Long Answer Questions-I (PYQ)

[4 Marks]

Q.1. Find the particular solution of the differential equation $log\left(\frac{dy}{dx}\right) = 3x + 4y$, given that $y = 0$ when $x = 0$.

Ans.

Given differential equation is $\log \frac{dy}{dx} = 3x + 4y$

$$\Rightarrow \quad \frac{dy}{dx} = e^{3x+4y}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{3x} . e^{4y}$$

$$\Rightarrow \quad \frac{dy}{e^{4y}} = e^{3x} . dx$$

$$\Rightarrow \quad e^{-4y} dy = e^{3x} dx$$

Integrating both sides, we get

$$\int e^{-4y} dy = \int e^{3x} dx$$

$$\Rightarrow \quad \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C_1$$

$$\Rightarrow \quad -3e^{-4y} = 4e^{3x} + 12C_1$$

$$\Rightarrow \quad 4e^{3x} + 3e^{-4y} = -12C_1$$

$$\Rightarrow \quad 4e^{3x} + 3e^{-4y} = C \qquad \ldots(i)$$

It is general solution.

Now for particular solution we put $x = 0$ and $y = 0$ in (i)

$$4 + 3 = C \quad \Rightarrow \quad C = 7.$$

Putting $C = 7$ in (i), we get

$$4e^{3x} + 3e^{-4y} = 7$$

It is required particular solution.

Q.2. Solve the following differential equation:

$$2x^2\frac{dy}{dx} - 2xy + y^2 = 0$$

Ans.

Given $2x^2\frac{dy}{dx} - 2xy + y^2 = 0$

$$\Rightarrow \quad 2x^2\frac{dy}{dx} = 2xy - y^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy - y^2}{2x^2} \qquad ...(i)$$

It is homogeneous differential equation.

Let $y = vx$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = v + x\frac{\mathrm{dv}}{\mathrm{dx}}$$

Equation (i) becomes

$$v + x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{2x.\mathrm{vx} - v^2x^2}{2x^2}$$

$$\Rightarrow \quad v + x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{2x^2\left(v - \frac{v^2}{2}\right)}{2x^2}$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} = v - \frac{v^2}{2} - v$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} = -\frac{v^2}{2}$$

$$\Rightarrow \quad \frac{\mathrm{dx}}{x} = -\frac{2\,\mathrm{dv}}{v^2}$$

Integrating both sides, we get

$$\Rightarrow \quad \int\frac{\mathrm{dx}}{x} = -2\int\frac{\mathrm{dv}}{v^2}$$

$$\Rightarrow \quad \log|x| + C = -2\frac{v^{-2+1}}{-2+1}$$

$$\Rightarrow \quad \log|x| + C = 2\,.\,\frac{1}{v}$$

Putting $v = \frac{y}{x}$, we get

$$\log|x| + C = \frac{2x}{y}$$

Q.3. Find the particular solution of the following differential equation:

$\frac{\mathrm{dy}}{\mathrm{dx}} = 1 + x^2 + y^2 + x^2y^2$, given that $y = 1$ when $x = 0$

Ans.

Given $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$

$\Rightarrow \quad \frac{dy}{dx} = (1 + x^2) + y^2(1 + x^2)$

$\Rightarrow \quad \frac{dy}{dx} = (1 + x^2)(1 + y^2)$

$\Rightarrow \quad (1 + x^2)dx = \frac{dy}{(1+y^2)}$

Integrating both sides, we get

$$\int (1 + x^2)dx = \int \frac{dy}{(1+y^2)}$$

$\Rightarrow \quad \int dx + \int x^2 dx = \int \frac{dy}{(1+y^2)}$

$\Rightarrow \quad x + \frac{x^3}{3} + C = \tan^{-1} y$

Putting $y = 1$ and $x = 0$, we get

$$\tan^{-1}(1) = 0 + 0 + C$$

$\Rightarrow \quad C = \tan^{-1}(1) = \frac{\pi}{4}$

Therefore, required particular solution is

$$\tan^{-1} y = x + \frac{x^3}{3} + \frac{\pi}{4}$$

Q.4. Find the particular solution of the differential equation:

$x(x^2 - 1)\frac{dy}{dx} = 1;\ y = 0;$ **when** $x = 2$

Ans.

Given differential equation is,

$$x(x^2-1)\frac{\mathrm{dy}}{\mathrm{dx}}=1$$

$$\Rightarrow \quad \mathrm{dy}=\frac{\mathrm{dx}}{x\,(x^2-1)}$$

$$\Rightarrow \quad \mathrm{dy}=\frac{\mathrm{dx}}{x\,(x-1)\,(x+1)}$$

Integrating both sides, we get,

$$\int \mathrm{dy}=\int\frac{\mathrm{dx}}{x\,(x-1)\,(x+1)}$$

$$\Rightarrow \quad y=\int\frac{\mathrm{dx}}{x(x-1)\,(x+1)} \qquad ...(i)$$

Let $\frac{1}{x(x-1)(x+1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$

$$\Rightarrow \quad \frac{1}{x(x-1)(x+1)}=\frac{A(x-1)\,(x+1)+\mathrm{Bx}\,(x+1)+\mathrm{Cx}\,(x-1)}{x(x-1)(x+1)}$$

$$\Rightarrow \quad 1=A(x-1)(x+1)+Bx(x+1)+Cx(x-1)$$

Putting $x=1$, we get $1=0+B.\,1.\,2+0 \quad\Rightarrow\quad B=\frac{1}{2}$

Putting $x=-1$, we get $1=0+0+C.(-1).(-2)\Rightarrow\quad C=\frac{1}{2}$

Putting $x=0$, we get $1=A\,(-1).1 \quad\Rightarrow\quad A=-1$

Hence, $\frac{1}{x(x-1)(x+1)}=\frac{-1}{x}+\frac{1}{2(x-1)}+\frac{1}{2(x+1)}$

From (i) $y = \int \left(-\frac{1}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}\right) \mathrm{dx}$

$\Rightarrow \quad y = -\int \frac{\mathrm{dx}}{x} + \frac{1}{2}\int \frac{\mathrm{dx}}{x-1} + \frac{1}{2}\int \frac{\mathrm{dx}}{x+1}$

$\Rightarrow \quad y = -\log x + \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1| + \log C_1$

$\Rightarrow \quad 2y = 2\log\frac{1}{x} + \log|x^2-1| + 2\log C_1$

$\Rightarrow \quad 2y = \log\left|\frac{x^2-1}{x^2}\right| + \log C_1^2 \qquad \ldots(ii)$

When $x = 2,\ y = 0$

$\Rightarrow \quad 0 = \log\left|\frac{4-1}{4}\right| + \log C_1^2$

$\Rightarrow \quad \log C_1^2 = -\log\frac{3}{4}$

Putting $\log C_1^2 = -\log\frac{3}{4}$ in (ii) we get

$2y = \log\left|\frac{x^2-1}{x^2}\right| - \log\frac{3}{4}$

$\Rightarrow \quad y = \frac{1}{2}\log\left|\frac{x^2-1}{x^2}\right| - \frac{1}{2}\log\frac{3}{4}$

Q.5. Solve the differential equation $(1+x^2))\frac{\mathrm{dy}}{\mathrm{dx}} + y = e^{\tan^{-1}x}$.

Ans.

Given differential equation is

$$(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{1}{1+x^2}y=\frac{e^{\tan^{-1}x}}{1+x^2} \qquad \ldots(i)$$

Equation (i) is of the form

$$\frac{dy}{dx}+\mathrm{Py}=Q,\ \text{where } P=\frac{1}{1+x^2},Q=\frac{e^{\tan^{-1}x}}{1+x^2}$$

$$\therefore \quad \mathrm{IF}=e^{\int \mathrm{Pdx}}=e^{\int \frac{1}{1+x^2}\mathrm{dx}}=e^{\tan^{-1}x}$$

Therefore, general solution of required differential equation is

$$y.e^{\tan^{-1}x}=\int e^{\tan^{-1}x}.\frac{e^{\tan^{-1}x}}{1+x^2}\mathrm{dx}+C$$

$$\Rightarrow \quad y.e^{\tan^{-1}x}=\int \frac{e^{2\tan^{-1}x}}{1+x^2}\mathrm{dx}+C \qquad \ldots(ii)$$

Let $\tan^{-1}x=z \quad \Rightarrow \quad \frac{1}{1+x^2}\mathrm{dx}=\mathrm{dz}$

(ii) becomes

$$y.e^{\tan^{-1} x} = \int e^{2z}\mathrm{dz} + C$$

$$\Rightarrow \quad y.e^{\tan^{-1} x} = \frac{e^{2z}}{2} + C$$

$$\Rightarrow \quad y.e^{\tan^{-1} x} = \frac{e^{2\tan^{-1} x}}{2} + C \qquad [\text{Putting } z = \tan^{-1} x]$$

$$\Rightarrow \quad y = \frac{e^{\tan^{-1} x}}{2} + C.e^{-\tan^{-1} x} \qquad [\text{Dividing both sides by } e^{\tan^{-1} x}]$$

It is required solution.

