CHAPTER

5

Permutation and Combination

- Fundamental Principle of Counting
- Factorial Notation
- Permutation
- Combination
- Circular Permutations
- All Possible Selections
- Division and Distribution
- Multinomial Theorem
- Principle of Inclusion and Exclusion

FUNDAMENTAL PRINCIPLE OF COUNTING

Multiplication Rule

If a work A can be done in m ways and another work B can be done in n ways and C is a work, which is done only when both A and B are completed, the number of ways of doing the work C is $m \times n$. In other words, if an operation can be performed in m different ways and corresponding to each of these there are n different ways of performing another operation, then both the operations can be performed in $m \times n$ different ways.

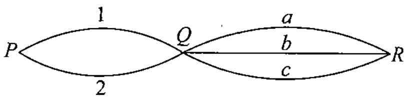


Fig. 5.1

First of all, we give an example to show the validity of the above principle. Suppose, there are three stations P, Q and R and we have three routes to go from P to Q and two routes to go from Q to R. We want to know the number of routes to go from P to R.

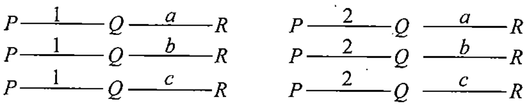


Fig. 5.2

For each path to go from P to Q, there are three paths to go from Q to R.

Thus, for going from P to R via Q, there will be $2 \times 3 = 6$ paths.

Proof of the Multiplication Rule of Fundamental Principle of Counting

The first operation can be performed in any one of the m ways and for each of these ways of performing the first operation, there are n ways of performing the second operation. Thus, if the first operation could be performed in one such way, there would have been $1 \times n = n$ ways of performing both the operations. But it is given that first operation can be performed in m ways and for each way of performing the first operation, second can be performed in n ways.

Therefore, the total number of ways of performing both the operations is $n + n + n + \cdots$ to m terms $= n \times m$.

Note: *If three operations can be separately performed in m, n and p ways, respectively, then the three operations together can be performed in m × n × p ways. Similar result holds for any number of operations.*

Addition Rule

If a work A can be done in m ways and another work B can be done in n ways and C is a work which is done only when either A or B is completed, then number of ways of doing the work C is $m + n$.

Example:

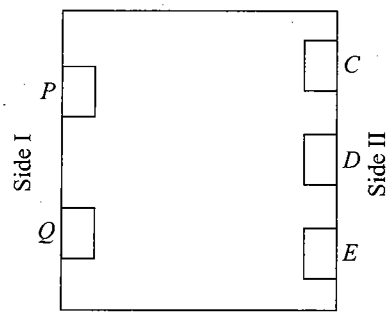


Fig. 5.3

Suppose, there are 5 doors in a room: 2 on one side and 3 on the other. A man has to go out of the room. The man can go out from any one of the 5 doors. Thus, the number of ways in which the man can go out is 5. Here, the work of going out through the doors on one side will be done in 2 ways and the work of going out through the doors on other side will be done in 3 ways. The work of going out will be done when the man goes out from either side I or side II. Thus, the work of going out can be done in $2 + 3 = 5$ ways.

Example 5.1 **Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word MAKE, where the repetition of the letters is not allowed.**

Sol. There are as many words as there are ways of filling in 4 vacant places XXXX by the 4 letters, keeping in mind that the repetition is not allowed. The first place can be filled in 4 different ways by anyone of the 4 letters M, A, K, E.

Following which, the second place can be filled in by anyone of the remaining 3 letters in 3 different ways, following which the third place can be filled in 2 different ways; following which, the fourth place can be filled in 1 way.

Thus, the number of ways in which the4 places can be filled, by the multiplication principle, is $4 \times 3 \times 2 \times 1 = 24$.

Hence, the required number of words is 24.

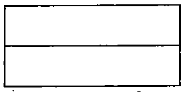
Example 5.2 **How many 2 digit even numbers can be formed from the digits1, 2, 3, 4, 5 if the digits can be repeated?**

Sol. There will be as many ways as there are ways of filling 2 vacant places XX in succession by the five given digits. Here, we start filling in unit's place, because the options for this place are 2 and 4 only and this can be done in 2ways; following which the ten's place can be filled by any of the 5 digits in 5 different ways as the digits can be repeated.

Therefore, by the multiplication principle, the required number of two digits even numbers is 2×5, i.e., 10.

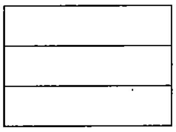
Example 5.3 **Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.**

Sol. A signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags. Now, let us count the possible number of signals consisting of 2 flags, 3 flags, 4 flags and 5 flags separately and then add the respective numbers.



There will be as many 2 flag signals as there are ways of filling in 2 vacant places in succession by the 5 flags available. By Multiplication rule, the number of ways is $5 \times 4 = 20$.

Similarly, there will be as many 3 flag signals as there are ways of filling in 3vacant places in succession by the 5 flags.



The number of ways is $5 \times 4 \times 3 = 60$. Continuing the same way, we find that the number of 4 flag signals $= 5 \times 4 \times 3 \times 2 = 120$ and the number of 5 flag signals $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

Therefore, the required no of signals $= 20 + 60 + 120 + 120 = 320$.

Example 5.4 **Poor Dolly's T.V. has only 4 channels; all of them quite boring, hence it is not surprising that she desires to switch (change) channel after every one minute. Then find the number of ways in which she can change the channels so that she is back to her original channel for the first time after 4 minutes.**

Sol. Let there be 4 channels C_1 C_2 C_3 and C_4

at t = 0 minute she is watching channel 1

$\therefore$ after 1st minute she has 3 choices to switch the channel (C_2, C_3, C_4)

after 2nd minute she has 2 choices to switch the channel

after 3rd minute she has 2 choices to switch the channel

but after the 4th minute she has only 1 choice to switch the channel i.e. C_1

$\therefore$ Total number of ways $= 3 \times 2 \times 2 = 12$

Example 5.5 **There are '*n*' locks and '*n*' matching keys. If all the locks and keys are to be perfectly matched, find the maximum number of trials required to open a lock.**

Sol. The maximum number of trials needed for the first key is n. For second key, it will be $n - 1$.

Now, for the r^{th} key, the maximum number of trials needed is $n - r + 1$. Thus, the required answer is

$$n + (n - 1) + \cdots + 1 = \frac{n(n+1)}{2}$$

Example 5.6 **Find the 2-digit number (having different digits), which is divisible by 5.**

Sol. Any number of required type ends in either 5 or 0. Hence, the two-digit number (with different digits) that ends in 5 is 8 and that of 0 is 9. Therefore, by addition principle, the required number is $8 + 9 = 17$.

Example 5.7 **Find the total number of ways in which n distinct objects can be put into two different boxes.**

Sol. Let the two boxes be B_1 and B_2. For each of the n objects, there are two choices, it is put in either box B_1 or box B_2. Therefore, by fundamental principle of counting, the total number of ways is $2 \times 2 \times \cdots \times 2$ (n times) $= 2^n$.

Example 5.8 **Three dice are rolled. Find the number of possible outcomes in which at least one dice shows 5.**

Sol. When a dice is rolled, there are six possible outcomes. So, the total number of outcomes when three dice are rolled is $6 \times 6 \times 6 = 6^3$.

Now, the number of possible outcomes in which at least one dice shows 5 is as follows.

Total number of possible outcomes – Number of possible outcomes in which 5 does not appear on any dice $= 6^3 - 5^3 = 91$

Example 5.9 **Find the number of distinct rational numbers x such that $0 < x < 1$ and $x = p/q$, where $p, q \in \{1, 2, 3, 4, 5, 6\}$.**

Sol. As $0 < x < 1$, we have $p < q$.

p	q
1	2, 3, 4, 5, 6
2	3, 4, 5, 6
3	4, 5, 6
4	5, 6
5	6

Thus, the number of rational numbers is $5+4+3+2+1=15$.

When p and q have a common factor, we get some rational numbers, which are not different from those already counted. Here, there are four such numbers: 2/4, 2/6, 3/6, 4/6.

Therefore, the required number of rational numbers is $15-4=11$.

Example 5.10 **Find the total number of integer '*n*' such that $2 \le n \le 2000$ and H.C.F. of n and 36 is 1.**

Sol. $36 = 2^2 \times 3^2$

If H.C.F. of integer 'n' and 36 is 1, then n should not be divisible by 2 or 3.

Let us first find the numbers that are divisible by 2 or 3.

The number of integers in the range [2, 2000] that are divisible by 2 is 1000 (2, 4, 6,..., 1998, 2000).

The number of integers in the range [2, 2000] that are divisible by 3 is 666 (3, 6, 9, ..., 1995, 1998).

The number of integers in the range [2, 2000] that are divisible by 6 is 333 (6, 12, 18,..., 1992, 1998).

Total number of integers divisible by 2 or 3 is $1000 + 666 - 33 = 1333$.

Thus, the total number of integers that are divisible by neither 2 nor 3 is $1999 - 1333 = 666$.

Example 5.11 **Find the number of polynomials of the form $x^3 + ax^2 + bx + c$ that are divisible by $x^2 + 1$, where $a, b, c \in \{1, 2, 3, ..., 9, 10\}$.**

Sol.

$$\begin{array}{r|l} & x+a \\ x^2+1 & x^3+ax^2+bx+c \\ & x^3+x \\ \hline & ax^2+(b-1)x+c \\ & ax^2+a \\ \hline & (b-1)x+c-a \end{array}$$

Now, remainder $(b-1)x+c-a$ must be zero for any x. Then,

$b - 1 = 0$ and $c - a = 0$

$\Rightarrow$ $b = 1$ and $c = a$

Now, c or a can be selected in 10 ways. Hence, number of polynomials are 10.

Example 5.12 **Find the number of diagonals in the polygon of n sides.**

Sol.

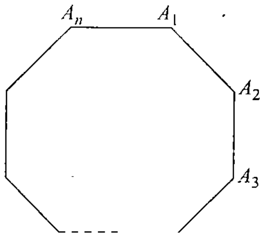


Fig. 5.4

For diagonal, we have to select any two vertices. The first vertex can be selected in n ways. Let A_1 be chosen as first vertex. Now, diagonal cannot be formed if any of A_2 and A_n is chosen. Hence, for A_1 another vertex can be selected in $n - 3$ ways from remaining $n - 3$ vertices.

Again, by principle of counting, the number of ways two vertices can be selected is $n(n - 3)$.

Now, when A_1 is chosen as the first vertex, sometimes A_4 is chosen as the second vertex.

Similarly, when A_4 is chosen as the first vertex, sometimes A_1 is chosen as the second vertex.

Hence, each pair is selected twice. Therefore, the total number of diagonals is $n(n - 3)/2$.

Example 5.13 **Find the total number of 'n'-digit numbers ($n > 1$), having the property that no two consecutive digits are same.**

Sol.

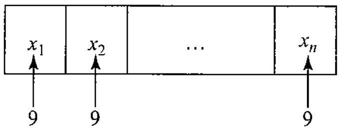


Fig. 5.5

The digit 'x_1' can be selected in 9 ways as '0' cannot be selected.

The digit 'x_2' can be selected in 9 ways as '0' can be selected but digit in position x_1 cannot be selected.

Similarly, each of the remaining digits can also be selected in 9 ways.

Thus, the total number of such numbers is 9^n.

Concept Application Exercise 5.1

1. Four buses run between Bhopal and Gwalior. If a man goes from Gwalior to Bhopal by a bus and comes back to Gwalior by another bus, find the total possible ways.
2. A gentleman wants to invite six friends. In how many ways can he send invitation cards to them, if he has three servants to carry the cards.
3. Find the total number of ways of answering five objective type questions, each question having four choices.
4. In how many ways first and second rank in Mathematics, first and second rank in Physics, first rank in Chemistry and first rank in English be given away to a class of 30 students.
5. Five persons entered the lift cabin on the ground floor of an 8-floor building. If, each of them can leave the cabin independently at any floor beginning with the first; find the total number of ways in which each of the five persons can leave the cabin: (i) at any one of the 7 floors and (ii) at different floors.
6. If $p, q \in \{1, 2, 3, 4\}$, then find the number of equations of the form $px^2 + qx + 1 = 0$ having real roots.
7. Find the number of non-zero determinant of order 2 with elements 0 or 1 only.
8. Find the number ordered pairs (x, y) if $x, y \in \{0, 1, 2, 3, ..., 10\}$ and if $|x - y| > 5$.
9. (a) If $a, b \in \{1, 2, 3, 4, 5, 6\}$, find the number of ways a and b can be selected if

$$\lim_{x \to 0}\left(\frac{a^x + b^x}{2}\right)^{\frac{2}{x}} = 6$$

(b) If $a, b, c \in \{1, 2, 3, 4, 5, 6\}$, find the number of ways a, b, c can be selected if $f(x) = x^3 + ax^2 + bx + c$ is an increasing function.

10. Find the number of ways in which two small squares can be selected on the normal chessboard if they are not in same row or same column.

FACTORIAL NOTATION

The product of first n natural number is denoted by $n!$ and is read as 'factorial n'. Thus,

$$n! = 1 \times 2 \times 3 \times 4 \cdots (n-1) \times n$$
$$= n(n-1)(n-2) \cdots 3 \times 2 \times 1$$

For example $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$, $4! = 1 \times 2 \times 3 \times 4 = 24$.

Some Results Related to Factorial *n*

- $n! = 1 \times 2 \times 3 \cdots (n-1)n = \{1 \times 2 \times 3 \cdots (n-1)\}n$

  Hence,
  $$n! = (n-1)!n = n(n-1)!$$
  Similarly,
  $$(n-1)! = (n-1)(n-2)!$$
  Thus,
  $$n! = n(n-1)!$$
  $$= n(n-1)(n-2)!$$
  $$= n(n-1)(n-2)(n-3)!$$
  and so on

- If n and r are positive integers, then
  $$\frac{n!}{r!} = \frac{1 \times 2 \times 3 \times \cdots \times n}{1 \times 2 \times 3 \times \cdots \times r}$$
  $$= (r+1)(r+2) \cdots (n-1)n$$
  $$= n(n-1)(n-2) \cdots (r+1)$$

- $$\frac{n!}{(n-r)!} = \frac{1 \times 2 \times 3 \times \cdots \times (n-1)n}{1 \times 2 \times \cdots \times (n-r)}$$
  $$= \frac{1 \times 2 \times 3 \times \cdots \times (n-r)(n-r+1)(n-r+2) \times \cdots \times (n-1)n}{1 \times 2 \times 3 \times \cdots \times (n-r)}$$
  $$= (n-r+1)(n-r+2) \cdots (n-1)n$$
  $$= n(n-1)(n-2) \cdots (n-r+2)(n-r+1)$$
  $$= n(n-1)(n-2) \cdots \text{to } r \text{ factors}$$

Exponent of Prime in *n!*

Let p be a given prime and n any positive integer. Then the maximum power of p present in $n!$ is $[n/p] + [n/p^2] + [n/p^3] + \cdots$ where $[x]$ denotes the greatest integer function. The proof of the above formula can be obtained using the fact that $[n/m]$ gives the number of integral multiples of m in $1, 2, \ldots, n$ for any positive integers n and m. The above formula does not work for composite numbers. For example, if we find the maximum power of 6 present in 32!, we find that the answer is not $[32/6] + [32/6^2] + \cdots = 5$, as 5 is the number of integral multiples of 6 in $1, 2, \ldots, 32$ and 6 can also be obtained by multiplying 2 and 3.

Hence, for the required number, we find the maximum powers of 2 and 3 (say r and s) present in 32! using the above formula $r = 31$ and $s = 14$. Hence, 2 and 3 will be combined 14 times (to form 6). Thus, maximum power of 6 present in 32! is 14.

Example 5.14 **Find n, if $(n+1)! = 12 \times (n-1)$.**

Sol. $(n+1)! = 12 \times (n-1)!$

$\Rightarrow (n+1) \times n \times (n-1)! = 12 \times (n-1)!$

$\Rightarrow n(n+1) = 12$

$\Rightarrow n^2 + n - 12 = 0$

$\Rightarrow (n+4)(n-3) = 0$

$\Rightarrow n = 3$

Example 5.15 **Prove that $(n!)^2 < n^n\, n! < (2n)!$ for all positive integers n.**

Sol. We have,

$$(n!)^2 = (n!)(n!) = (1 \times 2 \times 3 \times 4 \times \cdots \times (n-1)n)(n!)$$

Now, $1 \le n, 2 \le n, 3 \le n, \ldots, n \le n$

$\Rightarrow 1 \times 2 \times 3 \cdots (n-1)n \le n \times n \times n \cdots n$

$\Rightarrow n! \le n^n$

$\Rightarrow (n!)(n!) \le (n!)\,n^n$

$\Rightarrow (n!)^2 \le n^n (n!)$ (1)

Also,

$$(2n)! = 1 \times 2 \cdots n \times (n+1) \cdots (2n-1) \times (2n)$$

Now,

$$n+1 > n, n+2 > n, n+3 > n, \ldots, n+n > n$$

$\Rightarrow (n+1)(n+2)(n+3) \cdots (2n-1)(2n) > n^n$

$\Rightarrow n!(n+1)(n+2) \cdots (2n-1)(2n) > n!\,n^n$

$\Rightarrow (2n)! > n!\,n^n \Rightarrow n!\,n^n < (2n)!$ (2)

From (1) and (2), we get $(n!)^2 \le n^n (n!) < (2n)!$

Example 5.16 **Find the sum of the series $\sum_{r=1}^{n} r \times r!$**

Sol. Here, the general term of the series is

$$T_r = r \times r! = (r+1-r)r! = (r+1)r! - r! = (r+1)! - r!$$

Hence,

$$T_1 = 2! - 1!$$
$$T_2 = 3! - 2!$$
$$T_3 = 4! - 3!$$
$$T_n = (n+1)! - n!$$

Adding all the above terms, we have the sum of n, terms, i.e.,

$$S_n = (n+1)! - 1$$

Example 5.17 **Find the exponent of 3 in 100!**

Sol. $100! = 1 \times 2 \times 3 \times \cdots \times 98 \times 99 \times 100$

$= (1 \times 2 \times 4 \times 5 \times \cdots \times 98 \times 100)$
$(3 \times 6 \times 9 \times \cdots \times 96 \times 99)$

$= K \times 3^{33}(1 \times 2 \times 3 \times \cdots \times 32 \times 33)$

$= K \times 3^{33}(1 \times 2 \times 4 \times \cdots \times 31 \times 32)$
$(3 \times 9 \times 12 \times \cdots \times 30 \times 33)$

$= [K(1 \times 2 \times 4 \times \cdots \times 31 \times 32)] \times 3^{33}$
$\times (3 \times 9 \times 12 \times \cdots \times 30 \times 33)$

$= K_1 \times 3^{33} \times 3^{11}(1 \times 2 \times 3 \times \cdots \times 10 \times 11)$

$= K_1 \times (1 \times 2 \times 4 \times \cdots \times 10 \times 11)\, 3^{33} \times 3^{11}$
$(3 \times 6 \times 9 \times 12)$

$= K_2 \times 3^{33} \times 3^{11} \times 3^4 \times (1 \times 2 \times 3 \times 4)$

$= K_3 \times 3^{33} \times 3^{11} \times 3^4 \times 3$

$= K_3 \times 3^{49}$

Hence, exponent of 3 is 49.

Alternative solution:

Exponent of 3 in 100! is

$$\left[\frac{100}{3}\right]+\left[\frac{100}{3^2}\right]+\left[\frac{100}{3^3}\right]+\left[\frac{100}{3^4}\right]=33+11+4+1=49$$

Example 5.18 **Find the number of zeroes at the end of 130.**

Sol. Number of zeroes at the end of 130! is equal to the exponent of 10 in 130. Now, exponent of 10 is equal to exponent of 5 as exponent of 2 is higher than exponent of 5. Now, exponent of 5 is

$$\left[\frac{130}{5}\right]+\left[\frac{130}{5^2}\right]+\left[\frac{130}{5^3}\right]=26+5+1=32$$

Also, exponent of 10 is 32 and hence, there are 32 zeros at the end of 130. It should be noted that exponent of 2 is

$$\left[\frac{130}{2}\right]+\left[\frac{130}{2^2}\right]+\left[\frac{130}{2^3}\right]+\left[\frac{130}{2^4}\right]+\left[\frac{130}{2^5}\right]+\left[\frac{130}{2^6}\right]+\left[\frac{130}{2^7}\right]$$

$$=65+32+16+8+4+2+1=128$$

Hence, exponent of 10 is equal to exponent of 5.

Concept Application Exercise 5.2

1. Prove that $(2n)!/n = \{1 \times 3 \times 5 \cdots (2n-1)\}\, 2^n$.
2. Show that $1!+2!+3!+\cdots+n!$ cannot be a perfect square for any $n \in N$, $n \geq 4$.
3. Prove that $(n!+1)$ is not divisible by any natural number between 2 and n.
4. Find the remainder when $1!+2!+3!+4!+\cdots+n!$ is divided by 15, if $n \geq 5$.
5. Find the exponent of 80 in 200!

PERMUTATION

Each of the different arrangements that can be made by taking some or all of a number of given things or objects at a time is called a permutation. In permutation, order of appearance of things is taken into account.

Example:

The following six arrangements can be made with three distinct objects a, b, c taking two at a time: ab, ba, bc, cb, ac, and ca. Each of these arrangements is called a permutation.

Number of Permutations of *n* Different Things Taken *r* at a Time

To establish the formula $^nP_r = n!/(n-r)!$

Proof:

^nP_r is number of permutations of r things out of n different things, i.e., number of ways of filling up r vacant places with n different things. (In each place, exactly one object is put.)

Let the n different things be $a_1, a_2, a_3, \ldots, a_n$.

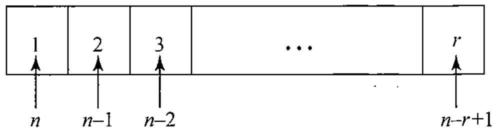


Fig. 5.6

First place can be filled up by any one of the n things a_1, $a_2, a_3, \ldots, a_n$ in n ways.

Number of things left after filling up the first place is $n-1$.

Second place can be filled up by any one of the remaining $n-1$ things in $n-1$ ways.

Number of things left after filling up the first and second places is $n-2$.

Third place can be filled up by any one of the remaining $n-2$ things in $n-2$ ways.

The number of ways of filling up the third place is $n-2$ and so on.

Finally, the number of ways of filling up the r^{th} place is $n-(r-1)=n-r+1$.

By the multiplication rule of counting, first, second, third, ..., r^{th} places can together be filled up in $n(n-1)(n-2)\cdots(n-r+1)$ ways. Hence,

$$^nP_r = n(n-1)(n-2)\cdots(n-r+1)$$

$$=\frac{[n(n-1)(n-2)\cdots(n-r+1)](n-r)\cdots 3\times 2\times 1}{(n-r)(n-r-1)\,3\times 2\times 1}$$

$$=\frac{n!}{(n-r)!}$$

Thus,

$$^nP_r=\frac{n!}{(n-r)!}$$

Number of Permutations of *n* Different Things Taken All at a Time Is *n!*

Proof:

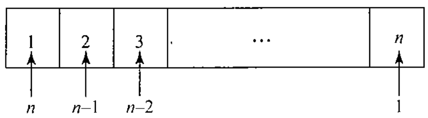


Fig. 5.7

By multiplication rule, number of ways of filling up the first, second, third, ... n^{th} places together is

$$n(n-1)(n-2)\cdots 2\times 1 = n!$$

Thus, $^nP_n = n!$

Note:

$$^nP_n = {}^nP_{n-1} = n!$$

Factorial Zero

From the formula ^nP_r, we have

$$^nP_n=\frac{n!}{0!} \qquad (1)$$

Also, number of permutations of n different things taken all at a time is $n!$ (2)

From Eq. (1), number of permutations of n different things taken all at a time,

$$^nP_n = \frac{n!}{0!} \quad (3)$$

From Eq. (2) and (3),

$$n! = \frac{n!}{0!} \quad (4)$$

Again, Eq. (4) will be valid only when 0! is taken as 1.

Thus, 0! has no meaning from the definition of factorial.

But in order to make the formula for $^nP_r = n!/(n-r)!$ valid for $r = n$, 0! is taken as 1.

Meaning of 1/(−k)! where k Is a Positive Integer

$$^nP_r = \frac{n!}{(n-r)!} \quad (1)$$

Putting $r = n + k$, we have

$$^nP_{n+k} = \frac{n!}{(-k)!} \quad (2)$$

But, number of ways of arranging $n + k$ out of n different things is 0.

$$\therefore \quad \frac{n!}{(-k!)} = 0, \text{ i.e., } \frac{1}{(-k)!} = 0$$

Note: Although $(-k)!$ has no meaning by the definition of factorial but if $1/(-k)!$ is taken as 0 (zero), then the formula $^nP_r = n!/(n-r)!$ will become valid even for $r > n$.

Example 5.19 **If $^{10}P_r = 5040$, find the value of r.**

Sol. $^{10}P_r = 5040$

$= 10 \times 504$

$= 10 \times 9 \times 8 \times 7$

$= {}^{10}P_4$

$\Rightarrow r = 4$

Example 5.20 **If $^9P_5 + 5\,{}^9P_4 = {}^{10}P_r$, find the value of r.**

Sol. $^{10}P_r = {}^9P_5 + 5\,{}^9P_4$

$$= \frac{9!}{(9-5)!} + 5 \times \frac{9!}{(9-4)!}$$

$$= \frac{9!}{4!} + 5 \times \frac{9!}{5!}$$

$$= \frac{9!}{4!} + \frac{9!}{4!}$$

$$= 2 \times \frac{9!}{4!}$$

$$= \frac{5 \times 2 \times 9!}{5 \times 4!}$$

$$= \frac{10 \times 9!}{5!}$$

$$= \frac{10!}{5!}$$

$$= {}^{10}P_5$$

$\Rightarrow r = 5$

Example 5.21 **If $^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$, then find the value of n.**

Sol. $^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$

$$\Rightarrow \frac{^{2n+1}P_{n-1}}{^{2n-1}P_n} = \frac{3}{5}$$

$$\Rightarrow \frac{(2n+1)!}{(n+2)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$

$$\Rightarrow \frac{(2n+1)\,(2n)}{(n+2)\,(n+1)n} = \frac{3}{5}$$

$\Rightarrow 10\,(2n+1) = 3\,(n+2)\,(n+1)$

$\Rightarrow 3n^2 - 11n - 4 = 0$

$\Rightarrow (n-4)\,(3n+1) = 0$

$\Rightarrow n = 4$

Example 5.22 **Prove that if $r \le s \le n$, then ^nP_s is divisible by ^nP_r.**

Sol. Let $s = r + k$ where $0 \le k \le s - r$. Then,

$$^nP_s = \frac{n!}{(n-s)!}$$

$= n\,(n-1)\,(n-2) \cdots (n-(s-1))$

$= n\,(n-1)\,(n-2) \cdots (n-(r+k-1))$

$= n\,(n-1)\,(n-2) \cdots (n-(r-1))\,(n-r)\,(n-(r+1)) \cdots (n-(r+k-1))$

$= \{n\,(n-1)\,(n-2) \cdots n-(r-1)\}\,\{(n-r)\,(n-(r+1)) \cdots (n-(r+k-1))\}$

$= {}^nP_r\,\{(n-r)\,(n-(r+1)) \cdots (n-(r+k-1))\}$

$= {}^nP_r \times \text{Integer}$

Hence, ^nP_s is divisible by ^nP_r.

Example 5.23 **Seven athletes are participating in a race. In how many ways can the first three athletes win the prizes?**

Sol. It is equivalent to filling 3 places (as prizes) with 7 persons The number of permutations of 7 objects taken three at a time is

$$^7P_3 = 7 \times 6 \times 5 = 210$$

Example 5.24 **In how many ways can 6 persons stand in a queue?**

Sol. The number of ways in which 6 persons can stand in a queue is same as filling six places with six persons. The number of permutations of six objects taken all at a time is

$$^6P_6 = 6! = 720$$

Example 5.25 **How many different signals can be given using any number of flags from 5 flags of different colours?**

Sol. The signals can be made by using one or more flags at a time.

The total number of signals when r flags are used at a time from 5 flags is equal to the number of arrangements of 5, taking r at a time, i.e., ${}^5P_r$.

Since r can take the values 1, 2, 3, 4, 5, hence, by the fundamental principle of addition, the total number of signals is

$$\begin{aligned}{}^5P_1+{}^5P_2+{}^5P_3+{}^5P_4+{}^5P_5 &= 5+(5\times4)+(5\times4\times3)+(5\times4\times3\times2)\\ &\quad+(5\times4\times3\times2\times1)\\ &= 5+20+60+120+120\\ &= 325\end{aligned}$$

Example 5.26 **Find the sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time.**

Sol. Total number of numbers formed with the digits 2, 3, 4, 5 taken all at a time is equal to the number of arrangements of 4 digits, taken all at a time, i.e., ${}^4P_4 = 4! = 24$.

To find the sum of these 24 numbers, we have to find the sum of the digits at unit's, ten's, hundred's and thousand's places in all these numbers.

Consider the digits in the unit's places in all these numbers.

If 2 is the digit in unit's place, remaining three places can be filled in 3! ways or we can say 2 occurs in unit's place 3! (= 6) times. Similarly, each digit occurs six times.

So, the total sum of the digits in the unit's place in all these numbers is $(2 + 3 + 4 + 5) \times 3! = 84$.

Similarly, sum of digits is 84 in ten's, hundred's and thousand's places.

Hence, the sum of all the numbers is $84\,(10^0 + 10^1 + 10^2 + 10^3) = 93324$.

