\ \square$

$= 1 \times 2 \times 3 \times 1 = 6$

Similarly, number of numbers ending with 3, 4 or 5 are 6 each.

$\therefore$ Sum of digits in unit's place

$= 6 \times 6 + 6 \times 3 + 6 \times 4 + 6 \times 5 = 6(18)$

Ans. 108

3. (d) A. $\frac{7!}{5!} = 42$

B. $\frac{12!}{10!2!} = 66$

C. $\frac{8!}{6! \times 2!} = 28$

4. (d) I. We have, $5\ {}^4P_r = 6\ {}^5P_{r-1}$

$\Rightarrow \quad 5 \times \frac{4!}{(4-r)!} = 6 \times \frac{5!}{(5-r+1)!}$

$\Rightarrow \quad (6-r)(5-r) = 6$

$\Rightarrow \quad r^2 - 11r + 24 = 0$

$\Rightarrow \quad (r-8)(r-3) = 0$

$\Rightarrow \quad r = 8, 3$

II. We have, ${}^5P_r = {}^6P_{r-1}$

$\frac{5!}{(5-r)!} = \frac{6!}{[6-(r-1)]!}$

$\Rightarrow \quad \frac{1}{(5-r)!} = \frac{6}{(7-r)(6-r)(5-r)!}$

$\Rightarrow \quad r^2 - 13r + 36 = 0$

$\Rightarrow \quad (r-4)(r-9) = 0$

$\Rightarrow \quad r = 4, 9$

$\Rightarrow \quad r = 4 \qquad [\because r \neq 9]$

5. (d) I. Here, there are 4 letters of which there are 2 O's and rest are different.

Therefore, the required number of arrangements

$= \frac{4!}{2!} = 12$

II. There are 9 letters of which there are 2 I's, 3T's and rest are different.

Therefore, the required number of arrangements

$= \frac{9!}{2!3!}$

6. (a) Total number of 4 digit natural numbers with distinct digits $= {}^{10}P_4 - {}^9P_3$

Total number of 3 digit natural numbers with distinct digit $= {}^{10}P_3 - {}^9P_2$

Total number of 2 digit natural numbers with distinct digits $= {}^{10}P_2 - {}^9P_1$

Total number of 1 digit natural number = 9

$\therefore$ Required number of natural numbers

$= ({}^{10}P_4 - {}^9P_3) + ({}^{10}P_3 - {}^9P_2) + ({}^{10}P_2 - {}^9P_1) + 9$

Ans. 5274

7. (b) Three girls can be seated together in a back row on adjacent seats in the following ways

1, 2, 3 or 2, 3, 4 of first van

and 1, 2, 3 or 2, 3, 4 of second van.

In each way the three girls can interchange among themselves in 3! ways. So, the total number of ways in which three girls can be seated together in a back row on adjacent seats $= 4 \times 3!$

Now, 9 boys are to be seated on remaining 11 seats, which can be done in ${}^{11}P_9$ ways. **Ans.** ${}^{11}P_9 \times 4 \times 3!$.

8. (b) The combination will be *AB*, *AC*, *AD*, *BC*, *BD* and *CD*. Here, *AB* and *BA* are the same combination as order does not alter the combination.

9. (c) Suppose he invites r friends at a time.

Then, number of parties $= {}^{20}C_r$

even. **Ans.** 10

10. (a) Since, a five-digit number is formed using digits {0, 1, 2, 3, 4 and 5} divisible by 3 *i.e.*, only possible when sum of digits is multiple of 3 which gives two cases.

***Case* I** {using digits 0, 1, 2, 4, 5}

Number of numbers $= {}^4C_1 \times {}^4C_1 \times {}^3C_1 \times {}^2C_1 \times {}^1C_1$

Ans. 96

***Case* II** {using digits 1, 2, 3, 4, 5}

Number of numbers $= {}^5C_1 \times {}^4C_1 \times {}^3C_1 \times {}^2C_1 \times {}^1C_1$

Ans. 120

11. LHS $= \frac{1}{9!} + \frac{1}{10!} + \frac{1}{11!}$

$= \frac{11 \times 10}{11!} + \frac{11}{11!} + \frac{1}{11!} = \frac{122}{11!} = \text{RHS}$ **Hence proved.**

12. $\because \frac{11 \times 10}{11!} + \frac{11}{11!} = \frac{x}{11!} \Rightarrow 121 = x$

14. (i) ${}_5P_2 = \frac{5!}{3!} = 20$

15. $\because {}^nP_4 = 12 \cdot {}^nP_2 \Rightarrow \frac{n!}{(n-4)!} = 12 \times \frac{n!}{(n-2)!}$