Q.6. Find the particular solution of the differential equation $e^x\sqrt{1-y^2}\mathrm{dx} + \frac{y}{x}\mathrm{dy} = 0$ **given that** y**= 1 when** x **= 0.**

Ans.

We have, $e^x\sqrt{1-y^2}\mathrm{dx} + \frac{y}{x}\mathrm{dy} = 0$

$$\Rightarrow \quad e^x\sqrt{1-y^2}\mathrm{dx} = -\frac{y}{x}\mathrm{dy}$$

$$\Rightarrow \quad \mathrm{xe}^x\,\mathrm{dx} = -\frac{y}{\sqrt{1-y^2}}\mathrm{dy}$$

$$\Rightarrow \quad \int \underset{I}{x}\,\underset{\mathrm{II}}{e^x}\,\mathrm{dx} = -\int \frac{y}{\sqrt{1-y^2}}\mathrm{dy}$$

$$\Rightarrow \quad \mathrm{xe}^x - \int e^x\mathrm{dx} = \frac{1}{2}\int\frac{\mathrm{dt}}{\sqrt{t}}, \text{ where } t = 1 - y^2 \qquad (\text{Using ILATE on LHS})$$

$$\Rightarrow \quad \mathrm{xe}^x - e^x = \frac{1}{2}\left(\frac{t^{1/2}}{1/2}\right) + C$$

$$\Rightarrow \quad \mathrm{xe}^x - e^x = \sqrt{t} + C$$

$$\Rightarrow \quad \mathrm{xe}^x - e^x = \sqrt{1-y^2} + C, \text{ is the required solution.}$$

Putting $y = 1$ and $x = 0$, we get

$$0e^0 - e^0 = \sqrt{1-1^2} + C$$

$$\Rightarrow \quad C = -1$$

Therefore, required particular solution is $\mathrm{xe}^x - e^x = \sqrt{1-y^2} - 1.$

Q.7. Solve: $x\frac{\mathrm{dy}}{\mathrm{dx}} = y - x\tan\left(\frac{y}{x}\right)$

Ans.

We have, $\quad x\frac{\mathrm{dy}}{\mathrm{dx}} = y - x\tan\left(\frac{y}{x}\right)$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x} - \tan\left(\frac{y}{x}\right) \qquad \ldots(i)$$

Clearly, the given differential equation is homogeneous.

Putting $y = \mathrm{vx}$ and $\frac{\mathrm{dy}}{\mathrm{dx}} = v + x\frac{\mathrm{dv}}{\mathrm{dx}}$ in (i), we get

$$v + x\frac{\mathrm{dv}}{\mathrm{dx}} = v - \tan v$$

$$v+x\frac{dv}{dx} = v-\tan v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \tan v$$

$$\Rightarrow \quad \cot v\,dv = \frac{-dx}{x},\ \text{if } x \neq 0 \qquad \text{[By separating the variables]}$$

Integrating both sides, we get

$$\int \cot v\,dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \log|\sin v| = -\log|x| + \log C$$

$$\Rightarrow \quad \log|\sin v| = \log C$$

$$\Rightarrow \quad \left|x\sin\frac{y}{x}\right| = |C|$$

Hence, $x\sin\frac{y}{x} = C$ is the required solution.

Q.8. Solve the differential equation:

$(\tan^{-1} y - x)\,dy = (1 + y^2)\,dx$

OR

Find the particular solution of the differential equation $(\tan^{-1} y - x)dy = (1 + y^2)dx$**, given that *x* = 1 when *y* = 0.**

Ans.

The given differential equation can be written as

$$\frac{\mathrm{dx}}{\mathrm{dy}}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2} \qquad \ldots(i)$$

Now, (i) is a linear differential equation of the form $\frac{\mathrm{dx}}{\mathrm{dy}}+\mathrm{Px}=Q$,

where, $P=\frac{1}{1+y^2}$ and $Q=\frac{\tan^{-1}y}{1+y^2}$

Therefore, $\mathrm{IF}=e^{\int\frac{1}{1+y^2}\mathrm{dy}}=e^{\tan^{-1}y}$

Thus, the solution of the given differential equation is

$$\mathrm{x}e^{\tan^{-1}y}=\int\left(\frac{\tan^{-1}y}{1+y^2}\right)e^{\tan^{-1}y}\mathrm{dy}+C \qquad \ldots(ii)$$

Let $I=\int\left(\frac{\tan^{-1}y}{1+y^2}\right)e^{\tan^{-1}y}\,\mathrm{dy}$

Substituting $\tan^{-1}y=t$ so that $\left(\frac{1}{1+y^2}\right)\mathrm{dy}=\mathrm{dt}$, we get

$$I=\int te^t\,\mathrm{dt}=te^t-\int 1.e^t\,\mathrm{dt}=te^t-e^t\equiv e^t(t-1)$$

or $\quad I=e^{\tan^{-1}y}(\tan^{-1}y-1)$

Substituting the value of I in equation (ii), we get

$$x.e^{\tan^{-1}y}=e^{\tan^{-1}y}(\tan^{-1}y-1)+C$$

or $\qquad x=(\tan^{-1}y-1)+2e^{-\tan^{-1}y}$

Which is the general solution of the given differential equation.

OR

For general solution same as above.

General solution is $x = (\tan^{-1} y - 1) + \mathrm{C}e^{-\tan^{-1} y}$

For particular solution putting $x = 1$, $y = 0$, we get

$$1 = (\tan^{-1} 0 - 1) + C.e^{-\tan^{-1} 0}$$
$$1 = -1 + C \qquad \Rightarrow \qquad C = 2$$

Therefore required particular solution is

$$x = (\tan^{-1} y - 1) + e^{-\tan^{-1} y}$$

Q.9. Solve the following differential equation:

$$\left[\frac{e^{2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{\mathrm{dx}}{\mathrm{dy}} = 1, x \neq 0$$

Ans.

Given $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{\mathrm{dx}}{\mathrm{dy}} = 1,\ x \neq 0$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} + \frac{1}{\sqrt{x}} \cdot y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

It is in the form $\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{Py} = Q$, where $P = \frac{1}{\sqrt{x}}, Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\therefore \quad \mathrm{IF} = e^{\int P\ \mathrm{dx}} = e^{\int \frac{1}{\sqrt{x}} \mathrm{dx}} = e^{\int x^{-\frac{1}{2}}\ \mathrm{dx}} = e^{\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}} = e^{2\sqrt{x}}$$

Therefore general solution is

$$y \cdot e^{2\sqrt{x}} = \int Q \times \text{IF} \, \mathrm{dx} + C$$

$$\Rightarrow \quad y \cdot e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot e^{2\sqrt{x}} \mathrm{dx} + C$$

$$\Rightarrow \quad y \cdot e^{2\sqrt{x}} = \int \frac{\mathrm{dx}}{\sqrt{x}} + C$$

$$\Rightarrow \quad y \cdot e^{2\sqrt{x}} = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$\Rightarrow \quad y \cdot e^{2\sqrt{x}} = 2\sqrt{x} + C$$

Q.10. Solve the differential equation

$(x^2 - 1)\frac{\mathrm{dy}}{\mathrm{dx}} + 2\mathrm{xy} = \frac{2}{x^2 - 1}$, where $x \in (-\infty, -1) \bigcup (1, \infty)$

Ans.

The given differential equation is $(x^2 - 1)\frac{\mathrm{dy}}{\mathrm{dx}} + 2\mathrm{xy} = \frac{2}{x^2-1}$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} + \frac{2x}{x^2 - 1} y = \frac{2}{(x^2 - 1)^2} \qquad \ldots(i)$$

This is a linear differential equation of the form

$\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{Py} = Q$, where $P = \frac{2x}{x^2-1}$ and $Q = \frac{2}{(x^2-1)^2}$

$$\therefore \quad \text{IF} = e^{\int P \, \mathrm{dx}} = e^{\int 2x/(x^2-1) \, \mathrm{dx}} = e^{\log\, (x^2-1)} = (x^2 - 1)$$

Multiplying both sides of (i) by $\text{IF} = (x^2 - 1)$, we get $(x^2 - 1)\frac{\mathrm{dy}}{\mathrm{dx}} + 2\mathrm{xy} = \frac{2}{x^2-1}$

Integrating both sides, we get

$$y(x^2-1)=\int \frac{2}{x^2-1}\,\mathrm{dx}+C \qquad [\text{Using } : y\ (\text{IF}\ )=\int Q\,.\,(\text{IF}\ )\mathrm{dx}+C]$$

$$\Rightarrow \quad y(x^2-1)=\frac{2}{2}\log\left|\frac{x-1}{x+1}\right|+C$$

$$\Rightarrow \quad y\,(x^2-1)=\log\left|\frac{x-1}{x+1}\right|+C$$

This is the required solution.