Example 5.27 **How many 4-letter words, with or without meaning, can be formed out of the letters in the word 'LOGARITHMS', if repetition of letters is not allowed?**

Sol. There are 10 letters in the word 'LOGARITHMS'. So, the number of 4-letter words is equal to number of arrangements of 10 letters, taken 4 at a time, i.e., ${}^{10}P_4 = 5040$.

Example 5.28 **Eleven animals of a circus have to be placed in eleven cages (one in each cage). If 4 of the cages are too small for 6 of the animals, then find the number of the ways of caging all the animals.**

Sol. Let the 6 animals be placed in 7 of larger cages. This can be done in ${}^7P_6$ ways. In each of these ways, one larger cage is left vacant. The remaining five animals can be placed in the remaining five cages in 5! ways. Hence, by the fundamental theorem, the required number of ways is ${}^7P_6 \times 5! = 604800$.

Example 5.29 **If $A = \{x \mid x$ is a prime number and $x < 30\}$, find the number of different rational numbers whose numerator and denominator belong to A.**

Sol. Here, $A = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$. A rational number is made by taking any two numbers in any order. Therefore, the required number of rational numbers is ${}^{10}P_2 + 1$ (including 1).

Example 5.30 **How many different numbers of 4 digits can be formed from the digits 0, 1, 2, ..., 9 if repetition is**
(i) allowed,
(ii) not allowed.

Sol. (i) Repetition is allowed.
First place is filled by any number from 1 to 9 as 0 cannot occur at first place and each of the remaining 3 places can be filled by any one of the digits from 0, 1, ..., 9, i.e., in 10 different ways.

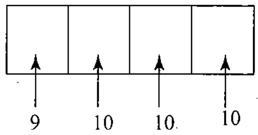


Fig. 5.8

Hence, the total number of 4-digit numbers that can be formed is $9 \times 10 \times 10 \times 10 = 9 \times 10^4$.

(ii) Repetition is not allowed.

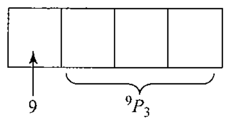


Fig. 5.9

The total number is $9 \times 9 \times 8 \times 7 = 4536$. Hence, the total number of 4-digit numbers that can be formed is $9 \times {}^9P_3$.

Example 5.31 **How many six-digit odd numbers, greater than 6,00,000, can be formed from the digits 5, 6, 7, 8, 9 and 0 if**
(i) repetition of digits is allowed,
(ii) repetition of digits is not allowed.

Sol. We have 6 digits, viz., 5, 6, 7, 8, 9 and 0 and we have to form numbers (integers) greater than 6,00,000, which are odd.

So the first place (lakh's position) should be ≥ 6 and the last position (i.e. unit) must be odd, i.e., 5, 7 or 9.

(i) When repetitions are allowed.

First place can be filled by 6, 7, 8 or 9 in 4 ways, last place can be filled by 5, 7 or 9 in 3 ways and each of the remaining 4 places (i.e. 2nd, 3rd, 4th, 5th) can be filled by any of the 6 digits in 6 ways. Hence, the total number will be $4 \times 6 \times 6 \times 6 \times 6 \times 3 = 15552$.

Note: *The above discussion can also be shown as follows:*

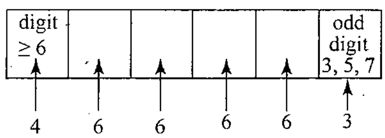


Fig. 5.10

The total number of numbers that can be formed is $4 \times 6 \times 6 \times 6 \times 6 \times 3 = 15552$.

(ii) When repetitions are not allowed.

Since we have restrictions on first and last places and no digit can be repeated, we have the following system: 1st place having numbers ≥ 6 and last place having odd numbers.

Digits in first place	Digits in last place
6	5, 7, 9
7	5, 9
8	5, 7, 9
9	5, 7
Total number of ways of filling last place	10 ways

Thus, the first and the last place can be filled in 10 ways, and the remaining four places can be filled by the remaining 4 digits in 4! = 24 ways. Hence, the total number of numbers that can be formed is $10 \times 24 = 240$.

Example 5.32 **A shelf contains 20 books of which 4 are single volume and the other form sets of 8, 5 and 3 volumes, respectively. Find the number of ways in which the books may be arranged on the shelf so that**

(i) volumes of each set will not be separated,

(ii) volumes of each set remain in their due order.

Sol. (i) Considering each set as single unit, permutations of 7 units is 7!.

Permutations of books of the set of 8 volumes among themselves is 8!.

Respective permutations of books of the set of 5 volumes is 5! and that of books of 3 volumes is 3!.

By the product rule, total number of permutations is

7! 8! 5! 3!.

(ii) Since the books in a set of books containing any number of volumes can be arranged in due order in 2 ways, the total number of permutations is $7! \times 2 \times 2 \times 2 = 8 \times 7! = 8$.

Number of Permutations of *n* Things Taken All Together When the Things Are Not All Different

To find the number of permutations of things taken all at a time when p of them are similar and are of one type, q of them are similar and are of second type, r of them are similar and are of third type and rest are all different.

Proof: Total number of things is n. p things are identical and are of one type, q things are identical and are of second type, r things are identical and are of third type, and rest are all different.

Let the required number of permutations be x.

Since p different things can be arranged among themselves in $p!$ ways, therefore, if we replace p identical things by p different things, which are also different from the rest of things, the number of permutations will become $x \times p!$.

Again, if we replace q identical things by q different things, the number of permutations will become $(x \times p! \times q!)$.

Again, if we replace r identical things by r different things, which are different from the rest, the number of permutations will become $(x\, p!\, q!)\, r!$.

Now, all the n things are different and therefore, number of permutations should be $n!$. Thus,

$$x\, p!\, q!\, r! = n!$$

$$\therefore x = \frac{n!}{p!q!r!}$$

Example 5.33 **How many words can be formed with the letters of the word 'MATHEMATICS' by rearranging them.**

Sol. Since there are 2 M's, 2 A's and 2 T's, the required number of ways is 11!/(2!2!2!).

Example 5.34 **Find the total number of nine-digit numbers that can be formed using the digits 2, 2, 3, 3, 5, 5, 8, 8, 8 so that the odd digit occupy the even places.**

Sol. Odd digits 3, 3, 5, 5 occupy four even places in 4!/(2!2!) = 6 ways. Rest five digits 2, 2, 8, 8, 8 occupy rest five places in 5!/(2!3!) = 10 ways. Hence, total number of ways is $6 \times 10 = 60$.

Example 5.35 **Find the number of permutation of all the letters of the word "MATHEMATICS" which starts with consonants only.**

Sol. (M M), (A A), (T T), H, E, I, C, S

Words starting with M or A or T are $\frac{10!}{2!2!}$

Words starting with H, E, I, C, S are $\frac{10!}{2!2!2!}$

Hence number of words are

$$3\frac{10!}{2!2!}+5\frac{10!}{2!2!2!}=\frac{10!}{2!2!}\left(3+\frac{5}{2}\right)=\frac{11!}{8}$$

Example 5.36 **There are six periods in each working day of a school. Find the number of ways in which 5 subjects can be arranged if each subject is allotted at least one period and no period remains vacant.**

Sol. Let the five subjects are a, b, c, d, e.

Since number of subjects are less than the number of periods, any one of the five subjects will occur twice.

If subject 'a' occur twice (a, a, b, c, d, e), then six subjects can be arranged in $\frac{6!}{2!}$ ways.

Similar number of ways when subject b, c, d and e occur twice.

Hence total number of ways are $5\times\frac{6!}{2!} = 1800$

Number of Permutations of *n* Different Things Taken *r* at a Time When Each Thing Can Be Repeated Any Number of Times

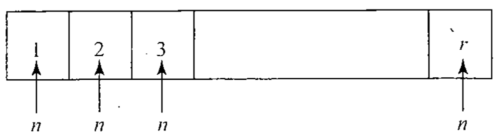


Fig. 5.11

By multiplication rule of fundamental principle of counting, number of ways in which first, second, third, ..., r^{th} places can together be filled up is $n \times n \times n \times \cdots r$ times $= n^r$.

Example 5.37 **How many 4-digit numbers can be formed by using the digits 1, 2, 3, 4, 5, 6, 7 if at least one digit is repeated.**

Sol. The numbers that can be formed when repetition of digits is allowed are 7^4.

The numbers that can be formed when all the digits are distinct or when repetition is not allowed are ${}^7P_4$.

Therefore, the numbers that can be formed when at least one digit is repeated are $7^4 - {}^7P_4$.

Example 5.38 **Find the total number of permutations of *n* different things taken not more than *r* at a time, when each thing may be repeated any number of times.**

Sol. Here, we have to arrange p things out of n, $1 \le p \le r$, and repetition is allowed. When $p = 1$, the number of permutations is n. When $p = 2$, the number of permutations is $n \times n = n^2$.

(Since repetition is allowed, first thing can be taken in n ways and the second thing can also be taken in n ways.)

When $p = 3$, the number of permutations is $n \times n \times n = n^3$. When $p = r$, the number of permutations is $n \times n \times n \cdots r$ times $= n^r$.

Hence, total number of permutations is

$$n + n^2 + n^3 + \cdots + n^r = \frac{n(n^r - 1)}{(n-1)} \quad \text{[sum of G.P.]}$$

Permutations Under Restrictions

When Particular Objects Are Never Together (Gap Method)

Example 5.39 **Number of ways in which 5 girls and 5 boys can be arranged in a row if no two boys are together.**

Sol. In the question, there is no condition for arranging the girls. Now, 5 girls can be arranged in 5! ways.

$$\times G \times G \times G \times G \times G \times$$

When girls are arranged, six gaps are generated as shown above with '×'.

Now, boys must occupy the places with '×' marked so that no two boys are together.

Therefore, five boys can be arranged in these six gaps in ${}^6P_5$ ways.

Hence, total number of arrangement is $5! \times {}^6P_5$.

Example 5.40 **Number of ways in which 5 girls and 5 boys can be arranged a in row if boys and girls are alternate.**

Sol. First five girls can be arranged in 5! ways, i.e.,

$$\times G \times G \times G \times G \times G$$

or, $\quad G \times G \times G \times G \times G \times$

Now, if girls and boys are alternate, then boys can occupy places with '×' mark in the diagram.

Hence, total number of arrangements is $5! \times 5! + 5! \times 5! = 2 \times 5! \times 5!$

When Particular Objects Are Always Together

Example 5.41 **If the best and the worst papers never appear together, find in how many ways six examination papers can be arranged.**

Sol. If the best and worst papers appear always together, the number of ways is $5! \times 2$. Therefore, required number of ways is as follows.

Total number of ways without any restrictions − number of ways when best and worst paper are together $= 6! - 5! \times 2 = 480$.

Example 5.42 **Find the number of arrangements of the letters of the word 'SALOON', if the two O's do not come together.**

Sol. The total number of arrangements is $6!/2! = 360$. The number of ways in which O's come together is $5! = 120$. Hence, the required number of ways is $360 - 120 = 240$.

Example 5.43 **Find the number of seven letter words that can be formed by using the letters of the word SUCCESS so that the two C are together but no two S are together.**

Sol. Considering CC as single object, U, CC, E can be arranged in 3! ways

$$\times \text{U} \times \text{CC} \times \text{E} \times$$

Now the three S are to be placed in the four available places (×)

Hence required no. of ways $= 3! \,.\, {}^4C_3 = 24$.

Example 5.44 **There are six teachers. Out of them two are primary teachers, two are middle teachers and two secondary teaches. They are to stand in a row, so as the primary teaches, middle teaches and secondary teaches are always in a set. Find the number of ways in which they can do so.**

Sol. There are 2 primary teachers. They can stand in a row in $2! = 2$ ways

There are 2 middle teachers. They can stand in a row in $2! = 2$ ways.

There are 2 secondary teachers. They can stand in a row in $2! = 2$ ways.

These three sets can be arranged In themselves in $= 3! = 6$ ways

Hence the required number of ways $= 2 \times 2 \times 2 \times 6 = 48$

Example 5.45 **There are 2 identical white balls, 3 identical red balls and 4 green balls of different shades. Find the number of ways in which they can be arranged in a row so**

that at least one ball is separated from the balls of the same colour.

Sol. Total number of arrangements without any restrictions = $\frac{9!}{2!\ 3!}$

Now number of ways when balls of the same color are together $= 3!\ 4!$

Now required number of ways

= Total number of arrangements

– number of ways when balls of the same colour are together

$= \frac{9!}{2!\ 3!} - 3!4! = 6\ (7! - 4!)$

Example 5.46 **Find the number of ways in which 6 boys and 6 girls can be seated in a row so that**

(i) all the girls sit together and all the boys sit together,

(ii) all the girls are never together.

Sol. (i)

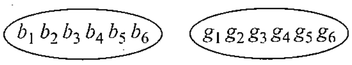


Fig. 5.12

Considering boys and girls as two units, the number of permutations is $2! \times 6! \times 6! = 2 \times (6!)^2$.

(ii) The total arrangements where all girls are not together is as follows: Total arrangement without any restrictions – arrangement when all girls are together = $(12)! - 7!\ 6!$.

Example 5.47 **The numbers ways in which the letters of the word 'ARRANGE' be arranged so that**

(i) the two R's are never together,

(ii) the two A's are together but not two R's,

(iii) neither two A's nor two R's are together.

Sol. The letters of word ARRANGE can be rewritten as

A R N G E

A R

So we have 2 A's and 2 R's, i.e., total 7 letters.

(i) Total number of words is $1/x + 1/y = 1/n$.

The number of words in which 2 R's are together [consider (R R) as one unit] is 6!/2!. e.g.,

(R R), A, A, N, G, E

Note that permutations of R R give nothing extra. Therefore, number of words in which the two R's are never together is

$$\frac{7!}{2!2!} - \frac{6!}{2!} = 900$$

(ii) The number of words in which both as are together is 6!/2! = 360, e.g.,

(A A), R, R, N, G, E

The number of words in which both A's and both R's are together is 5! = 120, e.g.,

(A A), (R R), N, G, E

Therefore, the number of words in which both A's are together but the two R's are not together is 360 – 120 = 240.

(iii) There are in all 900 words in each of which the two R's are never together. Consider any such word. Either the two A's are together or the two A's are not together. But the number of all such arrangements in which the two A's are together is 240. Hence, the number of all such arrangements in which the two A's are not together is 900 – 240 = 660.

Concept Application Exercise 5.3

1. Prove that ${}^nP_r - 5\,{}^{n-1}P_r + r\,{}^{n-1}P_{r2-1}$.
2. If ${}^nP_5 = 20\,{}^nP_3$, find the value of n.
3. **a.** If ${}^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$, find r.
   b. If ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800:1$, find r.
4. How many numbers can be formed from the digits 1, 2, 3, 4 when repetition is not allowed?
5. Find the 3-digit odd numbers that can be formed by using the digits 1, 2, 3, 4, 5, 6 when the repetition is allowed.
6. If the 11 letters $A, B, \ldots, K$ denote an arbitrary permutation of the integers (1, 2, …, 11), then $(A-1)(B-2)(C-3) \cdots (K-11)$ will be
   a. necessarily zero **b.** always odd
   c. always even **d.** none of these
7. In how many ways can 5 boys and 3 girls sit in a row so that no two girls are together?
8. In how many ways can 5 girls and 5 boys be arranged in row if all boys are together.
9. Find the number of words that can be made out of the letters of the word 'MOBILE' when consonants always occupy odd places.
10. Find the number of positive integers, which can be formed by using any number of digits from 0, 1, 2, 3, 4, 5 but using each digit not more than once in each number. How many of these integers are greater than 3000? What happened when repetition is allowed?

COMBINATION

Each of the different groups or selections, which can be made by taking some or all of a number of given things or objects at a time, is called a combination. In combination, order of appearance of things is not taken into account.

Example 1:

Three groups can be made with three different objects a, b, c taking two at a time, i.e., ab, bc, ac.

Here, ab and ba are the same group. It is also clear that for each combination (selection or group) of two things, number of permutations (arrangements) is 2!. For example, for combination ab, there are two permutations, i.e., ab and ba.

Example 2:

Four groups can be made with 4 different things a, b, c, d taking three at a time, i.e., abc, abd, acd, and bcd. Now, for each combination (group) of three things, number of permutations is 3!, i.e., 6. For example, for the group *abc*, there are 6 permutations (arrangements): *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*.

Number of Combinations of *n* Different Things Taking *r* at a Time ($r < n$)

To establish the formula

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

Proof: Let the number of combinations of n different things taken r at a time be ${}^nC_r$.

Now, each combination consists of r different things and these r things can be arranged among themselves in $r!$ ways.

Thus, for one combination of r different things, the number of arrangements is $r!$.

Hence, for ${}^nC_r$ combinations, number of arrangements is

$$r!\,{}^nC_r. \quad (1)$$

But number of permutations of n different things taken r at a time is

$$^nP_r. \quad (2)$$

From Eqs. (1) and (2), we get

$$r!\,{}^nC_r = {}^nP_r = \frac{n!}{(n-r)!}$$

$$\therefore \quad {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

Properties of ${}^nC_r$

1. ${}^nC_r = {}^nC_{n-r}$

Proof:

$${}^nC_r = \frac{n!}{r!(n-r)!} \quad (1)$$

$${}^nC_{n-r} = \frac{n!}{(n-r)!\,(n-(n-r))!} = \frac{n!}{r!(n-r)!} \quad (2)$$

From Eqs. (1) and (2), it follows that ${}^nC_r = {}^nC_{n-r}$.

2. If ${}^nC_x = {}^nC_y$, then either $x = y$ or $x + y = n$.

Proof:

$${}^nC_x = {}^nC_y = {}^nC_{n-y} \qquad [\because {}^nC_r = {}^nC_{n-r}]$$

From ${}^nC_x$ (i) and (ii),

$x = y$

From ${}^nC_x$ (i) and (iii),

$x = n - y$ or $x + y = n$

3. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Proof:

L.H.S. $= {}^nC_r + {}^nC_{r-1}$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!\,(n-(r-1))!}$$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{n+1}{r(n-r+1)}\right]$$

$$= \frac{(n+1)n!}{r(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{(n+1)!}{r!(n-r+1)!} = {}^{n+1}C_r = \text{R.H.S.}$$

4. $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$

Proof:

L.H.S. $= r\,{}^nC_r$

$$= r\frac{n!}{r!(n-r)!}$$

$$= r\frac{n!}{r(r-1)!(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}$$

$$= n\frac{(n-1)!}{(r-1)!(n-r)!}$$

$$= n\,{}^{n-1}C_{r-1} = \text{R.H.S.}$$

5. $$\frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}$$

Proof:

L.H.S. $$= \frac{{}^nC_r}{r+1}$$

$$= \frac{\frac{n!}{r!(n-r)!}}{r+1}$$

$$= \frac{n!}{(r+1)r!(n-r)!}$$

$$= \frac{n!}{(r+1)!(n-r)!}$$

$$= \frac{(n+1)n!}{(n+1)(r+1)!(n-r)!}$$

$$= \frac{1}{(n+1)}\frac{(n+1)!}{(r+1)!(n-r)!}$$

$$= \frac{{}^{n+1}C_{r+1}}{n+1} = \text{R.H.S.}$$

6. $$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$$

7. Maximum value of ^nC_r

We can observe that in the list of $^6C_0$, $^6C_1$, $^6C_2$, $^6C_3$, $^6C_4$, $^6C_5$, and $^6C_6$, the maximum value is $^6C_3$.

Also, in the list of $^7C_0$, $^7C_1$, $^7C_2$, $^7C_3$, $^7C_4$, $^7C_5$, $^7C_6$, and $^7C_7$, the maximum value is $^7C_3$ or $^7C_4$.

In general, when n is even, maximum value of ^nC_r is $^nC_{n/2}$ and when n is odd, maximum value of ^nC_r is $^nC_{(n-1)/2}$ or $^nC_{(n+1)/2}$.

8. The product of k consecutive positive integers is divisible by $k!$

Let the k consecutive integers be $m, m+1, m+2, \ldots, m+k-1$. Then,

$$m(m+1)(m+2)\cdots(m+k-1)$$
$$= \frac{(m-1)!m(m+1)\cdots(m+k-1)}{(m-1)!}$$
$$= \frac{(m+k-1)!}{(m-1)!}$$
$$= k!\frac{(m+k-1)!}{(m-1)!k!}$$
$$= (k!)\,^{(m+k-1)}C_k$$

Since $^{(m+k-1)}C_k$ is an integer, it follows that $k!$ divides $m(m+1)\cdots(m+k-1)$.

9. ^nC_r is divisible by n only if n is a prime number ($1 \le r \le n-1$)

For example, $^6C_2$ is not divisible by 6, but $^7C_4$ is divisible by 7.

Restricted Combinations

Number of Combinations of n Different Things Taken r at a Time when p Particular Things Are Always Included

Already p things are selected. The remaining $r-p$ things from the remaining $n-p$ things can be selected in $^{n-p}C_{r-p}$ ways.

Number of Combinations of n Different Things Taken r at a Time when p Particular Things Are Always to Be Excluded

Since p particular things are always to be excluded, therefore, we have to select r things out of remaining $n-p$ different things. This can be done in $^{n-p}C_r$ ways.

Example 5.48 **Prove that $^nC_r + 2\,^nC_{r-1} + {^nC_{r-2}} = {^{n+2}C_r}$.**

Sol. $^nC_r + 2\,^nC_{r-1} + {^nC_{r-2}}$
$$= {^nC_r} + {^nC_{r-1}} + {^nC_{r-1}} + {^nC_{r-2}}$$
$$= {^{n+1}C_r} + {^{n+1}C_{r-1}} = {^{n+2}C_r}$$

Example 5.49 **Prove that $^nC_r + {^{n-1}C_r} + \cdots + {^rC_r} = {^{n+1}C_{r+1}}$.**

Sol. $^rC_r + {^{r+1}C_r} + {^{r+2}C_r} + \cdots + {^{n-1}C_r} + {^nC_r}$
$$= {^{r+1}C_{r+1}} + {^{r+1}C_r} + {^{r+2}C_r} + \cdots + {^{n-1}C_r} + {^nC_r}$$
$$= {^{r+2}C_{r+1}} + {^{r+2}C_r} + \cdots + {^{n+1}C_r} + {^nC_r}$$
$$= {^{r+3}C_{r+1}} + \cdots + {^{n-1}C_r} + {^nC_r}$$

On adding similar way, we get

L.H.S. $= {^{n-1}C_{r+1}} + {^{n-1}C_r} + {^nC_r}$
$$= {^nC_{r+1}} + {^nC_r}$$
$$= {^{n+1}C_{r+1}} = \text{R.H.S.}$$

Example 5.50 **If $^{15}C_{3r} = {^{15}C_{r+3}}$, then find r.**

Sol. $^{15}C_{3r} = {^{15}C_{r+3}}$

$\Rightarrow$ $3r = r + 3$ or $3r + r + 3 = 15$

$\Rightarrow$ $r = 3/2$ or $r = 3$

$\Rightarrow$ $r = 3$ (as r is positive integer)

Example 5.51 **If $^nC_r = 84$, $^nC_{r-1} = 36$ and $^nC_{r+1} = 126$, then find the value of n.**

Sol. $\frac{n-r+1}{r} = \frac{84}{36} = \frac{7}{3}$ and $\frac{n-r}{r+1} = \frac{126}{84} = \frac{3}{2}$

$\therefore$ $\frac{7}{3}r - 1 = n - r = \frac{3}{2}(r+1)$

or

$14r - 6 = 9r + 9$ or $r = 3$

$\therefore$ $n = 9$

Example 5.52 **If $^nC_8 = {^nC_6}$, then find $^nC_2$.**

Sol. If $^nC_x = {^nC_y}$ and $x \ne y$, then $x + y = n$. Hence,

$^nC_8 = {^nC_6}$

$\Rightarrow$ $n = (8 + 6) = 14$

Now,
$$^nC_2 = {^{14}C_2} = \frac{14\times 13}{2} = 91$$

Example 5.53 **If the ratio $^{2n}C_3 : {^nC_3}$ is equal to 11:1, find n.**

Sol. We have,

$$^{2n}C_3 : {^nC_3} = 11:1$$
$$\Rightarrow \frac{^{2n}C_3}{^nC_3} = \frac{11}{1}$$
$$\Rightarrow \frac{\frac{(2n)!}{(2n-3)!(3!)}}{\frac{n!}{(n-3)!(3!)}} = \frac{11}{1}$$
$$\Rightarrow \frac{(2n)!}{(2n-3)!} \times \frac{(n-3)!}{n!} = \frac{11}{1}$$
$$\Rightarrow \frac{(2n)(2n-1)(2n-2)}{n(n-1)(n-2)} = \frac{11}{1}$$
$$\Rightarrow \frac{4(2n-1)}{n-2} = \frac{11}{1}$$
$$\Rightarrow 8n - 4 = 11n - 22$$
$$\Rightarrow 3n = 18 \Rightarrow n = 6$$

Example 5.54 **If ${}^{15}C_{3r}:{}^{15}C_{r+1} = 11:3$, find the value of '$r$'.**

Sol. ${}^{15}C_{3r}:{}^{15}C_{r+1} = 11:3$

Clearly, 'r' can be 0, 1, 2, 3, 4, 5 but possibilities of $r = 0$ or 5 are clearly ruled out (as ${}^{15}C_0 = {}^{15}C_{15} = 1$).

For $r = 1$,

$$^{15}C_{3r} = {}^{15}C_3 = \frac{15\times14\times13}{6} \text{ and } {}^{15}C_{r+1} = {}^{15}C_2 = \frac{15\times14}{2}$$

$\therefore\ {}^{15}C_{3r}:{}^{15}C_{r+1} = 13:3$

For $r = 2$,

$$^{15}C_{3r} = {}^{15}C_6 = \frac{15\times14\times13\times12\times11\times10}{6\times5\times4\times3\times2\times1}$$

$$^{15}C_{r+1} = {}^{15}C_3 = \frac{15\times14\times13}{6}$$

$\therefore\ {}^{15}C_{3r}:{}^{15}C_{r+1} \neq 11:3$

For $r = 3$,

$$^{15}C_{3r} = {}^{15}C_9 = \frac{15\times14\times13\times12\times11\times10}{6\times5\times4\times3\times2\times1}$$

$$^{15}C_{r+1} = {}^{15}C_4 = \frac{15\times14\times13\times12}{4\times3\times2\times1}$$

$\therefore\ {}^{15}C_{3r}:{}^{15}C_{r+1} = 11:3$

For $r = 4$,

$$^{15}C_{3r} = {}^{15}C_{12} = {}^{15}C_3 = \frac{15\times14\times13}{3\times2\times1}$$

$$^{15}C_{r+1} = {}^{15}C_5 = \frac{15\times14\times13\times12\times11}{5\times4\times3\times2\times1}$$

$\therefore\ {}^{15}C_{3r}:{}^{15}C_{r+1} = 5:33$

Thus, $r = 3$.

Example 5.55 **Twenty-eight games were played in a football tournament with each team playing once against each other. How many teams were there?**

Sol. Let the number of teams be n. Then number of matches to be played is ${}^nC_2 = 28$.

$\therefore\ \frac{n(n-1)}{2} = 28$

$\Rightarrow\ n^2 - n - 56 = 0$

$\Rightarrow\ (n-8)(n+7) = 0$

$\Rightarrow\ n = 8$ as $n \neq -7$

Example 5.56 **In a network of railways, a small island has 15 stations. Find the number of different types of tickets to be printed for each class, if every stations must have tickets for other stations.**

Sol. For each pair of stations, two different types of tickets are required.

Now, the number of selections of 2 stations from 15 stations $= {}^{15}C_2$.

$\therefore$ Required number of types of tickets $= 2\ {}^{15}C_2 = 2\ \frac{15!}{2!13!}$ $= 15\times14 = 210$.

Example 5.57 **In a certain algebraical exercise book there are 4 examples on arithmetical progressions, 5 examples on permutation and combination and 6 examples on binomial theorem. Find the number of ways a teacher can select for his pupils at least one but not more than 2 examples from each of these sets.**

Sol. Number of ways teacher can select examples from arithmetic progression $= ({}^4C_1 + {}^4C_2)$

Number of ways teacher can select examples from permutation and combinations $= ({}^5C_1 + {}^5C_2)$

Number of ways teacher can select examples from binomial theorem $= ({}^6C_1 + {}^6C_2)$

Hence total number of ways $= ({}^4C_1 + {}^4C_2)({}^5C_1 + {}^5C_2)({}^6C_1 + {}^6C_2)$

Example 5.58 **A person tries to form as many different parties as he can, out of his 20 friends. Each party should consist of the same number. How many friends should be invited at a time? In how many of these parties would the same friends be found?**

Sol. Let the person invite r number of friends at a time. Then, the number of parties is ${}^{20}C_r$, which is maximum when $r = 10$.

If a particular friend will be found in x parties, then x is the number of combinations out of 20 in which this particular friend must be included. Therefore, we have to select 9 more from 19 remaining friends. Hence, $x = {}^{19}C_9$.

Example 5.59 **In how many of the permutations of n things taken r at a time will three given things occur?**

Sol. According to the condition of the problem, we have to select $r - 3$ things from remaining $n - 3$ things and permute these r things. So the number of permutations is

$$^{(n-3)}C_{(r-3)}\ r! = \frac{(n-3)!\ r!}{(r-3)!\ (n-r)!}$$

Example 5.60 **Out of 10 consonants and 4 vowels, how many words can be formed each containing 3 consonants and 2 vowels?**

Sol. The number of ways of selection of three consonants from 10 is ${}^{10}C_3$. The number of ways of selection of two vowels from 4 is ${}^4C_2$. Permutation of these 5 letters (all distinct) is 5! Therefore, number of words that can be formed is ${}^{10}C_3 \times {}^4C_2 \times 5! = 86400$.

