Chapter 1

Probability

Contents

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Probability theory, like many other branches of mathematics, evolved out of practical considerations. It had its origin in the 16th century when an Italian physician and Mathematician Jerome Cardan (1501-1576) *wrote the first book on the subject* "Book on Games of Chance (Liber de Ludo Aleae)" *It was published in 1663 after his death.*

In 1654, a gambler Chevalier de Mere approached the well known French philosopher and mathematician Blaise Pascal (1623-1662) *for certain dice problems. Pascal became interested in these problems and discussed with famous* French mathematician Pierre de Fermat (1601-1665) *Both Pascal and Fermat solved the problem independently.*

Probability

1.1 Introduction

Numerical study of chances of occurrence of events is dealt in probability theory.

The theory of probability is applied in many diverse fields and the flexibility of the theory provides approximate tools for so great a variety of needs.

There are two approaches to probability viz. (i) Classical approach and (ii) Axiomatic approach.

In both the approaches we use the term 'experiment', which means an operation which can produce some well-defined outcome(*s*). There are two types of experiments:

(1) **Deterministic experiment :** Those experiments which when repeated under identical conditions produce the same result or outcome are known as deterministic experiments. When experiments in science or engineering are repeated under identical conditions, we get almost the same result everytime.

(2) **Random experiment :** If an experiment, when repeated under identical conditions, do not produce the same outcome every time but the outcome in a trial is one of the several possible outcomes then such an experiment is known as a probabilistic experiment or a random experiment.

In a random experiment, all the outcomes are known in advance but the exact outcome is unpredictable.

For example, in tossing of a coin, it is known that either a head or a tail will occur but one is not sure if a head or a tail will be obtained. So it is a random experiment.

1.2 Definitions of Various Terms

(1) **Sample space :** The set of all possible outcomes of a trial (random experiment) is called its sample space. It is generally denoted by S and each outcome of the trial is said to be a sample point.

Example : (i) If a dice is thrown once, then its sample space is $S = \{1, 2, 3, 4, 5, 6\}$

(ii) If two coins are tossed together then its sample space is $S = \{HT, TH, HH, TT\}$.

(2) **Event :** An event is a subset of a sample space.

(i) **Simple event :** An event containing only a single sample point is called an elementary or simple event.

Example : In a single toss of coin, the event of getting a head is a simple event.

Here $S = \{H, T\}$ and $E = \{H\}$

(ii) **Compound events :** Events obtained by combining together two or more elementary events are known as the compound events or decomposable events.

For example, In a single throw of a pair of dice the event of getting a doublet, is a compound event because this event occurs if any one of the elementary events (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) occurs.

(iii) **Equally likely events :** Events are equally likely if there is no reason for an event to occur in preference to any other event.

Example : If an unbiased die is rolled, then each outcome is equally likely to happen *i.e.*, all elementary events are equally likely.

(iv) **Mutually exclusive or disjoint events :** Events are said to be mutually exclusive or disjoint or incompatible if the occurrence of any one of them prevents the occurrence of all the others.

Example : E = getting an even number, F = getting an odd number, these two events are mutually exclusive, because, if E occurs we say that the number obtained is even and so it cannot be odd *i.e.*, F does not occur.

A_1 and A_2 are mutually exclusive events if $A_1 \cap A_2 = \phi$.

(v) **Mutually non-exclusive events :** The events which are not mutually exclusive are known as compatible events or mutually non exclusive events.

(vi) **Independent events :** Events are said to be independent if the happening (or non-happening) of one event is not affected by the happening (or non-happening) of others.

Example : If two dice are thrown together, then getting an even number on first is independent to getting an odd number on the second.

(vii) **Dependent events :** Two or more events are said to be dependent if the happening of one event affects (partially or totally) other event.

Example : Suppose a bag contains 5 white and 4 black balls. Two balls are drawn one by one. Then two events that the first ball is white and second ball is black are independent if the first ball is replaced before drawing the second ball. If the first ball is not replaced then these two events will be dependent because second draw will have only 8 exhaustive cases.

(3) **Exhaustive number of cases** : The total number of possible outcomes of a random experiment in a trial is known as the exhaustive number of cases.

Example : In throwing a die the exhaustive number of cases is 6, since any one of the six faces marked with 1, 2, 3, 4, 5, 6 may come uppermost.

(4) **Favourable number of cases :** The number of cases favourable to an event in a trial is the total number of elementary events such that the occurrence of any one of them ensures the happening of the event.

Example : In drawing two cards from a pack of 52 cards, the number of cases favourable to drawing 2 queens is ${}^4C_2$.

(5) **Mutually exclusive and exhaustive system of events :** Let S be the sample space associated with a random experiment. Let A_1, A_2, ...A_n be subsets of S such that

(i) $A_i \cap A_j = \phi$ for $i \neq j$ and (ii) $A_1 \cup A_2 \cup .... \cup A_n = S$

Then the collection of events $A_1, A_2, ..., A_n$ is said to form a mutually exclusive and exhaustive system of events.

If $E_1, E_2, ..., E_n$ are elementary events associated with a random experiment, then

(i) $E_i \cap E_j = \phi$ for $i \neq j$ and (ii) $E_1 \cup E_2 \cup .... \cup E_n = S$

So, the collection of elementary events associated with a random experiment always form a system of mutually exclusive and exhaustive system of events.

In this system, $P(A_1 \cup A_2 .... \cup A_n) = P(A_1) + P(A_2) + ... + P(A_n) = 1$.

Important Tips

- *Independent events are always taken from different experiments, while mutually exclusive events are taken from a single experiment.*
- *Independent events can happen together while mutually exclusive events cannot happen together.*
- *Independent events are connected by the word "and" but mutually exclusive events are connected by the word "or".*

Example: 1 Two fair dice are tossed. Let A be the event that the first die shows an even number and B be the event that second die shows an odd number. The two events A and B are **[IIT 1979]**

(a) Mutually exclusive (b) Independent and mutually exclusive

(c) Dependent (d) None of these

Solution: (d) They are independent events but not mutually exclusive.

Example: 2 The probabilities of a student getting I, II and III division in an examination are respectively $\frac{1}{10}, \frac{3}{5}$ and $\frac{1}{4}$. The probability that the student fail in the examination is **[MP PET 1997]**

(a) $\frac{197}{200}$ (b) $\frac{27}{200}$ (c) $\frac{83}{100}$ (d) None of these

Solution: (d) A denote the event getting I; B denote the event getting II;

C denote the event getting III; and D denote the event getting fail.

Obviously, these four events are mutually exclusive and exhaustive, therefore

$P(A) + P(B) + P(C) + P(D) = 1 \Rightarrow P(D) = 1 - 0.95 = 0.05$.

1.3 Classical definition of Probability

If a random experiment results in n mutually exclusive, equally likely and exhaustive outcomes, out of which m are favourable to the occurrence of an event A, then the probability of occurrence of A is given by

$$P(A) = \frac{m}{n} = \frac{\text{Number of outcomes favourable to } A}{\text{Number of total outcomes}}$$

It is obvious that $0 \leq m \leq n$. If an event A is certain to happen, then $m = n$, thus $P(A) = 1$.

If A is impossible to happen, then $m = 0$ and so $P(A) = 0$. Hence we conclude that

$$0 \leq P(A) \leq 1.$$

Further, if $\overline{A}$ denotes negative of A *i.e.* event that A doesn't happen, then for above cases m, n; we shall have

$$P(\overline{A}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)$$

$\therefore \quad P(A) + P(\overline{A}) = 1.$

Notations : For two events A and B,

(i) A' or $\overline{A}$ or A^C stands for the non-occurrence or negation of A.

(ii) $A \cup B$ stands for the occurrence of at least one of A and B.

(iii) $A \cap B$ stands for the simultaneous occurrence of A and B.

(iv) $A' \cap B'$ stands for the non-occurrence of both A and B.

(v) $A \subseteq B$ stands for "the occurrence of A implies occurrence of B".