Q.11. Find the particular solution of the differential equation $\frac{dy}{dx}=1+x+y+xy$ given that *y* = 0 when *x* = 1.

Ans.

Given differential equation is $\frac{\mathrm{dy}}{\mathrm{dx}}=1+x+y+\mathrm{xy}$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}}=(1+x)+y(1+x)$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}}=(1+x)(1+y)$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{1+y}=(1+x)\mathrm{dx}$$

Integrating both sides, we get $\log|1+y|\ =\int(1+x)\mathrm{dx}$

Integrating both sides, we get $\log|1+y|\ =\int(1+x)\mathrm{dx}$

$\Rightarrow \quad \log|1+y|\ =x+\frac{x^2}{2}+C$, it is general solution.

Putting $x=1$, $y=0$, we get

$$\log 1=1+\frac{1}{2}+C$$

$$\Rightarrow \quad 0=\frac{3}{2}+C$$

$$\Rightarrow \quad C=\frac{-3}{2}$$

Hence, particular solution is $\log|1+y|=x+\frac{x^2}{2}-\frac{3}{2}$.

Q.12. Solve the following differential equation:

$$x \log x \frac{dy}{dx} + y = 2 \log x$$

Ans.

We have the differential equation

$$x \log x \frac{dy}{dx} + y = 2 \log x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{x \log x} . y = \frac{2}{x}$$

It is linear differential equation of the form

$\frac{dy}{dx} + \mathrm{Py} = Q,$ where $P = \frac{1}{x \log x}$ and $Q = \frac{2}{x}$

Now, $\mathrm{IF} = e^{\int \mathrm{pdx}} = e^{\int \frac{1}{x \log x} dx} = e^{\log |\log x|} = \log x$

Hence, solution of given differential equation is $y \times \mathrm{IF} = \int Q \times \mathrm{IF}\ dx$

$$\Rightarrow \quad y \log x = \int \frac{2}{x} .\ \log x\ dx = 2 \int \frac{1}{x} . \log x\ dx = 2 \frac{(\log x)^2}{2} + C$$

$$\Rightarrow \quad y \log x = (\log x)^2 + C$$

Q.13. Solve the differential equation $x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x.$

Ans.

Given differential equation is $x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x$

$$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{1}{x . \log x} \right) . y = \frac{2}{x^2} \qquad \text{(Divide each term by } x \log x\text{)}$$

It is in the form $\frac{dy}{dx} + \mathrm{Py} = Q$ where $P = \frac{1}{x . \log x}, Q = \frac{2}{x^2}$

$$\therefore \quad \mathrm{IF} = e^{\int \mathrm{Pdx}} = e^{\int \frac{dx}{x \log x}}$$

Put log $x = z \Rightarrow \frac{dx}{x} = dz = e^{\int \frac{1}{z} dz} = e^{\log z} = z = \log x$

$\therefore$ General solution is

$$y. \log x = \int \log x . \frac{2}{x^2} dx + C$$

$$\Rightarrow \quad y \log x = 2 \int \frac{\log x}{x^2} dx + C$$

Let $\log x = z \Rightarrow \frac{1}{x} dx = dz,$

Also $\quad \log x = z \Rightarrow x = e^z$

$$\therefore \quad y \log x = 2 \int \frac{z}{e^z} dz + C$$

$$\Rightarrow \quad y \log x = 2 \int z . e^{-z} dz + C$$

$$\Rightarrow \quad y \log x = 2 \left[z . \frac{e^{-z}}{-1} - \int \frac{e^{-z}}{-1} dz \right] + C$$

$$\Rightarrow \quad y \log x = 2[-ze^{-z} + \int e^{-z} dz] + C$$

$$\Rightarrow \quad y \log x = -2ze^{-z} - 2e^{-z} + C$$

$$\Rightarrow \quad y \log x = -2 \log x\, e^{-\log x} - 2e^{-\log x} + C$$

$$\Rightarrow \quad y \log x = -2 \log x . \frac{1}{x} - \frac{2}{x} + C \qquad \left[\because e^{-\log x} = e^{\log \frac{1}{x}} = \frac{1}{x} \right]$$

$$\Rightarrow \quad y \log x = -\frac{2}{x}(1 + \log x) + C$$

Q.14. Solve the following differential equation:

$$\text{cosec } x \log y \frac{dy}{dx} + x^2 y^2 = 0$$

Ans.

We have, $\operatorname{cosec} x \log y \frac{\mathrm{dy}}{\mathrm{dx}} + x^2 y^2 = 0$

$\Rightarrow \quad \operatorname{cosec} x . \log y \frac{\mathrm{dy}}{\mathrm{dx}} = -x^2 y^2$

$\Rightarrow \quad \frac{\log\ y.\ \mathrm{dy}}{y^2} = -\frac{x^2\ \mathrm{dx}}{\operatorname{cosec}\ x}$

$\Rightarrow \quad \int y^{-2} . \log y \mathrm{dy} = -\int x^2 \sin x\ \mathrm{dx}$

$\Rightarrow \quad \log y . \frac{y^{-2+1}}{-2+1} - \int \frac{1}{y} . \frac{y^{-2+1}}{-2+1} \mathrm{dy} = -[x^2 (- \cos x) - \int 2x (- \cos x) \mathrm{dx}]$

$\Rightarrow \quad -\frac{1}{y} \log y + \int y^{-2} \mathrm{dy} = x^2 \cos x - 2 \int x \cos x\ \mathrm{dx}$

$\Rightarrow \quad -\frac{1}{y} \log y + \frac{y^{-2+1}}{-2+1} = x^2 \cos x - 2 [x \sin x - \int \sin x\ \mathrm{dx}]$

$\Rightarrow \quad -\frac{1}{y} \log y - \frac{1}{y} = x^2 \cos x - 2x \sin + 2(- \cos x) + C$

$\Rightarrow \quad -\frac{1}{y} (\log y + 1) = x^2 \cos x - 2x \sin x - 2 \cos x + C$

Q.15. Solve the following differential equation:

$\frac{\mathrm{dy}}{\mathrm{dx}} + y = \cos x - \sin x$

Ans.

Given differential equation,

$\frac{\mathrm{dy}}{\mathrm{dx}} + y = \cos x - \sin x$ is a linear differential equation of the type $\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{Py} = Q$, where

$P = 1,\ Q = \cos x - \sin x$

Here, $\mathrm{IF} = e^{\int 1.\,\mathrm{dx}} = e^{x}$

Its solution is given by

$\Rightarrow \quad \mathrm{ye}^{x} = \int e^{x}\,(\cos x - \sin x)\,\mathrm{dx}$

$\Rightarrow \quad \mathrm{ye}^{x} = \int e^{x} \cos x\,\mathrm{dx} - \int e^{x} \sin x\,\mathrm{dx}$ (Integrating by parts)

$\Rightarrow \quad \mathrm{ye}^{x} = e^{x} \cos x - \int -\sin \mathrm{x}e^{x}\,\mathrm{dx} - \int e^{x} \sin x\,\mathrm{dx}$

$\therefore \quad ye^{x} = e^{x} \cos x + C$

$\Rightarrow \quad y = \cos x + Ce^{-x}$

Q.16. Show that the differential equation $x\frac{\mathrm{dy}}{\mathrm{dx}}\sin\left(\frac{y}{x}\right) + x - y\sin\left(\frac{y}{x}\right) = 0$ **is homogeneous. Find the particular solution of this differential equation, given that x = 1 when** $y = \frac{\pi}{2}$.

Ans.

Given differential equation is $x\frac{\mathrm{dy}}{\mathrm{dx}}\sin\frac{y}{x} + x - y\sin\frac{y}{x} = 0$

Dividing both sides by $x \sin\frac{y}{x}$, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} + \operatorname{cosec}\frac{y}{x} - \frac{y}{x} = 0$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x} - \operatorname{cosec}\frac{y}{x} \qquad \ldots(i)$$

Let $F(x, y) = \frac{y}{x} - \operatorname{cosec}\frac{y}{x}$

$$\therefore \quad F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} - \operatorname{cosec}\frac{\lambda y}{\lambda x} = \lambda^{0}\left[\frac{y}{x} - \operatorname{cosec}\frac{y}{x}\right] = \lambda^{0} F(x, y)$$

Hence, differential equation (i) is homogeneous.