Example 5.61 **Find the maximum number of points of intersection of 6 circles.**

Sol. Two circles intersect maximum at two distinct points. Now, two circles can be selected in ${}^6C_2$ ways. Again, each selection of two circles gives two points of intersection. Therefore, the total number of points of intersection is ${}^6C_2 \times 2 = 30$.

Example 5.62 **There are 10 points on a plane of which no three points are collinear. If lines are formed joining**

these points, find the maximum points of intersection of these lines.

Sol. Two points are required to form a line. Then, the number of lines is equal to the number of ways two points are selected, i.e., ${}^{10}C_2 = 45$.

Now, two lines intersect at one point. Hence, the number of points of intersection of lines is ${}^{45}C_2$.

Example 5.63 **There are 10 points on a plane of which 5 points are collinear. Also, no three of the remaining 5 points are collinear. Then find (i) the number of straight lines joining these points; (ii) the number of triangles formed joining these points.**

Sol. (i) Line is formed joining two points. Hence, number of lines is ${}^{10}C_2$. But joining any points from 5 collinear points gives the same line. Again, 2 points are selected from 5 in ${}^5C_2$ ways or lines joining collinear points is taken ${}^5C_2$ (= 10) times. Then the number of straight lines = ${}^{10}C_2 - 10 + 1 = 36$.

(ii) For a triangle, three non-collinear points are required. Three points can be selected in ${}^{10}C_3$ ways. Now, the selection of three points from 5 collinear points does not form triangle. Hence, number of triangles is ${}^{10}C_3 - {}^5C_3$.

Example 5.64 **Find the total number of rectangles on the normal chessboard.**

Sol. To form a rectangle on a chessboard two vertical lines and two horizontal lines should be selected. There are 9 vertical lines and 9 horizontal lines found on the chessboard. Selection of 2 vertical and 2 horizontal lines can be done in ${}^9C_2 \times {}^9C_2$ ways, which is equivalent to the number of rectangles.

Example 5.65 **A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour?**

Sol. The selection of 6 balls, consisting of at least two balls of each colour from 5 red and 6 white balls, can be made in the following ways:

Red balls (5)	White balls (6)	Number of ways
2	4	${}^5C_2 \times {}^6C_4 = 150$
3	3	${}^5C_3 \times {}^6C_3 = 200$
4	2	${}^5C_4 \times {}^6C_2 = 75$
	Total	425

Example 5.66 **In a conference 10 speakers are present. If S_1 wants to speak before S_2 and S_2 wants to speak after S_3, then find number of ways all the 10 speakers can give their speeches with the above restriction if the remaining seven speakers have no objection to speak at any number.**

Sol. According to question the order of speakers S_1, S_2 and S_3 can be (not necessarily consecutive)

$S_1 S_2 S_3$ or $S_3 S_1 S_2$

For each order we can select three slots out of ten in ${}^{10}C_3$ ways.

After selecting these three slots in which speakers S_1, S_2, S_3 have only one way of arrangement as said, the remaining seven speakers can be arranged in seven slots in 7! ways.

Hence total number of arrangements = $2\ {}^{10}C_3 7!$

Example 5.67 **A delegation of four students is to be selected from a total of 12 students. In how many ways can the delegation be selected**

(i) if all the students are equally willing?
(ii) if two particular students have to be included in the delegation?
(iii) if two particular students do not wish to be together in the delegation?
(iv) if two particular students wish to be included together only in the delegation?
(v) if two particular students refuse to be together and two other particular students wish to be together only in the delegation?

Sol. (i) Formation of delegation means selection of 4 out of 12. Hence, the number of ways is ${}^{12}C_4 = 495$.

(ii) If two particular students are already selected, we need to select only 2 out of the remaining 10. Hence, the number of ways is ${}^{10}C_2 = 45$.

(iii) The number of ways in which both are selected is 45. Hence, the number of ways in which the two are not included together is $495 - 45 = 450$.

(iv) There are two possible cases:

(a) Both are selected. In this case, the number of ways in which the selection can be made is 45.

(b) Both are not selected. In this case, all the four students are selected from the remaining ten students. This can be done in ${}^{10}C_4 = 210$ ways.

Hence, the total number of ways of selection is $45 + 210 = 255$.

(v) We assume that students A and B wish to be selected together and students C and D do not wish to be together.

Cases	Number of selection
A, B always selected	${}^{10}C_2 = 45 = s_1$
A, B always excluded	${}^{10}C_4 = 210 = s_2$
A, B, C, D always selected	$1 = s_3$
A, B excluded and C, D included	${}^8C_2 = 28 = s_4$

The total number of ways is

$$s_1 - s_3 + s_2 - s_4$$
$$= 45 - 1 + 210 - 28$$
$$= 216$$

Example 5.68 **Find the total number of ways of selecting five letters from the word 'INDEPENDENT'.**

Sol. Given letters are I, (N, N, N), (D, D), (E, E, E), P, T
The choices are as follows:

Choice	Ways
All the letters are distinct [different letters are I, N, D E, P, T]	${}^6C_5 = 6$
3 distinct, 2 alike	${}^3C_1 \times {}^5C_3 = 30$
2 distinct, 3 alike	${}^2C_1 \times {}^5C_2 = 20$
2 alike, 2 alike, 1 distinct	${}^3C_2 \times {}^4C_1 = 12$
3 alike, 2 alike	${}^2C_1 \times {}^2C_1 = 4$
	Total = 72

Example 5.69 **In a plane, there are 5 straight lines which will pass through a given point, 6 others which all pass through another given point and 7 others which all pass through a third given point. Supposing no three lines intersect at any point and no two are parallel, find the number of triangles formed by the intersection of the straight line.**

Sol. Let 5 straight lines be passing through A, 6 passing through B and 7 passing through C. In all, there are 18 straight lines.
To find the number of triangles equivalent, we have to find the number of selection of 3 lines from these 18 lines, keeping in mind that selection of 3 lines from the lines passing through A, B or C will not give any triangle.

Hence, the required number of triangles is ${}^{18}C_3 - ({}^5C_3 + {}^6C_3 + {}^7C_3) = 751$.

Example 5.70 **A regular polygon of 10 sides is constructed. In how many ways can 3 vertices be selected so that no two vertices are consecutive?**

Sol. The required number of selections is given as
The number of selections without restriction
(the number of selections when 3 vertices are consecutive) – (the number of selections when 2 vertices are consecutive)

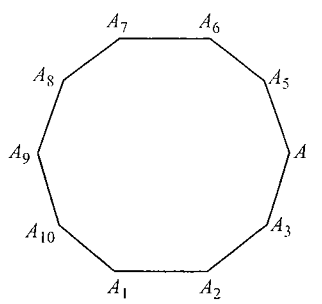


Fig. 5.13

Now, the number of selections of 3 vertices without restriction is ${}^{10}C_3$.

The number of selections of 3 consecutive vertices is 10 (by observation: $A_1A_2A_3, A_2A_3A_4, \ldots, A_{10}A_1A_2$).

The number of selections when two vertices are consecutive is $10 \times {}^6C_1$.

(After selecting two consecutive vertices in 10 ways, the third can be selected from 6 vertices.)

Therefore, the required number of selections is

$${}^{10}C_3 - 10 - 10 \times {}^6C_1 = \frac{10 \times 9 \times 8}{6} - 10 - 60 = 120 - 70 = 50$$

Concept Application Exercise 5.4

1. If ${}^{n+2}C_8 : {}^{n-2}P_4 = 57:16$, find n.
2. Find the ratio of ${}^{20}C_r$ and ${}^{25}C_r$ when each of them has the greatest possible value.
3. If there are 12 persons in a party, and if each two of them shake hands with each other, how many handshakes happen in the party?
4. On the occasion if Deepawali festival, each student in a class sends greeting cards to others. If there are 20 students in the class, find the total number of greeting cards exchanged by the students?
5. Out of 15 balls, of which some are white and the rest are black, how many should be white so that the number of ways in which the balls can be arranged in a row may be the greatest possible? It is assumed that the balls of same colour are alike.
6. A committee of 6 is chosen from 10 men and 7 women so as to contain at least 3 men and two women. In how many ways can this be done if two particular women refuse to serve on the same committee?
   a. 7850 **b.** 8700
   c. 7800 **d.** none of these
7. Find the number of ways in which the birthdays of six different persons will fall in exactly two calendar months.
8. A bag contains 50 tickets numbered 1, 2, 3, …, 50. Find the number of set of five tickets x_1, x_2, x_3, x_4, x_5 one has if $x_1 < x_2 < x_3 < x_4 < x_5$ and $x_3 = 30$.
9. Four visitors A, B, C, D arrived at a town that has 5 hotels. In how many ways, can they disperse themselves among 5 hotels.
10. In how many shortest ways can we reach from the point (0, 0, 0) to point (3, 7, 11) in space where the movement is possible only along the x-axis, y-axis and z-axis or parallel to them and change of axes is permitted only at integral points. (An integral point is one, which has its coordinate as integer.)
11. Out of 8 sailors on a boat, 3 can work only on one particular side and 2 only on the other side. Find the number of ways in which the sailors can be arranged on the boat.
12. For an examination, a candidate has to select 7 subjects from 3 different groups A, B, C, which contain 4, 5, 6 subjects, respectively. The number of different ways in which a candidate can make his selection if he has to select at least 2 subjects from each group is
    a. 2500 **b.** 2600
    c. 2700 **d.** 2800

CIRCULAR PERMUTATIONS

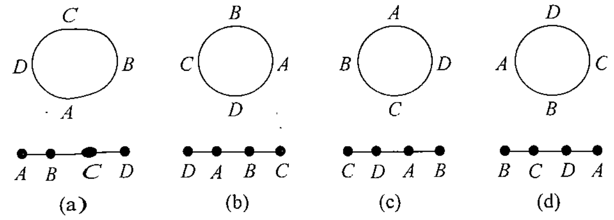


Fig. 5.14

Let us consider that persons A, B, C, D are sitting around a round table. If all of them (A, B, C, D) are shifted at one place in anticlockwise order, then we will get Fig. 5.13(b) from Fig. 5.14(a). Now, if we shift A, B, C, D in anticlockwise order, we will get Fig. 5.13(c). Again, if we shift them we will get Fig. 5.13(d); and in the next time, Fig. 5.13(a).

Thus, we see that if 4 persons are sitting at a round table, they can be shifted four times and the four different arrangements thus obtained will be the same, because anticlockwise order of A, B, C, D does not change.

But if A, B, C, D are sitting in a row and they are shifted in such an order that the last occupies the place of first, then the four arrangements will be different.

Thus, if there are 4 things, then for each circular arrangement number of linear arrangements is 4.

Similarly, if n different things are arranged along a circle, for each circular arrangement number of linear arrangements is n.

Therefore, the number of linear arrangements of n different things is $n \times$ (number of circular arrangements of n different things). Hence, the number of circular arrangements of n different things is

$(1/n) \times$ (number of linear arrangements of n different things)

$= n!/n = (n-1)!$

Clockwise and Anticlockwise Arrangements

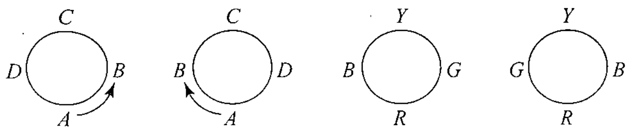


Fig. 5.15

Let the four persons A, B, C, D sit at a round table in anticlockwise as well as clockwise directions. These two arrangements are different. But if four flowers R (red), G (green), Y (yellow) and B (blue) be arranged to form a garland in anticlockwise and in clockwise order, then the two arrangements are same because if we see the garland from one side the four flowers R, G, Y, B will appear in anticlockwise direction and if seen from the other side the four flowers will appear in the clockwise direction. Here, the two arrangements will be considered as one arrangement because the order of flowers is not changing rather only the side of observation is changing. Here, two permutations will be counted as one.

Therefore, when clockwise and anticlockwise arrangements are not different, i.e., when observation can be made from both sides, the number of circular arrangements of n different things is $(n-1)!/2$.

Example 5.71 **Five boys and 5 girls sit alternately around a round table. In how many ways can this be done?**

Sol.

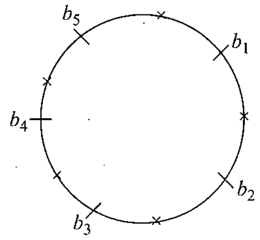


Fig. 5.16

Five boys can be arranged in a circle in 4! ways.

After that girls can be arranged in the five gaps shown as '×' in 5! ways. Hence, total number of ways is $4! \times 5! = 2880$.

Example 5.72 **A round-table conference is to be held among 20 delegates belonging from 20 different countries. In how many ways can they be seated if two particular delegates are (i) always to sit together; (ii) never to sit together.**

Sol. **(i)** Let the two particular delegates who wish to sit together be treated as one delegate. So we have 19 delegates who can be arranged on a round table in $(19-1)!$, i.e., 18! ways.

After this, the two particular delegates can be permuted between themselves in $2! = 2$ ways. Hence, by product rule, number of required arrangements is $2 \times (18)!$.

(ii) The total number of arrangements of 20 delegates on a round table is 19!.

Hence, the number of arrangements in which the two particular delegates never sit together is $19! - 2 \times 18! = 18!\,(19-2) = 17 \times 18!$.

Example 5.73 **A person invites a group of 10 friends at dinner and sits**

(i) 5 on a round table and 5 more on another round table,

(ii) 4 on one round table and 6 on the other round table.

Find the number of ways in each case in which he can arrange the guests.

Sol. **(i)** The number of ways of selection of 5 friends for first table is ${}^{10}C_5$. Remaining 5 friends are left for second table.

The total number of permutations of 5 guests on each table is 4!. Hence, the total number of arrangements is ${}^{10}C_5 \times 4! \times 4! = 10!/(5! \times 5!)4! \times 4! = 10!/25$.

(ii) The number of ways of selection of 6 guests is ${}^{10}C_6$. The number of ways of permutations of 6 guests on round table is 5!. The number of permutation of 4 guests on round table is 3!.

Therefore, total number of arrangements is

$$^{10}C_6 \times 5!\,3! = \frac{(10)!}{6! \times 4!}\, 5!\,3! = \frac{(10)!}{24}$$

Example 5.74 **Find the number of ways in which 10 different diamonds can be arranged to make a necklace.**

Sol. Since diamonds do not have natural order of left and right so clockwise and anticlockwise arrangements are taken as identical. Therefore, the number of arrangements of 10 different diamonds to make a necklace is $1/2 \times 9 = 181440$.

Example 5.75 **Six persons *A*, *B*, *C*, *D*, *E*, *F* are to be seated at a circular table. In how many ways can this be done if *A* should have either *B* or *C* on his right and *B* must always have either *C* or *D* on his right.**

Sol. Let the seat occupied by A be numbered as 1 and the remaining 5 seats be numbered as 2, 3, 4, 5, 6 in anticlockwise direction. There arise two cases:

Case I: B is on right of A, i.e., at number 2.
Then, seat number 3 can be occupied by C or D in $^2C_1$ ways and remaining 3 persons can have remaining 3 seats in 3! ways. Hence, the number of arrangements in this case is $2 \times 6 = 12$.

Case II: C is on the right of A, i.e., at number 2.
Then, B can occupy any seat from number 3 or 4 or 5. Then, D must be on the right of B, so we are left with two persons and 2 seats, which can be occupied in 2! ways. Hence, the number of arrangements in this case is $^3C_1 \times 2! = 6$. These cases are exclusive. So by sum rule total number of arrangements is $12 + 6 = 18$.

Example 5.76 **Find the number of ways in which six persons can be seated at a round table, so that all shall not have the same neighbours in any two arrangements.**

Sol. In this case, anticlockwise and clockwise arrangements are the same.

Hence, the number of ways of arrangements is $5!/2 = 60$.

Concept Application Exercise 5.5

1. In how many ways can 3 ladies and 3 gentlemen be seated around a round table so that any two and only two of the ladies sit together?
2. In how many ways can 15 members of a council sit along a circular table, when the secretary is to sit on one side of the chairman and the deputy secretary on the other side?
3. Find the number of ways in which 6 men and 5 women can dine at a round table if no two women are to sit together.
4. Find the number of ways that 8 beads of different colours be strung as a necklace.
5. Find the number of ways in which 8 different flowers can be strung to form a garland so that four particular flowers are never separated.

ALL POSSIBLE SELECTIONS

Total Number of Combinations of *n* Different Things Taken One or More at a Time

Method 1

For each thing there are two possibilities, whether it is selected or not selected.

Hence, the total number of ways is given by total possibilities of all the things which is equal to $2 \times 2 \times 2 \times \cdots n$ times $= 2^n$.

But this includes one case in which nothing is selected.

Hence, the total number of ways of selecting one or more of n different things is $2^n - 1$.

Method 2

Number of ways of selecting one, two, three, ..., n things from n different things is $^nC_1, {}^nC_2, {}^nC_3, \ldots, {}^nC_n$, respectively.

Hence, the total number of ways of selecting at least one thing is

$$^nC_1 + {}^nC_2 + \cdots + {}^nC_n = ({}^nC_0 + {}^nC_1 \cdots + {}^nC_n) - {}^nC_0 = 2^n - 1$$

Total Number of Selections of One or More Things from *p* Identical Things of One Type, *q* Identical Things of Another Type, *r* Identical Things of the Third Type and *n* Different Things

Since, the number of ways of selecting r things out of n identical things is 1 for all $r \le n$.

Hence, the number of ways of selecting zero or more things out of p identical things is

$$1 + 1 + 1 + \cdots (p + 1) \text{ times} = p + 1$$

Similarly, the number of ways of selecting zero or more things out of q and r identical things is $q + 1$ and $r + 1$, respectively.

Also the number of ways of selecting zero or more things out of n different things is $2 \times 2 \times 2 \times \cdots n$ times $= 2^n$.

Therefore, the number of ways of selecting zero or more things out of given things is $(p + 1)(q + 1)(r + 1)\, 2^n$.

But the number of ways of selecting zero thing out of given things is $1 \times 1 \times 1 \times 1^n = 1$.

Thus, the total number of ways of selecting one or more things out of given things is $(p + 1)(q + 1)(r + 1)\, 2^n - 1$.

Number of Divisors of *N*

- Every natural number N can always be put in the form $N = p_1^{\alpha_1}\; p_2^{\alpha_2} \;\ldots\; p_k^{\alpha_k}$ where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are non-negative integers.
- If $N = p_1^{\alpha_1}\; p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ then number of divisor of N is equivalent of number of ways of selecting zero or more objects from the groups of identical objects, $(p_1, p_1, \ldots, \alpha_1$ times), $(p_2, p_2, \ldots, \alpha_2$ times), $(p_k, p_k \ldots \alpha_k$ times) $= (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1)$ which includes 1 and N also.

Note:

All the divisors excluding 1 and N are called proper divisors.

- *Also number of divisors of N can been seen as number of different terms in the expansion of*

$$(p_1^0 + p_1^1 + p_1^2 \cdots + p_1^{\alpha_1}) \times (p_2^0 + p_2^1 + p_2^2 + \cdots + p_2^{\alpha_2}) \times \cdots \times (1 + p_k + p_k^2 + \cdots + p_k^{\alpha_k})$$

Hence, sum of the divisors of N is

$$(1 + p_1 + p_1^2 + \cdots + p_1^{\alpha_1})(1 + p_2 + p_2^2 + \cdots + p_2^{\alpha_2}) \cdots (1 + p_k + p_k^2 + \cdots + p_k^{\alpha_k})$$

$$= \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{\alpha_2+1} - 1}{p_2 - 1} \cdots \frac{p_k^{\alpha_k+1} - 1}{p_k - 1}$$

- *The number of ways of putting N as a product of two natural numbers is (1/2) $(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1)$ if N is not a perfect square.*

  If N is a perfect square, then this is (1/2) $[(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1) + 1]$.

Example 5.77 **There are p copies each of n different books. Find the number of ways in which a non-empty selection can be made from them.**

Sol. Number of selections of any number of copies of a book is $p + 1$ (because copies of the same book are identical things). Similar is the case for each book. Therefore, total number of selections is $(p + 1)^n$.

But this includes a selection, which is empty, i.e., zero copy of each book. Excluding this, the required number of non-empty selections is $(p + 1)^n - 1$.

Example 5.78 **A person is permitted to select at least one and at most n coins from a collection of $(2n + 1)$ distinct coins. If the total number of ways in which he can select coins is 255, find the value of n.**

Sol. We have,

$$^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n = 255 \quad (1)$$

Also the sum of binomial coefficients is

$$^{2n+1}C_0 + {}^{2n+1}C_1 + \cdots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + \cdots + {}^{2n+1}C_{2n+1} = (1 + 1)^{2n+1} = 2^{2n+1}$$

$$\Rightarrow \; ^{2n+1}C_0 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n) + {}^{2n+1}C_{2n+1} = 2^{2n+1}$$

$$\Rightarrow \; 1 + 2(255) + 1 = 2^{2n+1}$$

$$\Rightarrow \; 1 + 255 = 2^{2n}$$

$$\Rightarrow \; 2^{2n} = 2^8 \Rightarrow n = 4$$

Example 5.79 **Nishi has 5 coins each of the different denomination. Find the number different sums of money she can form.**

Sol. Number of different sums of money she can form is equal to number of ways she select one or more coins

$\therefore$ Required no. of ways $= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 2^5 - 1 = 31$.

Example 5.80 **Find the number of groups that can be made from 5 different green balls, 4 different blue balls and 3 different red balls, if at least 1 green and 1 blue ball is to be included.**

Sol. At least, one green ball can be selected out of 5 green balls in $2^5 - 1$, i.e., in 31 ways.

Similarly, at least one blue ball can be selected from 4 blue balls in $2^4 - 1 = 15$ ways. And at least one red or no red ball can be selected in $2^3 = 8$ ways.

Hence, the required number of ways is $31 \times 15 \times 8 = 3720$.

Example 5.81 **There are 3 books of mathematics, 4 of science, and 5 of literature. How many different collections can be made such that each collection consists of**

(i) one book of each subject,
(ii) at least one book of each subject,
(iii) at least one book of literature.

Sol. (i) $^3C_1 \times {}^4C_1 \times {}^5C_1 = 3 \times 4 \times 5 = 60$

(ii) $(2^3 - 1)(2^4 - 1)(2^5 - 1) = 7 \times 15 \times 31 = 3255$

(iii) $(2^5 - 1) \times 2^7 = 31 \times 128 = 3968$

Example 5.82 **Find the total number of proper factors of the number 35700. Also find**

(i) sum of all these factors,
(ii) sum of the odd proper divisors,
(iii) the number of proper divisors divisible by 10 and the sum of these divisors.

Sol. $35700 = 5^2 \times 2^2 \times 3^1 \times 7^1 \times 17^1$

The total number of factors is equal to the total number of selections from (5, 5), (2, 2), (3), (7) and (17), which is given by $3 \times 3 \times 2 \times 2 \times 2 = 72$.

These include 1 and 35700. Therefore, the number of proper divisors (excluding 1 and 35700) is $72 - 2 = 70$

Sum of all these factors (proper) is

$$(5^0 + 5^1 + 5^2)(2^0 + 2^1 + 2^2)(3^0 + 3^1)(7^0 + 7^1)(17^0 + 17^1) - 1 - 35700$$
$$= 31 \times 7 \times 4 \times 8 \times 18 - 1 - 35700 = 89291$$

Now, the sum of odd proper divisors is

$$(5^0 + 5^1 + 5^2)(3^0 + 3^1)(7^0 + 7^1)(17^0 + 17^1) - 1$$
$$= 31 \times 4 \times 8 \times 18 - 1 = 17856 - 1 = 17855$$

(Here, 2 as a factor and 1 as a divisor or are to be excluded.)

The number of proper divisors divisible by 10 is equal to number of selections from (5, 5), (2, 2), (3), (7), (17) consisting of at least one 5 and at least one 2 and 35700 is to be excluded and is given by $2 \times 2 \times 2 \times 2 \times 2 - 1 = 31$.

Sum of these divisors is

$$(5^1 + 5^2)(2^1 + 2^2)(3^0 + 3^1)(7^0 + 7^1)(17^0 + 17^1) - 35700$$
$$= 30 \times 6 \times 4 \times 8 \times 18 - 35700 = 67980$$

Example 5.83 **Find the number of ways in which the number 94864 can be resolved as a product of two factors.**

Sol. $94864 = 2^4 \times 7^2 \times 11^2$

Hence, the number of ways is

$$\frac{1}{2}[(4 + 1)(2 + 1)(2 + 1) + 1] = 23$$

Example 5.84 **Find the number of divisors of the number $N = 2^3 \cdot 3^5 \cdot 5^7 \cdot 7^9$ which are perfect square.**

Sol. Since the divisor is perfect square each prime factor must occur even number of times.

2 can be taken in 2 ways (2^0or 2^2)
3 can be taken in 3 ways (3^0 or 3^2 or 3^4)
Similarly 5 can be taken in 4 ways (5^0 or 5^2 or 5^4 or 5^6)
and 7 can be taken in 5 ways (7^0 or 7^2 or 7^4 or 7^6 or 7^8)
hence total divisors which are perfect squares

$$= 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

Example 5.85 **Find the number of ways in which the number 300300 can be split into 2 factors which are relatively prime.**

Sol. $300300 = 2^2 3^1 5^2 7^1 11^1 13^1$

Now we have to make factors which are relative prime.

$\Rightarrow 2^2, 3^1, 5^2, 7^1, 11^1, 13^1$ should behave as single identities.

So no. of divisors $(1 + 1)(1 + 1)(1 + 1)(1 + 1)(1 + 1)(1 + 1)$

$$= 2^6 = 64$$

No. of ways of splitting into 2 factors $= \frac{64}{2} = 32$

Concept Application Exercise 5.6

1. Out of 10 white, 9 black and 7 red balls, find the number of ways in which selection of one or more balls can be made (balls of the same color are identical).
2. In an election, number of candidates exceeds the number to be elected by 2. A man can vote in 56 ways. Find the number of candidates.
3. There are 5 historical monuments, 6 gardens and 7 shopping malls in a city. In how many ways a tourist can visit the city if he visits at least one shopping mall.
4. Find the number of divisors of 720. How many of these are even? Also find the sum of divisors.
5. Find the number of odd proper divisors of $3^p \times 6^m \times 21^n$.
6. In how many ways the number 7056 can be resolved as a product of 2 factors.

DIVISION AND DISTRIBUTION

Distinct Objects

Division of m + n Distinct Objects into Two Groups of the Size m and n ($m \neq n$)

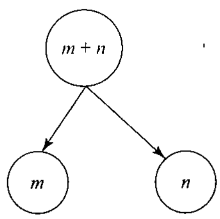


Fig. 5.17

The number of ways $m + n$ distinct objects are divided into two groups of the size m and n is equivalent to the number of ways m objects are selected out of $m + n$ objects to form one of the groups, which can be done in ${}^{m+n}C_m$ ways. The other group of n objects is formed by the remaining n objects. So, the number of ways is

$${}^{m+n}C_m = \frac{(m+n)!}{m!n!}$$

Now, distribution of $m + n$ distinct objects between two persons (if one gets m and the other gets n objects) is equal to number of ways to divide $m + n$ objects into two groups × number of ways in which these two groups can be given to two persons, which is equal to $\frac{(m+n)!}{m!n!}2!$

Division of m + n + p Distinct Objects Into Three Groups of the Size m, n and p ($m \neq n \neq p$)

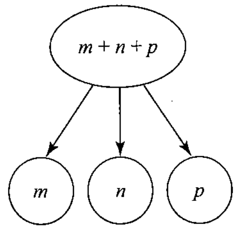


Fig. 5.18

For this division, let us first select m objects for first group which can be done in ${}^{m+n+p}C_m$ ways.

From the remaining $n + p$ objects, select n objects for second group, which can be done in ${}^{n+p}C_n$ ways.

The third group is formed from the remaining p objects. Hence, the total number of ways is

$$({}^{m+n+p}C_m)({}^{n+p}C_n) = \frac{(m+n+p)!}{m!\,n!\,p!}$$

Now, distribution of $m + n + p$ objects among three persons if they get m, n and p objects is equal to number of ways to divide $(m + n + p) \times$ (number of ways in which these three groups can be given to three persons) which is equal to

$$\frac{(m+n+p)!}{m!n!p!}3!$$

In general, division of $x_1 + x_2 + x_3 + \cdots + x_n$ into n groups of the size $x_1, x_2, x_3, \ldots, x_n$ ($x_1 \neq x_2 \neq \cdots \neq x_n$)

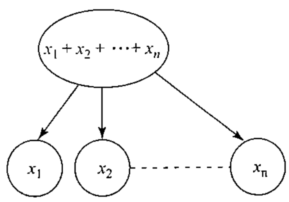


Fig. 5.19

The number of division ways is

$$\frac{(x_1+x_2+\cdots+x_n)!}{x_1!x_2!\cdots x_n!}$$

Now, distribution ways of these n groups among n persons is

$$\frac{(x_1+x_2+\cdots+x_n)!}{x_1!x_2!\cdots x_n!}n!$$

Division of Objects into Two Groups of Equal Size n

Consider the distribution of 4 objects a, b, c, d into two groups each of size 2. The number of ways of selection is $4!/(2!2!) = 6$.

But this answer is more similar to the one given in the division chart below:

No.	Group 1	Group 2
1	*ab*	*cd*
2	*ac*	*bd*
3	*ad*	*bc*
4	*bc*	*ad*
5	*bd*	*ac*
6	*cd*	*ab*

Clearly, in the above chart way number 1 and 6, 2 and 5, 3 and 4 are same division.

Then, the actual number of ways is

$$\frac{\frac{4!}{2!2!}}{2!}=3$$

as in each division way, say ab and cd, these groups can be arranged in 2! ways.