1.4 Some important remarks about Coins, Dice , Playing cards and Envelopes

(1) **Coins :** A coin has a head side and a tail side. If an experiment consists of more than a coin, then coins are considered to be distinct if not otherwise stated.

Number of exhaustive cases of tossing n coins simultaneously (or of tossing a coin n times) = 2^n.

(2) **Dice :** A die (cubical) has six faces marked 1, 2, 3, 4, 5, 6. We may have tetrahedral (having four faces 1, 2, 3, 4) or pentagonal (having five faces 1, 2, 3, 4, 5) die. As in the case of coins, if we have more than one die, then all dice are considered to be distinct if not otherwise stated.

Number of exhaustive cases of throwing n dice simultaneously (or throwing one dice n times) = 6^n.

(3) **Playing cards :** A pack of playing cards usually has 52 cards. There are 4 suits (Spade, Heart, Diamond and Club) each having 13 cards. There are two colours red (Heart and Diamond) and black (Spade and Club) each having 26 cards.

In thirteen cards of each suit, there are 3 face cards or coart cards namely king, queen and jack. So there are in all 12 face cards (4 kings, 4 queens and 4 jacks). Also there are 16 honour cards, 4 of each suit namely ace, king, queen and jack.

(4) **Probability regarding *n* letters and their envelopes :** If n letters corresponding to n envelopes are placed in the envelopes at random, then

(i) Probability that all letters are in right envelopes $= \frac{1}{n!}$.

(ii) Probability that all letters are not in right envelopes $= 1 - \frac{1}{n!}$.

(iii) Probability that no letter is in right envelopes $= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - ... + (-1)^n \frac{1}{n!}$.

(iv) Probability that exactly r letters are in right envelopes $= \frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - ..... + (-1)^{n-r}\frac{1}{(n-r)!}\right]$.

Example: 3 If $(1+3p)/3, (1-p)/4$ and $(1-2p)/2$ are the probabilities of three mutually exclusive events, then the set of all values of p is **[IIT 1986; AMU 2002; AIEEE 2003]**

(a) $\frac{1}{3} \le p \le \frac{1}{2}$ (b) $\frac{1}{3} < p < \frac{1}{2}$ (c) $\frac{1}{2} \le p \le \frac{2}{3}$ (d) $\frac{1}{2} < p < \frac{2}{3}$

Solution: (a) Since $\frac{(1+3p)}{3}, \frac{(1-p)}{4}$ and $\left(\frac{1-2p}{2}\right)$ are the probabilities of the three events, we must have

$0 \le \frac{1+3p}{3} \le 1, 0 \le \frac{1-p}{4} \le 1$ and $0 \le \frac{1-2p}{2} \le 1 \Rightarrow -1 \le 3p \le 2, -3 \le p \le 1$ and $-1 \le 2p \le 1$

$\Rightarrow \quad -\frac{1}{3} \le p \le \frac{2}{3}, -3 \le p \le 1$ and $-\frac{1}{2} \le p \le \frac{1}{2}$.

Also as $\frac{1+3p}{3}, \frac{1-p}{4}$ and $\frac{1-2p}{2}$ are the probabilities of three mutually exclusive events,

$0 \le \frac{1+3p}{3} + \frac{1-p}{4} + \frac{1-2p}{2} \le 1 \Rightarrow 0 \le 4 + 12p + 3 - 3p + 6 - 12p \le 12 \Rightarrow \frac{1}{3} \le p \le \frac{13}{3}$

Thus the required values of p are such that $\max\left\{-\frac{1}{3}, -3, -\frac{1}{2}, \frac{1}{3}\right\} \le p \le \min\left\{\frac{2}{3}, 1, \frac{1}{2}, \frac{13}{3}\right\} \Rightarrow \frac{1}{3} \le p \le \frac{1}{2}$.

Example: 4 The probability that a leap year selected randomly will have 53 Sundays is **[MP PET 1991, 93, 95]**

(a) $\frac{1}{7}$ (b) $\frac{2}{7}$ (c) $\frac{4}{53}$ (d) $\frac{4}{49}$

Solution: (b) A leap year contain 366 days *i.e.* 52 weeks and 2 days, clearly there are 52 Sundays in 52 weeks.
For the remaining two days, we may have any of the two days
(i) Sunday and Monday, (ii) Monday and Tuesday, (iii) Tuesday and Wednesday, (iv) Wednesday and Thursday,
(v) Thursday and Friday, (iv) Friday and Saturday and (vii) Saturday and Sunday.

Now for 53 Sundays, one of the two days must be Sundays, hence required probability $= \frac{2}{7}$.

Example: 5 Three identical dice are rolled. The probability that same number will appear on each of them will be
[SCRA 1991; MP PET 1989; IIT 1984; Rajasthan PET 2000, 02; DCE 2001]

(a) $\frac{1}{6}$ (b) $\frac{1}{36}$ (c) $\frac{1}{18}$ (d) $\frac{3}{28}$

Solution: (b) If three identical dice are rolled then total number of sample points $= 6 \times 6 \times 6 = 216$.
Favourable events (same number appear on each dice) are

(1, 1, 1) (2, 2, 2) ....(6, 6, 6). $\therefore$ Required probability $= \frac{6}{216} = \frac{1}{36}$.

1.5 Problems based on Combination and Permutation

(1) **Problems based on combination or selection :** To solve such kind of problems, we use $^nC_r = \frac{n!}{r!(n-r)!}$.

Example: 6 Three of the six vertices of a regular hexagon are chosen at random. The probability that the triangle with these three vertices is equilateral, is equal to **[IIT 1995; MP PET 2002]**

(a) $\frac{1}{2}$ (b) $\frac{1}{5}$ (c) $\frac{1}{10}$ (d) $\frac{1}{20}$

Solution: (c) Total number of triangles which can be formed = $^6C_3 = \frac{6 \times 5 \times 4}{1 \times 2 \times 3} = 20$

Number of equilateral triangles = 2. $\therefore$ Required probability $= \frac{2}{20} = \frac{1}{10}$.

Example: 7 Three distinct numbers are selected from 100 natural number. The probability that all the three numbers are divisible by 2 and 3 is **[IIT Screening 2004]**

(a) $\frac{4}{25}$ (b) $\frac{4}{35}$ (c) $\frac{4}{55}$ (d) $\frac{4}{1155}$

Solution: (d) The numbers should be divisible by 6. Thus the number of favourable ways is ${}^{16}C_3$ (as there are 16 numbers in first 100 natural numbers, divisible by 6). Required probability is $\frac{{}^{16}C_3}{{}^{100}C_3} = \frac{16 \times 15 \times 14}{100 \times 99 \times 98} = \frac{4}{1155}$.

Example: 8 Out of 21 tickets marked with numbers from 1 to 21, three are drawn at random. The chance that the numbers on them are in A.P., is **[Roorkee 1988; DCE 1999]**

(a) $\frac{10}{133}$ (b) $\frac{9}{133}$ (c) $\frac{9}{1330}$ (d) None of these

Solution: (a) Total number of ways $= {}^{21}C_3 = 1330$. If common difference of the A.P. is to be 1, then the possible groups are 1, 2, 3; 2, 3, 4; ......19, 20, 21.

If the common difference is 2, then possible groups are 1, 3, 5; 2, 4, 6; ..... 17, 19, 21.

Proceeding in the same way, if the common difference is 10, then the possible group is 1, 10, 21.

Thus if the common difference of the A.P. is to be ≥ 11, obviously there is no favourable case.

Hence, total number of favourable cases = 19 +17 + 15 + ...+ 3 + 1 =100

Hence, required probability $= \frac{100}{1330} = \frac{10}{133}$.

(2) **Problems based on permutation or arrangement :** To solve such kind of problems, we use ${}^nP_r = \frac{n!}{(n-r)!}$.

Example: 9 There are four letters and four addressed envelopes. The chance that all letters are not dispatched in the right envelope is

[Rajasthan PET 1997; MP PET 1999; DCE 1999]

(a) $\frac{19}{24}$ (b) $\frac{21}{23}$ (c) $\frac{23}{24}$ (d) $\frac{1}{24}$

Solution: (c) Required probability is 1 – P (they go in concerned envelopes) $= 1 - \frac{1}{4!} = \frac{23}{24}$.