Let $y = \mathrm{vx}$

$\Rightarrow \quad \frac{y}{x} = v$

$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = v + x\,.\frac{\mathrm{dv}}{\mathrm{dx}}$

Now, equation (i) becomes

$$v + x.\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\mathrm{vx}}{x} - \operatorname{cosec}\frac{\mathrm{vx}}{x}$$

$$v + x.\frac{\mathrm{dv}}{\mathrm{dx}} = v - \operatorname{cosec} v$$

$\Rightarrow \quad x.\frac{\mathrm{dv}}{\mathrm{dx}} = -\operatorname{cosec} v$

$\Rightarrow \quad -\sin v\,\mathrm{dv} = \frac{\mathrm{dx}}{x}$

$\Rightarrow \quad -\int \sin v\,\mathrm{dv} = \int \frac{\mathrm{dx}}{x}$

$\Rightarrow \quad \cos v = \log|x| + C$

$\Rightarrow \quad \cos\frac{y}{x} = \log|x| + C \qquad \ldots(ii)$

Putting $y = \frac{\pi}{2}$, $x = 1$ in (ii), we get

$\therefore \quad \cos\frac{\pi}{2} = \log 1 + C$

$\Rightarrow \quad 0 = 0 + C \quad \Rightarrow \quad C = 0$

Hence, particular solution is

$$\cos\frac{\pi}{2} = \log|x| + 0 \qquad i.e., \qquad \cos\frac{y}{x} = \log|x|$$

Q.17. Solve the following differential equation:

$$\cos^2 x\frac{\mathrm{dy}}{\mathrm{dx}} + y = \tan x$$

Ans.

Given differential equation is,

$$\cos^2 x.\frac{\mathrm{dy}}{\mathrm{dx}} + y = \tan x$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} + y\sec^2 x = \tan x.\sec^2 x$$

Given differential equation is a linear differential equation of the type $\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{Py} = Q$ where $P = \sec^2 x$, $Q = \tan x \, . \sec^2 x$.

$$\mathrm{IF} = e^{\int P \,\mathrm{dx}} = e^{\int \sec^2 x \,\mathrm{dx}} = e^{\tan x}$$

$\therefore$ Solution is given by

$$e^{\tan x} y = \int \tan x.\sec^2 x.e^{\tan x}\,\mathrm{dx}$$

Let $I = \int \tan x.\sec^2 x.e^{\tan x}\,\mathrm{dx}$

Put $\tan x = t$, $\sec^2 x\, dx = dt$, we get

$$I = \int t\, e^t\,\mathrm{dt}$$

$$\therefore \quad = te^t - \int e^t\mathrm{dt} = te^t - e^t + C \qquad \text{[Integrating by parts]}$$

$$= \tan x\, e^{\tan x} - e^{\tan x} + C$$

Hence, $e^{\tan x} y = e^{\tan x}(\tan x - 1) + C$

$$\Rightarrow \quad y = \tan x - 1 + C\, e^{-\tan x}$$

Q.18. Find the particular solution of the differential equation satisfying the given condition $\frac{dy}{dx} = y \tan x,$ given that *y* = 1 when *x* = 0.

Ans.

We have $\frac{dy}{dx} = y \tan x \quad \Rightarrow \quad \frac{dy}{y} = \tan x \; dx$

By integrating both sides, we get

$$\int \frac{dy}{y} = \int \tan x \, dx$$

$$\log y = \log |\sec x| + C \qquad \ldots(i)$$

By putting $x = 0$ and $y = 1$ (as given) in (i), we get

$$\log 1 = \log(\sec 0) + C \quad \Rightarrow \quad C = 0$$

$\therefore \; (i) \Rightarrow \; \log y = \log |\sec x|$

$\Rightarrow$ Hence, the particular solution is $y = \sec x$

Q.19. Solve the differential equation:

$$\sqrt{1 + x^2 + y^2 + x^2y^2} + xy\frac{dy}{dx} = 0$$

Ans.

Given $\sqrt{1 + x^2 + y^2 + x^2y^2} + xy\frac{dy}{dx} = 0$

By simplifying the equation, we get

$$xy\frac{dy}{dx} = -\sqrt{1 + x^2 + y^2 + x^2y^2}$$

$$\Rightarrow \; xy\frac{dy}{dx} = -\sqrt{(1 + x^2) + (1 + y^2)} = -\sqrt{(1 + x^2)}\sqrt{(1 + y^2)}$$

$$\Rightarrow \quad \frac{y}{\sqrt{(1+y^2)}}\mathrm{dy} = -\frac{\sqrt{(1+x^2)}}{x}\mathrm{dx}$$

Integrating both sides, we get

$$\int \frac{y}{\sqrt{(1+y^2)}}\mathrm{dy} = -\int \frac{\sqrt{(1+x^2)}}{x}\mathrm{dx} \qquad ...(i)$$

Let $1 + y^2 = t \quad \Rightarrow \quad 2y\ dy = dt,$ (For LHS)

and $1 + x^2 = m^2 \quad \Rightarrow \quad 2x\ dx = 2m\ dm \quad \Rightarrow \quad x\ dx = m\ dm$ (For RHS)

$$\therefore\ (i) \Rightarrow \quad \frac{1}{2}\int \frac{1}{\sqrt{t}}\mathrm{dt} = -\int \frac{m}{m^2-1}\,.m\ \mathrm{dm}$$

$$\Rightarrow \quad \frac{1}{2}\frac{t^{1/2}}{1/2} + \int \frac{m^2}{m^2-1}\ \mathrm{dm} = 0$$

$$\Rightarrow \quad \sqrt{t} + \int \frac{m^2+1-1}{m^2-1}\mathrm{dm} = 0$$

$$\Rightarrow \quad \sqrt{t} + \int \left(1 + \frac{1}{m^2-1}\right)\mathrm{dm} = 0$$

$$\Rightarrow \quad \sqrt{t} + m + \frac{1}{2}\log\left|\frac{m-1}{m+1}\right| = 0$$

Now, substituting these value of t and m, we get

$$\sqrt{1+y^2} + \sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + C = 0$$

Q.20. Show that the following differential equation is homogeneous and then solve it.

$$y\,\mathrm{dx} + x\log\left(\frac{y}{x}\right)\ \mathrm{dy} - 2x\,\mathrm{dy} = 0$$

Ans.

We have $y\,\mathrm{dx} + x\log\left(\frac{y}{x}\right)\,\mathrm{dy} - 2x\,\mathrm{dy} = 0$

Simplifying the above equation, we get

$$\left[x\log\left(\frac{y}{x}\right) - 2x\right]\mathrm{dy} = -y\,\mathrm{dx}$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} \qquad \ldots(i)$$

Let $F(x, y) = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$

$$F(\mu x, \mu y) = \frac{\mu y}{2\mu x + \mu x\log\left(\frac{\mu y}{\mu x}\right)} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} = \mu^{\circ} F(x, y)$$

$\therefore$ Function $F(x, y)$ is homogenous and hence the equation is homogeneous.

Let $\quad y = vx \quad \Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = v + x\frac{\mathrm{dv}}{\mathrm{dx}}$

Substituting in equation (i), we get

$$v + x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\mathrm{vx}}{2x - x\log v}$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{v}{2 - \log v} - v$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{v\log v - v}{2 - \log v}$$

$$\Rightarrow \quad \frac{2 - \log v}{v\log v - v}\mathrm{dv} = \frac{\mathrm{dx}}{x}$$

Integrating both sides, we get

$$\int \frac{2-\log v}{v\log v-v}\,\mathrm{dv}=\int \frac{\mathrm{dx}}{x}$$

$$\Rightarrow \quad \int \frac{1+(1-\log v)}{v(\log v-1)}\,\mathrm{dv}=\int \frac{\mathrm{dx}}{x}$$

$$\Rightarrow \quad \int \frac{\mathrm{dv}}{v(\log v-1)}-\int \frac{\mathrm{dv}}{v}=\int \frac{\mathrm{dx}}{x}$$

Let $\log v-1=m \quad \Rightarrow \quad \frac{1}{v}\mathrm{dv}=\mathrm{dm}$

$$\Rightarrow \quad \int \frac{1}{m}\,\mathrm{dm}-\int \frac{1}{v}\,\mathrm{dv}=\int \frac{\mathrm{dx}}{x}$$

$$\Rightarrow \quad \log|m|-\log|v|=\log|x|+\log|C|$$

$$\Rightarrow \quad \log\left|\frac{m}{v}\right|=\log|\mathrm{Cx}| \quad \Rightarrow \quad \frac{m}{v}=\mathrm{Cx}$$

$$\Rightarrow \quad (\log v-1)=vCx$$

$$\Rightarrow \quad \left[\log\left(\frac{y}{x}\right)-1\right]=\mathrm{Cy}$$

which is the required solution.