The division of $2n$ objects into two groups of equal size is

$$\frac{\frac{(2n)!}{n!n!}}{2!}=\frac{(2n)!}{n!n!2!}$$

Now, the distribution ways of these 2 groups between 2 persons is

$$\frac{(2n)!}{n!n!2!}2!=\frac{(2n)!}{n!n!}$$

Again, division of $3n$ objects into three groups of equal size n is

$$\frac{\frac{(3n)!}{n!n!n!}}{3!}=\frac{(3n)!}{n!n!n!3!}$$

as in each division way, say ab, cd and ef three groups can be arranged in 3! ways.

Now, distribution ways of these 3 groups among 3 persons is

$$\frac{(3n)!}{n!n!n!3!}3!=\frac{(3n)!}{n!n!n!}$$

Division of Distinct Objects into Multiple Equal Groups

Division of $12n$ objects into 5 groups of $2n$, $2n$, $2n$, $3n$, $3n$ size is

$$\frac{\frac{(12n)!}{(2n)!(2n)!(2n)!(3n)!(3n)!}}{3!2!}$$

Now, the distribution ways of these 5 groups among 5 persons is

$$\frac{\frac{(12n)!}{(2n)!(2n)!(2n)!(3n)!(3n)!}}{3!2!}5!$$

Distribution of n Distinct Objects in r Different Boxes, if Any Number of Objects Are Placed in Any Box (Empty Boxes Are Allowed)

Consider distribution of n distinct objects into two different boxes of any size

Number of objects in box 1	Number of objects in box 2	Number of ways
0	n	${}^nC_0$
1	$n-1$	${}^nC_1$
2	$n-2$	${}^nC_2$
...	...	...
n	0	${}^nC_n$

The total number of ways is ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$. Alternatively, each object has two possibilities; it can enter either into box 1 or box 2. The total number of possibilities for n objects is $2 \times 2 \times 2 \times \cdots n$ times $= 2^n$.

Consider distribution of n distinct objects into three groups of any size. For each object, there are three possibilities. Then the total number of possibilities for n objects is $3 \times 3 \times 3 \ldots n$ times $= 3^n$.

Similarly, for distribution of n distinct objects into r different boxes, if in any box any number of objects can be placed for each object, there are r possibilities. Then total number of possibilities for n objects is $r \times r \times r \cdots n$ times $= r^n$.

Distribution of n Distinct Objects Into r Different Boxes if Empty Boxes Are Not Allowed or in Each Box at Least One Object is Put (n > r)

The number of ways is given by

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - {}^rC_3(r-3)^n + \cdots + (-1)^{r-1}\cdot {}^rC_{r-1}\cdot 1$$

This formula can be derived from the principle of inclusion and exclusions using set theory.

Example 5.86 **Find the number of ways of dividing 52 cards among four players equally.**

Sol. We can divide 52 cards equally into four groups in $52!/(13!)^4(4!)$ ways.

Now, these four groups can be distributed among four players as 4! ways.

Therefore, the total number of ways of dividing the cards among four players equally is

$$\frac{52!}{(13!)^4(4!)} \times 4! = \frac{52!}{(13!)^4}$$

Example 5.87 **Find the number of ways to give 16 different things to three persons *A*, *B*, *C* so that *B* gets 1 more than *A* and *C* gets 2 more than *B*.**

Sol. Let A get n objects, then B gets $n + 1$ and C gets $n + 3$

Now $n + n + 1 + n + 3 = 16 \Rightarrow n = 4$

$\Rightarrow$ A, B, C gets 4, 5 and 7 objects respectively.

$\Rightarrow$ Number of ways of distribution is equal to number of ways 16 objects can be divided into three groups of size 4, 5 and 7.

Hence, number of ways = $\dfrac{16!}{4!5!7!}$.

Example 5.88 **In how many ways can 8 different books be distributed among 3 students if each receives at least 2 books?**

Sol. If each receives at least two books, then the division trees would be as shown below:

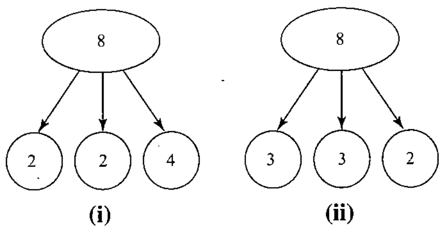


Fig. 5.20

The number of division ways for tree in Fig. 5.20 (i) is $8!/(2!)^2 4!2!$. The number of division ways for tree in Fig. 5.20 (ii) is $8!/(3!)^2 2!2!$. The total number of ways of distribution of these groups among 3 students is

$$\left[\frac{8!}{(2!)^2\, 4!\, 2!} + \frac{8!}{(3!)^2\, 2!\, 2!}\right] \times 3!$$

Example 5.89 ***n* different toys have to be distributed among *n* children. Find the number of ways in which these toys can be distributed so that exactly one child gets no toy.**

Sol. If exactly one child gets no toy, then exactly one child must get two toys and rest $n - 2$ gets one toy each.

The division tree will be as follows:

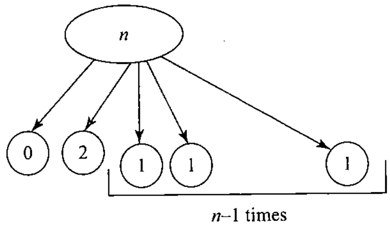


Fig. 5.21

The number ways of division in the groups as shown in the tree is

$$\frac{n!}{0!2!(1!)^{n-2}(n-2)!} = \frac{n!}{2!(n-2)!} = {}^nC_2$$

The number of ways of distribution of these n groups among n children is $n!$. Then, the total number of ways of distributions is ${}^nC_2 \times n!$.

Example 5.90 **Find the number of ways in which *n* different prizes can be distributed among *m* (< *n*) persons if each is entitled to receive at most *n* − 1 prizes.**

Sol. The total number of ways is $m \times m \times \cdots n$ times $= m^n$. The number of ways in which one gets all the prizes is m. Therefore, the required number of ways is $m^n - m$.

Distribution of Identical Objects

When identical objects are distributed for different boxes, the number of objects is only important. It is of no use to consider about the boxes that what objects they hold.

Suppose 4 identical objects are distributed in 2 distinct boxes if empty boxes are allowed.

Number of objects in box 1	Number of objects in box 2	Number of ways
0	4	1
1	3	1
2	2	1
3	1	1
4	0	1

Here, the total number of ways is 5.

Distribution of n Identical Objects in r Different Boxes if Empty Boxes Are Not Allowed

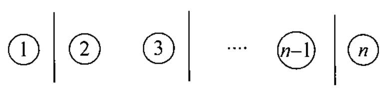


Fig. 5.22

Let n identical objects be put on the floor in line. Now, to form r groups, we require to put $r - 1$ identical partitions. If empty boxes are not allowed, we cannot put partition to the left of the first object and to the right of n^{th} object. Hence, there are $n-1$ gaps available to put $r-1$ partitions. Now, $r-1$ gaps can be selected from $n-1$ gaps in ${}^{n-1}C_{r-1}$ ways. This is equivalent to number of ways of distributing n objects in r boxes if empty boxes are not allowed.

Distribution of n Identical Objects in r Different Boxes if Empty Boxes Are Allowed

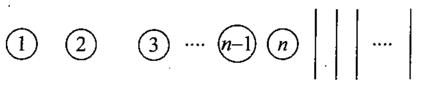


Fig. 5.23

Here, the number of ways are equivalent to arranging n identical objects and $r-1$ identical partitions, which automatically takes care of empty boxes. The number of ways is

$$\frac{(n+r-1)!}{n!(r-1)!} = {}^{n+r-1}C_{r-1}$$

Number of Non-Negative Integral Solutions of the Equation $x_1 + x_2 + \cdots + x_r = n$

This is equivalent to the number of ways of distributing n identical objects into r different boxes if empty boxes are allowed which is ${}^{n+r-1}C_{r-1} = {}^{n+r-1}C_n$.

Number of Positive Integral Solutions of the Equation $x_1 + x_2 + \cdots + x_r = n$

This is equivalent to the number of ways of distributing n identical objects into r different boxes if empty boxes are not allowed which is ${}^{n-1}C_{r-1}$.

Example 5.91 **Find the number of non-negative integral solutions of the equations $x + y + z = 10$.**

Sol. Here, the number of solutions is equivalent to the number of ways. Ten identical objects are distributed in 3 distinct boxes if empty boxes are allowed, which is ${}^{10+3-1}C_3 = {}^{12}C_3$.

Example 5.92 **Find the number of positive integral solutions of the equations $x + y + z = 12$.**

Sol. Here, the number of solutions is equivalent to number of ways. Twelve identical objects are distributed in 3 distinct boxes if empty boxes are not allowed, which is ${}^{12-1}C_{3-1} = {}^{11}C_2 = 55$.

Example 5.93 **Find the number of non-negative integral solutions of the equation $x + y + z + 2w = 20$.**

Sol. Let $w = 0$. Then, the equation reduces to $x + y + z = 20$. Number of non-negative integral solutions is ${}^{20+4-1}C_{4-1} = {}^{23}C_3$. If $w = 1$, then the equation reduces to $x + y + z = 18$. Number of non-negative integral solutions is ${}^{18+4-1}C_{4-1} = {}^{21}C_3$.

Similarly, we have $w = 2, 3, \ldots, 10$.

Therefore, the total number of solutions is ${}^{23}C_3 + {}^{21}C_3 + {}^{19}C_3 + \cdots + {}^{5}C_3 + {}^{3}C_3$.

Example 5.94 **Find the number of non-negative integral solutions of $x + y + z + w \le 20$.**

Sol. Let,

$$x + y + z + w + t = 20 \quad (1)$$

where $t \ge 0$.

Now, we find the non-negative integral solutions of Eq. (1). The total number of such solutions is ${}^{20+5-1}C_{5-1}$

Example 5.95 **In how many ways can a party of 6 men be selected out of 10 Hindus, 8 Muslims and 6 Christians. If the party consists of at least one person of each religion, find the number of ways of selection. (Consider only the religion of the person.)**

Sol. Let x, y and z be the number of Hindus, Muslims and Christians, respectively, who are selected. Then,

$$x + y + z = 6 \quad (1)$$

Also, at least one person must be selected from each religion when we have to find non-negative integral solutions of Eq. (1).

Therefore, the number of non-negative solutions of Eq. (1) is ${}^{6-1}C_{3-1} = {}^{5}C_2 = 10$.

Example 5.96 **Find the total number of positive integral solutions for (x, y, z) such that $xyz = 24$. Also find out the total number of integral solutions.**

Sol. $24 = 2^3 \times 3$

Now, consider three boxes x, y, z. 3 can be put in any of the three boxes.

Also, 2, 2, 2 can be distributed in the three boxes in ${}^{3+3-1}C_{3-1} = {}^{5}C_2$ ways. Hence, the total number of positive integral solutions is equal to the number of distributions which is given by $3 \times {}^{5}C_2 = 30$.

Note: *If any box remains empty, say x, than $x = 1$. To find integral solutions where negative integers are also allowed.*

Any two of the factors in each factorization may be negative. Therefore, the number of ways to associate negative sign in each case is ${}^{3}C_2 = 3$. Hence, the total number of integral solutions is $30 + 3 \times 30 = 120$.

Example 5.97 **In how many ways can 10 persons take seats in a row of 24 fixed seats so that no two persons take consecutive seats.**

Sol. As no two persons take consecutive seats, there will be at least one vacant seat between any two persons sitting before the first person and after the last person. Let the number of vacant seats before the first person $= x_0$, and the number of vacant seats between the first and the second persons be x_1, etc., as shown in the figure.

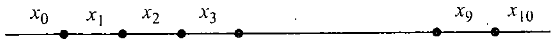


Fig. 5.24

Clearly, the total number of vacant seats is 24 – 10 = 14.

$\therefore \quad x_0 + x_1 + x_2 + \cdots + x_9 + x_{10} = 14$

where $x_0 \geq 0, x_1 \geq 1, x_2 \geq 1, x_3 \geq 1, \ldots, x_9 \geq 1, x_{10} \geq 0$.

Let $x_0 = y_0, x_1 = y_1 + 1, x_2 = y_2 + 1, \ldots, x_9 = y_9 + 1,$ $x_{10} = y_{10}$. Then, the equation becomes

$$y_0 + (y_1 + 1) + (y_2 + 1) + \cdots + (y_9 + 1) + y_{10} = 14$$

$$\Rightarrow \quad y_0 + y_1 + y_2 + \cdots + y_9 + y_{10} = 14 - 9 = 5$$

Therefore, the required number of ways = the number of non-negative integral solutions of the equation $y_0 + y_1 + y_2 + \cdots + y_9 + y_{10} = 5$, i.e., ${}^{5+11-1}C_{11-1} = {}^{15}C_{10}$.

But 10 persons can arrange among themselves in 10! ways. Hence, the required number of ways is

$${}^{15}C_{10} \times 10! = \frac{15!}{10!5!} \times 10! = \frac{15!}{5!}.$$

Example 5.98 **Find the number of distinct throws which can be thrown with *n* six-faced normal dice, which are indistinguishable among themselves.**

Sol. Consider six faces as six beggars and *n* identical dice to be identical coins.

Now, number of distribution is ${}^{n+6-1}C_{6-1} = {}^{n+5}C_5$. If a beggar (say face 6) gets no coin, then it is equivalent to 6, which does not appear on the dice.

Example 5.99 **In how many ways 3 boys and 15 girls can sit together in a row such that between any 2 boys at least 2 girls sit.**

Sol.

		B		*B*		*B*
x		*y*		*z*		*w*

Fig. 5.25

First three boys can be arranged in 3! ways. After arranging the boys, four gaps are created. Let in these gaps *x*, *y*, *z* and *w* girls sit as shown in the diagram. Let us first find out the distribution ways of girls in the four gaps. As given in question, *y*, $z \geq 2$ and $x, w \geq 0$, we have to find the integral solutions of the equation $x + y + z + w = 15$ with the above condition. Let,

$$y = y_1 + 2 \text{ and } z = z_1 + 2 \text{ (where } y_1, z_1 \geq 0)$$

$$\Rightarrow \quad x + y_1 + z_1 + w = 11$$

Number of solutions of above equation is ${}^{11+4-1}C_{4-1} = {}^{14}C_3$. After it is decided as in which gap how many girls will sit, they can be arranged in 15! ways.

Hence, the total number of ways is 3! 15! ${}^{14}C_3$.

Concept Application Exercise 5.7

1. Find the number of ways in which 22 different books can be given to 5 students, so that two students get 5 books each and all the remaining students get 4 books each.
2. Find the number of ways in which 16 constables can be assigned to patrol 8 villages, 2 for each.
3. In how many ways can 10 different prizes be given to 5 students if one particular boy must get 4 prizes and rest of the students can get any number of prizes?
4. In how many different ways can a set *A* of 3*n* elements be partitioned into 3 subsets of equal number of elements? (The subsets *P*, *Q*, *R* form a partition if $P \cup Q \cup R = A$, $P \cap R = \phi$, $Q \cap R = \phi$, $R \cap P = \phi$.)
5. In how many ways can Rs. 16 be divided into 4 persons when none of them gets less than Rs. 3?
6. Find the number of ways of selecting 10 balls out of an unlimited number of identical white, red and blue balls.
7. If *a*, *b*, *c*, *d* are odd natural numbers such that $a + b + c + d = 20$, then find the number of values of the ordered quadruplet (*a*, *b*, *c*, *d*).
8. Roorkee University has to send 10 professors to 5 centres for its entrance examination, 2 to each centre. Two of the centres are in Roorkee and the others are outside. Two of the professors prefer to work in Roorkee while three prefer to work outside. In how many ways can this be made if the preferences are to be satisfied?
9. In how many ways, two different natural numbers can be selected, which are less than or equal to 100 and differ by almost 10.

MULTINOMIAL THEOREM

Consider the equation $x_1 + x_2 + \cdots + x_r = n$, where $a_i \leq x_i \leq b_i$; $x_i \in 1$; $i = 1, 2, \ldots, r$.

In order to find the number of solutions of the given equation satisfying the given conditions, we observe that the number of solutions is the same as the coefficient of x^n in the product

$$(x^{a_1} + x^{a_1+1} + x^{a_1+2} + \cdots + x^{b_1}) \times (x^{a_2} + x^{a_2+1} + x^{a_2+2} + \cdots + x^{b_2})$$

$$\times (x^{a_3} + x^{a_3+1} + x^{a_3+2} + \cdots + x^{b_3}) \times \cdots \times (x^{a_r} + x^{a_r+1} + x^{a_r+2} + \cdots + x^{b_r})$$

For example, if we have to find the number of non-negative integral solutions of $x_1 + x_2 + \cdots + x_r = n$, then as above, the required number is the coefficient of x^n in $(x^0 + x^1 + \cdots + x^n)(x^0 + x^1 + \cdots + x^n) \ldots (x^0 + x^1 + \cdots + x^n)$ (*r* – brackets)

$= \text{Coefficient of } x^n \text{ in } (1 + x + x^2 + \cdots + x^n)^r$

$= \text{Coefficient of } x^n \text{ in } (1 + x + x^2 + \cdots)^r$

$= \text{Coefficient of } x^n \text{ in } (1 - x)^{-r}$

$= \text{Coefficient of } x^n$ in

$$\left[1 + (-r)(-x) + \frac{(-r)(-r-1)}{2!}(-x)^2\right.$$

$$+\frac{(-r)(-r-1)(-r-2))3!}{3!}(-x)^3\Bigg]$$

$= \text{Coefficient of } x^n \text{ in } [1 + {}^rC_1 x + {}^{r+1}C_2 x^2 + {}^{r+2}C_3 x^3 + \cdots]$

$= {}^{n+r-1}C_n$

$= {}^{n+r-1}C_{r-1}$

Note: *If there are l objects of one kind, m objects of second kind, n objects of third kind and so on, then the number of ways of choosing r objects out of these objects is the coefficient of* x^r *in the expansion of* $(1 + x + x^2 + x^3 + \cdots + x^l) \times (1 + x + x^2 + \cdots + x^m) \times (1 + x + x^2 + \cdots + x^n)$.

Further, if one object of each kind is to be included, then the number of ways of choosing r objects out of these objects is the coefficient of x^r *in the expansion of* $(x + x^2 + x^3 + \cdots + x^l) \times (x + x^2 + x^3 + x^3 + \cdots + x^m) \times (x + x^2 + x^3 + \cdots + x^n)$.

Different Cases of Multinomial Theorem

Case I

If upper limit of a variable is more than or equal to the sum required, than the upper limit of that variable can be taken as infinite.

Example 5.100 **In how many ways the sum of upper faces of four distinct dices can be six.**

Sol. Here, the number of required ways will be equal to the number of solutions of $x_1 + x_2 + x_3 + x_4 = 6$, i.e., $1 \le x_1, x_2, x_3, x_4 \le 6$. Since the upper limit is six, which is equal to the sum required, so upper limit can be taken as infinite. So, number of solutions is equal to the coefficient of t^5 in $(1 + t + t^2 + \cdots)^4$

$= \text{Coefficient of } t^5 \text{ in } (1 - t)^{-4}$

$= {}^{6+4-1}C_{4-1}$

$= {}^9C_3 = 84$

Case II

If the upper limit of a variable is less than the sum required and the lower limit of all the variables is non-negative, then the upper limit of that variable is that given in the problem.

Example 5.101 **In how many different ways can 3 persons *A*, *B*, C having 6 one-rupee coin, 7 one-rupee coin, 8 one-rupee coin, respectively, donate 10 one-rupee coin collectively?**

Sol. The number of ways in which they can donate Rs. 10 is same as the number of solutions to the equation $x_1 + x_2 + x_3 = 10$ subject to the condition $0 \le x_1 \le 6$, $0 \le x_2 \le 7$, $0 \le x_3 \le 8$.

Therefore, the required number of ways = Coefficient of x^{10} in $(1 + x + x^2 + \cdots + x^6)(1 + x + x^2 + \cdots + x^7)(1 + x + x^2 + \cdots + x^8)$

$= \text{Coefficient of } x^{10} \text{ in } (1 - x^7)(1 - x^8)(1 - x^9)(1 - x)^{-3}$

$= \text{Coefficient of } x^{10} \text{ in } (1 - x^7 - x^8 - x^9)(1 + {}^3C_1 x + {}^4C_2 x^2 + {}^5C_2 x^2 + {}^5C_3 x^3 + \cdots + {}^{12}C_{10} x^{10})$ (ignoring powers higher than 10).

$= {}^{12}C_2 - {}^5C_3 - {}^4C_2 - {}^3C_1$

$= 66 - 10 - 6 - 3 = 47$

Example 5.102 **In an examination, the maximum mark for each of the three papers is 50 and the maximum mark for the fourth paper is 100. Find the number of ways in which the candidate can score 605 marks in aggregate.**

Sol. Let the marks scored by the candidate in four papers be x_1, x_2, x_3, x_4. Then, $x_1 + x_2 + x_3 + x_4 = 150$ (i.e., 60% of 250 is 150), where $0 \le x_1, x_2, x_3 \le 50$, $0 \le x_4 \le 100$.

The number of solutions of the equation is same as the coefficient of x^{150} in $(1 + x + x^2 + \cdots + x^{50})^3 (1 + x + x^2 + \cdots + x^{100})$ which is given by

$$\text{Coefficient of } x^{150} \text{ in } \left(\frac{1-x^{51}}{1-x}\right)^3 \left(\frac{1-x^{101}}{1-x}\right)$$

$= \text{Coefficient of } x^{150} \text{ in } (1 - x^{51})^3 (1 - x^{101})(1 - x)^{-4}$

$= \text{Coefficient of } x^{150} \text{ in } (1 - 3x^{51} + 3x^{102} - x^{101})(1 - x)^{-4}$

$= {}^{153}C_3 - 3 \times {}^{102}C_3 + 3 \times {}^{51}C_3 - {}^{52}C_3$

$= 110551$

Case III

When coefficients of variables are not uniform. The number of solutions of $ax_1 + bx_2 + cx_3 = n$ is subject to this condition in $a_1 \le x_1 \le b_1$, $a_2 \le x_2 \le b_2$, $a_3 \le x_3 \le b_3$. The coefficient of t^n is

$$\{(t^a)^{a_1} + (t^a)^{a_1+1} + \cdots + (t^a)^{b_1}\} \times \{(t^b)^{a_2} + (t^b)^{a_2+1} + \cdots + (t^b)^{b_2}\} \times \{(t^c)^{a_3} + (t^c)^{a_3+1} + \cdots + (t^c)^{b_3}\}$$

Example 5.103 **Find the number of non-negative integral solutions of $x_1 + x_2 + x_3 + 4x_4 = 20$.**

Sol. The number of non-negative integral solutions of the given equation is equal to the coefficient of x^{20} in $(1 - x)^{-1}(1 - x)^{-1}(1 - x)^{-1} \times (1 - x^4)^{-1}$, which is given by

Coefficient of x^{20} in $(1 - x)^{-3}(1 - x^4)^{-1}$

$= \text{Coefficient of } x^{20} \text{ in } (1 + {}^3C_1 x + {}^4C_2 x^2 + {}^5C_3 x^3 + {}^6C_4 x^4 + \cdots)(1 + x^4 + x^8 + \cdots)$

$= 1 + {}^6C_4 + {}^{10}C_8 + {}^{14}C_{12} + {}^{18}C_{16} + {}^{22}C_{20} = 536$

Example 5.104 **In how many ways can 15 identical blankets be distributed among six beggars such that everyone gets at least one blanket and two particular beggars get equal blankets and another three particular beggars get equal blankets.**

Sol. The number of ways of distributing blankets is equal to the number of solutions of the equation $3a + 2b + c = 15$ $a, b, c \geq 1$, which is equal to coefficient of t^{15} in $(t^3 + t^6 + t^9 + t^{12} + \cdots) \times (t^2 + t^4 + \cdots)(t + t^2 + \cdots)$, which is given by

Coefficient of t^9 in $(1 + t^3 + t^6 + t^9)(1 + t^2 + t^4 + t^6 + t^8) \times (1 + t + t^2 + \cdots + t^9)$ (neglecting higher powers)
$= $ Coefficient of t^9 in $(1 + t^2 t^3 + t^4 + t^5 + 2t^6 + t^7 + 2t^8 + 2t^9)(1 + t + t^2 + \cdots + t^9) = 1 + 1 + 1 + 1 + 1 + 2 + 1 + 2 + 2 = 12$

Case IV

In the case when variables are distinct, we introduce some new variables to remove the condition of distinctness.

Example 5.105 **In how many ways can 14 identical toys be distributed among three boys so that each one gets at least one toy and no two boys get equal number of toys.**

Sol. Let the boys get a, b, c toys. Now, $a + b + c = 14$, $a, b, c \geq 1$ and a, b and c are distinct.

Let $a < b < c$ and $x_1 = a$, $x_2 = b - a$, $x_3 = c - b$. So,

$$3x_1 + 2x_2 + x_3 = 13, \quad x_1, x_2, x_3 \geq 1$$

Therefore, the number of solutions is equal to the coefficient of t^{14} in $(t^3 + t^6 + t^9 + \cdots)(t^2 + t^4 + \cdots)(t + t^2 + \cdots)$

$= $ Coefficient of t^8 in $(1 + t^3 + t^6)(1 + t^2 + t^4 + t^6 + t^8)(1 + t + t^2 + \cdots + t^8)$ (neglecting higher powers)

$= $ Coefficient of t^8 in $(1 + t^2 + t^3 + t^4 + t^5 + 2t^6 + t^7 + 2t^8) \times (1 + t + t^2 + \cdots + t^8)$

$= 1 + 1 + 1 + 1 + 1 + 2 + 1 + 2 = 10$

Now, three distinct numbers can be assigned to three boys in 3! ways.

So, corresponding to each solution, we have six ways of distribution. So, total numbers of ways is $10 \times 6 = 60$.

Case V

When the required sum is not fixed.

To find the number of solutions of the equation,

$$x_1 + x_2 + \cdots + x_m \leq n \qquad (1)$$

We introduce a dummy variable x_{m+1} so that

$$x_1 + x_2 + \cdots + x_{m+1} = n, x_{m+1} \geq 0 \qquad (2)$$

Hence, the number of solutions of Eqs. (1) and (2) will be same.

To find the number of solution of $x_1 + x_2 + \cdots + x_n \geq n$ (when the values of $x_1, x_2, \ldots, x_n$ are restricted), first find the number of solutions of $x_1 + x_2 + \cdots + x_n \leq n - 1$ and then subtract it from the total number of solutions of $x_1 + x_2 + \cdots + x_n \leq n - 1$; and then subtract it from the total number of solutions.

Example 5.106 **In how many ways can we get a sum of at most 17 by throwing six distinct dice.**

Sol. Let $x_1, x_2, \ldots, x_6$ be the number that appears on the six dice. Let us find the number of ways to get the sum less than or equal to 17. This will be same as finding the number of solutions to the inequality $x_1 + x_2 + x_3 + \cdots + x_6 \leq 17$. Introducing a dummy variable x_7 $(x_7 \geq 0)$, the inequality becomes an equation

$$x_1 + x_2 + \cdots + x_6 + x_7 = 17$$

Here, $1 \leq x_i \leq 6$ where $i = 1, 2, \ldots, 6$ and $x_7 \geq 0$. Therefore,

Number of solutions = Coefficient of x^{17} in $(x + x^2 + \cdots + x^6)^6 \times (1 + x + x^2 + \cdots)$

= Coefficient of x^{11} in $(1 - x^6)^6 (1 - x)^{-7}$

= Coefficient of x^{11} in $(1 - 6x^6)(1 - x)^{-7} = {}^{17}C_6 - 6\,{}^{11}C_5$

Example 5.107 **In how many ways can we get a sum greater than 17 by throwing six distinct dice.**

Sol. Let $x_1, x_2, \ldots, x_6$ be the number that appears on the six dice. Here, $1 \leq x_i \leq 6$, $\forall i \in \{1, 2, 3, 4, 5, 6\} \Rightarrow$ total number of cases is 6^6. In the above example, we have calculated the number of ways to get the sum less than or equal to 17, which is

$${}^{17}C_{11} - 6 \times {}^{11}C_5.$$

Hence, the number of ways to get a sum greater than 17 is $6^6 - ({}^{17}C_{11} - 6\,{}^{11}C_5)$.

Concept Application Exercise 5.8

1. In how many ways can 30 marks be allotted to 8 questions if each question carries at least 2 marks?
2. Find the number of positive integral solutions of the inequality $3x + y + z \leq 30$.
3. Find the number of integers between 1 and 100000 having the sum of the digits 18.
4. Find the number of integral solutions of $x_1 + x_2 + x_3 = 24$ subjected to the condition that $1 \leq x_1 \leq 5$, $12 \leq x_2 \leq 18$ and $-1 \leq x_3$.