Example: 10 The letters of the word 'ASSASSIN' are written down at random in a row. The probability that no two S occur together is

[BIT Ranchi 1990; IIT 1983]

(a) $\frac{1}{35}$ (b) $\frac{1}{14}$ (c) $\frac{1}{15}$ (d) None of these

Solution: (b) Total ways of arrangements $= \frac{8!}{2!.4!}$. $\bullet w \bullet x \bullet y \bullet z \bullet$

Now 'S' can have places at dot's and in places of w, x, y, z we have to put 2A's, one I and one N.

Therefore, favourable ways $= {}^5C_4\left(\frac{4!}{2!}\right)$. Hence, required probability $= \frac{5.4!2!4!}{2!8!} = \frac{1}{14}$.

1.6 Odds In favour and Odds against an Event

As a result of an experiment if "a" of the outcomes are favourable to an event E and "b" of the outcomes are against it, then we say that odds are a to b in favour of E or odds are b to a against E.

Thus odds in favour of an event $E = \frac{\text{Number of favourable cases}}{\text{Number of unfavourable cases}} = \frac{a}{b} = \frac{a/(a+b)}{b/(a+b)} = \frac{P(E)}{P(\overline{E})}$.

Similarly, odds against an event $E = \frac{\text{Number of unfavourable cases}}{\text{Number of favourable cases}} = \frac{b}{a} = \frac{P(\overline{E})}{P(E)}$.

Important Tips

☞ *If odds in favour of an event are a : b, then the probability of the occurrence of that event is* $\frac{a}{a+b}$ *and the probability of non-occurrence of that event is* $\frac{b}{a+b}$.

☞ *If odds against an event are a : b, then the probability of the occurrence of that event is* $\frac{b}{a+b}$ *and the probability of non-occurrence of that event is* $\frac{a}{a+b}$.

Example: 11 Two dice are tossed together. The odds in favour of the sum of the numbers on them as 2 are **[Rajasthan PET 1**

(a) 1 : 36 (b) 1 : 35 (c) 35 : 1 (d) None of these

Solution: (b) If two dice are tossed, total number of events = 6 × 6 = 36.

Favourable event is (1, 1). Number of favourable events = 1

$\therefore$ odds in favour $=\frac{1}{36-1}=\frac{1}{35}$.

Example: 12 A party of 23 persons take their seats at a round table. The odds against two persons sitting together are **[Rajasthan PET 1999]**

(a) 10 : 1 (b) 1 : 11 (c) 9 : 10 (d) None of these

Solution: (a) $P=\frac{(21)!\,2!}{(22)!}=\frac{1}{11}=\frac{1}{1+10}$. $\therefore$ odd against = 10 : 1.

1.7 Addition Theorems on Probability

Notations : (i) $P(A+B)$ or $P(A\cup B)$ = Probability of happening of A or B

= Probability of happening of the events A or B or both

= Probability of occurrence of at least one event A or B

(ii) $P(AB)$ or $P(A\cap B)$ = Probability of happening of events A and B together.

(1) **When events are not mutually exclusive :** If A and B are two events which are not mutually exclusive, then $P(A\cup B)=P(A)+P(B)-P(A\cap B)$ or $P(A+B)=P(A)+P(B)-P(AB)$.

For any three events A, B, C

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$$

or $P(A+B+C)=P(A)+P(B)+P(C)-P(AB)-P(BC)-P(CA)+P(ABC)$.

(2) **When events are mutually exclusive :** If A and B are mutually exclusive events, then

$n(A\cap B)=0 \Rightarrow P(A\cap B)=0$

$\therefore P(A\cup B)=P(A)+P(B)$.

For any three events A, B, C which are mutually exclusive,

$P(A\cap B)=P(B\cap C)=P(C\cap A)=P(A\cap B\cap C)$ = 0 $\therefore P(A\cup B\cup C)=P(A)+P(B)+P(C)$.

The probability of happening of any one of several mutually exclusive events is equal to the sum of their probabilities, *i.e.* if $A_1, A_2 .....A_n$ are mutually exclusive events, then

$$P(A_1+A_2+...+A_n)=P(A_1)+P(A_2)+.....+P(A_n) \text{ i.e. } P(\sum A_i)=\sum P(A_i).$$

(3) **When events are independent :** If A and B are independent events, then $P(A\cap B)=P(A).P(B)$

$\therefore P(A\cup B)=P(A)+P(B)-P(A).P(B)$.

(4) **Some other theorems**

(i) Let A and B be two events associated with a random experiment, then

(a) $P(\overline{A} \cap B) = P(B) - P(A \cap B)$ (b) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

If $B \subset A$, then

(a) $P(A \cap \overline{B}) = P(A) - P(B)$ (b) $P(B) \le P(A)$

Similarly if $A \subset B$, then

(a) $(\overline{A} \cap B) = P(B) - P(A)$ (b) $P(A) \le P(B)$.

Note : ❑ *Probability of occurrence of neither A nor B is* $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$.

(ii) **Generalization of the addition theorem :** If $A_1, A_2, \ldots\ldots, A_n$ are n events associated with a random experiment, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{\substack{i,j=1 \\ i \ne j}}^{n} P(A_i \cap A_j) + \sum_{\substack{i,j,k=1 \\ i \ne j \ne k}}^{n} P(A_i \cap A_j \cap A_k) + \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots\ldots \cap A_n).$$

If all the events $A_i (i = 1, 2 \ldots, n)$ are mutually exclusive, then $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$

i.e. $P(A_1 \cup A_2 \cup \ldots. \cup A_n) = P(A_1) + P(A_2) + \ldots. + P(A_n)$.

(iii) **Booley's inequality :** If $A_1, A_2, \ldots. A_n$ are n events associated with a random experiment, then

(a) $P\left(\bigcap_{i=1}^{n} A_i\right) \ge \sum_{i=1}^{n} P(A_i) - (n-1)$ (b) $P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$

These results can be easily established by using the Principle of Mathematical Induction.

Important Tips

Let *A, B,* and *C* are three arbitrary events. Then

Verbal description of event	*Equivalent Set Theoretic Notation*
(i) Only A occurs	*(i)* $A \cap \overline{B} \cap \overline{C}$
(ii) Both A and B, but not C occur	*(ii)* $A \cap B \cap \overline{C}$
(iii) All the three events occur	*(iii)* $A \cap B \cap C$
(iv) At least one occurs	*(iv)* $A \cup B \cup C$
(v) At least two occur	*(v)* $(A \cap B) \cup (B \cap C) \cup (A \cap C)$
(vi) One and no more occurs	*(vi)* $(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)$
(vii) Exactly two of A, B and C occur	*(vii)* $(A \cap B \cap \overline{C}) \cup (\overline{A} \cap B \cap C) \cup (A \cap \overline{B} \cap C)$
(viii) None occurs	*(viii)* $\overline{A} \cap \overline{B} \cap \overline{C} = \overline{A \cup B \cup C}$
(ix) Not more than two occur	*(ix)* $(A \cap B) \cup (B \cap C) \cup (A \cap C) - (A \cap B \cap C)$
(x) Exactly one of A and B occurs	*(x)* $(A \cap \overline{B}) \cup (\overline{A} \cap B)$

Example: 13 A box contains 6 nails and 10 nuts. Half of the nails and half of the nuts are rusted. If one item is chosen at random, what is the probability that it is rusted or is a nail **[MP PET 1992, 2000]**

(a) 3/16 (b) 5/16 (c) 11/16 (d) 14/16

Solution: (c) Let *A* be the event that the item chosen is rusted and *B* be the event that the item chosen is a nail.

$\therefore\ P(A) = \frac{8}{16}, P(B) = \frac{6}{16}$ and $P(A \cap B) = 3/16$

Required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{8}{16} + \frac{6}{16} - \frac{3}{16} = \frac{11}{16}$.

