

Integration by Partial Fraction

Q.1. Evaluate : $\int [(x - 1)/\{(x - 3)(x - 2)^2\}] dx$.

Solution : 1

We are given $\int [(x - 1)/\{(x - 3)(x - 2)^2\}] dx$.

Now, let $(x - 1)/\{(x - 3)(x - 2)^2\} = A/(x - 3) + B/(x - 2) + C/(x - 2)^2$

Then $x - 1 = A(x - 2)^2 + B(x - 3)(x - 2) + C(x - 3)$

Put $x = 2$, $1 = C(2 - 3) = -C \Rightarrow C = -1$.

Put $x = 3$, $2 = A(3 - 2)^2 = A \Rightarrow A = 2$.

Put $x = 0$, $-1 = A(0 - 2)^2 + B(0 - 3)(0 - 2) + C(0 - 3) = 4A + 6B - 3C$

$= 4 \times 2 + 6B - 3 \times (-1) = 6B + 11 \Rightarrow 6B = -12 \Rightarrow B = -2$.

Putting the values of A, B and C we get

$$\int [(x - 1)/\{(x - 3)(x - 2)^2\}] dx = \int [2/(x - 3)] dx - \int [2/(x - 2)] dx - \int [1/(x - 2)^2] dx$$

$$= 2 \log (x - 3) - 2 \log (x - 2) + 1/(x - 2) + c.$$

Q.2. Evaluate : $\int dx/(2x^2 + x - 1)$.

Solution : 2

Let $I = \int dx/(2x^2 + x - 1)$;

Let $1/(2x^2 + x - 1) = 1/[(2x - 1)(x + 1)]$

$$= 1/[(2x - 1)(1/2 + 1)] + 1/[\{2(-1) - 1\}(x + 1)]$$

$$= (2/3)/(2x - 1) - (1/3)/(x + 1) .$$

$$\text{Therefore, } I = (2/3) \int dx/(2x - 1) - (1/3) \int dx/(x + 1)$$

$$= (2/3)[(1/2) \log |2x - 1| - (1/3) \log |x + 1| + c$$

$$= (1/3) \log |(2x - 1)/(x + 1)| + c.$$

Q.3. Evaluate : $\int [x/\{(x^2 - a^2)(x^2 - b^2)\}].dx.$

Solution : 3

$$\text{Let } I = \int [x/\{(x^2 - a^2)(x^2 - b^2)\}].dx$$

$$[\text{Put } x^2 = t \text{ then } 2x dx = dt \Rightarrow x dx = (1/2)dt]$$

$$\text{Therefore, } I = (1/2) \int dt/[(t - a^2)(t - b^2)] ;$$

$$\text{Now } 1/[(t - a^2)(t - b^2)] = 1/[(t - a^2)(a^2 - b^2)] - 1/[(t - b^2)(a^2 - b^2)]$$

$$\text{Therefore, } I = 1/\{2(a^2 - b^2)\} \int [1/(t - a^2) - 1/(t - b^2)].dt$$

$$= 1/\{2(a^2 - b^2)\} [\log |t - a^2| - \log |t - b^2|] + c$$

$$= 1/\{2(a^2 - b^2)\} [\log |x^2 - a^2| - \log |x^2 - b^2| + c$$

$$= 1/\{2(a^2 - b^2)\} \log |(x^2 - a^2)/(x^2 - b^2)| + c.$$

Q.4. Evaluate : $\int [\cos x/\{(1 + \sin x)(2 + \sin x)\}].dx.$

Solution : 4

$$\text{Let } I = \int [\cos x/\{(1 + \sin x)(2 + \sin x)\}].dx$$

$$[\text{Put } \sin x = t \text{ then } \cos x dx = dt]$$

$$\text{Then } I = \int dt/\{(1 + t)(2 + t)\}$$

$$\begin{aligned}
&= \int [1/\{(1+t)(2-1)\} + 1/\{(1-2)(2+t)\}] dt \\
&= \int [1/(1+t) - 1/(2+t)] dt \\
&= \log |1+t| - \log |2+t| \\
&= \log |(1+t)/(2+t)| + c \\
&= \log |(1+\sin x)/(2+\sin x)| + c.
\end{aligned}$$

Q.5. Evaluate : $\int dx/(\sin x + \sin 2x)$.

Solution : 5

$$\begin{aligned}
\text{Let } I &= \int dx/(\sin x + \sin 2x) \\
&= \int dx/(\sin x + 2 \sin x \cos x) \\
&= \int dx/\{\sin x (1 + 2\cos x)\} \\
&= \int [\sin x/\{\sin^2 x(1 + 2\cos x)\}] dx \\
&= \int [\sin x/\{(1 - \cos^2 x)(1 + 2\cos x)\}] dx \\
&\text{[Put } \cos x = t \text{ then } -\sin x dx = dt] \\
\text{Then } I &= - \int dt/\{(1 - t^2)(1 + 2t)\} \\
&= - \int dt/\{(1+t)(1-t)(1+2t)\}; \\
\text{Now } 1/\{(1+t)(1-t)(1+2t)\} \\
&= 1/[(1+t)\{1-(-1)\}\{1+2(-1)\}] + 1/[(1+1)(1-t)\{1+2(1)\}] + 1/[(1-1/2)\{1-(-1/2)\}(1+2t)] \\
&= (-1/2)\{1/(1+t)\} + (1/6)\{1/(1-t)\} + (4/3)\{1/(1+2t)\} \\
\text{Therefore, } I &= (1/2) \int dt/(1+t) - (1/6) \int dt/(1-t) - (4/3) \int dt/(1+2t) \\
&= (1/2) \log |1+t| + (1/6) \log |1-t| - (4/3)(1/2) \log |1+2t| + c \\
&= (1/2) \log |1+\cos x| + (1/6) \log |1-\cos x| - (2/3) \log |1+2\cos x| + c.
\end{aligned}$$

Q.6. Evaluate : $\int dx/[x(x^5 + 1)]$.

Solution : 6

$$\text{Let } I = \int dx/[x(x^5 + 1)]$$

$$= \int [x^4/\{x^5 (x^5 + 1)\}].dx \text{ [Multiplying Nr \& Dr by } x^4]$$

$$[\text{Put } x^5 = t \text{ then } x^4 dx = (1/5)dt]$$

$$\text{Therefore, } I = (1/5) \int dt/[t(t + 1)]$$

$$= (1/5) \int [1/t - 1/(t + 1)].dt$$

$$= (1/5) [\log |t| - \log |t + 1|] + c$$

$$= (1/5) \log |t/(t + 1)| + c$$

$$= (1/5) \log |x^5/(x^5 + 1)| + c.$$