PRINCIPLE OF INCLUSION AND EXCLUSION

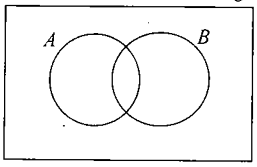


Fig. 5.26

In the above Venn's diagram, we get

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A_1' \cap A_2') = n(U) - n(A \cup B)$

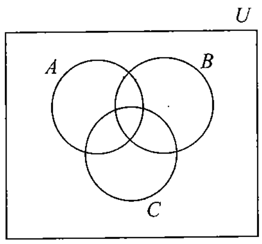


Fig. 5.27

In the above Venn's diagram, we get

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

$$n(A_1' \cap A_2' \cap A_3') = n(U) - n(A \cup B \cup C)$$

In general, we have

$$n(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) + \cdots + (-1)^n \sum n(A_1 \cap A_2 \cap \cdots \cap A_n)$$

Example 5.108 **Find the numbers of positive integers from 1 to 1000, which are divisible by at least 2, 3 or 5.**

Sol. Let A be the set of positive integers form 1 to 1000, which is divisible by k. Obviously, we have to find $n(A_2 \cup A_3 \cup A_5)$. If $[\cdot]$ denotes the greatest integer function, then

$$n(A_2) = \left[\frac{1000}{2}\right] = 500$$

$$n(A_3) = \left[\frac{1000}{3}\right] = 333$$

$$n(A_5) = \left[\frac{1000}{5}\right] = 200$$

Hence, $n(A_2 \cap A_3) = 166$, $n(A_3 \cap A_5) = 66$, $n(A_2 \cap A_5) = 100$, $n(A_2 \cap A_3 \cap A_5) = 33$.

Hence, $n(A_2 \cup A_3 \cup A_5) = 500 + 333 + 200 - 166 - 66 - 100 + 33 = 734$.

Note that the number of positive integers from 1 to 1000, which are not divisible by 2, 3 or 5 is $1000 - n(A_2 \cup A_3 \cup A_5) = 266$.

Example 5.109 **Find the number of ways in which two Americans, two British, one Chinese, one Dutch and one Egyptian can sit on a round table so that persons of the same nationality are separated.**

Sol. Total number of person is 6!. When A_1, A_2 are together,

$$n(A) = 5!\,2! = 240$$

When B_1, B_2 together,

$$n(B) = 5!\,2! = 240$$

$$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B) = 240 + 240 - 96 = 384$$

Hence,

$$n(\bar{A} \cap \bar{B}) = \text{Total persons} - n(A \cup B) = 6! - 384 = 720 - 384 = 336$$

Example 5.110 **Find the number of permutations of letters *a, b, c, d, e, f, g* taken all together if neither '*beg*' nor '*cad*' pattern appear.**

Sol. The total number of permutations without any restrictions is 7!.

(*b e g*) *a c d f*

Fig. 5.28

The number of permutations in which '*beg*' pattern always appears is 5!. Likewise, there are some cases in which '*cad*' pattern also appears.

(*c a d*) *b e f g*

Fig. 5.29

The number of permutations in which '*cad*' pattern always appears is 5!. Likewise, there are some cases in which '*beg*' pattern also appears.

(*c a d*) (*b e g*) *f*

Fig. 5.30

The number of permutations in which '*beg*' and '*cad*' patterns appears is 3!.

Therefore, the total number of permutations in which '*beg*' and '*cad*' patterns do not appear is 7! − 5! −5! + 3!.

Example 5.111 **Find the number of *n* digit numbers, which contain the digits 2 and 7, but not the digits 0, 1, 8, 9.**

Sol. The total number without any restrictions containing digits 2, 3, 4, 5, 6, 7 is $n(S) = 6^n$.

The total number of numbers that contain 3, 4, 5, 6, 7 is $n(A) = 5^n$.

The total number of numbers that contain 2, 3, 4, 5, 6 is $n(B) = 5^n$.

The total number of numbers that contain 3, 4, 5, 6 is $n(A \cap B) = 4^n$.

The total number of numbers that do not contain digits 2 and 7 is $5^n + 5^n - 4^n$. The total number of numbers that contain 2 and 7 is $6^n - 5^n - 5^n + 4^n$.

Derangement

There are n letters and n corresponding envelopes. The number of ways in which letters can be placed in the envelopes (one letter in each envelope) so that no letter is placed in correct envelope is

$$n!\left[1-\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{(-1)^n}{n!}\right]$$

Proof: n letters are denoted by 1, 2, 3, ..., n. Let A_i denote the set of distribution of letters in envelopes (one letter in each envelop) so that the i^{th} letter is placed in the corresponding envelope. Then,

$n(A_i) = 1 \times (n-1)!$ [since the remaining $n-1$ letters can be placed in $n-1$ envelopes in $(n-1)!$ ways]

Then, $n(A_i \cap A_j)$ represents the number of ways where letters i and j can be placed in their corresponding envelopes. Then,

$$n(A_i \cap A_j) = 1 \times 1 \times (n-2)!$$

Also,

$$n(A_i \cap A_j \cap A_k) = 1 \times 1 \times 1 \times (n-3)!$$

Hence, the required number is

$$n(A_1' \cup A_2' \cup \cdots \cup A_n') = n! - n(A_1 \cup A_2 \cup \cdots \cup A_n)$$

$$= n! - \left[\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) + \cdots + (-1)^n \sum n(A_1 \cap A_2 \cdots \cap A_n)\right]$$

$$= n! - [{}^nC_1(n-1)! - {}^nC_2(n-2)! + {}^nC_3(n-3)! + \cdots + (-1)^{n-1} \times {}^nC_n 1]$$

$$= n! - \left[\frac{n!}{1!(n-1)!}(n-1)! - \frac{n!}{2!(n-2)!}(n-2)! + \cdots + (-1)^{n-1}\right]$$

$$= n!\left[1-\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{(-1)^n}{n!}\right]$$

Example 5.112 **There are four balls of different colours and four boxes of colours same as those of the balls. Find the number of ways in which the balls, one in each box, could be placed in such a way that a ball does not go to box of its own colour.**

Sol. Number of derangements in such problems is given by

$$n!\left\{1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots+(-1)^n\frac{1}{n!}\right\}$$

Hence, the required number of derangements is

$$4!\left\{\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right\} = 12-4+1 = 9$$

Distribution of n Distinct Objects into r Distinct Boxes if in Each Box at Least One Object is Placed

The number of ways in which n distinct objects can be distributed among r persons if each gets at least one object is

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - {}^rC_3(r-3)^n + \cdots + (-1)^{r-1}\,{}^rC_{r-1}1$$

Proof: Let A_i denote the set of distribution of objects if i^{th} person gets no object. Then,

$n(A_i) = (r-1)^n$ [as now, n objects can be distributed among $r-1$ persons in $(r-1)^n$ ways]

Then, $n(A_i \cap A_j)$ represents number of distribution ways in which persons i and j get no object. Then,

$$n(A_i \cap A_j) = (r-2)^n$$

Also,

$$n(A_i \cap A_j \cap A_k) = (r-3)^n$$

The required number is

$$n(A_1' \cup A_2' \cup \cdots \cup A_r') = n! - n(A_1 \cup A_2 \cup \cdots \cup A_r)$$

$$= r^n - \left[\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) + \cdots (-1)^n \sum n(A_1 \cap A_2 \cdots \cap A_r)\right]$$

$$= r^n - [{}^rC_1(r-1)^n - {}^rC_2(r-2)^n + {}^rC_3(r-3)^n - \cdots + {}^rC_{r-1}1]$$

$$= r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - {}^rC_3(r-3)^n + \cdots + (-1)^{r-1}\,{}^rC_{r-1}1$$

Example 5.113 **Find the number of ways in which 5 distinct balls can be distributed in three different boxes if no box remains empty.**

Sol. By above formula, the number of ways is $3^5 - {}^3C_1 \times (3-1)^5 + {}^3C_2(3-2)^5 = 243 - 96 + 3 = 150$.

Example 5.114 **If $n(A) = 5$ and $n(B) = 3$, then find the number of onto functions from A to B.**

Sol. We know that in onto function, each image must be assigned at least one pre-image.

This is equivalent to number of ways in which 5 different objects (pre-images) can be distributed in 3 different boxes (images) if no box remains empty. The total number is given by $3^5 - {}^3C_1(3-1)^5 + {}^3C_2(3-2)^5 = 243 - 96 + 3 = 150$.

EXERCISES

Subjective Type

Solutions on page 5.43

1. Prove by combinatorial argument that ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$.
2. Prove that $(n!)!$ is divisible by $(n!)^{(n-1)!}$.
3. If n_1 and n_2 are five-digit numbers, find the total number of ways of forming n_1 and n_2 so that these numbers can be added without carrying at any stage.
4. n_1 and n_2 are four-digit numbers. Find the total number of ways of forming n_1 and n_2 so that n_2 can be subtracted from n_1 without borrowing at any stage.
5. How many five-digit numbers can be made having exactly two identical digits?
6. An ordinary cubical dice having six faces marked with alphabets A, B, C, D, E and F is thrown n times and the list of n alphabets showing up are noted. Find the total number of ways in which among the alphabets A, B, C, D, E and F only three of them appear in the list.
7. Find the number of three-digit numbers from 100 to 999 including all numbers which have any one digit that is the average of the other two.
8. The members of a chess club took part in a round robin competition in which each player plays with other once. All members scored the same number of points, except four juniors whose total score were 17.5. How many members were there in the club? Assume that for each win a player scores 1 point: 1/2 for draw point and zero for losing.
9. There are $2n$ guests at a dinner party. Supposing that the master and mistress of the house have fixed seats opposite one another and that there are two specified guests who must not be placed next to one another, show that the number of ways in which the company can be placed is $(2n-2)! \times (4n^2 - 6n + 4)$.
10. In how many ways can two distinct subsets of the set A of k $(k \geq 2)$ elements be selected so that they have exactly two common elements?
11. Prove that the number of ways to select n objects from $3n$ objects of which n are identical and the rest are different is

$$2^{2n-1} + \frac{1}{2}\frac{2n!}{(n!)^2}$$

12. There are n straight lines in a plane, in which no two are parallel and no three pass through the same point. Their points of intersection are joined. Show that the number of fresh lines thus introduced is

$$\frac{1}{8}n(n-1)(n-2)(n-3)$$

13. There are n points in a plane, in which no three are in a straight line except 'm' which are all in a straight line. Find the number of (a) different straight lines, (b) different triangles, (c) different quadrilaterals that can be formed with the given points as vertices.
14. Find the number of ways of disturbing n identical objects among n persons if at least $n-3$ persons get none of these objects.
15. The streets of a city are arranged like the lines of a chessboard. There are m streets running from north to south and n streets from east to west. Find the number of ways in which a man can travel from north-west to south-east corner, covering the shortest possible distance.
16. A batsman scores exactly a century by hitting fours and sixes in twenty consecutive balls. In how many different ways can he do it if some balls may not yield runs and the order of boundaries and overboundaries are taken into account?
17. In how many ways can $2t + 1$ identical balls be placed in three distinct boxes so that any two boxes together will contain more balls than the third?
18. Sohan has x children by his first wife. Geeta has $(x + 1)$ children by her first husband. They marry and have children of their own. The whole family has 24 children. Assuming that two children of the same parents do not fight, prove that the maximum possible number of fights that can take place is 191.
19. Let $S(n)$ denote the number of ordered pairs (x, y) satisfying $1/x + 1/y = 1/n$ where $n > 1$ and $x, y, n \in N$.
    (i) Find the value of $S(6)$.
    (ii) Show that if n is prime, then $S(n) = 3$ always.
20. Six apples and six mangoes are to be distributed among ten boys so that each boy receives at least one fruit. Find the number of ways in which the fruits can be distributed.

Objective Type

Solutions on page 5.47

Each question has four choices a, b, c and d, out of which *only one* is correct.

1. If $\alpha = {}^mC_2$, then ${}^{\alpha}C_2$ is equal to

   a. ${}^{m+1}C_4$ **b.** ${}^{m-1}C_4$
   c. $3\ {}^{m+2}C_4$ **d.** $3\ {}^{m+1}C_4$

2. If ${}^nC_3 + {}^nC_4 > {}^{n+1}C_3$, then

   a. $n > 6$ **b.** $n > 7$
   c. $n < 6$ **d.** none of these

3. The value of $\sum_{r=0}^{n-1} {}^nC_r / ({}^nC_r + {}^nC_{r+1})$ equals

   a. $n + 1$ **b.** $n/2$
   c. $n + 2$ **d.** none of these

4. In a city no two persons have identical set of teeth and there is no person without a tooth. Also no person has more than 32 teeth. If we disregard the shape and size of tooth and consider only the positioning of the teeth, the maximum population of the city is

   a. 2^{32} **b.** $(32)^2 - 1$
   c. $2^{32} - 1$ **d.** 2^{32-1}

5. In a room, there are 12 bulbs of the same wattage, each having a separate switch. The number of ways to light the room with different amount of illumination is

   a. $12^2 - 1$ **b.** 2^{12}
   c. $2^{12} - 1$ **d.** 12^2

6. The number of possible outcomes in a throw of n ordinary dice in which at least one of the dice shows an odd number is

   a. $6^n - 1$ **b.** $3^n - 1$
   c. $6^n - 3^n$ **d.** none of these

7. Let A be a set of n (≥ 3) distinct elements. The number of triplets (x, y, z) of the A elements in which at least two coordinates is equal to

a. ${}^nP_3$ b. $n^3 - {}^nP_3$
c. $3n^2 - 2n$ d. $3n^2(n - 1)$

8. The total number of flags with three horizontal strips in order, which can be formed using 2 identical red, 2 identical green and 2 identical white strips is equal to

a. 4! b. 3 × (4!)
c. 2 × (4!) d. none of these

9. The number of five-digit numbers that contain 7 exactly once is

a. (41) (9^3) b. (37) (9^3)
c. (7) (9^4) d. (41) (9^4)

10. A variable name in certain computer language must be either an alphabet or an alphabet followed by a decimal digit. The total number of different variable names that can exist in that language is equal to

a. 280 b. 290
c. 286 d. 296

11. The number less than 1000 that can be formed using the digits 0, 1, 2, 3, 4, 5 when repetition is not allowed is equal to

a. 130 b. 131
c. 156 d. 155

12. Total number of six-digit numbers that can be formed, having the property that every succeeding digit is greater than the preceding digit, is equal to

a. ${}^9C_3$ b. ${}^{10}C_3$
c. ${}^9P_3$ d. ${}^{10}P_3$

13. Numbers greater than 1000 but not greater than 4000, which can be formed with the digits 0, 1, 2, 3, 4 (repetition of digits is allowed), are

a. 350 b. 375
c. 450 d. 576

14. The total number of five-digit numbers of different digits in which the digit in the middle is the largest is

a. $\sum_{n=4}^{9} {}^nP_4$ b. 33(3!)
c. 30(3!) d. none of these

15. The number of four-digit numbers that can be made with the digits 1, 2, 3, 4 and 5 in which at least two digits are identical is

a. $4^5 - 5!$ b. 505
c. 600 d. none of these

16. The number of nine-non-zero digits such that all the digits in the first four places are less than the digit in the middle and all the digits in the last four places are greater than that in the middle is

a. 2(4!) b. 3(7!)/2
c. 2(7!) d. ${}^4P_4 \times {}^4P_4$

17. Total number of words that can be formed using all letters of the word 'BRIJESH' that neither begins with 'I' nor ends with 'B' is equal to

a. 3720 b. 4920
c. 3600 d. 4800

18. The total number of six-digit natural numbers that can be made with the digits 1, 2, 3, 4, if all digits are to appear in the same number at least once is

a. 1560 b. 840
c. 1080 d. 480

19. Total number of six-digit numbers in which all and only odd digits appear is

a. $\frac{5}{2}(6!)$ b. 6!
c. $\frac{1}{2}(6!)$ d. none of these

20. Total number less than 3×10^8 and can be formed using the digits 1, 2, 3 is equal to

a. $\frac{1}{2}(3^9 + 4 \times 3^8)$ b. $\frac{1}{2}(3^9 - 3)$
c. $\frac{1}{2}(7 \times 3^8 - 3)$ d. $\frac{1}{2}(3^9 - 3 + 3^8)$

21. If all the permutations of the letters in the word 'OBJECT' are arranged (and numbered serially) in alphabetical order as in a dictionary, then the 717th word is

a. TOJECB b. TOEJBC
c. TOCJEB d. TOJCBE

22. In a three-storey building, there are four rooms on the ground floor, two on the first and two on the second floor. If the rooms are to be allotted to six persons, one person occupying one room only, the number of ways in which this can be done so that no floor remains empty is

a. ${}^8P_6 - 2(6!)$ b. ${}^8P_6$
c. ${}^8P_5$ (6!) d. none of these

23. The total number not more than 20 digits that are formed by using the digits 0, 1, 2, 3 and 4 is

a. 5^{20} b. $5^{20} - 1$
c. $5^{20} + 1$ d. none of these

24. The number of ways in which we can select four numbers from 1 to 30 so as to exclude every selection of four consecutive numbers is

a. 27378 b. 27405
c. 27399 d. none of these

25. The number of words of four letters containing equal number of vowels and consonants, where repetition is allowed, is

a. 105^2 b. 210 × 243
c. 105 × 243 d. 150 × 21^2

26. The number of different seven digit numbers that can be written using only the three digits 1, 2 and 3 with the condition that the digit 2 occurs twice in each number is

a. ${}^7P_2\, 2^5$ b. ${}^7C_2\, 2^5$
c. ${}^7C_2\, 5^2$ d. none of these

27. The total number of three-letter words that can be formed from the letter of the word 'SAHARANPUR' is equal to

a. 210 b. 237
c. 247 d. 227

28. The number of ways to fill each of the four cells of the table with a distinct natural number such that the sum of the numbers is 10 and the sums of the numbers placed diagonally are equal is

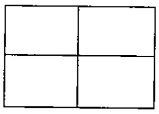


Fig. 5.31

a. 4 b. 8
c. 24 d. 6

29. The number of ways of arranging m positive and n $(< m+1)$ negative signs in a row so that no two negative signs are together is

a. ${}^{m+1}P_n$ b. ${}^{n+1}P_m$
c. ${}^{m+1}C_n$ d. ${}^{n+1}C_m$

30. Three boys of class X, four boys of class XI and five boys of class XII sit in a row. The total number of ways in which these boys can sit so that all the boys of same class sit together is equal to

a. $(3!)^2(4!)(5!)$ b. $(3!)(4!)^2(5!)$
c. $(3!)(4!)(5!)$ d. $(3!)(4!)(5!)^2$

31. A library has 'a' copies of one book, 'b' copies each of two books, 'c' copies each of three books, an single copy of 'd' books. The total number of ways in which these books can be arranged in a shelf is equal to

a. $\dfrac{(a+2b+3c+d)!}{a!(b!)^2(c!)^3}$ b. $\dfrac{(a+2b+3c+d)!}{a!(2b!)(c!)^3}$
c. $\dfrac{(a+b+3c+d)!}{(c!)^3}$ d. $\dfrac{(a+2b+3c+d)!}{a!(2b)!(3c)!}$

32. The sum of the digits in the unit's place of all numbers formed with the help of 3, 4, 5, 6 taken all at a time is

a. 18 b. 432
c. 108 d. 144

33. The sum of all the numbers of four different digits that can be made by using the digits 0, 1, 2 and 3 is

a. 26664 b. 39996
c. 38664 d. none of these

34. The sum of all four-digit numbers that can be formed by using the digits 2, 4, 6, 8 (when repetition of digits is not allowed) is

a. 133320 b. 533280
c. 53328 d. none of these

35. The number of ordered pairs of integers (x, y) satisfying the equation $x^2 + 6x + y^2 = 4$ is

a. 2 b. 8
c. 6 d. none of these

36. The number of five-digit telephone numbers having at least one of their digits repeated is

a. 90000 b. 100000
c. 30240 d. 69760

37. How many numbers can be made with the digits 3, 4, 5, 6, 7, 8 lying between 3000 and 4000, which are divisible by 5 while repetition of any digit is not allowed in any number?

a. 60 b. 12
c. 120 d. 24

38. The number of ways in which ten candidates $A_1, A_2, \ldots, A_{10}$ can be ranked such that A_1 is always above A_{10} is

a. 5! b. 2(5!)
c. 10! d. $\dfrac{1}{2}(10!)$

39. In the decimal system of numeration of six-digit numbers in which the sum of the digits is divisible by 5 is

a. 180000 b. 540000
c. 5×10^5 d. none of these

40. To fill 12 vacancies there are 25 candidates of which five are from scheduled caste. If three of the vacancies are reserved for scheduled caste candidates while the rest are open to all, the number of ways in which the selection can be made is

a. ${}^5C_3 \times {}^{22}C_9$ b. ${}^{22}C_9 - {}^5C_3$
c. ${}^{22}C_3 + {}^5C_3$ d. none of these

41. If the difference of the number of arrangements of three things from a certain number of dissimilar things and the number of selections of the same number of things from them exceeds 100, then the least number of dissimilar things is

a. 8 b. 6
c. 5 d. 7

42. Among 10 persons, A, B, C are to speak at a function. The number of ways in which it can be done if A wants to speak before B and B wants to speak before C is

a. 10!/24 b. 9!/6
c. 10!/6 d. none of these

43. In how many ways can a team of 11 players be formed out of 25 players, if six out of them are always to be included and five always to be excluded

a. 2020 b. 2002
c. 2008 d. 8002

44. In how many ways can a team of six horses be selected out of a stud of 16, so that there shall always be three out of $A\ B\ C\ A'\ B'\ C'$, but never $A\ A'$, $B\ B'$ or $C\ C'$ together

a. 840 b. 1260
c. 960 d. 720

45. There are two bags each containing m balls. If a man has to select equals number of balls from both the bags the number of ways in which he can do so if he must choose at least one ball from each bag is

a. m^2 b. ${}^{2m}C_m$
c. ${}^{2m}C_m - 1$ d. none of these

46. The number of ways in which the letters of the word 'PERSON' can be placed in the squares of the given figure so that no row remains empty is

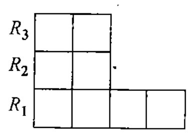


Fig. 5.32

a. $24 \times 6!$ b. $26 \times 6!$
c. $26 \times 7!$ d. $27 \times 6!$

47. The number of words of four letters that can be formed from the letters of the word 'EXAMINATION' is

a. 1464 b. 2454
c. 1678 d. none of these

48. The letters of word 'ZENITH' are written in all possible ways. If all these words are written in the order of a dictionary, then the rank of the word 'ZENITH' is

a. 716 b. 692
c. 698 d. 616

49. A class contains three girls and four boys. Every Saturday, five go on a picnic (a different group of students is sent every week). During the picnic, each girl in the group is given a doll by the

accompanying teacher. If all possible groups of five have gone for picnic once, the total number of dolls that the girls have got is

a. 21 b. 45
c. 27 d. 24

50. A teacher takes three children from her class to the zoo at a time, but she does not take the same three children to the zoo more than once. She finds that she went to the zoo 84 times more than a particular child has gone to the zoo. The number of children in her class is

a. 12 b. 10
c. 60 d. none of these

51. Number of ways in which a lawn-tennis mixed double be made from seven married couples if no husband and wife play in the same set is

a. 240 b. 420
c. 720 d. none of these

52. In a class tournament, all participants were to play different games with one another, Two players fell ill after having played three games each. If the total number of games played in the tournament is equal to 84, the total number of participants in the beginning was equal to

a. 10 b. 15
c. 12 d. 14

53. A person always prefers to eat 'parantha' and 'vegetable dish' in his meal. How many ways can he make his platter in a marriage party if there are three types of paranthas, four types of 'vegetable dish', three types of 'salads' and two types of 'sauces'?

a. 3360 b. 4096
c. 3000 d. none of these

54. The number of even divisors of the number $N = 12600 = 2^3 3^2 5^2 7$ is

a. 72 b. 54
c. 18 d. none of these

55. A candidate is required to answer six out of 10 questions, which are divided into two groups, each containing five questions. He is not permitted to attempt more than four questions from either group. The number of different ways in which the candidate can choose six questions is

a. 50 b. 150
c. 200 d. 250

56. There were two women participating in a chess tournament. Every participant played two games with the other participates. The number of games that the men played among themselves proved to exceed by 66 number of games that the men played with the women. The number of participants is

a. 6 b. 11
c. 13 d. none of these

57. Two teams are to play a series of five matches between them. A match ends in a win, loss or draw for a team. A number of people forecast the result of each match and no two people make the same forecast for the series of matches. The smallest group of people in which one person forecasts correctly for all the matches will contain n people, where n is

a. 81 b. 243
c. 486 d. none of these

58. Ten IIT and 2 DCE students sit in a row. The number of ways in which exactly 3 IIT students sit between 2 DCE students is

a. ${}^{10}C_3 \times 2! \times 3! \times 8!$ b. $10! \times 2! \times 3! \times 8!$
c. $5! \times 2! \times 9! \times 8!$ d. none of these

59. A team of four students is to be selected from a total of 12 students. The total number of ways in which the team can be selected such that two particular students refuse to be together and other two particular students wish to be together only is equal to

a. 220 b. 182
c. 226 d. none of these

60. In an election, the number of candidates is one greater than the persons to be elected. If a voter can vote in 254 ways, the number of candidates is

a. 7 b. 10
c. 8 d. 6

61. Two players P_1 and P_2 play a series of '$2n$' games. Each game can result in either a win or a loss for P_1. The total number of ways in which P_1 can win the series of these games is equal to

a. $\frac{1}{2}(2^{2n} - {}^{2n}C_n)$ b. $\frac{1}{2}(2^{2n} - 2\times {}^{2n}C_n)$
c. $\frac{1}{2}(2^{n} - {}^{2n}C_n)$ d. $\frac{1}{2}(2^{n} - 2\times {}^{2n}C_n)$

62. In an examination of nine papers, a candidate has to pass in more papers than the number of papers in which he fails in order to be successful. The number of ways in which he can be unsuccessful is

a. 255 b. 256
c. 193 d. 319

63. A student is allowed to select at most n books from a collection of $(2n + 1)$ books. If the total number of ways in which he can select at least one book is 63, then the value of n is

a. 2 b. 3
c. 4 d. 5

64. In a group of 13 cricket players, four are bowlers. Find out in how many ways can they form a cricket team of 11 players in which at least 2 bowlers are included.

a. 55 b. 72
c. 78 d. None of these

65. A person predicts the outcome of 20 cricket matches of his home team. Each match can result in a either win, loss or tie for the home team. Total number of ways in which he can make the predictions so that exactly 10 predictions are correct is equal to

a. ${}^{20}C_{10} \times 2^{10}$ b. ${}^{20}C_{10} \times 3^{20}$
c. ${}^{20}C_{10} \times 3^{10}$ d. ${}^{20}C_{10} \times 2^{20}$

66. The number of different ways in which five 'alike dashes' and eight 'alike dots' can be arranged using only seven of these 'dashes' and 'dots' is

a. 350 b. 120
c. 1287 d. none of these

67. Let there be $n \geq 3$ circles in a plane. The value of n for which the number of radical centres is equal to the number of radical axes is (assume that all radical axes and radical centre exist and are different)

a. 7 b. 6
c. 5 d. none of these

68. The number of ways of choosing a committee of two women and three men from five women and six men, if Mr. A refuses to serve on the committee if Mr. B is a member and Mr. B can only serve, if Miss C is the member of the committee is

a. 60 **b.** 84
c. 124 **d.** none of these

69. The last digit of $(1! + 2! + \cdots + 2005!)^{500}$ is

a. 9 **b.** 2
c. 7 **d.** 1

70. *ABCD* is a convex quadrilateral and 3, 4, 5 and 6 points are marked on the sides *AB*, *BC*, *CD* and *DA*, respectively. The number of triangles with vertices on different sides is

a. 270 **b.** 220
c. 282 **d.** 342

71. There are 10 points in a plane of which no three points are collinear and four points are concyclic. The number of different circles that can be drawn through at least three points of these points is

a. 116 **b.** 120
c. 117 **d.** none of these

72. n lines are drawn in a plane such that no two of them are parallel and no three of them are concurrent. The number of different points at which these lines will cut is

a. $\sum_{k=1}^{n-1} k$ **b.** $n(n-1)$
c. n^2 **d.** none of these

73. The number of triangles that can be formed with 10 points as vertices, n of them being collinear, is 110. Then n is

a. 3 **b.** 4
c. 5 **d.** 6

74. There are three coplanar parallel lines. If any p points are taken on each of the lines, the maximum number of triangles with vertices on these points is

a. $3p^2(p-1)+1$ **b.** $3p^2(p-1)$
c. $p^2(4p-3)$ **d.** none of these

75. The maximum number of points of intersection of five lines and four circles is

a. 60 **b.** 72
c. 62 **d.** none of these

76. If m parallel lines in a plane are intersected by a family of n parallel lines, the number of parallelograms that can be formed is

a. $\frac{1}{4}mn(m-1)(n-1)$ **b.** $\frac{1}{2}mn(m-1)(n-1)$
c. $\frac{1}{4}m^2n^2$ **d.** none of these

77. The number of integral solutions of $x + y + z = 0$ with $x \geq -5$, $y \geq -5$, $z \geq -5$ is

a. 134 **b.** 136
c. 138 **d.** 140

78. The number of ways in which we can get a score of 11 by throwing three dice is

a. 18 **b.** 27
c. 45 **d.** 56

79. In how many different ways can the first 12 natural numbers be divided into three different groups such that numbers in each group are in A.P.?

a. 1 **b.** 5
c. 6 **d.** 4

80. Fifteen identical balls have to be put in five different boxes. Each box can contain any number of balls. The total number of ways of putting the balls into the boxes so that each box contains at least two balls is equal to

a. ${}^9C_5$ **b.** ${}^{10}C_5$
c. ${}^6C_5$ **d.** ${}^{10}C_6$

81. If n objects are arranged in a row, then the number of ways of selecting three of these objects so that no two of them are next to each other is

a. ${}^{n-2}C_3$ **b.** ${}^{n-3}C_2$
c. ${}^{n-3}C_3$ **d.** none of these

82. The number of ways to give 16 different things to three persons *A*, *B*, *C* so that *B* gets one more than *A* and *C* gets two more than *B*, is

a. $\frac{16!}{4!5!7!}$ **b.** $4!\,5!\,7!$
c. $\frac{16!}{3!5!8!}$ **d.** none of these