Example: 14 The probability that a man will be alive in 20 years is $\frac{3}{5}$ and the probability that his wife will be alive in 20 years is $\frac{2}{3}$. Then the probability that at least one will be alive in 20 years is **[Bihar CEE 1994]**

(a) $\frac{13}{15}$ (b) $\frac{7}{15}$ (c) $\frac{4}{15}$ (d) None of these

Solution: (a) Let A be the event that the husband will be alive 20 years. B be the event that the wife will be alive 20 years. Clearly A and B are independent events. $\therefore\ P(A \cap B) = P(A)P(B)$.

Given $P(A) = \frac{3}{5}, P(B) = \frac{2}{3}$.

The probability that at least one of them will be alive 20 years is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A).P(B) = \frac{3}{5} + \frac{2}{3} - \frac{3}{5} \cdot \frac{2}{3} = \frac{9 + 10 - 6}{15} = \frac{13}{15}.$$

Example: 15 Let A and B be two events such that $P(A) = 0.3$ and $P(A \cup B) = 0.8$. If A and B are independent events, then $P(B) =$

[IIT 1990; UPSEAT 2001, 02]

(a) $\frac{5}{6}$ (b) $\frac{5}{7}$ (c) $\frac{3}{5}$ (d) $\frac{2}{5}$

Solution: (b) Here $P(A \cup B) = 0.8$, $P(A) = 0.3$ and A and B are independent events.

Let $P(B) = x$. $\therefore\ P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow P(A \cup B) = P(A) + P(B) - P(A).P(B)$

$\Rightarrow 0.8 = 0.3 + x - 0.3x \Rightarrow x = \frac{5}{7}$.

Example: 16 A card is chosen randomly from a pack of playing cards. The probability that it is a black king or queen of heart or jack is

[Rajasthan PET 1998]

(a) 1/52 (b) 6/52 (c) 7/52 (d) None of these

Solution: (c) Let A, B, C are the events of choosing a black king, a queen of heart and a jack respectively.

$\therefore\ P(A) = \frac{2}{52}, P(B) = \frac{1}{52}, P(C) = \frac{4}{52}$

These are mutually exclusive events, $\therefore\ P(A \cup B \cup C) = \frac{2}{52} + \frac{1}{52} + \frac{4}{52} = \frac{7}{52}$.

Example: 17 If A and B are events such that $P(A \cup B) = 3/4, P(A \cap B) = 1/4, P(\overline{A}) = 2/3$, then $P(\overline{A} \cap B)$ is **[AIEEE 2002]**

(a) 5/12 (b) 3/8 (c) 5/8 (d) 1/4

Solution: (a) $P(A \cup B) = \frac{3}{4}, P(A \cap B) = \frac{1}{4},\ P(\overline{A}) = \frac{2}{3} \Rightarrow P(A) = \frac{1}{3}$.

$\therefore\ P(A \cap B) = P(A) + P(B) - P(A \cup B) \Rightarrow \frac{1}{4} = \frac{1}{3} + P(B) - \frac{3}{4} \Rightarrow P(B) = \frac{2}{3}$.

$P(\overline{A} \cap B) = P(B) - P(A \cap B) = \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12}$.

Example: 18 The probability that A speaks truth is $\frac{4}{5}$, while this probability for B is $\frac{3}{4}$. The probability that they contradict each other when asked to speak on a fact is **[AIEEE 2004]**

(a) $\frac{4}{5}$ (b) $\frac{1}{5}$ (c) $\frac{7}{20}$ (d) $\frac{3}{20}$

Solution: (c) Let E be the event that B speaks truth and F be the event that A speaks truth.

Now $P(E)=\frac{75}{100}=\frac{3}{4}$ and $P(F)=\frac{80}{100}=\frac{4}{5}$.

$\therefore$ P (A and B contradict each other)

= P [(B tells truth and A tells lie) or (B tells lie and A tells truth)]

$=P[(E\cap\overline{F})\cup(\overline{E}\cap F)]=P(E).P(\overline{F})+P(\overline{E}).P(F)=\frac{3}{4}\times\frac{1}{5}+\frac{1}{4}\times\frac{4}{5}=\frac{7}{20}$.

Example: 19 A student appears for tests I, II and III. The student is successful if he passes either in tests I and II or tests I and III. The probabilities of the student passing in tests I, II, III are p, q and $\frac{1}{2}$ respectively. If the probability that the student is successful is $\frac{1}{2}$, then **[IIT 1986]**

(a) $p=1, q=0$ (b) $p=\frac{2}{3}, q=\frac{1}{2}$

(c) There are infinitely many values of p and q (d) All of the above

Solution: (c) Let A, B and C be the events that the student is successful in test I, II and III respectively, then P (the student is successful)

$=P[(A\cap B\cap C')\cup(A\cap B'\cap C)\cup(A\cap B\cap C)]=P(A\cap B\cap C')+P(A\cap B'\cap C)+P(A\cap B\cap C)$

$=P(A).P(B).P(C')+P(A).P(B').P(C)+P(A).P(B).P(C)$ [$\because$ A, B, C are independent]

$=pq\left(1-\frac{1}{2}\right)+p(1-q)\left(\frac{1}{2}\right)+pq\left(\frac{1}{2}\right)=\frac{1}{2}p(1+q) \Rightarrow \frac{1}{2}=\frac{1}{2}p(1+q) \Rightarrow p(1+q)=1$.

This equation has infinitely many values of p and q.

Example: 20 A man and his wife appear for an interview for two posts. The probability of the husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that only one of them will be selected.**[AISSE 1**

(a) $\frac{1}{7}$ (b) $\frac{2}{7}$ (c) $\frac{3}{7}$ (d) None of these

Solution: (b) The probability of husband is not selected $=1-\frac{1}{7}=\frac{6}{7}$; The probability that wife is not selected $=1-\frac{1}{5}=\frac{4}{5}$

The probability that only husband is selected $=\frac{1}{7}\times\frac{4}{5}=\frac{4}{35}$; The probability that only wife is selected $=\frac{1}{5}\times\frac{6}{7}=\frac{6}{35}$

Hence, required probability $=\frac{6}{35}+\frac{4}{35}=\frac{10}{35}=\frac{2}{7}$.

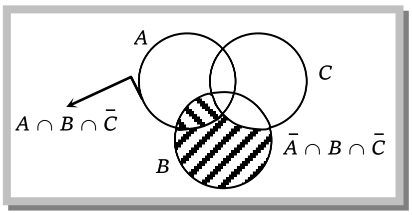
Example: 21 If $P(B)=\frac{3}{4}$, $P(A\cap B\cap\overline{C})=\frac{1}{3}$ and $P(\overline{A}\cap B\cap\overline{C})=\frac{1}{3}$, then $P(B\cap C)$ is **[IIT Screening 2003]**

(a) 1/12 (b) 1/6 (c) 1/15 (d) 1/9

Solution: (a) From Venn diagram, we can see that

$P(B\cap C)=P(B)-P(A\cap B\cap\overline{C})-P(\overline{A}\cap B\cap\overline{C})$

$=\frac{3}{4}-\frac{1}{3}-\frac{1}{3}=\frac{1}{12}$.



Example: 22 A purse contains 4 copper coins and 3 silver coins, the second purse contains 6 copper coins and 2 silver coins. If a coin is drawn out of any purse, then the probability that it is a copper coin is**[Ranchi BIT 199**

(a) 4/7 (b) 3/4 (c) 37/56 (d) None of these

Solution: (c) Required probability $=\frac{1}{2}\cdot\frac{4}{7}+\frac{1}{2}\cdot\frac{6}{8}=\frac{37}{56}$.

Example: 23 The probability of happening an event A is 0.5 and that of B is 0.3. If A and B are mutually exclusive events, then the probability of happening neither A nor B is **[IIT 1980; DCE 2000]**

(a) 0.6 (b) 0.2 (c) 0.21 (d) None of these

Solution: (b) $P(\overline{A}\cap\overline{B})=P(\overline{A\cup B})=1-P(A\cup B)$

Since A and B are mutually exclusive, so $P(A\cup B)=P(A)+P(B)$

Hence, required probability $=1-(0.5+0.3)=0.2$.