Q.21. Solve the differential equation:

$$(x^2+1)\frac{\mathrm{dy}}{\mathrm{dx}}+2\mathrm{xy}=\sqrt{x^2+4}$$

Ans.

We have $(x^2+1)\frac{\mathrm{dy}}{\mathrm{dx}}+2\mathrm{xy}=\sqrt{x^2+4}$

Simplifying the above equation, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}}+\frac{2x}{x^2+1}y=\frac{\sqrt{x^2+4}}{(x^2+1)}$$

This is a linear differential equation of the form $\frac{\mathrm{dy}}{\mathrm{dx}}+\mathrm{Py}=Q$

where, $P=\frac{2x}{x^2+1}, Q=\frac{\sqrt{x^2+4}}{(x^2+1)}$

$$\therefore \quad \mathrm{IF}=e^{\int P\,\mathrm{dx}}=e^{\int \frac{2x}{x^2+1}\mathrm{dx}}=e^{\log\,(x^2+1)}=(x^2+1)$$

Its solution is given by

$$(x^2+1)y=\int (x^2+1).\frac{\sqrt{x^2+4}}{(x^2+1)}\mathrm{dx}=\int\sqrt{x^2+4}\,\mathrm{dx}$$

$$\Rightarrow \quad (x^2+1)y=\frac{x}{2}\sqrt{x^2+4}+\frac{4}{2}\log|x+\sqrt{x^2+4}|+C$$

Q.22. Find the particular solution of the following differential equation satisfying the given condition:

$(3x^2+y)\frac{\mathrm{dx}}{\mathrm{dy}}=x,\ x>0,$ **when** $x = 1,\ y = 1$

Ans.

We are given

$$(3x^2+y)\frac{\mathrm{dx}}{\mathrm{dy}}=x, x>0$$

$$\Rightarrow \quad \frac{\mathrm{dx}}{\mathrm{dy}}=\frac{x}{3x^2+y}$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}}=\frac{3x^2+y}{x}=3x+\frac{y}{x}$$

$$\Rightarrow \quad \frac{\mathrm{dx}}{\mathrm{dy}}-\frac{1}{x}y=3x$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = -\frac{1}{x}, Q = 3x$

$\therefore \quad IF = e^{\int P\,dx} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

Its solution is given by

$\therefore \quad \frac{y}{x} = \int \frac{1}{x} 3x\,dx = 3\int dx$

$\Rightarrow \quad \frac{y}{x} = 3x + C$

$\Rightarrow \quad y = 3x^2 + Cx$

Putting $x = 1$, $y = 1$, we get

$\Rightarrow \quad 1 = 3 + C$

$\Rightarrow \quad C = -2$

$\therefore \quad y = 3x^2 - 2x$

Q.23. $(x^2 + y^2)\,dy = xy\,dx$**. If** $y(1) = 1$ **and** $y(x_0) = e$**, then find the value of** x_0**.**

Ans.

Given differential equation is $(x^2 + y^2)dy = xy\ dx$

It is also written as

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2} \qquad \ldots(i)$$

Now, to solve let $y = vx$.

Differentiating $y = vx$ with respect to x, we get

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2}$$

Putting $y = \mathrm{vx}$ and $\frac{\mathrm{dy}}{\mathrm{dx}} = v + x\frac{\mathrm{dy}}{\mathrm{dx}}$ in (i), we get

$$v + x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{x.\,\mathrm{vx}}{x^2+(\mathrm{vx})^2}$$

$$v + x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\mathrm{vx}^2}{x^2+v^2x^2}$$

$$\Rightarrow \quad v + x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\mathrm{vx}^2}{x^2(1+v^2)}$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{v}{(1+v^2)} - v$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{v - v - v^3}{(1+v^2)}$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{-v^3}{(1+v^2)}$$

$$\Rightarrow \quad \frac{(1+v^2)\,\mathrm{dv}}{v^3} = -\frac{\mathrm{dx}}{x}$$

Integrating both sides, we get

$$\int \frac{(1+v^2)\,\mathrm{dv}}{v^3} = -\int \frac{\mathrm{dx}}{x}$$

$$\Rightarrow \quad \int \frac{\mathrm{dv}}{v^3} + \int \frac{\mathrm{dv}}{v} = -\log|x| + C$$

$$\Rightarrow \quad -\frac{1}{2v^2} + \log|v| = -\log|x| + C$$

$$\Rightarrow \quad -\frac{x^2}{2y^2} + \log\left|\frac{y}{x}\right| = -\log|x| + C$$

$$\Rightarrow \quad -\frac{x^2}{2y^2} + \log|y| - \log|x| = -\log|x| + C$$

$$\Rightarrow \quad -\frac{x^2}{2y^2} + \log|y| = C \qquad \ldots(ii)$$

Given, $x = 1, y = 1$

$$\Rightarrow \quad -\frac{1}{2 \times 1} + \log|1| = C$$

$$\Rightarrow \quad -\frac{1}{2} = C \qquad [\because \log 1 = 0]$$

Now (ii) becomes

$$-\frac{x^2}{2y^2} + \log|y| = -\frac{1}{2}$$

$$\Rightarrow \quad \log|y| = \frac{x^2}{2y^2} - \frac{1}{2}$$

$$\Rightarrow \quad \log|y| = \frac{x^2 - y^2}{2y^2}$$

Putting $x = x_0$ and $y = e$ in (iii), we get

$$\log|e| = \frac{x_0^2 - e^2}{2e^2}$$

$$\Rightarrow \quad 1 = \frac{x_0^2 - e^2}{2e^2}$$

$$\Rightarrow \quad x_0^2 - e^2 = 2e^2$$

$$\Rightarrow \quad x_0^2 = 3e^2$$

$$\Rightarrow \quad x_0 = \sqrt{3}e$$

Q.24. Find the particular solution of the differential equation.

$\frac{dy}{dx} + y \tan x = 3x^2 + x^3 \tan x$, $x \neq \frac{\pi}{2}$, **given that** $y = 0$ **when** $x = \frac{\pi}{3}$.

Ans.

Given, $\frac{dy}{dx} + y \tan x = 3x^2 + x^3 \tan\ x$

$\Rightarrow \quad \frac{dy}{dx} + \tan x . y = 3x^2 + x^3 \tan x$

Comparing the given differential equation with linear from

$$\frac{dy}{dx} + Py = Q, \text{ we get}$$

$$P = \tan x, Q = 3x^2 + x^3 \tan x.$$

$\therefore \quad IF = e^{\int \tan x \ dx} = e^{\log\ \sec\ x} = \sec\ x.$

Therefore, general solution is given by

$$y . \sec x = \int (3x^2 + x^3 \tan x) . \sec x \ dx + C$$

$\Rightarrow \quad y . \sec x = \int 3x^2 \sec x \ dx + \int x^3 \tan x . \ \sec x \ dx \ + \ C$

$\Rightarrow \quad y \sec x = \int 3x^2 \sec x \ dx + x^3 . \sec x - \int 3x^2 . \sec\ x \ dx + C$

$\Rightarrow \quad y \sec x = x^3 \sec x + C$

$\Rightarrow \quad y = x^3 + C \cos x$

Now $x = \frac{\pi}{3}, y = 0$

$\therefore \quad 0 = \left(\frac{\pi}{3}\right)^3 + C . \cos\left(\frac{\pi}{3}\right)$

$\Rightarrow \quad 0 = \frac{\pi^3}{27} + \frac{C}{2}$

$\Rightarrow \quad C = - \frac{2\pi^3}{27}$

Hence required particular solution is

$$y = x^3 - \frac{2\pi^3}{27}\cos x.$$

Q.25. Show that the differential equation $(x - y)\frac{\mathrm{dy}}{\mathrm{dx}} = x + 2y$ **is homogeneous and solve it.**

Ans.

Given, $(x - y)\frac{\mathrm{dy}}{\mathrm{dx}} = x + 2y$

By simplifying the above equation, we get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{x+2y}{x-y} \qquad \ldots(i)$$

Let $F(x,y) = \frac{x+2y}{x-y}$

then $F(\lambda x,\lambda y) = \frac{\lambda x+2\lambda y}{\lambda x-\lambda y} = \frac{\lambda(x+2y)}{\lambda(x-y)} = \lambda \circ F(x,y)$

$F(x, y)$ is homogeneous function and hence given differential equation is homogeneous.