83. The number of ways in which we can distribute mn students equally among m sections is given by

a. $\frac{(mn)!}{n!}$ **b.** $\frac{(mn)!}{(n!)^m}$
c. $\frac{(mn)!}{m!n!}$ **d.** $(mn)^m$

84. $2m$ white counters and $2n$ red counters are arranged in a straight line with $(m + n)$ counters on each side of a central mark. The number of ways of arranging the counters, so that the arrangements are symmetrical with respect to the central mark, is

a. ${}^{m+n}C_m$ **b.** ${}^{2m+2n}C_{2m}$
c. $\frac{1}{2}\frac{(m+n)!}{m!n!}$ **d.** none of these

85. A person buys eight packets of TIDE detergent. Each packet contains one coupon, which bears one of the letters of the word TIDE. If he shows all the letters of the word TIDE, he gets one free packet. If he gets exactly one free packet, then the number of different possible combinations of the coupons is

a. ${}^7C_3$ **b.** ${}^8C_4$
c. ${}^8C_3$ **d.** 4^4

86. There are three copies each of four different books. The number of ways in which they can be arranged in a shelf is

a. $\frac{12!}{(3!)^4}$ **b.** $\frac{12!}{(4!)^3}$
c. $\frac{21!}{(3!)^4 4!}$ **d.** $\frac{12!}{(4!)^3 3!}$

87. The number of ways in which 12 books can be put in three shelves with four on each shelf is

a. $\frac{12!}{(4!)^3}$ **b.** $\frac{12!}{(3!)(4!)^3}$
c. $\frac{12!}{(3!)^3 4!}$ **d.** none of these

88. The total number of ways in which $2n$ persons can be divided into n couples is

a. $\dfrac{2n!}{n!n!}$ **b.** $\dfrac{2n!}{(2!)^n}$

c. $\dfrac{2n!}{n!(2!)^n}$ **d.** none of these

89. Let $x_1, x_2, \ldots, x_k$ be the divisors of positive integer 'n' (including 1 and n). If $x_1 + x_2 + \cdots + x_k = 75$, then $\sum_{i=1} 1/x_i$ is equal to

a. $\dfrac{75}{n^2}$ **b.** $\dfrac{75}{n}$

c. $\dfrac{75}{k}$ **d.** none of these

90. Let $A = \{x_1, x_2, x_3, \ldots, x_7\}$, $B = \{y_1, y_2, y_3\}$. The total number of functions $f: A \to B$ that are on to and there are exactly three element x in A such that $f(x) = y_2$ is equal to

a. 490 **b.** 510

c. 630 **d.** none of these

91. The total number of ways in which n^2 number of identical balls can be put in n numbered boxes (1, 2, 3, ..., n) such that i^{th} box contains at least i number of balls is

a. ${}^{n^2}C_{n-1}$ **b.** ${}^{n^2-1}C_{n-1}$

c. ${}^{\frac{n^2+n-2}{2}}C_{n-1}$ **d.** none of these

92. The total number of ways in which 15 identical blankets can be distributed among four persons so that each of them gets at least two blankets is equal to

a. ${}^{10}C_3$ **b.** ${}^{9}C_3$

c. ${}^{11}C_3$ **d.** none of these

93. Number of ways in which 25 identical things be distributed among five persons if each gets odd number of things is

a. ${}^{25}C_4$ **b.** ${}^{12}C_8$

c. ${}^{14}C_{10}$ **d.** ${}^{13}C_3$

94. Number of ways in which Rs. 18 can be distributed amongst four persons such that nobody receives less than Rs. 3 is

a. 4^2 **b.** 2^4

c. 4! **d.** none of these

95. In how many ways can 17 persons depart from railway station in 2 cars and 3 autos, given that 2 particular persons depart by same car (4 persons can sit in a car and 3 persons can sit in an auto)?

a. $\dfrac{15!}{2!4!(3!)^3}$ **b.** $\dfrac{16!}{(2!)^2 4!(3!)^3}$

c. $\dfrac{17!}{2!4!(3!)^3}$ **d.** $\dfrac{15!}{4!(3!)^3}$

96. The total number of ways of selecting six coins out of 20 one-rupee coins, 10 fifty-paise coins and 7 twenty-five paise coins is

a. 28 **b.** 56

c. ${}^{37}C_6$ **d.** none of these

97. Let $f(n, k)$ denote the number of ways in which k identical balls can be coloured with n colours so that there is at least one ball of each colour. Then $f(2n, n)$ must be equal to

a. ${}^{2n}C_n$ **b.** ${}^{2n-1}C_{n+1}$

c. ${}^{2n-1}C_n$ **d.** none of these

98. The total number of ways in which three distinct numbers in $A.P.$ can be selected from the set $\{1, 2, 3, \ldots, 24\}$ is equal to

a. 66 **b.** 132

c. 198 **d.** none of these

99. The total number of ways of selecting two number from the set $\{1, 2, 3, 4, \ldots, 3n\}$ so that their sum is divisible by 3 is equal to

a. $\dfrac{2n^2-n}{2}$ **b.** $\dfrac{3n^2-n}{2}$

c. $2n^2 - n$ **d.** $3n^2 - n$

100. Among the 8! permutations of the digits 1, 2, 3, ..., 8, consider those arrangements which have the following property. If we take any five consecutive positions, the product of the digits in these positions is divisible by 5. The number of such arrangements is equal to

a. 7! **b.** 2. (7!)

c. ${}^{7}C_4$ **d.** none of these

101. The total number of divisors of 480, that are of the form $4n + 2$, $n \geq 0$, is equal to

a. 2 **b.** 3

c. 4 **d.** none of these

102. The total number of times, the digit '3' will be written, when the integers having less than 4 digits are listed is equal to

a. 300 **b.** 310

c. 302 **d.** 306

103. Straight lines are drawn by joining m points on a straight line to n points on another line. Then excluding the given points, the number of point of intersections of the lines drawn is (no two lines drawn are parallel and no three lines are concurrent)

a. $\dfrac{1}{4}mn(m-1)(n-1)$ **b.** $\dfrac{1}{2}mn(m-1)(n-1)$

c. $\dfrac{1}{2}m^2n^2$ **d.** $\dfrac{1}{4}m^2n^2$

104. In a polygon, no three diagonals are concurrent. If the total number of points of intersection of diagonals interior to the polygon is 70, then the number of diagonals of the polygon is

a. 20 **b.** 28

c. 8 **d.** none of these

105. Two packs of 52 cards are shuffled together. The number of ways in which a man can be dealt 26 cards so that he does not get two cards of the same suit and same denomination is

a. ${}^{52}C_{26} \cdot 2^{26}$ **b.** ${}^{104}C_{26}$

c. $2 \cdot {}^{52}C_{26}$ **d.** none of these

106. There are $(n + 1)$ white and $(n + 1)$ black balls each set numbered 1 to $n + 1$. The number of ways in which the balls can be arranged in a row so that the adjacent balls are of different colours is

a. $(2n + 2)!$ **b.** $(2n + 2)! \times 2$

c. $(n + 1)! \times 2$ **d.** $2\{(n + 1)!\}^2$

107. The number of three-digit numbers of the form xyz such that $x < y$ and $z \leq y$ is

a. 276 b. 285

c. 240 d. 244

108. A is a set containing 'n' different elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen. The number of ways of choosing P and Q so that $P \cap Q$ contains exactly two elements is

a. ${}^nC_3 \times 2^n$ b. ${}^nC_2 \times 3^{n-2}$

c. 3^{n-2} d. none of these

109. Messages are conveyed by arranging four white, one blue and three red flags on a pole. Flags of the same colour are alike. If a message is transmitted by the order in which the colours are arranged, the total number of messages that can be transmitted if exactly six flags are used is

a. 45 b. 65

c. 125 d. 185

110. 20 persons are sitting in a particular arrangement around a circular table. Three persons are to be selected for leaders. The number of ways of selection of three persons such that no two were sitting adjacent to each other is

a. 600 b. 900

c. 800 d. none of these

111. A seven-digit number without repetition and divisible by 9 is to be formed by using seven digits out of 1, 2, 3, 4, 5, 6, 7, 8. 9. The number of ways in which this can be done is

a. 9! b. 2(7!)

c. 4 (7!) d. none of these

112. n is selected from the set {1, 2, 3, ..., 10} and the number $2^n + 3^n + 5^n$ is formed. Total number of ways of selecting n so that the formed number is divisible by 4 is equal to

a. 50 b. 49

c. 48 d. none of these

113. The number of distinct natural numbers up to a maximum of four digits and divisible by 5, which can be formed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 each digit not occurring more than once in each number is

a. 1246 b. 952

c. 1106 d. none of these

114. A man has three friends. The number of ways he can invite one friend everyday for dinner on six successive nights so that no friend is invited more than three times is

a. 640 b. 320

c. 420 d. 510

115. There are four letters and four directed envelopes. The number of ways in which all the letters can be put in the wrong envelope is

a. 8 b. 9

c. 16 d. none of these

116. A bag contains four one-rupee coins, two twenty-five paisa coins and five ten-paisa coins. In how many ways can an amount, not less than Re 1 be taken out from the bag? (Consider coins of the same denominations to be identical.)

a. 71 b. 72

c. 73 d. 80

117. In a certain test, there are n questions. In the test 2^{n-i} students gave wrong answers to at least i questions, where $i = 1, 2, \ldots, n$. If the total number of wrong answers given is 2047, then n is equal to

a. 10 b. 11

c. 12 d. 13

118. Rajdhani Express going from Bombay to Delhi stops at five intermediate stations, 10 passengers enter the train during the journey with 10 different ticket of two classes. The number of different sets of tickets they may have is

a. ${}^{15}C_{10}$ b. ${}^{20}C_{10}$

c. ${}^{30}C_{10}$ d. none of these

119. A train timetable must be compiled for various days of the week so that two trains twice a day depart for three days, one train daily for two days and three trains once a day for two days. How many different timetables can be compiled?

a. 140 b. 210

c. 133 d. 72

120. The total number of positive integral solution of $15 < x_1 + x_2 + x_3 \le 20$ is equal to

a. 685 b. 785

c. 1125 d. none of these

Multiple Correct Answers Type

Solutions on page 5.55

Each question has 4 choices a, b, c and d, out of which *one or more* answers are correct.

1. Number of ways in which three numbers in A.P. can be selected from 1, 2, 3, ..., n is

a. $\left(\frac{n-1}{2}\right)^2$ if n is even b. $\frac{n(n-2)}{4}$ if n is even

c. $\frac{(n-1)^2}{4}$ if n is odd d. none of these

2. Kanchan has 10 friends among whom two are married to each other. She wishes to invite five of them for a party. If the married couples refuse to attend separately, then the number of different ways in which she can invite five friends is

a. ${}^8C_5$ b. $2 \times {}^8C_3$

c. ${}^{10}C_5 - 2 \times {}^8C_4$ d. none of these

3. A forecast is to be made of the results of five cricket matches, each of which can be a win or a draw or a loss for Indian team. Let,

p = number of forecasts with exactly 1 error

q = number of forecasts with exactly 3 errors and

r = number of forecasts with all five errors

Then the correct statement(s) is/are

a. $2q = 5r$ b. $8p = q$

c. $8p = 5r$ d. $2(p + r) > q$

4. Ten persons numbered 1, 2, ..., 10 play a chess tournament, each player playing against every other player exactly one game. It is known that no game ends in a draw. If $w_1, w_2, \ldots, w_{10}$ are the number of games won by players 1, 2, 3, ..., 10, respectively, and

$l_1, l_2, \ldots, l_{10}$ are the number of games lost by the players 1, 2, ..., 10, respectively, then,

a. $\Sigma w_i = \Sigma l_i = 45$ b. $w_i + l_i = 9$

c. $\Sigma w l_i^2 = 81 + \Sigma l_i^2$ d. $\Sigma w_i^2 = \Sigma l_i^2$

5. The number of ways of choosing triplet (x, y, z) such that $z \geq \max\{x, y\}$ and $x, y, z \in \{1, 2, \ldots, n, n+1\}$ is

a. ${}^{n+1}C_3 + {}^{n+2}C_3$ b. $n(n+1)(2n+1)/6$

c. $1^2 + 2^2 + \cdots + n^2$ d. $2({}^{n+2}C_3) - {}^{n+1}C_2$

6. Number of ways in which 200 people can be divided in 100 couples is

a. $\dfrac{(200)!}{2^{100}(100)!}$ b. $1 \times 3 \times 5 \cdots 199$

c. $\left(\dfrac{101}{2}\right)\left(\dfrac{102}{2}\right)\cdots\left(\dfrac{200}{2}\right)$ d. $\dfrac{(200)!}{(100)!}$

7. If a seven-digit number made up of all distinct digits 8, 7, 6, 4, 3, x and y is divisible by 3, then

a. maximum value of $x - y$ is 9

b. maximum value of $x + y$ is 12

c. minimum value of xy is 0

d. minimum value of $x + y$ is 3

8. If n is number of necklaces which can be formed using 17 identical pearls and two identical diamonds and similarly m is number of necklaces which can be formed using 17 identical pearls and different diamonds, then

a. $n = 9$

b. $m = 18$

c. $n = 18$

d. $m = 9$

9. Let $f(n)$ be the number of regions in which n coplanar circles can divide the plane. If it is known that each pair of circles intersect in two different point and no three of them have common point of intersection, then

a. $f(20) = 382$

b. $f(n)$ is always an even number

c. $f^{-1}(92) = 10$

d. $f(n)$ can be odd

10. Given that the divisors of $n = 3^p \cdot 5^q \cdot 7^r$ are of the form $4\lambda + 1$, $\lambda \geq 0$. Then

a. $p + r$ is always even

b. $p + q + r$ is always odd

c. q can be any integer

d. if p is odd then r is even

11. Number of ways of selecting three integers from $\{1, 2, 3, \ldots, n\}$ if their sum is divisible by 3 is

a. $3({}^{n/3}C_3) + (n/3)^3$ if $n = 3k, k \in N$

b. $2({}^{(n-1)/3}C_3) + ({}^{(n+2)/3}C_3) + ((n-1)/3)^2(n+2)$, if $n = 3k+1$, $k \in N$

c. $2({}^{(n-1)/3}C_3) + ({}^{(n+2)/3}C_3) + ((n-1)/3)^2(n+2)$, if $n = 3k+2$, $k \in N$

d. independent of n

12. Number of points of intersection of n straight lines if n satisfies

$${}^{n+5}P_{n+1} = \frac{11(n-1)}{2} \times {}^{n+3}P_n \text{ is}$$

a. 15 b. 28

c. 21 d. 10

13. Number of shortest ways in which we can reach from the point (0, 0, 0) to point (3, 7, 11) in space where the movement is possible only along the x-axis, y-axis and z-axis or parallel to them and change of axes is permitted only at integral points (an integral point is one which has its coordinate as integer) is

a. equivalent to number of ways of dividing 21 different objects in three groups of size 3, 7, 11

b. equivalent to coefficient of y^3z^7 in the expansion of $(1 + y + z)^{21}$

c. equivalent to number of ways of distributing 21 different objects in three boxes of size 3, 7, 11

d. equivalent to number of ways of arranging 21 objects of which 3 are alike of one kind, 7 are alike of second type and 11 are alike of third type

14. Number of ways in which 30 identical things are distributed among six persons is

a. ${}^{17}C_5$ if each gets odd number of things

b. ${}^{16}C_{11}$ if each gets odd number of things.

c. ${}^{14}C_5$ if each gets even number of things (excluding 0)

d. ${}^{15}C_{10}$ if each gets even number of things (excluding 0)

15. If N denotes the number of ways of selecting r objects out of n distinct objects $(r \geq n)$ with unlimited repetition but with each object included at least once in selection, then N is equal to

a. ${}^{r-1}C_{r-n}$ b. ${}^{r-1}C_n$

c. ${}^{r-1}C_{n-1}$ d. none of these

16. A is a set containing n elements. A subset P_1 of A is chosen. The set A is reconstructed by replacing the elements of P_1. Next, a subset P_2 of A is chosen and again the set is reconstructed by replacing the elements of P_2. In this way, m (>1) subsets $P_1, P_2, \ldots, P_m$ of A are chosen. The number of ways of choosing $P_1, P_2, \ldots, P_m$ is

a. $(2^m - 1)^n$ if $P_1 \cap P_2 \cap \cdots \cap P_m = \phi$

b. 2^{mn} if $P_1 \cup P_2 \cup \cdots \cup P_m = A$

c. 2^{mn} if $P_1 \cap P_2 \cap \cdots \cap P_m = \phi$

d. $(2^m - 1)^n$ if $P_1 \cup P_2 \cup \cdots \cup P_m = A$

17. If $10! = 2^p \cdot 3^q \cdot 5^r \cdot 7^s$, then

a. $2q = p$

b. $pqrs = 64$

c. number of divisors of 10! is 280

d. number of ways of putting 10! as a product of two natural numbers is 135

18. If $P = 21(21^2 - 1^2)(21^2 - 2^2)(21^2 - 3^2) \cdots (21^2 - 10^2)$, then P is divisible by

a. 22! b. 21!

c. 19! d. 20!

19. Let n is of four-digit integer in which all the digits are different. If x is number of odd integers and y is number of even integers, then

a. $x < y$
b. $x > y$
c. $x + y = 4500$
d. $|x - y| = 54$

Reasoning Type

Solutions on page 5.57

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.
b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.
c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.
d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** The number of positive integral solutions of $abc = 30$ is 27.

   Statement 2: Number of ways in which three prizes can be distributed among three persons is 3^3.

2. **Statement 1:** Number of ways in which 10 identical toys can be distributed among three students if each receives at least two toys is ${}^9C_2$.

   Statement 2: Number of positive integral solutions of $x + y + z + w = 7$ is ${}^6C_3$

3. **Statement 1:** $(n^2)!/(n!)^n$ is a natural number for all $n \in N$.

   Statement 2: Number of ways in which n^2 objects can be distributed among n persons equally is $(n^2)!/(n!)^n$.

4. **Statement 1:** The number of ways of writing 1400 as a product of two positive integers is 12.

   Statement 2: 1400 is divisible by exactly three prime factors.

5. **Statement 1:** Let $E = \{1, 2, 3, 4\}$ and $F = \{a, b\}$. Then the number of onto functions from E to F is 14.

   Statement 2: Number of ways in which four distinct objects can be distributed into two different boxes is 14 if no box remains empty.

6. **Statement 1:** Number of ways in which India can win the series of 11 matches is 2^{10}. (if no match is drawn).

   Statement 2: For each match there are two possibilities, either India wins or loses.

7. **Statement 1:** Number of ways in which Indian team (11 players) can bat, if Yuvraj wants to bat before Dhoni and Pathan wants to bat after Dhoni is $11!/3!$.

   Statement 2: Yuvraj, Dhoni and Pathan can be arranged in batting order in $3!$ ways.

8. **Statement 1:** When number of ways of arranging 21 objects of which r objects are identical of one type and remaining are identical of second type is maximum, then maximum value of ${}^{13}C_r$ is 78.

   Statement 2: ${}^{2n+1}C_r$ is maximum when $r = n$.

9. **Statement 1:** Total number of five-digit numbers having all different digits and divisible by 4 can be formed using the digits $\{1, 3, 2, 6, 8, 9\}$ is 192.

   Statement 2: A number is divisible by 4, if the last two digits of the number are divisible by 4.

10. **Statement 1:** Number of ways in which 30 can be partitioned into three unequal parts, each part being a natural number is 61.

    Statement 2: Number of ways of distributing 30 identical objects in three different boxes is ${}^{30}C_2$.

11. **Statement 1:** If $p, q < r$, the number of different selections of $p + q$ things taking r at a time, where p things are identical and q other things are identical, is $p + q - r + 1$.

    Statement 2: If $p, q > r$, the number of different selections of $p + q$ things taking r at a time, where p things are identical and q other things are identical, is $r - 1$.

12. **Statement 1:** The number of ways in which three distinct numbers can be selected from the set $\{3^1, 3^2, 3^3, ..., 3^{100}, 3^{101}\}$ so that they form a G.P. is 2500.

    Statement 2: If a, b, c are in A.P., then $3^a, 3^b, 3^c$ are in G.P.

13. **Statement 1:** Number of ways in which two persons A and B select objects from two different groups each having 20 different objects such that B selects always more objects than A (including the case when A selects no object) is $(2^{40} - {}^{40}C_{20})/2$.

    Statement 2: The sum $\sum_{0 \le i < j \le n} {}^nC_i {}^nC_j = (2^{2n} - {}^{2n}C_n)/2$.

14. **Statement 1:** Number of ways of selecting 10 objects from 42 objects of which, 21 objects are identical and remaining objects are distinct is 2^{20}.

    Statement 2: ${}^{42}C_0 + {}^{42}C_1 + {}^{42}C_2 + \cdots + {}^{42}C_{21} = 2^{41}$.

15. **Statement 1:** Number of terms in the expansion of $(x + y + z + w)^{50}$ is ${}^{53}C_3$.

    Statement 2: Number of non-negative solution of the equation $p + q + r + s = 50$ is ${}^{53}C_3$.

16. **Statement 1:** The number of ways in which n persons can be seated at a round table, so that all shall not have the same neighbours in any two arrangements is $(n - 1)!/2$.

    Statement 2: Number of ways of arranging n different beads in circles in which is $(n - 1)!/2$.

17. **Statement 1:** Number of zeros at the end of 50! is equal to 12.

    Statement 2: Exponent of 2 in 50! is 47.

Linked Comprehension Type

Solutions on page 5.59

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *is only one* is correct.

For Problems 1–3

We have to choose 11 players for cricket team from eight batsmen, six bowlers, four all rounders and two wicket keepers in the following conditions.

1. The number of selections when almost one all rounder and one wicket keeper will play

   a. ${}^4C_1 \times {}^{14}C_{10} + {}^2C_1 \times {}^{14}C_{10} + {}^4C_1 \times {}^2C_1 \times {}^{14}C_9 + {}^{14}C_{11}$

   b. ${}^4C_1 \times {}^{15}C_{11} + {}^{15}C_{11}$

c. ${}^{4}C_1 \times {}^{15}C_{10} + {}^{15}C_{11}$

d. none of these

2. Number of selections when two particular batsmen do not want to play when a particular bowler will play

a. ${}^{17}C_{10} + {}^{19}C_{11}$ b. ${}^{17}C_{10} + {}^{19}C_{11} + {}^{17}C_{11}$

c. ${}^{17}C_{10} + {}^{20}C_{11}$ d. ${}^{19}C_{10} + {}^{19}C_{11}$

3. Number of selections when a particular batsman and a particular wicket keeper do not want to play together

a. $2\,{}^{18}C_{10}$ b. ${}^{19}C_{11} + {}^{18}C_{10}$

c. ${}^{19}C_{10} + {}^{19}C_{11}$ d. none of these

For Problems 4–6

Twelve12 persons are to be arranged around two round tables such that one table can accommodate seven persons and another five persons only. Answer the following questions.

4. Number of ways in which these 12 persons can be arranged is

a. ${}^{12}C_5 6!4!$ b. $6!4!$

c. ${}^{12}C_5 6!4!$ d. none of these

5. Number of ways of arrangements if two particular persons A and B do not want to be on the same table is

a. ${}^{10}C_4\, 6!\, 4!$ b. $2\,{}^{10}C_6\, 6!\, 4!$

c. ${}^{11}C_6\, 6!\, 4!$ d. none of these

6. Number of ways of arrangement if two particular persons A and B want to be together and consecutive is

a. ${}^{10}C_7\, 6!\, 3!2! + {}^{10}C_5\, 4!5!2!$ b. ${}^{10}C_5\, 6!\, 3! + {}^{10}C_7\, 4!5!$

c. ${}^{10}C_7\, 6!\, 2! + {}^{10}C_5\, 5!2!$ d. none of these

For Problems 7–9

Five balls are to be placed in three boxes. Each box should hold all the five balls so that no box remains empty.

7. Number of ways if balls are different but boxes are identical is

a. 30 b. 25

c. 21 d. 35

8. Number of ways if balls and boxes are identical is

a. 3 b. 1

c. 2 d. none of these

9. Number of ways if balls as well as boxes are identical but boxes are kept in a row is

a. 10 b. 15

c. 20 d. 6

For Problems 10 – 12

Let $f(n)$ denote the number of different ways in which the positive integer n can be expressed as the sum of 1s and 2s. For example, $f(4) = 5$, since $4 = 2 + 2 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 = 1 + 1 + 1 + 1$. Note that order of 1s and 2s is important.

10. The value of $f(6)$ is

a. 12 b. 13

c. 18 d. 21

11. The value of $f(f(6))$ is

a. 400 b. 350

c. 377 d. none of these

12. $f : N \to N$ is

a. one-one and onto

b. one-one and into

c. many-one and onto

d. many-one and into

For Problems 13–15

There are m seats in the first row of a theatre, of which n are to be occupied.

13. The number of ways of arranging n persons if no two persons sit side by side is

a. $\dfrac{(m-n+1)!}{(m-3n+1)!}$ b. $\dfrac{(m-n+1)!}{(m-2n)!}$

c. $\dfrac{(m-n+1)!}{(m-2n+1)!}$ d. $\dfrac{(m-n+2)!}{(m-2n-1)!}$

14. If n is even, the number of ways of arranging n persons if each person has exactly one neighbour is

a. $({}^{n}P_{n/2})\,({}^{m-n+1}P_{n/2})$ b. $({}^{n}P_{n})\,({}^{m-n+1}P_{n/2})$

c. $({}^{n}P_{n/2})\,({}^{m-n+1}P_{n})$ d. none of these

15. The number of ways of arranging n persons, if out of any two seats located symmetrically in the middle of the row at least one is empty is

a. $({}^{m/2}C_n)(2^n) - 1$ b. ${}^{m/2}P_n$

c. $({}^{m/2}P_n)(2^n - 1)$ d. $({}^{m/2}P_n)(2^n)$

For Problems 16–18

Consider the letters of the word 'MATHEMATICS'.

16. Possible number of words taking all letters at a time such that at least one repeating letter is at odd position in each word is

a. $\dfrac{11!}{2!2!2!} - \dfrac{9!}{2!2!}$ b. $\dfrac{9!}{2!2!2!}$

c. $\dfrac{9!}{2!\,2!}$ d. $\dfrac{11!}{2!\,2!\,2!}$

17. Possible number of words taking all letters at a time such that in each word both M's are together and both T's are together but both A's are not together is

a. $\dfrac{11!}{2!\,2!\,2!} - \dfrac{10!}{2!\,2!}$ b. $7!\,{}^{8}C_2$

c. $\dfrac{6!\,4!}{2!\,2!}$ d. $\dfrac{9!}{2!\,2!\,2!}$

18. Possible number of words in which no two vowels are together is

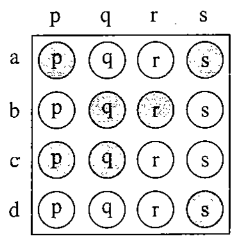
a. $7!\ {}^{8}C_4 \dfrac{4!}{2!}$ b. $\dfrac{7!}{2!}\,{}^{8}C_4 \dfrac{4!}{2!}$

c. $\dfrac{7!}{2!\,2!}\,{}^{8}C_4 \dfrac{4!}{2!}$ d. $\dfrac{7!}{2!\,2!\,2!}\,{}^{8}C_4 \dfrac{4!}{2!}$

Matrix–Match Type

Solutions on page 5.60

Each question contains statements given in two columns, which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct match are a→p, a→s, , b→q, b→r, c→p, c→q and d→s, then the correctly bubbled 4 × 4 matrix should be as follows:



1

Column I	Column II
a. Number of straight lines joining any two of 10 points of which four points are collinear	**p.** 30
b. Maximum number of points of intersection of 10 straight lines in the plane	**q.** 60
c. Maximum number of points of intersection of six circles in the plane	**r.** 40
d. Maximum number of points of intersection of six parabolas	**s.** 45

2. Consider a 6 × 6 chessboard. Then match the following columns.

Column I	Column II
a. Number of rectangles	**p.** $^{10}C_5$
b. Number of squares	**q.** 441
c. Number of ways three squares can be selected if they are not in same row or column	**r.** 91
d. In how many ways eleven '+' sign can be arranged in the squares if no row remains empty	**s.** 2400

3. Consider the convex polygon, which has 35 diagonals. Then match the following column.

Column I	Column II
a. Number of triangles joining the vertices of the polygon	**p.** 210
b. Number of points of intersections of diagonal which lies inside the polygon	**q.** 120
c. Number of triangles in which exactly one side is common with that of polygon	**r.** 10
d. Number of triangles in which exactly two sides are common with that of polygon	**s.** 60

4. A function is defined as $f: \{a_1, a_2, a_3, a_4, a_5, a_6\} \to \{b_1, b_2, b_3\}$.

Column I	Column II
a. Number of subjective functions	**p.** is divisible by 9
b. Number of functions in which $f(ai) \neq bi$	**q.** is divisible by 5
c. Number of invertible functions	**r.** is divisible by 4
d. Number of many one functions	**s.** is divisible by 3
	t. not possible

5.

Column I	Column II
a. Total number of function $f: \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ that are on to and $f(i) \neq i$ is equal to	**p.** divisible by 11
b. If $x_1x_2x_3 = 2 \times 5 \times 7^2$, then the number of solution set for (x_1, x_2, x_3) where $x_i \in N$, $x_i > 1$ is	**q.** divisible by 7
c. Number of factors of 3780 are divisible by either 3 or 2 or both is	**r.** divisible by 3
d. Total number of divisors of $n = 2^5 \times 3^4 \times 5^{10}$ that are of the form $4\lambda + 2$, $\lambda \geq 1$ is	**s.** divisible by 4

6.