1.8 Conditional Probability

Let A and B be two events associated with a random experiment. Then, the probability of occurrence of A under the condition that B has already occurred and $P(B)\neq 0$, is called the conditional probability and it is denoted by $P(A/B)$.

Thus, $P(A/B)$ = Probability of occurrence of A, given that B has already happened.

$$=\frac{P(A\cap B)}{P(B)}=\frac{n(A\cap B)}{n(B)}.$$

Similarly, $P(B/A)$ = Probability of occurrence of B, given that A has already happened.

$$=\frac{P(A\cap B)}{P(A)}=\frac{n(A\cap B)}{n(A)}.$$

Note : ❑ Sometimes, $P(A/B)$ is also used to denote the probability of occurrence of A when B occurs. Similarly, $P(B/A)$ is used to denote the probability of occurrence of B when A occurs.

(1) **Multiplication theorems on probability**

(i) If A and B are two events associated with a random experiment, then $P(A\cap B)=P(A).P(B/A)$, if $P(A)\neq 0$ or $P(A\cap B)=P(B).P(A/B)$, if $P(B)\neq 0$.

(ii) **Extension of multiplication theorem :** If $A_1,A_2,....,A_n$ are n events related to a random experiment, then $P(A_1\cap A_2\cap A_3\cap....\cap A_n)=P(A_1)P(A_2/A_1)P(A_3/A_1\cap A_2)....P(A_n/A_1\cap A_2\cap...\cap A_{n-1})$,

where $P(A_i/A_1\cap A_2\cap...\cap A_{i-1})$ represents the conditional probability of the event A_i, given that the events $A_1,A_2,.....,A_{i-1}$ have already happened.

(iii) **Multiplication theorems for independent events :** If A and B are independent events associated with a random experiment, then $P(A\cap B)=P(A).P(B)$ *i.e.,* the probability of simultaneous occurrence of two independent events is equal to the product of their probabilities.

By multiplication theorem, we have $P(A\cap B)=P(A).P(B/A)$.

Since A and B are independent events, therefore $P(B/A)=P(B)$. Hence, $P(A\cap B)=P(A).P(B)$.

(iv) **Extension of multiplication theorem for independent events :** If $A_1,A_2,....,A_n$ are independent events associated with a random experiment, then $P(A_1\cap A_2\cap A_3\cap...\cap A_n)=P(A_1)P(A_2)...P(A_n)$.

By multiplication theorem, we have

$$P(A_1 \cap A_2 \cap A_3 \cap ... \cap A_n) = P(A_1)P(A_2 / A_1)P(A_3 / A_1 \cap A_2)...P(A_n / A_1 \cap A_2 \cap ... \cap A_{n-1})$$

Since $A_1, A_2, ..., A_{n-1}, A_n$ are independent events, therefore

$$P(A_2 / A_1) = P(A_2), P(A_3 / A_1 \cap A_2) = P(A_3), ..., P(A_n / A_1 \cap A_2 \cap ... \cap A_{n-1}) = P(A_n)$$

Hence, $P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1)P(A_2)....P(A_n)$.

(2) **Probability of at least one of the *n* independent events :** If $p_1, p_2, p_3, ..., p_n$ be the probabilities of happening of n independent events $A_1, A_2, A_3, ..., A_n$ respectively, then

(i) Probability of happening none of them

$$= P(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 ...... \cap \overline{A}_n) = P(\overline{A}_1).P(\overline{A}_2).P(\overline{A}_3).....P(\overline{A}_n) = (1-p_1)(1-p_2)(1-p_3)....(1-p_n).$$

(ii) Probability of happening at least one of them

$$= P(A_1 \cup A_2 \cup A_3 .... \cup A_n) = 1 - P(\overline{A}_1)P(\overline{A}_2)P(\overline{A}_3)....P(\overline{A}_n) = 1 - (1-p_1)(1-p_2)(1-p_3)...(1-p_n).$$

(iii) Probability of happening of first event and not happening of the remaining

$$= P(A_1)P(\overline{A}_2)P(\overline{A}_3).....P(\overline{A}_n) = p_1(1-p_2)(1-p_3).......(1-p_n)$$

Example: 24 If $4P(A) = 6, P(B) = 10, P(A \cap B) = 1$, then $P\left(\frac{B}{A}\right) =$ **[MP PET 2003]**

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) $\frac{7}{10}$ (d) $\frac{19}{60}$

Solution: (a) $P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{(1/10)}{(1/4)} = \frac{2}{5}$.

Example: 25 A coin is tossed three times in succession. If E is the event that there are at least two heads and F is the event in which first throw is a head, then $P\left(\frac{E}{F}\right) =$ **[MP PET 1996]**

(a) $\frac{3}{4}$ (b) $\frac{3}{8}$ (c) $\frac{1}{2}$ (d) $\frac{1}{8}$

Solution: (a) $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$n(E) = 4, n(F) = 4$ and $n(E \cap F) = 3$

$\therefore P\left(\frac{E}{F}\right) = \frac{P(E \cap F)}{P(F)} = \frac{3/8}{4/8} = \frac{3}{4}$.

Example: 26 Two cards are drawn one by one from a pack of cards. The probability of getting first card an ace and second an honour card is (before drawing second card first card is not placed again in the pack)**[UPSEAQT 19**

(a) 1/26 (b) 5/52 (c) 5/221 (d) 4/13

Solution: (c) $P(E_1) = \frac{4}{52} = \frac{1}{13}, P\left(\frac{E_2}{E_1}\right) = \frac{15}{51} = \frac{5}{17}$

$P(E_1 \cap E_2) = P(E_1).P\left(\frac{E_2}{E_1}\right) = \frac{1}{13}.\frac{5}{17} = \frac{5}{221}$.

Example: 27 If A and B are two events such that $P(A) \neq 0$ and $P(B) \neq 1$, then $P\left(\frac{\overline{A}}{\overline{B}}\right) =$

[IIT 1982; RPET 1995, 2000; DCE 2000; UPSEAT 2001]

(a) $1 - P\left(\frac{A}{B}\right)$ (b) $1 - P\left(\frac{\overline{A}}{B}\right)$ (c) $\frac{1 - P(A \cup B)}{P(\overline{B})}$ (d) $\frac{P(\overline{A})}{P(\overline{B})}$

Solution: (c) $P\left(\frac{\overline{A}}{\overline{B}}\right)=\frac{P(\overline{A}\cap\overline{B})}{P(\overline{B})}=\frac{P(\overline{A\cup B})}{P(\overline{B})}=\frac{1-P(A\cup B)}{P(\overline{B})}$.

Example: 28 If A and B are two events such that $P(A\cup B)=P(A\cap B)$, then the true relation is **[IIT 1985]**

(a) $P(A)+P(B)=0$ (b) $P(A)+P(B)=P(A)P\left(\frac{B}{A}\right)$

(c) $P(A)+P(B)=2P(A)P\left(\frac{B}{A}\right)$ (d) None of these

Solution: (c) $P(A\cup B)=P(A)+P(B)-P(A\cap B)\Rightarrow P(A\cap B)=P(A)+P(B)-P(A\cap B)$ $\{\because P(A\cap B)=P(A\cup B)\}$

$\Rightarrow 2P(A\cap B)=P(A)+P(B)\Rightarrow 2P(A).\frac{P(A\cap B)}{P(A)}=P(A)+P(B)\Rightarrow 2P(A)P\left(\frac{B}{A}\right)=P(A)+P(B).$

Example: 29 Let E and F be two independent events. The probability that both E and F happens is $\frac{1}{12}$ and the probability that neither E nor F happens is $\frac{1}{2}$, then **[IIT 1993]**

(a) $P(E)=\frac{1}{3},P(F)=\frac{1}{4}$ (b) $P(E)=\frac{1}{2},P(F)=\frac{1}{6}$ (c) $P(E)=\frac{1}{6},P(F)=\frac{1}{2}$ (d) None of these

Solution: (a) We are given $P(E\cap F)=\frac{1}{12}$ and $P(\overline{E}\cap\overline{F})=\frac{1}{2}$

$\Rightarrow P(E).P(F)=\frac{1}{12}$ .....(i) and $P(\overline{E}).P(\overline{F})=\frac{1}{2}$ .....(ii)

$\Rightarrow \{1-P(E)\}\{(1-P(F)\}=\frac{1}{2}\Rightarrow 1+P(E)P(F)-P(E)-P(F)=\frac{1}{2}\Rightarrow 1+\frac{1}{12}-[P(E)+P(F)]=\frac{1}{2}$

$\Rightarrow P(E)+P(F)=\frac{7}{12}$ .....(iii)

On solving (i) and (iii), we get $P(E)=\frac{1}{3},\frac{1}{4}$ and $P(F)=\frac{1}{4},\frac{1}{3}$.