Now, let $y = vx \quad \Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = v + x\frac{\mathrm{dv}}{\mathrm{dx}}$

Substituting these values in equation (i), we get

$$v + x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{x+2\,\mathrm{vx}}{x-\mathrm{vx}}$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{1+2v}{1-v} - v = \frac{1+2v-v+v^2}{1-v} = \frac{1+v+v^2}{1-v}$$

$$\Rightarrow \quad \frac{1-v}{1+v+v^2}\mathrm{dv} = \frac{\mathrm{dx}}{x}$$

By integrating both sides, we get

$$\int \frac{1-v}{1+v+v^2}\mathrm{dv} = \int \frac{\mathrm{dx}}{x} \qquad \ldots(ii)$$

LHS $\int \frac{1-v}{v^2+v+1}\mathrm{dv}$

Let $1 - v = A(2v+1) + B = 2Av + (A+B)$

Comparing coefficients of both sides, we get

$$2A = -1, \; A + B = 1$$

or $\quad A = -\frac{1}{2}, \; B = \frac{3}{2}$

$$\therefore \quad \int \frac{1-v}{v^2+v+1}\mathrm{dv} = \int \frac{-\frac{1}{2}(2v+1)+\frac{3}{2}}{v^2+v+1}\mathrm{dv}$$

$$= -\frac{1}{2}\int \frac{2v+1}{v^2+v+1}\mathrm{dv} + \frac{3}{2}\int \frac{\mathrm{dv}}{v^2+v+1}$$

$$= -\frac{1}{2}\int \frac{2v+1}{v^2+v+1}\mathrm{dv} + \frac{3}{2}\int \frac{\mathrm{dv}}{\left(v+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= -\frac{1}{2}\log\left|v^2+v+1\right| + \frac{3}{2} \times \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{v+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

Now, substituting it in equation (ii), we get

$$-\frac{1}{2}\log\left|v^2+v+1\right|+\sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)=\log x+C$$

$$\Rightarrow\quad -\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right|+\sqrt{3}\tan^{-1}\left(\frac{\frac{2y}{x}+1}{\sqrt{3}}\right)=\log x+C$$

$$\Rightarrow\quad -\frac{1}{2}\log\left|x^2+\mathrm{xy}+y^2\right|+\frac{1}{2}\log x^2+\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)=\log x+C$$

$$\Rightarrow\quad -\frac{1}{2}\log\left|x^2+\mathrm{xy}+y^2\right|+\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)=C$$

Q.26. Solve the following differential equation:

$$(x^3+x^2+x+1)\frac{\mathrm{dy}}{\mathrm{dx}}=2x^2+x$$

Ans.

We have $\quad (x^3+x^2+x+1)\frac{\mathrm{dy}}{\mathrm{dx}}=2x^2+x$

$$\Rightarrow\quad \frac{\mathrm{dy}}{\mathrm{dx}}=\frac{2x^2+x}{x^3+x^2+x+1}$$

$$\Rightarrow\quad \mathrm{dy}=\frac{2x^2+x}{(x^2+1)(x+1)}\mathrm{dx}$$

Integrating both sides, we get

$$\int\mathrm{dy}=\int\frac{2x^2+x}{(x^2+1)(x+1)}\,\mathrm{dx}\qquad\ldots(i)$$

$$\Rightarrow\quad \frac{2x^2+x}{(x^2+1)(x+1)}=\frac{A}{x+1}+\frac{\mathrm{Bx}+C}{x^2+1}=A(x^2+1)(\mathrm{Bx}+C)(x+1)\qquad\text{[By partial fraction]}$$

$$\Rightarrow \quad 2x^2 + x = x^2 (A + B) + x (B + C) + (A + C)$$

Comparing coefficients of both the sides, we get

$A + B = 2,\ B + C = 1$ and $A + C = 0$

$$\Rightarrow \quad B = \frac{3}{2}, A = \frac{1}{2}, C = \frac{-1}{2}$$

$$\therefore \ (i) \Rightarrow \ y = \int \left[\frac{\frac{1}{2}}{x+1} + \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1} \right] dx$$

$$= \frac{1}{2} \int \frac{1}{x+1} dx + \frac{3}{2} \int \frac{x}{x^2+1} dx - \frac{1}{2} \int \frac{1}{x^2+1} dx$$

$$y = \frac{1}{2} \log |x + 1| + \frac{3}{4} \log |x^2 + 1| - \frac{1}{2} \tan^{-1} x + C$$

Q.27. Solve the following differential equation:

$$(1 + y^2)(1 + \log x)\ dx + x\ dy = 0$$

Ans.

We have $(1 + y^2)(1 + \log x)\ dx + x\ dy = 0$

$$x\ dy = -(1 + y^2)(1 + \log x)\ dx$$

$$\Rightarrow \quad \frac{dy}{1+y^2} = -\frac{1+\log x}{x} dx$$

Integrating both sides, we get

$$\int \frac{dy}{1+y^2} = -\int \frac{1+\log x}{x} dx$$

$$\Rightarrow \quad \tan^{-1} y = -\int z\, dz \qquad \left[\text{Let } 1 + \log x = z \Rightarrow \frac{1}{x} dx = dz\right]$$

$$\Rightarrow \quad \tan^{-1} y = -\frac{z^2}{2} + C$$

$$\Rightarrow \quad \tan^{-1} y = -\frac{1}{2} (1 + \log x)^2 + C$$

Q.28. Solve the following differential equation:

$x\,dy - \left(y + 2x^2\right)\,dx = 0$

Ans.

We have $x\,dy - \left(y + 2x^2\right)\,dx = 0$

The given differential equation can be written as

$\Rightarrow \quad x\frac{dy}{dx} - y = 2x^2 \quad$ or $\quad \frac{dy}{dx} - \frac{1}{x}.y = 2x$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{-1}{x}, Q = 2x$

$$\text{IF} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$$

$\therefore$ Solution is $y.\frac{1}{x} = \int 2x.\frac{1}{x}dx$

$\Rightarrow \quad y.\frac{1}{x} = 2x + C \quad$ or $\quad y = 2x^2 + Cx$

Q.29. Solve the differential equation, $xdx + (y - x^3)\,dx = 0$.

Ans.

We have $x\,dy + (y - x^3)\,dx = 0$

$\Rightarrow \quad \mathrm{xdy} =- (y - x^3)\mathrm{dx}$

$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{-y+x^3}{x}$

$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{-y}{x} + x^2$

$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} + \left(\frac{1}{x}\right).y = x^2$

It is in the form of $\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{Py} = Q,$ where $P = \frac{1}{x}$ and $Q = x^2$

$\therefore \quad \mathrm{IF} = e^{\int \frac{1}{x}\mathrm{dx}} = e^{\log x} = x$

Hence, solution is $y.x = \int x.x^2\mathrm{dx} + C$

$\mathrm{xy} = \frac{x^4}{4} + C \quad \Rightarrow \quad y = \frac{x^3}{4} + \frac{C}{x}$

Q.30. Find the particular solution of the following differential equation.

$x\frac{\mathrm{dy}}{\mathrm{dx}} - y + x \sin\left(\frac{y}{x}\right) = 0$, **given that when $x = 2$, $y = \pi$**

Ans.

Given differential equation is $x\frac{\mathrm{dy}}{\mathrm{dx}} - y + x\sin\left(\frac{y}{x}\right) = 0$

$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} - \frac{y}{x} + \sin\left(\frac{y}{x}\right) = 0 \qquad ...(i)$

It is homogeneous differential equation.

Let $\frac{y}{x} = v \Rightarrow y = \mathrm{vx}$

$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = v + x\frac{\mathrm{dv}}{\mathrm{dx}}$

Putting these values in (i), we get

$$v + x\frac{\mathrm{dv}}{\mathrm{dx}} - v + \sin v = 0$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} + \sin v = 0$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} = -\sin v$$

$$\Rightarrow \quad \frac{\mathrm{dv}}{\sin v} = \frac{-\,\mathrm{dx}}{x}$$

$$\Rightarrow \quad \operatorname{cosec} v\,\mathrm{dv} = -\frac{\mathrm{dx}}{x}$$

Integrating both sides, we get

$$\Rightarrow \quad \int \operatorname{cosec} v\,\mathrm{dv} = -\int \frac{\mathrm{dx}}{x}$$

$$\Rightarrow \quad \log|\operatorname{cosec} v - \cot v| = -\log|x| + C$$

$$\Rightarrow \quad \log\left|\operatorname{cosec}\frac{y}{x} - \cot\frac{y}{x}\right| + \log|x| = C \qquad \ldots (i)$$

Putting $x = 2$, $y = \text{p}$ we get

$$\Rightarrow \quad \log\left|\operatorname{cosec}\frac{\pi}{2} - \cot\frac{\pi}{2}\right| + \log 2 = C$$

$$\Rightarrow \quad \log 1 + \log 2 = C \qquad [\because \log 1 = 0]$$

$$\Rightarrow \quad C = \log 2$$

Hence, particular solution, is

$$\log\left|\operatorname{cosec}\frac{y}{x} - \cot\frac{y}{x}\right| + \log|x| = \log 2$$

$$\Rightarrow \quad \log\left|x.\left(\operatorname{cosec}\frac{y}{x} - \cot\frac{y}{x}\right)\right| = \log 2$$

$$\Rightarrow \quad x\left(\operatorname{cosec}\frac{y}{x} - \cot\frac{y}{x}\right) = 2$$

Q.31. Find the particular solution of the differential equation:

$(1 - y^2)(1 + \log x)\ dx + 2xy\ dy = 0$ **given that** $y = 0$ **when** $x = 1$.