Column I	Column II
a. Four dice (six faced) are rolled. The number of possible outcomes in which at least one dice shows 2 is	**p.** 210
b. Let A be the set of 4-digit number $a_1a_2a_3a_4$ where $a_1 > a_2 > a_3 > a_4$. Then $n(A)$ is equal to	**q.** 480
c. The total number of three-digit numbers, the sum of whose digits is even, is equal to	**r.** 671
d. The number of four-digit numbers that can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7 so that each number contains digit 1 is	**s.** 450

7.

Column I	Column II
a. The number of five-digit numbers having the product of digits 20 is	(p) > 70
b. A closest has five pairs of shoes. The number of ways in which four shoes can be drawn from it such that there will be no complete pair is	(q) < 60
c. Three ladies have each brought their one child for admission to a school. The principal wants to interview the six persons one by one subject to the condition that no mother is interviewed before her child. The number of ways in which interviews can be arranged is	(r) ∈ (50, 110)
d. The figures 4, 5, 6, 7, 8 are written in every possible order. The number of numbers greater than 56000 is	(s) ∈ (40, 70)

8.

Column I	Column II
a. If a denotes the number of permutations of $x + 2$ things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of $x - 11$ things taken all at a time such that $a = 182\, bc$, then the value of x is product of	**p.** 6
b. The number of six-digit numbers that can be made with the digits 0, 1, 2, 3, 4 and 5 so that even digits occupy odd places is product of	**q.** 5
c. The number of five-digit numbers that can be made using the digits 1 and 2 and in which at least one digit is different is product of	**r.** 4
d. In a polygon the number of diagonals is 54. The number of sides of the polygon is product of	**s.** 3

Integer Type

Solutions on page 5.62

1. If ${}^nP_r = {}^nP_{r+1}$ and ${}^nC_r = {}^nC_{r-1}$ then the value of $n + r$ is.

2. A person has 'n' friends. The minimum value of 'n' so that a person can invite a different pair of friends every day for four weeks in a row is.

3. There are n distinct white and n distinct black balls. If the number of ways of arranging them in a row so that neighboring balls are of different colors is 1152 then value of 'n' is.

4. Numbers from 1 to 1000 are divisible by 60 but not by 24 is.

5. Number of ways in which the letters of the word 'ABBCABBC can be arranged such that the word ABBC does not appear is any word, is N then the value of $(N^{1/2} - 10)$ is.

6. A class has three teachers, Mr. P, Ms. Q and Mrs. R and six students A,B,C,D,E,F. Number of ways in which they can be seated in a line of 9 chairs, if between any two teachers there are exactly two students, is $k!(18)$, then the value of k is.

7. Consider the five points comprising of the vertices of a square and the intersection point of its diagonals. How many triangles can be formed using these points?

8. If number of selections of 6 different letters that can be made from the words 'SUMAN' and 'DIVYA' so that each selection contains 3 letters from each word, is N^2 then the value of N is.

9. There are 20 books on Albegra and Calculus in one library. For the greatest number of selections each of which consists of 5 books on each topic possible number of Algebra books are N then the value of $N/2$ is.

10. Number of permutations of 1, 2, 3, 4, 5, 6, 7, 8 and 9 taken all at a time are such that the digit
    1 appearing somewhere to the left of 2
    3 appearing to the left of 4 and
    5 somewhere to the left of 6, is $k \times 7!$ Then the value of k is.

11. The number of n digit numbers which consists of the digits 1 and 2 only if each digit is to be used at least once, is equal to 510 then n is equal to.

12. Number of ways in which 7 people can occupy six seats, 3 seats on each side in a first class railway compartment if two specified persons are to be always included and occupy adjacent seats on the same side, is $(5!) \cdot k$ then k has the value equal to.

13. There are 2 women participating in a chess tournament. Every participant played 2 games with the other participants. The number of games that the men played between themselves exceeded by 66 as compared to the number of games that the men played with the women. If the number of participants is n then the value of $n - 6$ is.

14. The number of three digit numbers having only two consecutive digits identical is N, then the value of $(N/2)^{1/2}$ is.

15. Number of 4 digit numbers of the form $N = abcd$ which satisfy following three conditions
    (i) $4000 \le N < 6000$
    (ii) N is a multiple of 5
    (iii) $3 \le b < c \le 6$
    is equal to N then the value of $N/3$ is.

16. Number of ways in which 5 A's and 6 B's can be arranged in a row which reads the same backwards and forwards, is N then the value of $N/2$ is.

17. If N is the number of different paths of length-12 which leads from A to B in the grid which do not pass through M, then the value of $[N/10]$, where [.] represents the greatest integer function, is.

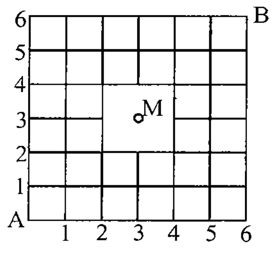


Fig. 5.33

18. There are 720 permutations of the digits 1, 2, 3, 4, 5, 6. Suppose these permutations are arranged from smallest to largest numerical values, beginning from 1 2 3 4 5 6 and ending with 6 5 4 3 2 1. Then the digit in unit place of number at 267th position is.

19. If N is the number of ways in which a person can walk up a stairway which has 7 steps if he can take 1 or 2 steps up the stairs at a time, then the value of $N/3$ is.

20. There are 3 men and 7 women taking a dance class. If N is number of different ways in which each man be paired with a woman partner, and the four remaining women be paired into two pairs each of two, then the value of $N/90$ is.

21. Let P_n denotes the number of ways in which three people can be selected out of 'n' people sitting in a row, if no two of them are consecutive. If, $P_{n+1} - P_n = 15$ then the value of 'n' is.

22. A man has 3 friends. If N is number of ways he can invite one friend everyday for dinner on 6 successive nights so that no friend is invited more than 3 times then the value of $N/170$ is.

23. If N is the number of ways in which 3 distinct numbers can be selected from the set $\{3^1, 3^2, 3^3, ...3^{10}\}$ so that they form a G.P. then the value of $N/5$ is.

24. Let $f(n) = \sum_{r=0}^{n} \sum_{k=r}^{n} \binom{k}{r}$. Find the total number of divisors of $f(9)$.

Archives

Solutions on page 5.64

Subjective Type

1. (i) In how many ways can a pack of 52 cards be divided equally among four players?

(ii) In how many ways can you divide these cards in four sets, three of them having 17 cards each and the fourth one just one card? **(IIT-JEE, 1979)**

2. Six Xs have to be placed in the squares of figure below in such a way that each row contains at least one X. In how many different ways can this be done?

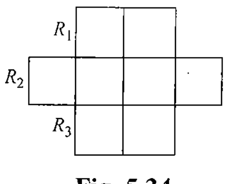


Fig. 5.34

(IIT-JEE, 1978)

3. Five balls of different colours are to be placed in the boxes of different size. Each box can hold all five. In how many different ways can we place the balls so that no box remains empty? **(IIT-JEE, 1981)**

4. m men and n women are to be seated in a row so that no two women sit together. If $m > n$, then show that the number of ways in which they can be seated is $m!(m+1)!/(m-n+1)!$. **(IIT-JEE, 1983)**

5. Seven relatives of a man comprises four ladies and three gentlemen; his wife has also seven relatives—three of them are ladies and four gentlemen. In how many ways can they invite 3 ladies and 3 gentlemen at a dinner party so that there are three man's relatives and three wife's relatives? **(IIT-JEE, 1985)**

6. A box contains two white balls, three black balls and four red balls. In how many ways can three balls be drawn from the box if at least one black ball is to be included in the draw? **(IIT-JEE, 1986)**

7. A student is allowed to select almost n books from a collection of $2n + 1$ books. If the total number of ways in which he can select at least one book is 63, find the value of n. **(IIT-JEE, 1987)**

8. A number of 18 guests have to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three others on the other side. Determine the number of ways in which the sitting arrangements can be made. **(IIT-JEE, 1991)**

9. A committee of 12 is to be formed from nine women and eight men. In how many ways can this be done if at least five women have to be included in a committee? In how many of these committees

a. the women hold majority?

b. the men hold majority? **(IIT-JEE, 1994)**

10. Prove by permutation or otherwise that $(n^2)!/(n!)^n$ is an integer $(n \in I^+)$. **(IIT-JEE, 2004)**

Objective Type

Fill in the blanks

1. In a certain test, a_i students gave wrong answers to at least i questions, where $i = 1, 2, ..., k$. No student gave more than k wrong answers. The total number of wrong answers given is——. **(IIT-JEE, 1982)**

2. The sides AB, BC and CA of a triangle ABC have 3, 4 and 5 interior points, respectively on them. The number of triangles that can be constructed using these interior points as vertices is ——. **(IIT-JEE, 1984)**

3. The total number of ways in which six '+' and four '–' signs can be arranged in a line such that no two '–' signs occur together is ——. **(IIT-JEE, 1988)**

4. There are four balls of different colours and four boxes of colours same as those of the balls. The number of ways in which the balls, one each in a box, could be placed such that a ball does not go to a box of its own colour is ——. **(IIT-JEE, 1992)**

True or false

1. The product of any r consecutive natural numbers is always divisible by $r!$. **(IIT-JEE, 1985)**

Multiple choice questions with one correct answer

1. If $^nC_{r-1} = 36$, $^nC_r = 84$ and $^nC_{r+1} = 126$, then r is

   a. 1 **b.** 2 **c.** 3 **d.** none of these

   (IIT-JEE, 1979)

2. Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. The number of words, which have at least one letter repeated is

   a. 59720 **b.** 79260
   c. 69760 **d.** none of these

3. The value of the expression $^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$ is equal to

   a. $^{47}C_5$ **b.** $^{52}C_5$
   c. $^{52}C_4$ **d.** none of these

4. Eight chairs are numbered 1 to 8. Two women and three men wish to occupy one chair each. First, the women choose the chairs from amongst the chairs marked 1 to 4, and then the men select the chairs from amongst the remaining. The number of possible arrangements is

   a. $^6C_3 \times {}^4C_2$ **b.** $^4P_2 \times {}^4P_3$
   c. $^4C_2 + {}^4P_3$ **d.** none of these

   (IIT-JEE, 1982)

5. In a group of boys, two boys are brothers and six more boys are present in the group. In how many ways can they sit if the brothers are not to sit along with each other?

   a. $2 \times 6!$ **b.** $^7P_2 \times 6!$
   c. $^7C_2 \times 6!$ **d.** none of these

   (IIT-JEE, 1982)

6. A five-digit number divisible by 3 is to be formed using the digits 0, 1, 2, 3, 4 and 5, without repetition. The total number of ways this can be done is

   a. 216 **b.** 240
   c. 600 **d.** 3125 **(IIT-JEE, 1989)**

7. An n-digit number is a positive number with exactly n digits. Nine hundred distinct n-digit numbers are to be formed using only the three digits 2, 5 and 7. The smallest value of n for which this is possible is

   a. 6 **b.** 7
   c. 8 **d.** 9 **(IIT-JEE, 1998)**

8. How many different nine-digit numbers can be formed from the number 223355888 by rearranging its digits so that the odd digits occupy even positions?

   a. 16 **b.** 36
   c. 60 **d.** 180 **(IIT-JEE, 2000)**

9. Let T_n denote the number of triangles, which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then n equals

   a. 5 **b.** 7
   c. 6 **d.** 4 **(IIT-JEE, 2001)**

10. The number of arrangements of the letters of the word 'BANANA' in which the two Ns do not appear adjacently is

    a. 40 **b.** 60
    c. 80 **d.** 100 **(IIT-JEE, 2002)**

11. A rectangle with sides $2m - 1$ and $2n - 1$ is divided into squares of unit length by drawing parallel lines as shown in the diagram, then the number of rectangles possible with odd side lengths is

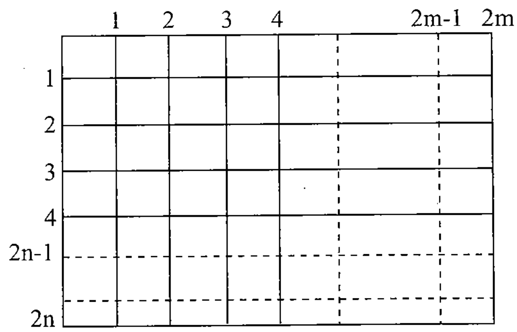


Fig. 5.35

    a. $(m + n - 1)^2$ **b.** 4^{m+n-1}
    c. m^2n^2 **d.** $m(m + 1)n(n + 1)$

    (IIT-JEE, 2000)

12. If r, s, t are prime numbers and p, q are the positive integers such that the LCM of p, q is $r^2t^4s^2$, then the number of ordered pair (p, q) is

    a. 252 **b.** 254
    c. 225 **d.** 224 **(IIT-JEE, 2006)**

13. The letters of the word 'COCHIN' are permuted and all the permutations are arranged in an alphabetical order as in an English dictionary. The number of words that appear before the word 'COCHIN' is

    a. 360 **b.** 192
    c. 96 **d.** 48

    (IIT-JEE, 2007)

14. Let $S = \{1, 2, 3, 4\}$. The total number of unordered pairs of disjoint subsets of S is equal to

    a. 25 **b.** 34
    c. 42 **d.** 41

    (IIT-JEE, 2010)

ANSWERS AND SOLUTIONS

Subjective Type

1. Consider $n + 1$ different toys Then,

$$\begin{pmatrix}\text{no. of ways of}\\ \text{selecting } r \text{ toys out of}\\ (n+1) \text{ different toys}\end{pmatrix}$$

$$= \begin{pmatrix}\text{no. of ways of selecting}\\ (r-1) \text{ toys out of } n \text{ toys}\\ \text{when } T_0 \text{ is always included}\end{pmatrix}$$

$$+ \begin{pmatrix}\text{no. of ways of}\\ \text{selecting } r \text{ toys when } T_0\\ \text{is always excluded}\end{pmatrix}$$

$$= {}^nC_{r-1} + {}^nC_r$$

Hence, ${}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$.

2. Now, $(n!)!$ is the product of the positive integers from 1 to $n!$ We write the integers from 1 to $n!$ in $(n-1)!$ rows as follows:

$$1\times2\times3\cdots n$$
$$(n+1)(n+2)(n+3)\cdots(2n)$$
$$(2n+1)(2n+2)\cdots(3n)$$
$$(3n+1)(3n+2)\cdots(4n)$$
$$\vdots$$
$$(n!-n+1)(n!-n+2)\cdots(n(n-1)!)$$

Each of these $(n-1)!$ rows contain n consecutive positive integers. The product of consecutive integers in each row is divisible by $n!$ Thus, the product of all the integers from 1 to n! is divisible by $(n!)^{(n-1)!}$.

3. $n_1 = x_1, x_2, x_3, x_4, x_5$

$n_2 = y_1, y_2, y_3, y_4, y_5$

and n_2 can be added without carrying at any stage if $x_i + y_i \le 9$.

Value of x_5	Value of y_5
0	0, 1, 2, ..., 9
1	0, 1, 2, ..., 8
2	0, 1, 2, ..., 7
3	0, 1, 2, 3, 4, 5, 6
4	0, 1, 2, 3, 4, 5
5	0, 1, 2, 3, 4
6	0, 1, 2, 3
7	0, 1, 2
8	0, 1
9	0

Thus, x_5 and y_5 can be selected collectively by $10 + 9 + 8 + \cdots + 1 = 55$ ways. Similarly, each pair (x_4, y_4), (x_3, y_3), (x_2, y_2) can be selected in 55 ways. But pair (x_1, y_1) can be selected in $1 + 2 + 3 + \cdots + 8 = 36$ ways as in this pair we cannot have 0 or 9. Thus, total number of ways is $36(55)^3$.

4. $n_1 = x_1, x_2, x_3, x_4, x_5$

$n_2 = y_1, y_2, y_3, y_4, y_5$

n_1 and n_2 can be subtracted without borrowing at any stage if $x_i \le y_i$.

Value of x_5	Value of y_5
9	0, 1, 2, ..., 9
8	0, 1, 2, ..., 8
7	0, 1, 2, ..., 7
6	0, 1, 2, 3, 4, 5, 6
5	0, 1, 2, 3, 4, 5
4	0, 1, 2, 3, 4
3	0, 1, 2, 3
2	0, 1, 2
1	0, 1
0	0

Thus, x_5 and y_5 can be selected collectively by $10 + 9 + 8 + \cdots + 1 = 55$ ways. Similarly, each pair (x_4, y_4), (x_3, y_3), (x_2, y_2) can be selected in 55 ways. But, pair (x_1, y_1) can be selected in $1 + 2 + 3 + \cdots + 9 = 36$ ways as in this pair we cannot have 0.

Thus, total number of ways is $45(55)^3$.

5. **Case I:** Two identical digits are 0, 0.

The number of ways to select three more digits is ${}^9C_3$. The number of arrangements of these five digits is $5!/2! - 4! = 36$.

Hence, the number of such numbers is

$${}^9C_3 \times 36 = 3024 \qquad (1)$$

Case II: Two identical digits are (1, 1) or (2, 2) or ... or (9, 9).

If 0 is included, then number of ways of selection of two more digits is ${}^8C_2$. The number of ways of arrangements of these five digits is $5!/2! - 4!/2! = 48$. Therefore, Number of such numbers is ${}^8C_2 \times 48$. If 0 is not included, then selection of three more digits is ${}^8C_3$. Therefore, Number of such numbers is ${}^8C_3 \times 5!/2! = {}^8C_3 \times 60$. Hence, total number of five-digit numbers with identical digits (1, 1), ..., (9, 9) is

$$9 \times ({}^8C_2 \times 48 + {}^8C_3 \times 60) = 42336 \qquad (2)$$

From Eqs. (1) and (2), the required number of numbers is $3024 + 42336 = 45360$.

6. Out of six faces, three can be selected in ${}^6C_3$ ways.

Consider one such selection, say ABC. Each of the 'n' places can be filled in three ways. So total number of ways is 3^n.

But this includes those ways also, which contain exactly one alphabet or exactly two alphabets which are to be subtracted.

Now, number of ways which contain only one letter is 3 and number of ways containing exactly two alphabets is ${}^3C_2\,(2^n - 2)$.

Hence, the number of ways is $3^n - {}^3C_2\,(2^n - 2) - 3$. So, required number of ways is ${}^6C_3\,[3^n - {}^3C_2\,(2^n - 2) - 3]$.

7.

One of the digits	**Pattern**	**Number of numbers**
1	102, 111	$4+1=5$
2	240, 231, 222	$4+6+1=11$
3	306, 315, 324, 333	$4+6+6+1=17$
4	408, 417, 426, 435, 444	$4+6+6+6+1$ $=23$
5	519, 528, 537, 546, 555	$6+6+6+6+1$ $=25$
6	639, 648, 657, 666	$6+6+6+1=19$
7	759, 768, 777	$6+6+1=13$
8	879, 888	$6+1=7$
9	999	1
	Total	121

Alternative solution:

Consider two sets.

(1) 1, 3, 5, 7, 9

(2) 0, 2, 4, 6, 8

The required number of ways = [any two from set (1) + any two from set (2) (excluding zero)] 3! + [0 along with any one from set (2)] × 4 + all three alike

$= ({}^5C_2 + {}^4C_2) \times 3! + {}^4C_1\, 4 + 9$

$= 121$

8. Let the number of members be n. Total number of points is ${}^nC_2$.

Therefore, ${}^nC_2 - 17 \times 5 = (n-4)x$ (where x is the number of point scored by each player)

$\Rightarrow \quad n(n-1) - 35 = 2(n-4)x$

$\Rightarrow \quad 2x = \dfrac{n(n-1)-35}{n-4}$ (where x takes the values 0.5, 1, 1.5, etc.)

$= \dfrac{n^2-n-35}{n-4}$

$= \dfrac{n(n-4)+3(n-4)-23}{n-4}$

$= (n+3) - \dfrac{23}{n-4}$

$\Rightarrow \quad \dfrac{23}{n-4}$ must be an integer

$\Rightarrow \quad n = 27$

9. Excluding the two specified guests, $2n$ persons can be divided into two groups one containing n and the other containing $n-2$ in $(2n-2)!/[n!(n-2)!]$ ways and can sit on either side of master and mistress in 2! ways and can arrange themselves in $n!\,(n-2)!$ Now, the two specified guests where $n-2$ guests are seated will have $n-1$ gaps and can arrange themselves in 2! ways. The number of ways when G_1G_2 will always be together is

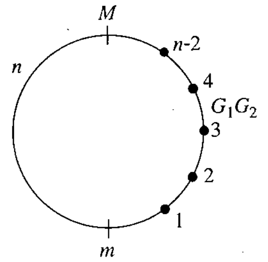


Fig. 5.36

$\dfrac{(2n-2)!}{n!(n-2)!}\; 2!\; n!\; (n-2)!\; (n-1) \times 2!$

$= (2n-2)!\; 4(n-1)$

Hence, the number of ways when G_1G_2 are never together is

$(2n)! - 4(n-1)(2n-2)! = (2n-2)!\,[2n(2n-1) - 4(n-1)]$

$= (2n-2)!\,[4n^2 - 6n + 4]$

10. Let the two subsets be called A and B. The elements for the two sets will be selected as follows.

First, two elements out of k elements for both the sets can be selected in ${}^kC_2$ ways. Now, remaining r elements for the subset A are selected from $k-2$ elements and any number of elements for B from the remaining $k-2-r$ elements.

Here r can vary from 0 to $k-2$. For a fixed r, the number of selections is ${}^{k-2}C_r \times 2^{k-2-r}$, because the number of selections of any number of things from n things is 2^n. Then, the total number of selections is $\sum_{r=0}^{k-2} {}^{k-2}C_r \times 2^{k-2-r} - 1$, excluding the case when both the subsets are equal having only the two common elements.

But, every pair of A, B is appearing twice like $\{a_1, a_2, a_3\}$, $\{a_1, a_2, a_4, a_5, a_6\}$ and $\{a_1, a_2, a_4, a_5, a_6\}$, $\{a_1, a_2, a_3\}$. Hence, the required number of ways is

$$\frac{1}{2} \times {}^kC_2 \left(\sum_{r=0}^{k-2} {}^{k-2}C_r \times 2^{k-2-r} - 1 \right)$$

$$= \frac{k(k-1)}{2} \times \frac{1}{2} \times [({}^{k-2}C_0 \times 2^{k-2} + {}^{k-2}C_1 \times 2^{k-3} + {}^{k-2}C_2 \times 2^{k-4} + \cdots + {}^{k-2}C_{k-2}) - 1]$$

$$= \frac{k(k-1)}{4}\,[(2+1)^{k-2} - 1]$$

$$= \frac{k(k-1)}{4}\,(3^{k-2} - 1)$$

11. The number of selections of r objects from n identical objects and $n-r$ objects from $2n$ different objects is $1 \times {}^{2n}C_{n-r} = {}^{2n}C_{n-r}$.

Here r varies form 0 to n. Therefore, the required number of selections is

$$\sum_{r=0}^{n} {}^{2n}C_{n-r} = {}^{2n}C_n + {}^{2n}C_{n-1} + {}^{2n}C_{n-2} + \cdots + {}^{2n}C_0$$

$= {}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + \cdots + {}^{2n}C_n$ (writing in reverse order)

$$= \frac{1}{2}\{2 \cdot {}^{2n}C_0 + 2 \cdot {}^{2n}C_1 + 2 \cdot {}^{2n}C_2 + \cdots + 2 \cdot {}^{2n}C_n\}$$

$$= \frac{1}{2}\{({}^{2n}C_0 + {}^{2n}C_{2n}) + ({}^{2n}C_1 + {}^{2n}C_{2n-1}) + \cdots + ({}^{2n}C_{n-1} + {}^{2n}C_{n+1}) + 2 \cdot {}^{2n}C_n\}$$

$$= \frac{1}{2}\left\{{}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + {}^{2n}C_3 + \cdots + {}^{2n}C_{2n} + {}^{2n}C_n\right\}$$

$$= \frac{1}{2}\left\{2^{2n} + {}^{2n}C_n\right\}$$

$$= 2^{2n-1} + \frac{1}{2}\frac{2n!}{(n!)^2}$$

12. Since no two lines are parallel and no three this pass through the same point, their points of intersection, i.e., number of ways of selecting two lines from n lines is ${}^nC_2 = N$ (say). It should also be noted that on each line there will be $(n-1)$ points of intersection made by the remaining $(n-1)$ lines.

Now we have to find number of new lines formed by these points of intersections. Clearly, a straight line is formed by joining two points so the problem is equivalent to select two points from N points. But each old line repeats itself ${}^{(n-1)}C_2$ times [selection of two points from $(n-1)$ points on this line]. Hence, the required number of new lines is

$${}^NC_2 - n\cdot{}^{(n-1)}C_2 = \frac{1}{2}\times\frac{1}{2}n(n-1)\left[\frac{1}{2}n(n-1)-1\right] - \frac{1}{2}n(n-1)(n-2)$$

$$= \frac{1}{8}n(n-1)(n^2-n-2) - \frac{1}{2}n(n-1)(n-2)$$

$$= \frac{1}{8}n(n-1)[n^2-n-2-4n+8]$$

$$= \frac{1}{8}n(n-1)(n-2)(n-3)$$

13. (a) A straight line can be formed by joining any two points. so number of straight lines (i.e., selection of two points from n) is ${}^nC_2$. But, selection of two points from m collinear points gives no extra line. Hence, number of distinct straight lines is

$${}^nC_2 - ({}^mC_2 - 1) = \frac{1}{2}n(n-1) - \frac{1}{2}m(m-1) + 1 \qquad (1)$$

(b) Formation of triangles is equivalents to section of three points from n points. As m points are collinear, selection of three points from m collinear points gives no triangles. Hence, the number of triangles is

$${}^nC_3 - {}^mC_3 = \frac{1}{6}\left[n(n-1)(n-2) - m(m-1)(m-2)\right]$$

(c) Four points determine a quadrilateral. But of these four points, not more than two is to be selected from the four collinear points. Now, number of selections of four points from all n is ${}^nC_4$. The number of selections of three points from m collinear and one from rest is ${}^mC_3\, {}^{n-m}C_1$. The number of selections of four points from m collinear $= {}^mC_4$.

Hence, the number of quadrilaterals is ${}^nC_4 - {}^mC_3 \times {}^{(n-m)}C_1 - {}^mC_4$.

14. The number of ways of disturbing n identical things among n person when at least $n-3$ persons get none of these of the objects is $A + B + C$ (suppose), where A is the number of ways when exactly $n-3$ persons get none of these, i.e., ${}^{n-1}C_2\,{}^nC_3$; B is the number of ways when exactly $n-2$ persons get none of these, i.e., ${}^{n-1}C_1\,{}^nC_2$; C is the number of ways when exactly $n-1$ persons get none of these, i.e., ${}^nC_1$. Hence,

$$A + B + C = {}^{n-1}C_2\,{}^nC_3 + {}^{n-1}C_1\,{}^nC_2 + {}^nC_1$$

$$= \frac{(n-1)(n-2)}{2}\frac{n(n-1)(n-2)}{6} + \frac{(n-1)n(n-1)}{2} + \frac{n}{1}$$

$$= \frac{n(n-1)^2}{2}\left[\frac{(n-2)^2}{6} + 1\right] + n$$

$$= \frac{n(n-1)^2}{2}\left[\frac{n^2-4n+10}{6}\right] + n$$

15. Let the city be represented by a rectangle whose sides are of length a and b north-south and west-east, respectively. Man has to go from P to Q. For this, he will have to travel a distance a vertically downward and a distance b horizontally from left to right.

Let $a_1, a_2, \ldots, a_{m-1}$ denote the distances between consecutive streets drawn horizontally beginning with the street passing through P and $b_1, b_2, \ldots, b_{n-1}$ denote the distances between consecutive streets drawn vertically beginning with the street passing through P.

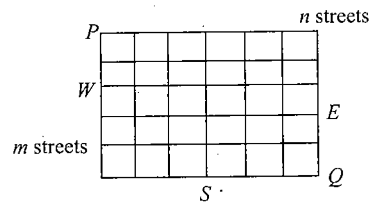


Fig. 5.37

Each arrangement of $(m + n - 2)$ things $a_1, a_2, \ldots, a_{m-1}, b_1, b_2, \ldots, b_{n-1}$ in a row so that order of a_i's does not change and order b_i's does not change corresponds to one path to go from P to Q.

Therefore, the required number is equal to number of arrangements of $m + n - 2$ things such that order of a_i's does not change and order of b_i's does not change, which is equal to $(m + n - 2)!/(m-1)!(n-1)!$.

Alternative solution:

Let the city be represented by a rectangle whose sides are of length a and b.

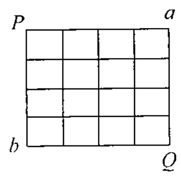


Fig. 5.38

For each path total distance covered in horizontal direction is a and that in vertical direction is b. a is the sum of lengths of $(m-1)$ horizontal line segments and b is the sum of lengths of $(n-1)$ vertical line segments.

Each path to go from P to Q will be a arrangement of $(m + n - 2)$ line segments of which $(m-1)$ are horizontal and $(n-1)$ are vertical.

Therefore, the required number is equal to number of arrangements of $m + n - 2$ different things of which $m - 1$ are of one kind and $n - 1$ are of another kind, which is given by $(m + n - 2)!/(m-1)!(n-1)!$.