Example: 30 Let p denotes the probability that a man aged x years will die in a year. The probability that out of n men $A_1,A_2,A_3,.....A_n$ each aged x, A_1 will die in a year and will be the first to die, is **[MNR 1987; UPSEAT 2000]**

(a) $\frac{1}{n}[1-(1-p)^n]$ (b) $[1-(1-p)^n]$ (c) $\frac{1}{n-1}[1-(1-p)^n]$ (d) None of these

Solution: (a) Let E_i denotes the event that A_i dies in a year.

Then $P(E_i)=p$ and $P(E_i')=1-p$ for i = 1, 2, ...n

P (none of $A_1,A_2,.....A_3$ dies in a year) $=P(E_1'\cap E_2'\cap.....E_n')=P(E_1')P(E_2')....P(E_n')=(1-p)^n$,

because $E_1,E_2,.......E_n$ are independent.

Let E denote the event that at least one of $A_1,A_2,.....A_n$ dies in a year.

Then $P(E)=1-P(E_1'\cap E_2'\cap.....\cap E_n')=1-(1-p)^n$

Let F denote the event that A_1 is the first to die.

Then $P(F/E)=\frac{1}{n}$. Also, $P(F)=P(E).P(F/E)=\frac{1}{n}[1-(1-p)^n]$.

Example: 31 A problem of mathematics is given to three students whose chances of solving the problem are 1/3, 1/4 and 1/5 respectively. The probability that the question will be solved is **[BIT Ranchi 1991; MP PET 1990]**

(a) 2/3 (b) 3/4 (c) 4/5 (d) 3/5

Solution: (d) The probabilities of students not solving the problem are $1-\frac{1}{3}=\frac{2}{3},1-\frac{1}{4}=\frac{3}{4}$ and $1-\frac{1}{5}=\frac{4}{5}$.

Therefore the probability that the problem is not solved by any one of them $=\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}=\frac{2}{5}$.

Hence, the probability that problem is solved $=1-\frac{2}{5}=\frac{3}{5}$.

Example: 32 The probability of happening an event A in one trial is 0.4. The probability that the event A happens at least once in three independent trials is **[IIT 1980; Kurukshetra CEE 1998; DCE 2001]**

(a) 0.936 (b) 0.784 (c) 0.904 (d) 0.216

Solution: (b) Here $P(A)=0.4$ and $P(\overline{A})=0.6$

Probability that A does not happen at all $=(0.6)^3$. Thus required probability $=1-(0.6)^3=0.784$.

1.9 Total Probability and Baye's rule

(1) **The law of total probability :** Let S be the sample space and let $E_1, E_2, ...E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or .... or E_n, then $P(A)=P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+...+P(E_n)P(A/E_n)$.

(2) **Baye's rule :** Let S be a sample space and $E_1, E_2, ...E_n$ be n mutually exclusive events such that $\bigcup_{i=1}^{n} E_i = S$ and $P(E_i)>0$ for i = 1, 2, ..., n. We can think of (E_i's as the causes that lead to the outcome of an experiment. The probabilities $P(E_i)$, i = 1, 2, ..., n are called prior probabilities. Suppose the experiment results in an outcome of event A, where $P(A)$ > 0. We have to find the probability that the observed event A was due to cause E_i, that is, we seek the conditional probability $P(E_i/A)$. These probabilities are called posterior probabilities, given by Baye's rule as $P(E_i/A)=\frac{P(E_i).P(A/E_i)}{\sum_{k=1}^{n} P(E_k)P(A/E_k)}$.

Example: 33 In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total bolts. Of their output 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product. Then the probability that the bolt drawn is defective is

(a) 0.0345 (b) 0.345 (c) 3.45 (d) 0.0034

Solution: (a) Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = the bolts is manufactured by machine A; E_2 = the bolts is manufactured by machine B; E_3 = the bolts is manufactured by machine C, and A = the bolt is defective.

Then $P(E_1)=\frac{25}{100}=\frac{1}{4}, P(E_2)=\frac{35}{100}, P(E_3)=\frac{40}{100}$.

$P(A/E_1)$ = Probability that the bolt drawn is defective given the condition that it is manufactured by machine A = 5/100.

Similarly $P(A/E_2)=\frac{4}{100}$ and $P(A/E_3)=\frac{2}{100}$.

Using the law of total probability, we have $P(A)=P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)$

$$=\frac{25}{100}\times\frac{5}{100}+\frac{35}{100}\times\frac{4}{100}+\frac{40}{100}\times\frac{2}{100}=0.0345.$$

Example: 34 A lot contains 20 articles. The probability that the lot contains 2 defective articles is 0.4 and the probability that the lot contains exactly 3 defective articles is 0.6. Articles are drawn at random one by one without replacement and tested till all the defective articles are found. The probability that the testing procedure ends at the twelfth testing is

(a) $\frac{9}{1900}$ (b) $\frac{19}{1000}$ (c) $\frac{99}{1900}$ (d) $\frac{19}{900}$

Solution: (c) The testing procedure may terminate at the twelfth testing in two mutually exclusive ways.

(I) When lot contains 2 defective articles, (II) When lot contains 3 defective articles.

Consider the following events.

A = Testing procedure ends at the twelfth testing.

A_1 = Lot contains 2 defective articles.

A_2 = Lot contains 3 defective articles.

Required probability

$= P(A) = P(A \cap A_1) \cup (A \cap A_2) = P(A \cap A_1) + P(A \cap A_2) = P(A_1)P(A/A_1) + P(A_2)P(A/A_2)$

Now, $P(A/A_1) =$ Probability that first 11 draws contain 10 non-defective and one defective and 12th draw contains a defective article.

$= \frac{{}^{18}C_{10} \times {}^{2}C_1}{{}^{20}C_{11}} \times \frac{1}{9}$.

And $P(A/A_2)$ = Probability that first 11 draws contain 9 non defective and 2 defective articles and 12th draw contains a defective article = $\frac{{}^{17}C_9 \times {}^{3}C_2}{{}^{20}C_{11}} \times \frac{1}{9}$

Hence, required probability $= 0.4 \times \frac{{}^{18}C_{10} \times {}^{2}C_1}{{}^{20}C_{11}} \times \frac{1}{9} + 0.6 \times \frac{{}^{17}C_9 \times {}^{3}C_2}{{}^{20}C_{11}} \times \frac{1}{9} = \frac{99}{1900}$.

Example: 35 A bag A contains 2 white and 3 red balls and bag B contains 4 white and 5 red balls. One ball is drawn at random from a randomly chosen bag and is found to be red. The probability that it was drawn from B is **[BIT Ranchi 1988; IIT 1976]**

(a) $\frac{5}{14}$ (b) $\frac{5}{16}$ (c) $\frac{5}{18}$ (d) $\frac{25}{52}$

Solution: (d) Let E_1 be the event that the ball is drawn from bag A, E_2 the event that it is drawn from bag B and E that the ball is red.

We have to find $P(E_2/E)$.

Since both the bags are equally likely to be selected,

we have $P(E_1) = P(E_2) = \frac{1}{2}$. Also $P(E/E_1) = 3/5$ and $P(E/E_2) = 5/9$.