Ans.

We have

$$(1 - y^2)(1 + \log x)\ dx + 2xy\ dy = 0,$$

$\Rightarrow \quad 2xy\ dy = -(1 - y^2)(1 + \log x)\ dx$

$\Rightarrow \quad \frac{2y\ \mathrm{dy}}{1-y^2} = -\frac{(1+\log x)\,\mathrm{dx}}{x}$

Integrating both sides, we get

$\Rightarrow \quad \int \frac{2y\ \mathrm{dy}}{1-y^2} = -\int \frac{(1+\log x)}{x}\mathrm{dx}$

$\Rightarrow \quad -\log|1 - y^2| = -\int \frac{(1+\log x)}{x}\mathrm{dx}$

$\Rightarrow \quad -\log|1 - y^2| = -\int z\ \mathrm{dz} \qquad \left[\text{Let } 1 + \log x = z \quad \Rightarrow \quad \frac{1}{x}\mathrm{dx} = \mathrm{dz}\right]$

$$\Rightarrow \quad \log|1-y^2| = \frac{z^2}{2} + C$$

$$\Rightarrow \quad \log|1-y^2| = \frac{(1+\log x)^2}{2} + C$$

Putting $x = 1$ and $y = 0$, we get

$$\Rightarrow \quad \log 1 = \frac{(1+\log 1)^2}{2} + C$$

$$\Rightarrow \quad 0 = \frac{1}{2} + C \quad \Rightarrow \quad C = -\frac{1}{2}$$

Hence particular solution is

$$\log|1-y^2| = \frac{(1+\log x)^2}{2} - \frac{1}{2}$$

Q.32. Find the general solution of the following differential equation:

$$(1+y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$$

Ans.

We have $(1+y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$

$$\Rightarrow \quad (x - e^{\tan^{-1} y})\frac{dy}{dx} = -(1+y^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1+y^2}{x-e^{\tan^{-1} y}}$$

$$\Rightarrow \quad \frac{dx}{dy} = -\frac{x-e^{\tan^{-1} y}}{1+y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} = -\frac{x}{1+y^2} + \frac{e^{\tan^{-1} y}}{1+y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{1+y^2}x = -\frac{e^{\tan^{-1} y}}{1+y^2}$$

It is in the form $\frac{dx}{dy} + Px = Q$.

Where $P = \frac{1}{1+y^2}$ and $Q = \frac{e^{\tan^{-1} y}}{1+y^2}$

$$\therefore \quad IF = e^{\int P.\ dy} = e^{\int \frac{1}{1+y^2}\ dy}$$
$$= e^{\tan^{-1} y}$$

Therefore, general solution is

$$x \,.\, e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1+y^2} \,.\, e^{\tan^{-1} y} dy + C$$

$$\Rightarrow \quad x \,.\, e^{\tan^{-1} y} = \int e^{z} \,.\, e^{z} dz + C \qquad \text{Let } \tan^{-1} y = z$$

$$\Rightarrow \quad x \,.\, e^{\tan^{-1} y} = \int e^{2z} dz + C \qquad \frac{1}{1+y^2} dy = dz$$

$$\Rightarrow \quad x \,.\, e^{\tan^{-1} y} = \frac{e^{2z}}{2} + C$$

$$\Rightarrow \quad x \,.\, e^{\tan^{-1} y} = \frac{e^{2\tan^{-1} y}}{2} + C$$

$$\Rightarrow \quad x = \frac{1}{2}\, e^{\tan^{-1} y} + C \,.\, e^{-\tan^{-1} y}$$

Q.33. Find the particular solution of differential equation

$: \frac{dy}{dx} = - \frac{x+y \cos x}{1+\sin x}$

given that *y* = 1 when *x* = 0.

Ans.

We have

$$\frac{\mathrm{dy}}{\mathrm{dx}}=-\frac{x+y\cos x}{1+\sin\ x}$$

$$\Rightarrow\quad \frac{\mathrm{dy}}{\mathrm{dx}}=-\frac{x}{1+\sin\ x}-\frac{y\cos x}{1+\sin\ x}$$

$$\Rightarrow\quad \frac{\mathrm{dy}}{\mathrm{dx}}+\frac{\cos x}{1+\sin\ x}y=-\frac{x}{1+\sin\ x}$$

Comparing it with linear form of differential equation $\frac{\mathrm{dy}}{\mathrm{dx}}+\mathrm{Py}=Q$, we get

$$P=\frac{\cos x}{1+\sin x},\ Q=-\frac{x}{1+\sin x}$$

Now $\mathrm{IF}=e^{\int\frac{\cos x}{1+\sin x}\mathrm{dx}}=e^{\log|1+\sin x|}=1+\sin x$

Therefore, general solution is

$$\begin{aligned} y(1+\sin x) &= \int-\frac{x}{1+\sin x}(1+\sin x)\mathrm{dx}+C \\ &= -\int x\,\mathrm{dx}+C \\ y(1+\sin x) &= -\frac{x^2}{2}+C \end{aligned}$$

Given $y=1$ and $x=0$

$$1(1+\sin 0)=0+\mathrm{C}$$

$$\Rightarrow\quad C=1$$

Hence, particular solution is

$$\Rightarrow\quad y(1+\sin x)=-\frac{x^2}{2}+1$$

$$y=\frac{2-x^2}{2(1+\sin x)}$$

Q.34. Solve the following differential equation :

$$\left(\cot^{-1} y + x\right) dy = \left(1 + y^2\right) dx$$

Ans.

We have

$$\left(\cot^{-1} y + x\right) dy = \left(1 + y^2\right) dx$$

This can be written as

$$\frac{\mathrm{dx}}{\mathrm{dy}} = \frac{\cot^{-1} y + x}{1+y^2} = \frac{\cot^{-1} y}{1+y^2} + \frac{x}{1+y^2}$$

$$\Rightarrow \quad \frac{\mathrm{dx}}{\mathrm{dy}} - \frac{1}{1+y^2} . x = \frac{\cot^{-1} y}{1+y^2}$$

It is linear differential equation of the form

$$\frac{\mathrm{dx}}{\mathrm{dy}} + \mathrm{Px} = Q, \text{ where } P = \frac{-1}{1+y^2} \text{ and } Q = \frac{\cot^{-1} y}{1+y^2}$$

$$\therefore \quad \mathrm{IF} = e^{\int -\frac{1}{1+y^2} \mathrm{dy}} = e^{\cot^{-1} y}$$

Therefore, required solution of differential equation is

$$x . e^{\cot^{-1} y} = \int \frac{\cot^{-1} y}{1+y^2} . e^{\cot^{-1} y} \mathrm{dy} + C$$

$$\Rightarrow \quad x . e^{\cot^{-1} y} = I + C \qquad \ldots (i)$$

Where, $I = \int \frac{\cot^{-1} y}{1+y^2} . e^{\cot^{-1} y} \mathrm{dy}$

Let $\cot^{-1} y = t$

$$-\frac{1}{1+y^2}\,dy = dt$$

$$\Rightarrow \quad \frac{1}{1+y^2}\,dy = -\,dt$$

$$\Rightarrow \quad I = -\int t\,.\,e^t dt = -\left[t.e^t - \int e^t\,dt\right] = -t\,.\,e^t + e^t$$

$$= e^t\,(1-t) = e^{\cot^{-1} y}\,(1-\cot^{-1} y)$$

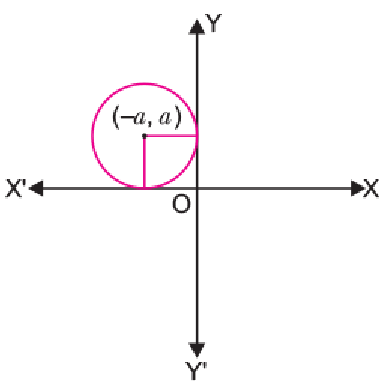
Hence, required solution is

$$x\,.\,e^{\tan^{-1} y} = e^{\cot^{-1} y}\,(1-\cot^{-1} y) + C$$

$$x = (1-\cot^{-1} y) + Ce^{-\cot^{-1} y}$$

Q.35. Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

Ans. Let *C* denotes the family of circles in the second quadrant and touching the coordinate axes. Let (–*a*, *a*) be the coordinate of the centre of any member of this family (see figure).