$\Rightarrow 1 = 12 \times \frac{1}{(n-2)(n-3)} \Rightarrow (n-3)(n-3) = 12$

$\Rightarrow (n-2)(n-3) = 4 \times 3 \Rightarrow n = 6$

19. $\because {}^nC_{12} = {}^nC_5$

$\Rightarrow {}^nC_{n-12} = {}^nC_5 \Rightarrow n - 12 = 5 \Rightarrow n = 17$

20. ${}^nC_8 = {}^nC_6 \Rightarrow {}^nC_{n-8} = {}^nC_6 \Rightarrow n - 8 = 6 \Rightarrow n = 14$

21. Solve as Q. 19.

22. Given, $(n+1)! = 12 \times (n-1)!$

$\Rightarrow (n+1)\,n = 12 \Rightarrow n^2 + n - 12 = 0$

$\Rightarrow (n+4)(n-3) = 0$

$\therefore n = 3$ [$\because n$ cannot be negative]

23. Solve as Q. 22.

24. $\frac{{}^{15}C_r}{{}^{15}C_{r-1}} = \frac{11}{5} \Rightarrow \frac{15!}{(15-r)!\,r!} \times \frac{(15-r+1)! \times (r-1)!}{15!} = \frac{11}{5}$

$\Rightarrow \frac{(16-r)}{r} = \frac{11}{5} \Rightarrow 80 - 5r = 11r \Rightarrow r = 5$

28. Total number of sequences of answers are possible

$= 4 \times 4 \times 4 \times 3 \times 3 \times 3$

29. Each person gives a vote for any of the four persons.

30. Only three-digit number exists between 100 and 1000. But each digit only any of two numbers 2 or 9 can be place.

$\therefore$ Total number of ways $= 2 \times 2 \times 2 = 8$

31. If repetition is not allowed, the any of 4 digits can be filled by a number is ${}^9C_4$ *i.e.* 3024.

32. All letters in a given word are distinct.

The number of ways in which 3-letter words can be formed from 8 letters $= {}^8P_3 = 336$

33. The letters in the word 'MATHEMATICS' are (MM, AA, TT, H, E, I, C, S).

$\therefore$ Total arragements $= \frac{11!}{2!\,2!\,2!}$

34. Total number of words $= \frac{8!}{3!\,2!\,3!} = 560$

35. We consider N and E as one letter and remaining four letters are (P, C, I, L).

$\therefore$ Total number of words $= 5! \times 2!$

[$\because$ N and E can be arranged in 2! ways]

36. In a word TRIANGLE, all letters are distrinct.

When we fix the letters T and E in the beginning and end, then rest of 6 letters can be arranged in 6! ways.

37. Required number of ways $= {}^{15}C_{11}$

38. Required number of ways $= {}^7C_3 \times {}^5C_2$

39. Required number of chords $= {}^7C_2$

40. Total number of flags possible $= {}^8P_3$

41. Total number of 4 letters code $= {}^{10}C_4$

42. In 5-digit telephone number, first two digits are fixed and rest of the three place number appears 8, 7, 6 ways.

$\therefore$ Required number of ways $= 8 \times 7 \times 6$

43. 0 do not comes in hundred's place, so only 4 numbers are possible in hundred's place. Since, the digits are repeated, so remaining two digits, 5 numbers are possible in each place.

$\therefore$ Total number of ways $= 4 \times 5 \times 5 = 100$

45. Total possible outcomes $= 6 \times 6 = 36$

46. Since, each ring can be worn in any four fingers.

47. Each toy can be ditributed to any of the children.

48. Three different jobs can be assigned to three different persons $= {}^3P_3 = 3!$

49. Required number of ways $= {}^{35}P_2$

50. We know that, if any number is divisible by 10, then 0 should be in unit's place. Since, no digit is repeated, therefore in rest of places, the numbers can be placed in 5! ways.

52. There are three different types of books. They can be arranged in 3! ways.

$\therefore$ Required number of ways $= 3! \times 6! \times 4! \times 5!$

53. Firstly, we fix the alternate position of 6 boys in 6! ways i.e.

$- B_1 - B_2 - B_3 - B_4 - B_5 - B_6$

In out of 7 blank places, four girls can be seated in ${}^7P_4$ ways.

$\therefore$ Required number of ways $= 6! \times {}^7P_4$

54. In the given, word, vowels are A, E and consonents are S, T, R, N, G.

In odd places, vowels can be arrange in ${}^4P_2$ and in remaining five places, five letters can be arranged in 5 ways.