Hence by Baye's theorem, we have $P(E_2/E) = \frac{P(E_2)P(E/E_2)}{P(E_1)P(E/E_1) + P(E_2)P(E/E_2)} = \frac{\frac{1}{2}\cdot\frac{5}{9}}{\frac{1}{2}\cdot\frac{3}{5} + \frac{1}{2}\cdot\frac{5}{9}} = \frac{25}{52}$.

Example: 36 A man is known to speak the truth 3 out of 4 times. He throws a die and reports that it is a six. The probability that it is actually a six, is

(a) $\frac{3}{8}$ (b) $\frac{1}{5}$ (c) $\frac{3}{4}$ (d) None of these

Solution: (a) Let E denote the event that a six occurs and A the event that the man reports that it is a '6', we have

$P(E) = \frac{1}{6}, P(E') = \frac{5}{6}, P(A/E) = \frac{3}{4}$ and $P(A/E') = \frac{1}{4}$.

By Baye's theorem, $P(E/A)=\dfrac{P(E).P(A/E)}{P(E).P(A/E)+P(E')P(A/E')}=\dfrac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4}+\frac{5}{6}\times\frac{1}{4}}=\dfrac{3}{8}$.

Example: 37 A pack of playing cards was found to contain only 51 cards. If the first 13 cards which are examined are all red, then the probability that the missing cards is black, is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{^{25}C_{13}}{^{51}C_{13}}$

Solution: (b) Let A_1 be the event that the black card is lost, A_2 be the event that the red card is lost and let E be the event that first 13 cards examined are red.

Then the required probability $=P\left(\frac{A_1}{E}\right)$. We have $P(A_1)=P(A_2)=\frac{1}{2}$; as black and red cards were initially equal in number.

Also $P\left(\frac{E}{A_1}\right)=\frac{^{26}C_{13}}{^{51}C_{13}}$ and $P\left(\frac{E}{A_2}\right)=\frac{^{25}C_{13}}{^{51}C_{13}}$.

The required probability $=P\left(\frac{A_1}{E}\right)=\dfrac{P(E/A_1)P(A_1)}{P(E/A_1)P(A_1)+P(E/A_2)P(A_2)}=\dfrac{\frac{1}{2}\cdot\frac{^{26}C_{13}}{^{51}C_{13}}}{\frac{1}{2}\cdot\frac{^{26}C_{13}}{^{51}C_{13}}+\frac{1}{2}\cdot\frac{^{25}C_{13}}{^{51}C_{13}}}=\dfrac{2}{3}$.

1.10 Binomial Distribution

(1) **Geometrical method for probability :** When the number of points in the sample space is infinite, it becomes difficult to apply classical definition of probability. For instance, if we are interested to find the probability that a point selected at random from the interval [1, 6] lies either in the interval [1, 2] or [5, 6], we cannot apply the classical definition of probability. In this case we define the probability as follows:

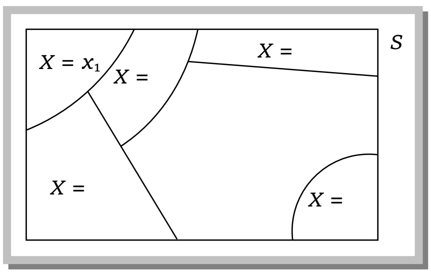
$$P\{x\in A\}=\frac{\text{Measure of region } A}{\text{Measure of the sample space } S},$$

where measure stands for length, area or volume depending upon whether S is a one-dimensional, two-dimensional or three-dimensional region.

(2) **Probability distribution :** Let S be a sample space. A random variable X is a function from the set S to R, the set of real numbers.

For example, the sample space for a throw of a pair of dice is $S=\begin{matrix}\{11, & 12, & \cdots, & 16\\ 21, & 22, & \cdots, & 26\\ \vdots & \vdots & \ddots & \vdots\\ 61, & 62, & \cdots, & 66\}\end{matrix}$

Let X be the sum of numbers on the dice. Then $X(12)=3, X(43)=7$, etc. Also, $\{X = 7\}$ is the event {61, 52, 43, 34, 25, 16}. In general, if X is a random variable defined on the sample space S and r is a real number, then $\{X = r\}$ is an event. If the random variable X takes n distinct values $x_1, x_2, ..., x_n$, then $\{X=x_1\}$, $\{X=x_2\}, ..., \{X=x_n\}$ are mutually exclusive and exhaustive events.



Now, since $(X = x_i)$ is an event, we can talk of $P(X = x_i)$. If $P(X = x_i) = P_i(1 \le i \le n)$, then the system of numbers.

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$$

is said to be the probability distribution of the random variable *X*. The expectation (mean) of the random variable *X* is defined as $E(X) = \sum_{i=1}^{n} p_i x_i$

and the variance of *X* is defined as $\text{var}(X) = \sum_{i=1}^{n} p_i(x_i - E(X))^2 = \sum_{i=1}^{n} p_i x_i^2 - (E(X))^2$.

(3) **Binomial probability distribution :** A random variable *X* which takes values 0, 1, 2, ..., *n* is said to follow binomial distribution if its probability distribution function is given by $P(X = r) = {}^nC_r p^r q^{n-r}, r = 0, 1, 2, \ldots\ldots, n$

where *p*, *q* > 0 such that *p* + *q* = 1.

The notation *X* ~ *B*(*n*, *p*) is generally used to denote that the random variable *X* follows binomial distribution with parameters *n* and *p*.

We have $P(X = 0) + P(X = 1) + \ldots + P(X = n) = {}^nC_0 p^0 q^{n-0} + {}^nC_1 p^1 q^{n-1} + \ldots + {}^nC_n p^n q^{n-n} = (q + p)^n = 1^n = 1$.

Now probability of

(a) Occurrence of the event exactly *r* times

$P(X = r) = {}^nC_r q^{n-r} p^r$.

(b) Occurrence of the event at least *r* times

$$P(X \ge r) = {}^nC_r q^{n-r} p^r + \ldots + p^n = \sum_{X=r}^{n} {}^nC_X p^X q^{n-X} .$$

(c) Occurrence of the event at the most *r* times

$$P(0 \le X \le r) = q^n + {}^nC_1 q^{n-1} p + \ldots + {}^nC_r q^{n-r} p^r = \sum_{X=0}^{r} p^X q^{n-X} .$$

(iv) If the probability of happening of an event in one trial be *p*, then the probability of successive happening of that event in *r* trials is p^r.

Note : ❑ If n trials constitute an experiment and the experiment is repeated *N* times, then the frequencies of 0, 1, 2, ..., *n* successes are given by $N.P(X = 0), N.P(X = 1), N.P(X = 2), \ldots., N.P(X = n)$.

(i) **Mean and variance of the binomial distribution :** The binomial probability distribution is

X	0	1	2	n
$P(X)$	${}^nC_0 q^n p^0$	${}^nC_1 q^{n-1} p$	${}^nC_2 q^{n-2} p^2$.....	${}^nC_n q^0 p^n$

The mean of this distribution is $\sum_{i=1}^{n} X_i p_i = \sum_{X=1}^{n} X.\,^nC_X q^{n-X} p^X = np$,

the variance of the Binomial distribution is $\sigma^2 = npq$ and the standard deviation is $\sigma = \sqrt{(npq)}$.

(ii) **Use of multinomial expansion :** If a die has m faces marked with the numbers 1, 2, 3, ...m and if such n dice are thrown, then the probability that the sum of the numbers exhibited on the upper faces equal to p is given by the coefficient of x^p in the expansion of $\frac{(x+x^2+x^3+....+x^m)^n}{m^n}$.

Example: 38 A random variable X has the probability distribution :

***X* :**	1	2	3	4	5	6	7	8
***P*(*X*) :**	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

For the events E = {X is a prime number} and F = {X < 4}, the probability $P(E \cup F)$ is **[AIEEE 2004]**

(a) 0.50 (b) 0.77 (c) 0.35 (d) 0.87

Solution: (b) E = {X is a prime number}

$P(E) = P(2) + P(3) + P(5) + P(7) = 0.62$, $F = \{x < 4\}$

$P(F) = P(1) + P(2) + P(3) = 0.50$ and $P(E \cap F) = P(2) + P(3) = 0.35$

$\therefore\; P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.62 + 0.50 - 0.35 = 0.77$.