Equation representing the family C is

$$(x+a)^2+(y-a)^2=a^2 \qquad ...(i)$$

or $\quad x^2+y^2+2ax-2ay+a^2=0 \qquad ...(ii)$

Differentiating equation (ii) with respect to x, we get

$$2x+2y\frac{dy}{dx}+2a-2a\frac{dy}{dx}=0$$

or $\quad x+y\frac{dy}{dx}=a\left(\frac{dy}{dx}-1\right)$

or $\quad a=\frac{x+yy'}{y'-1} \qquad \left(y'=\frac{dy}{dx}\right)$

Substituting the value of a in equation (i), we get

$$\left[x+\frac{x+yy'}{y'-1}\right]^2+\left[y-\frac{x+yy'}{y'-1}\right]^2=\left[\frac{x+yy'}{y'-1}\right]^2$$

or $\quad [xy'-x+x+yy']^2+[yy'-y-x-yy']^2=[x+yy']^2$

or $\quad (x+y)^2y'^2+(x+y)^2=(x+yy')^2$

or $\quad (x+y)^2\left[(y')^2+1\right]=[x+yy']^2$, is the required differential equation representing the given family of circles.

Long Answer Questions-I (OIQ)

[4 Marks]

Q.1. Solve: $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

Ans.

We have, $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

$\Rightarrow \quad \sec^2 x \tan y \, dx = -\sec^2 y \tan x \, dy$

$\Rightarrow \quad \frac{\sec^2 x}{\tan x}\mathrm{dx} = -\frac{\sec^2 y}{\tan y}\mathrm{dy}$

$\Rightarrow \quad \int \frac{\sec^2 x}{\tan x}\mathrm{dx} = -\int \frac{\sec^2 y}{\tan y}\mathrm{dy}$

$\Rightarrow \quad \log|\tan x| = -\log|\tan y| + \log C$

$\Rightarrow \quad \log|(\tan x)(\tan y)| = \log \mathrm{C}$

$\Rightarrow \quad |\tan x \tan y| = C$

Clearly, it is defined for $x \in |\mathrm{R} - \{(2n+1)\pi/2 : n \in Z\}$

Hence, $|\tan x \tan y| = C$, where $x \in |\mathrm{R} - \{(2n+1)\pi/2 : n \in Z\}$ is the solution of the given differential equation.

Q.2. **Solve:** $(x+3y^2)\frac{\mathrm{dy}}{\mathrm{dx}} = y\,(y > 0)$

Ans.

Given differential equation is $(x+3y^2)\frac{\mathrm{dy}}{\mathrm{dx}} = y,\, y0)$

We can write this as

$$\frac{\mathrm{dx}}{\mathrm{dy}} = \frac{x+3y^2}{y} = \frac{1}{y}.x + 3y$$

$$\Rightarrow \quad \frac{\mathrm{dx}}{\mathrm{dy}} + \left(-\frac{1}{y}\right).x = 3y$$

This is a linear equation of the form

$$\frac{\mathrm{dx}}{\mathrm{dy}} + \mathrm{Px} = Q, \text{ where } P = -\frac{1}{y}, Q = 3y$$

So, $\mathrm{IF} = e^{\int -\frac{1}{y}\mathrm{dy}} = e^{-\log y} = e^{\log y^{-1}} = y^{-1} = \frac{1}{y}$

Multiplying both sides by IF, we get

$$\frac{1}{y} \times \frac{dx}{dy} - \frac{1}{y^2} x = 3$$

$$\Rightarrow \quad \frac{d}{dy}\left(\frac{1}{y} . x\right) = 3$$

Integrating both sides, with respect to y, we get

$$\frac{1}{y} . x = 3y + C$$

Hence, $x = 3y^2 + Cy$ is the required solution.

Q.3. Solve $\frac{dy}{dx} + y \sec x = \tan x.$

Ans.

The given differential equation is

$$\frac{dy}{dx} + (\sec x) y = \tan x \qquad ...(i)$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = \sec x$ and $Q = \tan x$

$$\therefore \quad IF = e^{\int P \, dx} = e^{\int \sec x \, dx} = e^{\log (\sec x + \tan x)} = (\sec x + \tan x)$$

Multiplying both sides of (i) by IF = $(\sec x + \tan x)$, we get

$$(\sec x + \tan x)\frac{dy}{dx} + y \sec x (\sec x + \tan x) = \tan x (\sec x + \tan x)$$

Integrating both sides, we get

$$y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)\mathrm{dx} + C \quad [\text{Using} : y(\mathrm{IF}) = \int Q.(\mathrm{IF})\mathrm{dx} + C]$$

$\Rightarrow \quad y(\sec x + \tan x) = \int (\tan x \sec x + \tan^2 x)\mathrm{dx} + C$

$\Rightarrow \quad y(\sec x + \tan x) = \int (\tan x \sec x + \sec^2 x - 1)\mathrm{dx} + C$

$\Rightarrow \quad y(\sec x + \tan x) = \sec x + \tan x - x + C$, which is the required solution.

Q.4. Solve : $\frac{\mathrm{dy}}{\mathrm{dx}} = e^{x-y} + x^2 e^{-y}$

Ans.

We have,

$$\frac{\mathrm{dy}}{\mathrm{dx}} = e^{x-y} + x^2 e^{-y}$$

$\Rightarrow \quad dy = (e^{x-y} + x^2 e^{-y})\, dx$

$\Rightarrow \quad e^y\, dy = (e^x + x^2)\, dx$

Integrating both sides, we get

$\Rightarrow \quad \int e^y \mathrm{dy} = \int (e^x + x^2)\mathrm{dx}$

$\Rightarrow \quad e^y = e^x + \frac{x^3}{3} + C$, which is the required solution.

Q.5. Form the differential equation representing the family of curves $y^2 - 2ay + x^2 = a^2$, where a is an arbitrary constant.

Ans.

Given family of curves $y^2 - 2ay + x^2 = a^2$ ...(i)

Differentiating with respect to x, we get

$$2y\frac{dy}{dx} - 2a\frac{dy}{dx} + 2x = 0$$

$$\Rightarrow \quad (y-a)\frac{dy}{dx} + x = 0$$

$$(y-a)\frac{dy}{dx} = -x$$

$$\Rightarrow \quad y - a = -x.\frac{dx}{dy}$$

$$\Rightarrow \quad a = \left(y + x\frac{dx}{dy}\right)$$

Substituting the value of a, in (i), we get

$$y^2 - 2\left(y + x\frac{dx}{dy}\right)y + x^2 = \left(y + x\frac{dx}{dy}\right)^2$$

$$\Rightarrow \quad y^2 - 2y^2 - 2\,xy\,\frac{dx}{dy} + x^2 = y^2 + x^2\left(\frac{dx}{dy}\right)^2 + 2\,xy\,\frac{dx}{dy}$$

$$\Rightarrow \quad (x^2 - y^2) - 2\,xy\,\frac{dx}{dy} = y^2 + x^2\left(\frac{dx}{dy}\right)^2 + 2\,xy\,\frac{dx}{dy}$$

$$\Rightarrow \quad (x^2 - 2y^2) - 4\,xy\,\frac{dx}{dy} = x^2\left(\frac{dx}{dy}\right)^2 \qquad ...(ii)$$

Let $\frac{dy}{dx} = p \;\Rightarrow\; \frac{dx}{dy} = \frac{1}{p}$

Therefore, (ii) becomes, $(x^2 - 2y^2) - 4\,xy\,\frac{1}{p} = x^2\left(\frac{1}{p}\right)^2$

$$\Rightarrow \quad p^2\left(x^2 - 2y^2\right) - 4xyp = x^2$$

$\Rightarrow \quad p^2\left(x^2 - 2y^2\right) - 4xyp - x^2 = 0$, where $p = \frac{dy}{dx}$.'

Q.6. Find the general solution of the following differential equation:

$$x \cos\left(\frac{y}{x}\right) \frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x$$

Ans.

Given differential equation is $x \cos\left(\frac{y}{x}\right) \frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x$

$$\frac{dy}{dx} = \frac{y \cos y/x + x}{x \cos y/x} \qquad \ldots(i)$$

It is homogeneous differential equation.

Let $y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$(i) \quad \Rightarrow \quad v + x\frac{dv}{dx} = \frac{vx \cos v + x}{x \,.\, \cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v \cos v + 1}{\cos v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v \cos v + 1 - v \cos v}{\cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1}{\cos v}$$

$$\Rightarrow \quad \cos v \, dv = \frac{dx}{x}$$

Integrating both sides

$$\Rightarrow \quad \sin v = \log|x| + C$$

$$\Rightarrow \quad \sin \frac{y}{x} = \log|x| + C$$