$\therefore$ Required number of ways $= {}^4P_2 \times 5!$

56. Required number of ways = Total number of arrangements – When two R's together, the total arrangements $= \frac{7!}{2!\,2!} - \frac{6!}{2!}$

57. Firstly, we fix 2! identical books in alterante position in 1 way. There are 22 space so that 19 identical books can be placed in ${}^{22}C_{19}$ ways.

$\therefore$ Required number of ways $= {}^{22}C_{19}$

58. The letters of given word are $A, A, E, I, I, M, N, O, T, X$ i.e. word starting with A are formed with the letters $2I$'s, $2N$'s, A, E, X, M, T, O (total 10 letters).

Hence, number of words formed by these letters

$= \frac{10}{2!\,2!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 907200$

59. Required number of ways = ${}^2C_1 \times {}^5C_2 + {}^2C_2 \times {}^5C_1$

60. Required number of ways = ${}^{12}C_2 - {}^5C_2 + 1$

61. Required number of ways = ${}^5C_2 \times {}^7C_1 + {}^5C_3$

62. Required number of ways = ${}^nC_1 + {}^nC_2 + ... + {}^nC_n$

$= {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n - {}^nC_0 = 2^n - 1$

63. In automobile licence plates, two different letters can be arranging in ${}^{26}P_2$ ways and next three different digits can be arranging in ${}^{10}P_3$ ways.

$\therefore$ Total number of ways $= {}^{26}P_2 \times {}^{10}P_3$

$= 26 \times 25 \times 10 \times 9 \times 8 = 468000$

64. For odd number, first place should be odd number and rest of places may any of the number.

Also, less than 1000 means, number should be one digit, two digits and three digits odd numbers.

Number of one digit odd number = 2

Number of two digits odd number = ${}^2C_1 \times 4 = 8$

Number of three digits odd number = ${}^2C_1 \times 4 \times 4 = 32$

65. (i) Total number of ways = $2! \times 9!$

(ii) Total number of ways = $10! - 2! \times 9!$

66. Total number of ways = Arrangement of 6 differnt colours – Arrangement of different balls when white and black are together = $6! - 2! \times 5!$

67. In thousand's place, digit should be 3 and in unit's place, digit should be 5. For middle of the two digits, the arrangement of two digits from 4 digits is ${}^4P_2$ ways.

68. In thousand's place, only number (5 or 9) can be placed.

$\therefore$ Total number of ways $= {}^2C_1 \times \frac{4!}{2!}$

69. The letters in a word SUR ITI are {I, I, R, S, T, U}.

When word start with I, then number of arrangements $= 5!$

When word start with R, then number of arrangements $= \frac{5!}{2!}$

When word start with SI, then number of arrangement $= 4!$

When word start with SR, then number of arrangements $= \frac{4!}{2!}$

When word start with STI, then number of arrangements $= \frac{4!}{2!}$

When word start with SUI, then number of arrangements = 3!

When word start with SURI, the number of arrangements = 2!

Required rank $= 5! + \frac{5!}{2} + 4! + \frac{4!}{2!} + \frac{4!}{2} + 3! + 2!$

$= 120 + 60 + 24 + 12 + 12 + 6 + 2 = 236$

70. (i) Number of different selections = ${}^{10}C_4$

(ii) Number of different selections = ${}^{10-2}C_{4-2} = {}^8C_2$

(iii) Number of different selections = ${}^8C_4$

71. (i) Number of ways = ${}^7C_5$

(ii) Number of ways = ${}^7C_1 \times {}^4C_4 + {}^7C_2 \times {}^4C_3 + {}^7C_3 \times {}^4C_2 + {}^7C_4 \times {}^4C_1$

(iii) Number of ways = ${}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1$

72. Number of ways = ${}^6C_4 \times {}^7C_6 + {}^6C_5 \times {}^7C_5 + {}^6C_6 \times {}^7C_4$

74. (i) Total number of different routes = 4×4

(ii) Total number of different routes = 4

(iii) Total number of different routes = 4×3

75. The letters in the word INDEPENDENCE are (I, N, N, N, D, D, E, E, E, E, P, C).

$\therefore$ Total number of arrangement $= \frac{12!}{3! \times 2! \times 4!}$

(i) Total number of words start with P $= \frac{11!}{3! \times 2! \times 4!}$

(ii) Total number of words, when all the vowels occur together $= \frac{8! \times 5!}{3! \times 2! \times 4!}$

[consider all vowels at one unit]

(iii) Total number of ways = Total arrangements – Number of arrangements when vowels occurs together $= \frac{12!}{3! \times 2! \times 4!} - \frac{8! \times 5!}{3! \times 2! \times 4!}$

76. Number of four different digits formed = ${}^7C_4$

(i) Even number should be in unit's place and remaining any three digits, any different number can be placed.