Example: 39 8 coins are tossed simultaneously. The probability of getting at least 6 heads is **[AISSE 1985; MNR 1985; MP PE**

(a) $\frac{57}{64}$ (b) $\frac{229}{256}$ (c) $\frac{7}{64}$ (d) $\frac{37}{256}$

Solution: (d) The required probability $= {}^8C_6\left(\frac{1}{2}\right)^6 . \left(\frac{1}{2}\right)^2 + {}^8C_7\left(\frac{1}{2}\right)^7 . \left(\frac{1}{2}\right) + {}^8C_8\left(\frac{1}{2}\right)^8 = \frac{37}{256}$.

Example: 40 An unbiased die with faces marked 1, 2, 3, 4, 5 and 6 is rolled four times. Out of four face values obtained the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5, is

[IIT 1993; DCE 2000; Roorkee 2000]

(a) $\frac{16}{81}$ (b) $\frac{1}{81}$ (c) $\frac{80}{81}$ (d) $\frac{65}{81}$

Solution: (a) P(minimum face value is not less than 2 and maximum face value is not greater than 5)

= P(2 or 3 or 4 or 5) $= \frac{4}{6} = \frac{2}{3}$.

Hence required probability $= {}^4C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^0 = \frac{16}{81}$.

Example: 41 One hundred identical coins each with probability p of showing up heads are tossed once. If $0 < p < 1$ and the probability of heads showing on 50 coins is equal to that of heads showing on 51 coins, then the value of p is

[IIT 1988; CEE 1993; MP PET 2001]

(a) $\frac{1}{2}$ (b) $\frac{49}{101}$ (c) $\frac{50}{101}$ (d) $\frac{51}{101}$

Solution: (d) We have ${}^{100}C_{50}p^{50}(1-p)^{50} = {}^{100}C_{51}p^{51}(1-p)^{49}$ or $\frac{1-p}{p} = \frac{100\,!}{51\,!.\,49\,!} \times \frac{50\,!.\,50\,!}{100\,!} = \frac{50}{51}$ or $51 - 51p = 50p \Rightarrow p = \frac{51}{101}$.

Example: 42 The mean and the variance of a binomial distribution are 4 and 2 respectively. Then the probability of 2 successes is

[AIEEE 2004]

(a) $\frac{28}{256}$ (b) $\frac{219}{256}$ (c) $\frac{128}{256}$ (d) $\frac{37}{256}$

Solution: (a) $\left.\begin{array}{l} np = 4 \\ npq = 2 \end{array}\right\} \Rightarrow q = \frac{1}{2}, p = \frac{1}{2}, n = 8$

$p(X = 2) = {}^8C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^6 = 28.\frac{1}{2^8} = \frac{28}{256}$.

Example: 43 A man takes a step forward with probability 0.4 and backward with probability 0.6. The probability that at the end of eleven steps he is one step away from the starting point is

(a) ${}^{11}C_6(0.24)^5$ (b) ${}^{11}C_6(0.4)^6(0.6)^5$ (c) ${}^{11}C_6(0.6)^6(0.4)^5$ (d) None of these

Solution: (a) The man will be one step away from the starting point if (i) either he is one step ahead or (ii) one step behind the starting point.

$\therefore$ The required probability = $P(i) + P(ii)$

The man will be one step ahead at the end of eleven steps if he moves six step forward and five steps backward.

The probability of this event is ${}^{11}C_6(0.4)^6(0.6)^5$.

The man will be one step behind at the end of eleven steps if he moves six steps backward and five steps forward.

The probability of this event is ${}^{11}C_6(0.6)^6(0.4)^5$.

Hence the required probability $= {}^{11}C_6(0.4)^6(0.6)^5 + {}^{11}C_6(0.6)^6(0.4)^5 = {}^{11}C_6(0.4)^5(0.6)^5(0.4 + 0.6) = {}^{11}C_6(0.24)^5$.

Example: 44 A person can kill a bird with probability 3/4. He tries 5 times. What is the probability that he may not kill the bird

[Rajasthan PET 1997]

(a) 243/1024 (b) 781/1024 (c) 1/1024 (d) 1023/1024

Solution: (c) Probability to kill a bird $p = \frac{3}{4}$, $p + q = 1$

$\Rightarrow q = 1 - p = 1 - \frac{3}{4} = \frac{1}{4}$ and $n = 5$.

Probability that he may not kill the bird,

$P(X = 0) = {}^5C_0\left(\frac{3}{4}\right)^0 \cdot \left(\frac{1}{4}\right)^{5-0} = \frac{1}{1024}$.

Example: 45 If X follows a binomial distribution with parameters n = 8 and $p = \frac{1}{2}$, then $P(|X - 4| \le 2)$ equals

(a) $\frac{118}{128}$ (b) $\frac{119}{128}$ (c) $\frac{117}{128}$ (d) None of these

Solution: (b) We have, $P(|X-4| \le 2) = P(-2 \le X - 4 \le 2) = P(2 \le X \le 6) = P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)$

$= {}^8C_2\left(\frac{1}{2}\right)^8 + {}^8C_3\left(\frac{1}{2}\right)^8 + {}^8C_4\left(\frac{1}{2}\right)^8 + {}^8C_5\left(\frac{1}{2}\right)^8 + {}^8C_6\left(\frac{1}{2}\right)^8 = \frac{1}{2^8}[28 + 56 + 70 + 56 + 28] = \frac{238}{2^8} = \frac{119}{128}$.

Example: 46 Three six faced fair dice are thrown together. The probability that the sum of the numbers appearing on the dice is $k(3 \le k \le 8)$, is

(a) $\frac{(k-1)(k-2)}{432}$ (b) $\frac{k(k-1)}{432}$ (c) $\frac{k^2}{432}$ (d) None of these

Solution: (a) The total number of cases $= 6 \times 6 \times 6 = 216$

The number of favourable ways

= Coefficient of x^k in $(x + x^2 + ... + x^6)^3$

= Coefficient of x^{k-3} in $(1-x^6)^3(1-x)^{-3}$

= Coefficient of x^{k-3} in $(1-x)^{-3}$ $\{0 \le k-3 \le 5\}$

= Coefficient of x^{k-3} in $(1 + {}^3C_1x + {}^4C_2x^2 + {}^5C_3x^3 + ...) = {}^{k-1}C_2 = \frac{(k-1)(k-2)}{2}$

Thus the probability of the required event is $\frac{(k-1)(k-2)}{432}$.

Example: 47 If three dice are thrown simultaneously, then the probability of getting a score of 7 is **[Kurukshetra CEE 1998]**

(a) 5/216 (b) 1/6 (c) 5/72 (d) None of these

Solution: (c) $n(S) = 6 \times 6 \times 6$

$n(E)$ = The number of solutions of $x + y + z = 7$,

where $1 \le x \le 5, 1 \le y \le 5, 1 \le z \le 5$

= Coefficient of x^7 in $(x + x^2 + ... + x^5)^3$

= Coefficient of x^4 in $(1 + x + ... + x^4)^3$ = Coefficient of x^4 in $\left(\frac{1-x^5}{1-x}\right)^3$

= Coefficient of x^4 in $(1 - 3x^5 + 3x^{10} - x^{15})(1-x)^{-3}$

= Coefficient of x^4 in $(1 - 3x^5 + 3x^{10} - x^{15})({}^2C_0 + {}^3C_1x + {}^4C_2x^2 + {}^5C_3x^3 + {}^6C_4x^4 + ...)$

$= {}^6C_4 = \frac{6!}{4!2!} = \frac{6 \times 5}{2} = 15$.

$\therefore \; p(E) = \frac{n(E)}{n(S)} = \frac{15}{6 \times 6 \times 6} = \frac{5}{72}$.