$\therefore$ Total number of ways = ${}^6P_4 \times {}^3C_1$

(ii) Any four digit number is divisible by 25, if last two digits should be divisible by 25 i.e. 25, 75.

(iii) Any number is divisible by 4, if last two digits is divisible by 4 i.e. last digits can be 12, 16, 24, 32, 36, 52, 56, 54, 72, 76.

$\therefore$ Total number of ways = ${}^5P_2 \times {}^{10}C_1$

$\therefore$ Total number of ways $= 5P_2 \times {}^2C_1$

(iv) Total number of words $= \frac{10}{3! \times 2! \times 4!}$

77. (i) (a) We have five student

$\therefore$ Total number of arrangement is $5! = 120$

(ii) (b) Ajay and Yojana sit together

Total number of arrangement $= 4! \times 2! = 24 \times 2 = 48$

(iii) (d) Total number of arrangement of Yojana and Rahul sitting in extreme position is $2! \times 3! = 2 \times 6 = 12$

(iv) (a) Number of arrangement Shyam in middle is

$$4! = 24$$

(v) (a) Number of arrangement Yojana and Rahul not sit together is

$$\frac{4!}{2!} \times 3! = 72$$

78. (i) (c) The letters in the word 'REPUBLIC' are all distinct.

There are 8 letters in the given word. So, the number of arrangements are 8! i.e. 40320.

(ii) (b) The vowels in a given word are 'E, I, U'. If we start a word from vowel, we can choose 1 vowel from 3 vowels in ${}^3C_1$ ways. Further, remaining 7 letters can be arranged in 7! ways.

∴ Total number of arrangements start with a vowel

$$= {}^3C_1 \times 7! = 3 \times 5040 = 15120$$

(iii) (c) Combination

(iv) (b) Since, number of arrangements are 40320.

On comparing, we get

$$a = 4,\ b = 0,\ c = 3,\ d = 2,\ e = 0$$

So, $a + b + c + d + e = a + b + c + d + e$

$$= 4 + 0 + 3 + 2 + 0 = 9$$

(v) (c) Since, number of arrangements are 15120

On comparing, we get

$$a = 1,\ b = 5,\ c = 1,\ d = 2,\ e = 0$$

$$\therefore\ (a + b) - (d + e) = (1 + 5) - (2 + 0)$$

$$= 6 - 2 = 4$$

79. (i) (b) Given that, total number of players is 17.

We have to select 11 players including exactly 4 bowlers. Hence, 4 bowlers will be selected from 5 bowlers and remaining 7 players will be selected from 12 batsmen.

Now, 4 bowlers out of 5 bowlers can be selected in ${}^5C_4$ ways.

7 players out of 12 players can be selected in ${}^{12}C_7$ ways.

∴ Total number of ways selecting 11 players

$$= {}^5C_4 \times {}^{12}C_7 = {}^5C_1 \times {}^{12}C_5 \quad [\because {}^nC_r = {}^nC_{n-r}]$$

$$= 5 \times \frac{12!}{5!7!}$$

$$= 5 \times \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= 5 \times 11 \times 9 \times 8 = 55 \times 72$$

$$= 3960$$

Hence, he can select the team of 11 players in 3960 ways.

(ii) (b) Combination

(iii) (b) Given that, total number of players is 17.

We have to select 11 players including exactly 3 bowlers. Hence, 3 bowlers will be selected from 5 bowlers and remaining 8 players will be selected from 12 batsmen.

Now, 3 bowlers out of 5 bowlers can be selected in ${}^5C_3$ ways.

8 players out of 12 players can be selected in ${}^{12}C_8$ ways.

∴ Total number of ways selecting 11 players

$$= {}^5C_3 \times {}^{12}C_8$$

$$= \frac{5!}{2!3!} \times \frac{12!}{8!4!}$$

$$= \frac{5 \times 4}{2} \times \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2}$$

$$= 10 \times 11 \times 5 \times 9$$

$$= 4950$$

(iv) (b) Here, $a = 4,\ b = 9,\ c = 5,\ d = 0$

$$\therefore a + b + c + d = 4 + 9 + 5 + 0$$

$$= 18$$

(v) (a) Here, $a = 3,\ b = 9,\ c = 6$ and $d = 0$

$$\therefore\ a + b + c + d = 3 + 9 + 6 + 0 = 18$$