16. Let the batsman hit x fours, y sixes and let z balls may not yield runs. Then we have,

$$4x + 6y + 0z = 100 \quad (1)$$

where

$$x + y + z = 20 \quad (2)$$

From Eqs. (1) and (2),

$$4(20 - y - z) + 6y = 100$$

$\Rightarrow$ $2y - 4z = 20$

$\Rightarrow$ $y - 2z = 10$

$\Rightarrow$ $y = 10 + 2z$

Hence, z can be 0, 1, 2, 3 as any of x, y, z cannot exceed 20. So, we have following type of distribution of runs:

Number of fours, x	Number of sixes, y	Number of zeros, z	Number of ways of arranging x, y and z
10	10	0	$\frac{20!}{10!\,10!}$
7	12	1	$\frac{20!}{7!\,12!}$
4	14	2	$\frac{20!}{4!\,14!\,2!}$
1	16	3	$\frac{20!}{16!\,3!}$

Hence, the total number of ways is

$$20!\left(\frac{1}{16!\,3!} + \frac{1}{14!\,4!\,2!} + \frac{1}{12!\,7!} + \frac{1}{10!\,10!}\right)$$

17. The total number of ways to place the balls disregarding the constrains is ${}^{2t+1+3-1}C_{3-1} = {}^{2t+3}C_2$.

The total number of ways to place the balls so that the first box will have more balls than the other two is ${}^{t+3-1}C_{3-1} = {}^{t+2}C_3$.

[We place $t + 1$ balls in the first box and then divide the rest of t balls in the three boxes arbitrarily.]

The same result applies to the case of 2nd box holding more balls than 1st or 3rd combined and also for the 3rd box containing more balls than 1st and 2nd combined. Hence, required number of ways is

$${}^{2t+3}C_2 - 3\,{}^{t+2}C_2 = \frac{t}{2}(t+1)$$

18. The whole family has 24 children. Children of Geeta and Sohan are $24 - x - (x + 1) = 23 - 2x$ in number. When Sohan's child fights with a Geeta's child, there are $x(x + 1)$ fights. When Sohan's child fights with Sohan and Geeta's children, there are $x(23 - 2x)$ fights. Again, when Geeta's child fights with Sohan and Geeta's children, there are $(x + 1)(23 - 2x)$ fights. Therefore, total number of fights is

$$F(x) = x(x + 1) + x(23 - 2x) + (x + 1)(23 - 2x)$$

$$= 23 + 45x - 3x^2$$

$$= 3\left[\frac{23}{3} + \frac{225}{4} - \left(x - \frac{15}{2}\right)^2\right]$$

For $F(x)$ to be maximum,

$$x - \frac{15}{2} = 0,$$

i.e.,

$$x = 7.5$$

Since x is an integer, therefore, $x = 7$ or 8. For both $x = 7$ and $x = 8$. Total number of fights is $F(x) = 191$.

19. Given,

$$nx + ny = xy$$

$\Rightarrow$ $xy - nx - ny + n^2 = n^2$

$\Rightarrow$ $(x - n)(y - n) = n^2$

Hence, $(x - n)$ and $(y - n)$ are two integral factors of n^2. Obviously, if d is one divisor of n^2, then for each sub-divisor there will be an ordered pair (x, y). Let $S(n)$ be the number of divisors of n^2.

(i) For $n = 6$, we have $d = 1, 2, 3, 6, 9, 12, 18, 36$.

$\therefore$ $S(6) = 9$

(ii) If n is prime, then $d = 1$, n and n^2; hence $S(n) = 3$.

20. When 12 fruits are distributed subject to the given condition, either nine boys get one fruit each and the remaining one boy gets three fruits or eight boys get one fruit each and the remaining two boys get two fruit each. Three fruits to a boy can be given in the four ways.

Apples	3	2	1	0
Mangoes	0	1	2	3

After giving 3 apples to a single boy, the 3 remaining apples and 6 oranges can be distributed to the 9 boys in $({}^9C_3)({}^6C_6)$ ways. Thus, number of ways three fruits can be given to a particular boy is

$$({}^9C_3)({}^6C_6) + ({}^9C_4)({}^5C_5) + ({}^9C_5)({}^4C_4) + ({}^9C_6)({}^3C_3)$$

$$= 2[{}^9C_3 + {}^9C_4]$$

One particular boy can be chosen in ${}^{10}C_1 = 10$ ways. Therefore, three fruits can be given to a single boy in $({}^{10}C_1)(2)[{}^9C_3 + {}^9C_4] = 20({}^{10}C_4) = 4200$ ways.

We can give two fruits to two boys, say P and Q, in the following ways:

P	**Apples**	2	1	0	2	1	0	2	1	0
	Mangoes	0	1	2	0	1	2	0	1	2
Q	**Apples**	2	2	2	1	1	1	0	0	0
	Mangoes	0	0	0	1	1	1	2	2	2

The remaining eight fruits can be distributed among eight boys in the following ways:

$$({}^8C_2)({}^5C_6) + ({}^8C_3)({}^5C_5) + ({}^8C_4)({}^4C_4) + ({}^8C_3)({}^5C_5) + ({}^8C_4)({}^4C_4) + ({}^8C_5)({}^3C_3) + ({}^8C_4)({}^4C_4) + ({}^8C_5)({}^5C_5) + ({}^8C_6)({}^2C_2)$$

$$= 2({}^8C_2) + 4({}^8C_3) + 4({}^8C_4)$$

Two boys out of 10 can be chosen in ${}^{10}C_2$ ways. Therefore, number of ways two boys can get two fruits each is

$({}^{10}C_2)\ [2({}^8C_2) + 4\ ({}^8C_3) + 3\ ({}^8C_4)]$

$$= \frac{10!}{2!\,8!}\left[2\frac{8!}{2!\,6!}+4\frac{8!}{3!\,5!}+3\times\frac{8!}{4!\,4!}\right]$$

$$= \frac{10!}{2!\,6!}\left[14+\frac{15}{4}\right] = 22050.$$

Hence, the required number of ways is 4200 + 22050 = 26250.

Objective Type

1. d. $\alpha = {}^mC_2 \Rightarrow \alpha = \dfrac{m(m-1)}{2}$

$$\therefore\ {}^{\alpha}C_2 = \frac{\alpha(\alpha-1)}{2} = \frac{1}{2}\frac{m(m-1)}{2}\left\{\frac{m(m-1)}{2}-1\right\}$$

$$= \frac{1}{8}m(m-1)(m-2)(m+1)$$

$$= \frac{1}{8}(m+1)m(m-1)(m-2) = 3\ {}^{m+1}C_4$$

2. a. ${}^nC_3 + {}^nC_4 > {}^{n+1}C_3$

$\Rightarrow\ {}^{n+1}C_4 > {}^{n+1}C_3\ (\because {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1})$

$\Rightarrow\ \dfrac{{}^{n+1}C_4}{{}^{n+1}C_3} > 1$

$\Rightarrow\ \dfrac{n-2}{4} > 1$

$\Rightarrow\ n > 6$

3. b. $\displaystyle\sum_{r=0}^{n-1}\frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}}$

$$= \sum_{r=0}^{n-1}\frac{1}{1+\dfrac{{}^nC_{r+1}}{{}^nC_r}}$$

$$= \sum_{r=0}^{n-1}\frac{1}{1+\dfrac{n-r}{r+1}}$$

$$= \sum_{r=0}^{n-1}\frac{r+1}{n+1} = \frac{1}{n+1}\sum_{r=0}^{n-1}(r+1)$$

$$= \frac{1}{(n+1)}[1+2+\cdots+n] = \frac{n}{2}$$

4. c. We have 32 places for teeth. For each place, we have two choices either there is a tooth or there is no tooth. Therefore, the number of ways to fill up these places is 2^{32}. As there is no person without a tooth, the maximum population is $2^{32} - 1$.

5. c. For each bulb there are two possibilities. It will be switched either on or off. Hence, total number of ways in which the room can be illuminated is $2^{12} - 1$.

6. c. The total number of ways is $6 \times 6 \times \cdots$ to n times $= 6^n$. The total number of ways to show only even numbers is $3 \times 3 \times \cdots$ to n times $= 3^n$. Therefore, the required number of ways is $6^n - 3^n$.

7. b. Total number of triplets without restriction is $n \times n \times n$. The number of triplets with all different coordinates is ${}^nP_3$. Therefore, the required number of triplets is $n^3 - n(n-1)(n-2)$.

8. a. All strips are of different colours, then number of flags is $= 3! = 5$. When two strips are of same colour, then number of flags is ${}^3C_1 \times (3!/2) \times {}^2C_1 = 18$. Total number of flags is $6 + 18 = 24 = 4!$

9. a. If 7 is used at first place, the number of numbers is 9^4 and for any other four places it is 8×9^3.

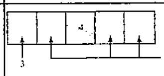
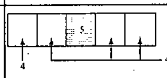
10. c. Total number of variables if only alphabet is used is 286. Total number of variables if alphabets and digits both are used is 26×10. Hence, the total number of variables is $26(1 + 10) = 286$.

11. b. The number of one-digit numbers is 6.
The number of two-digit numbers is $5 \times 5 = 25$.
The number of thee-digit numbers is $5 \times 5 \times 4 = 100$.
Hence, the total number are is 131.

12. a. $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$, when the number is $x_1x_2x_3x_4x_5x_6$
Clearly no digit can be zero. Also, all the digits are distinct. So, let us first select six digits from the list of digits 1, 2, 3, 4, 5, 6, 7, 8, 9 which can be done in ${}^9C_6$ ways.
After selecting these digits they can be put only in one order. Thus, total number of such numbers is ${}^9C_6 \times 1 = {}^9C_3$.

13. b. Numbers greater than 1000 and less than or equal to 4000 will be of 4 digits and will have either 1 (except 1000) or 2 or 3 in the 1st place with 0 in each of remaining places. After fixing 1st place, the 2nd place can be filled by any of the 5 digits. Similarly the 3rd place can be filled up in 5 ways and 4th place can be filled up in 5 ways. Thus, there will be $5 \times 5 \times 5 = 125$ ways in which 1 will be in first place but this also includes 1000. Hence, there will be 124 numbers having 1 in the first place. Similarly, 125 for each 2 or 3. One number will be there in which 4 will be in the first place, i.e., 4000. Hence, the required number of ways is 124 + 125 + 125 + 1 = 375.

14. d.

Middle digit	Digits available for remaining four places	Pattern	Number of ways filling remaining four places
4	0, 1, 2, 3		$3 \times {}^3P_3$
5	0, 1, ..., 4		$4 \times {}^4P_3$
6	0, 1, ..., 5	...	$5 \times {}^5P_3$
7	0, 1, ..., 6	...	$6 \times {}^6P_3$
8	0, 1, ..., 7	...	$7 \times {}^7P_3$
9	0, 1, ..., 8	...	$8 \times {}^8P_3$

15. **b.** The number of numbers when repetition is allowed is 5^4.
The number of numbers when digits cannot be repeated is ${}^5P_5$.
Therefore, the required number of numbers is $5^4 - 5!$.

16. **d.** According to given conditions, numbers can be formed by the following format:

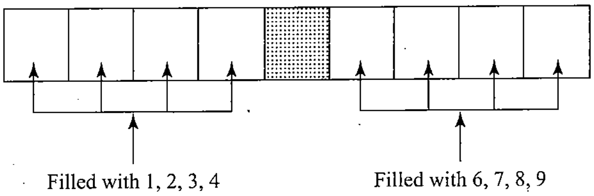


Fig. 5.39

The required number of numbers is ${}^4P_4 \times {}^4P_4$.

17. **a.** Total number of words without any restriction is 7!
Total number of words beginning with I is 6!
Total number of words ending with B is 6!
Total number of words beginning with I and ending with B is 5!
Thus the total number of required words is $7! - 6! - 6! + 5! = 7! - 2(6!) + 5!$.

18. **a.** There can be two types of numbers.
(i) Any one of the digits 1, 2, 3, 4 appears thrice and the remaining digits only once, i.e., of the type 1, 2, 3, 4, 4, 4, etc. Number of ways of selection of digit which appears thrice is ${}^4C_1$.

Then the number of numbers of this type is $(6!/3!) \times {}^4C_1 = 480$.

(ii) Any two of the digits 1, 2, 3, 4 appears twice and the remaining two only once, i.e., of the type 1, 2, 3, 3, 4, 4, etc. The number of ways of selection of two digits which appear twice is ${}^4C_2$. Then the number of numbers of this type is $[6!/(2!2!) \times {}^4C_2]$. Therefore, the required number of numbers is $480 + 1080 = 1560$.

19. **a.** Clearly, one of the odd digits 1, 3, 5, 7, 9 will be repeated. The number of selections of the sixth digit is ${}^5C_1 = 5$. Then the required number of numbers is $5 \times (6!/2!)$.

20. **c.** Formed number can be utmost of nine digits. Total number of such numbers is

$$3 + 3^2 + 3^3 + \cdots + 3^8 + 2 \times 3^8$$

$$= \frac{3(3^8-1)}{3-1} + 2\times 3^8 = \frac{3^9 - 3 + 4\times 3^8}{2} = \frac{7\times 3^8 - 3}{2}$$

21. **d.** The order of letters of the word 'OBJECT' is B C E J O T
Words starting with B can be formed in 5! ways.
Words starting with C can be formed in 5! ways.
Words starting with E can be formed in 5! ways.
Words starting with J can be formed in 5! ways.
Words starting with O can be formed in 5! ways.
Words starting with TB can be formed in 4! ways.
Words starting with TC can be formed in 4! ways.
Words starting with TE can be formed in 4! ways.
Words starting with TJ can be formed in 4! ways.
Words starting with TOB can be formed in 3! ways.
Words starting with TOC can be formed in 3! ways.
Words starting with TOE can be formed in 3! ways.
Words starting with TOJB can be formed in 2! ways.
Words starting with TOJC can be formed in 2! ways.
Therefore, the total number of words is 718 words.
Hence 717[th] word is TOJCBE.

22. **a.** The number of ways of allotment without any restriction is ${}^8P_6$. Now, it is possible that all rooms of 2[nd] floor or 3[rd] floor are not occupied. Thus, there are two ways in which one floor remains unoccupied. Hence, the number of ways of allotment in which a floor is unoccupied is $2 \times 6!$ Hence, number of ways in which none of the floor remains unoccupied is ${}^8P_6 - 2(6!)$.

23. **a.** Each position can be filled in 5 ways. Hence, the total number of numbers is 5^{20}.

24. **a.** The number of ways of selecting four numbers from 1 to 30 without any restriction is ${}^{30}C_4$. The number of ways of selecting four consecutive [i.e. (1, 2, 3, 4), (2, 3, 4, 5), ..., (27, 28, 29, 30)] number is 27. Hence, the number of ways of selecting four integers which excludes consecutive four selections is

$${}^{30}C_4 - 27 = \frac{30\times 29\times 28\times 27}{24} - 27 = 27378$$

25. **d.** Let us first select two places for vowel, which can be selected from 4 places in ${}^4C_2$ ways. Now this places can be filled by vowels in $5\times 5 = 25$ ways as repetition is allowed. The remaining two places can be filled by consonants in 21×21 ways. Then the total number of words is ${}^4C_2 \times 25 \times 21^2 = 150 \times 21^2$.

26. **b.** Other than 2, remaining five places can be filled by 1 and 3 for each place. The number of ways for five places is $2 \times 2 \times 2 \times 2 \times 2 = 2^5$. For 2, selecting 2 places out of 7 is ${}^7C_2$. Hence, the required number of ways is ${}^7C_2 \times 2^5$.

27. **c.** 1 S, 3A, 1 H, 2 R, 1 N, 1 P, 1 U
When all letters are different corresponding ways is ${}^7C_3 \times 3! = {}^7C_3 = 210$. When two letters are of one kind and other is different, corresponding number of ways is ${}^2C_1 \times {}^6C_1 \times (3!/2!) = 36$. When all letters are alike, corresponding number of ways is 1. Thus, total number of words that can be formed is $210 + 36 + 1 = 247$.

28. **b.** The natural numbers are 1, 2, 3, 4. Clearly, in one diagonal we have to place 1, 4 and in the other 2, 3.

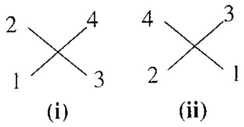


Fig. 5.40

The number of ways in (i) is $2! \times 2! = 4$.
The number of ways in (ii) is $2! \times 2! = 4$.
Therefore, the total number of ways is 8.

29. **c.** First arrange m positive signs. The number of ways is just 1 (as all + signs are identical). Now, $m + 1$ gaps are created of which n are to be selected for placing '–' signs. Then the total number of ways of doing so is ${}^{m+1}C_n$. After selecting the gaps '–' signs can be arranged in one way only.

30. **a.** We can think of three packets. One consisting of three boys of class X, other consisting of 4 boys of class XI and last one consisting of 5 boys of class XII. These packets can be arranged in 3! ways and contents of these packets can be further arranged in 3! 4! and 5! ways, respectively. Hence, the total number of ways is 3!× 3! × 4! × 5!

31. a. The total number of books is $a + 2b + 3c + d$. The total number of ways in which these books can be arranged in a shelf (in same row) is

$$\frac{(a+2b+3c+d)!}{a!(b!)^2(c!)^3}$$

32. c. Required sum is 3!(3 + 4 + 5 + 6) = 6 × 18 = 108.
[If we fix 3 in the unit place, other three digits can be arranged in 3! ways. Similarly for 4, 5, 6.]

33. c. The number of numbers with 0 in the unit's place is 3! = 6.

The number of numbers with 1 or 2 or 3 in the unit's place is 3! − 2! = 4. Therefore the sum of the digits in the unit's place is 6 × 0 + 4 × 1 + 4 × 2 + 4 × 3 = 24.

Similarly, for the ten's and hundred's places, the number of numbers with 1 or 2 in the thousand's place is 3! Therefore, the sum of the digits in the thousand's place is 6 × 1 + 6 × 2 + 6 × 3 = 36.

Hence, the required sum is 36 × 1000 + 24 × 100 + 24 × 10 + 24.

34. a. Total numbers ending with 2 is 3! as after fixing 2 in the unit's place other three places can be filled by 3! ways. Thus, 2 appears in the unit's place 3! times.

Similarly, all other digits 4, 6 and 8 also appear 3! times. Then sum of the digits in the unit's place is 6(2 + 4 + 6 + 8) = 120 units. Similarly, sum of digits in ten's place is 120 tens and that in hundred's place is 120 hundreds, etc. Hence, sum of all the 24 numbers is $120(1 + 10 + 10^2 + 10^3) = 120 \times 1111$ = 133320.

35. b. $(x + 3)^2 + y^2 = 13$

$\Rightarrow x + 3 = \pm 2, y = \pm 3$ or $x + 3 = \pm 3, y = \pm 2$

36. d. Using the digits 0, 1, 2, ..., 9 the number of five digit telephone numbers which can be formed is 10^5 (since repetition is allowed). The number of five digit telephone, numbers which have none of the digits repeated is $^{10}P_5 = 30240$. Therefore, the required number of telephone numbers is $10^5 - 30240$ = 69760.

37. b. 3 must be at thousand's place and since the number should be divisible by 5, or 5 must be at unit's place. Now, we have to fill two places (tens and hundreds), i.e., $^4P_2 = 12$.

38. d. Two positions for A_1 and A_{10} can be selected in $^{10}C_2$ ways. Rest 8 students can be ranked in 8! ways. Hence total number of ways is $^{10}C_2 \times 8! = (1/2)(10!)$.

39. a.

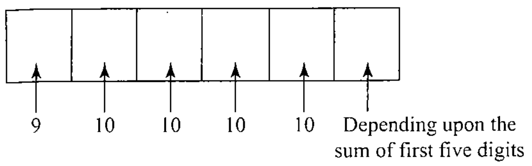


Fig. 5.41

First place from left cannot be filled with 0. Next four places can be filled with any of the 10 digits. After filling the first five places, the last place can be filled in following ways:

Sum of digits in first five places	Digit in the unit's place
$5k$	0 or 5
$5k + 1$	4 or 9
$5k + 2$	3 or 8
$5k + 3$	2 or 7
$5k + 4$	1 or 6

Thus, in any case the last place can be filled in two ways.
Hence, the required number of numbers is $9 \times 10^4 \times 2$.

40. a. The selection can be made in $^5C_3 \times {^{22}C_9}$ ways.
(Since 3 vacancies are filled from 5 candidates in $^5C_3$ ways and now remaining candidates are 22 and remaining seats are 9.)

41. d. Here,

$$^nP_3 - {^nC_3} > 100$$

$$\Rightarrow \frac{n!}{(n-3)!} - \frac{n!}{3!(n-3)!} > 100$$

$$\Rightarrow \frac{5}{6}n(n-1)(n-2) > 100$$

$$\Rightarrow n(n-1)(n-2) > 120$$

$$\Rightarrow n(n-1)(n-2) > 6 \times 5 \times 4$$

$$\Rightarrow n = 7, 8, \ldots$$

42. c. Places for A, B, C can be chosen in $^{10}C_3$ ways. Remaining 7 persons can speak in 7! ways. Hence, the number of ways in which they can speak is $^{10}C_3 \times 7! = 10!/6$.

43. b. Since 5 players are always to be excluded and 6 players always to be included, therefore 5 players are to be chosen from 14. Hence required number of ways is $^{14}C_5 = 2002$.

44. c.

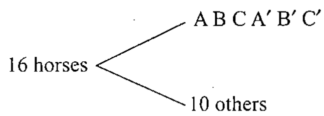


Fig. 5.42

The number of ways is $^{10}C_3$ × number of ways of choosing out of $ABC\,A'B'C'$, so that AA', BB' or CC' are not together

$= {^{10}C_3}$ (one from each of pairs AA', BB', CC')

$= {^{10}C_3} \times 8$

$= \dfrac{10 \times 9 \times 8}{1 \times 2 \times 3}\, 8 = 960$

45. b. The number of ways of selecting r $(0 \le r \le m)$ balls out of m is ^mC_r. Therefore, the number of ways if selecting r balls from each of the bag is $(^mC_r)^2$. Further the number of ways of selecting equal number of balls from each of the two bags, choosing at least one from each bag, is

$$(^mC_1)^2 + (^mC_2)^2 + \cdots + (^mC_m)^2 = {^{2m}C_m} - 1$$

$$[\because (^mC_0)^2 + (^mC_1)^2 + \cdots + (^mC_m)^2 = {^{2m}C_m}]$$

46. b. There are 6 different letters. We have to select 6 squares, taking at least one from each row and then arranging in each selection. Let us first select places in each row such that no row remains empty.

R_1	R_2	R_3	Number of selections
1	1	4	$^2C_1\,{^2C_1}\,{^4C_4} = 4$
1	2	3	$^2C_1\,{^2C_2}\,{^4C_3} = 8$
2	1	3	$^2C_2\,{^2C_1}\,{^4C_3} = 8$
2	2	2	$^2C_2\,{^2C_2}\,{^4C_2} = 6$

Therefore, the total number of selections of 6 squares is $4 + 8 + 8 + 6 = 26$. For each selection of 6 squares, the number of arrangements of 6 letters is $6! = 720$. Hence, the required number of ways is $26 \times 720 = 18720$.

47. **b.** There are 11 letters A, A; I, I; N, N; E, X, M, T, O. For the selection of 4 letters we have the following possibilities:

(A) 2 alike, 2 alike

(B) 2 alike, 2 different

(C) All four different

(A) There are 3 pairs of 2 letters. So, the number of ways of selection of 2 pairs is ${}^3C_2$ and permutation of these 4 letters is $4! / 2!\,2!$ Therefore, the number of words in this case is ${}^3C_2 \times 4!/2!\,2! = 18$.

(B) We have to select one pair from 3 pairs and 2 distinct letters from remaining 7 distinct letters. For illustration, let us select both A, A; then we have I, N, E, X, M, T, O, i.e., 7 as remaining distinct letters. Hence, the number of selections is ${}^3C_1 \times {}^7C_2$ and these 4 (2 same, 2 distinct) can be permuted in $4!/2!$ ways. Therefore, number of words is ${}^3C_1 \times {}^7C_2 \times 4!/2!$ $= 3 \times 21 \times 12 = 756$.

(C) There are 8 distinct letters so number of words of 4 letters is ${}^8C_4 \times 4! = 1680$. By sum rule, the total number of words is $18 + 756 + 1680 = 2454$.

48. **d.** The total number of words is $6! = 720$. Let us write the letters of word ZENITH alphabetically, i.e., EHINTZ.

For ZENITH word start with	Word starting with	Number of words
Z	E	5!
	H	5!
	I	5!
	N	5!
	T	5!
ZEN	ZEH	3!
	ZEI	3!
ZENI	ZENH	2!
ZENIT	ZENIH	1
	Total number of words before ZENITH	615

Hence, there are 615 words before ZENITH, so the rank of ZENITH is 616.

49. **b.**

Number of girls	Number of boys	Number of groups going to picnic	Total number of dolls
1	4	${}^3C_1\,{}^4C_4$	$1({}^3C_1\,{}^4C_4) = 3$
2	3	${}^3C_2\,{}^4C_3$	$2({}^3C_2\,{}^4C_3) = 24$
3	2	${}^3C_3\,{}^4C_2$	$3({}^3C_3\,{}^4C_2) = 18$
		Total	45

50. **b.** The number of times the teacher goes to the zoo is ${}^nC_3$. The number of times a particular child goes to the zoo is equal to number of ways two other children can be selected who accompany a particular child, i.e., ${}^{n-1}C_2$. From the question,

$${}^nC_3 - {}^{n-1}C_2 = 84$$

or

$$(n-1)(n-2)(n-3) = 6 \times 84 = 9 \times 8 \times 7 \Rightarrow n - 1 = 9$$

51. **b.** We first select 2 men out of 7 in ${}^7C_2$ ways. Now we exclude the wives of these two selected men and so select 2 ladies from remaining 5 ladies in ${}^5C_2$ ways. Let A, B be two men and X, Y be the ladies playing in one set. Then we can have

(i) A and X plying against B and Y.

(ii) A and Y playing against B and X.

Then the total number of ways is ${}^7C_2 \times {}^5C_2 \times 2 = 21 \times 10 \times 2$ $= 420$.

52. **b.** Suppose there 'n' players in the beginning. The total number of games to be played was equal to ${}^nC_2$ and each player would have played $n - 1$ games.

Let us assume that A and B fell ill. Now the total number of games of A and B is $(n-1) + (n-1) - 1 = 2n - 3$. But they have played 3 games each. Then their remaining number of games is $2n - 3 - 6 = 2n - 9$. Given,

$${}^nC_2 - (2n - 9) = 84$$

$$\Rightarrow n^2 - 5n - 150 = 0$$

$$\Rightarrow n = 15$$

Alternative solution:

The number of games excluding A and B is ${}^{n-2}C_2$. But before leaving A and B played 3 games each. Then,

$${}^{n-2}C_2 + 6 = 84$$

Solving this equation, we get $n = 15$.

53. **a.** The number of ways he can select at least one parantha is $2^3 - 1 = 7$. The number of ways he can select at least one vegetable dish is $2^4 - 1 = 15$. The number of ways he can select zero or more items from salads and sauces is 2^5.

Hence, the total number of ways is $7 \times 15 \times 32 = 3360$.

54. **b.** Number of even divisors is equal to number of ways in which one or more '2', zero or more '3', zero or more '5' and zero or more '7' can be selected, and is given by $(3)(2 + 1)(2 + 1)(1 + 1)$ $= 54$.

55. **c.** The number of ways the candidate can choose questions under the given conditions is enumerated below.

Group 1	Group 2	Number of ways
4	2	$({}^5C_4)({}^5C_2) = 50$
3	3	$({}^5C_3)({}^5C_3) = 100$
2	4	$({}^5C_2)({}^5C_4) = 50$
	Total number of ways	200

56. **c.** Let there be n men participants. Then the number of games that the men play between themselves is $2 \times {}^nC_2$ and the number of games that the men played with the women is $2 \times (2n)$.

$$\therefore\ 2 \times {}^nC_2 - 2 \times 2n = 66 \text{ (by hypothesis)}$$

$$\Rightarrow n^2 - 5n - 66 = 0 \Rightarrow n = 11$$

Hence, the number of participants is 11 men + 2 women = 13.

57. **b.** The smallest number of people = total number of possible forecasts

= total number of possible results

$= 3 \times 3 \times 3 \times 3 \times 3$

58. a. Three IIT students who will be between the IIT students can be selected in ${}^{10}C_3$ ways. Now, two DCE students having three IIT students between them can be arranged in 2!×3! ways. Finally, a group of above five students and the remaining seven students together can be arranged in 8! ways. Hence, total number of ways is ${}^{10}C_3 \times 2! \times 3! \times 8!$

59. c. Let S_1 and S_2 refuse to be together and S_3 and S_4 want to be together only. The total number of ways when S_3 and S_4 are selected is$({}^8C_2 + {}^2C_1 \times {}^8C_1) = 44$. The total ways when S_3 and S_4 are not selected is $({}^8C_4 + {}^2C_1 \times {}^8C_3) = 182$. Thus, the total number of ways is 44 + 182 = 226.

60. c. Let there be n eandidates. Then,

$${}^nC_1 + {}^nC_2 + \cdots + {}^nC_{n-1} = 254$$
$$\Rightarrow 2^n - 2 = 254$$
$$\Rightarrow 2^n = 2^8 \Rightarrow n = 8$$

61. b. 'P_1' must win at least $n + 1$ games. Let 'P_1' win $n + r$ games ($r = 1, 2, \ldots, n$). Therefore, corresponding number of ways is ${}^{2n}C_{n+r}$. The total number of ways is

$$\sum_{r=1}^{n} {}^{2n}C_{n+r} = {}^{2n}C_{n+1} + {}^{2n}C_{n+2} + \cdots + {}^{2n}C_{2n}$$
$$= \frac{2^{2n}}{2} - {}^{2n}C_n$$
$$= \frac{1}{2}(2^{2n} - 2 \times {}^{2n}C_n)$$

62. b. The candidate is unsuccessful if he fails in 9 or 8 or 7 or 6 or 5 papers. Therefore, the number of ways to be unsuccessful is

${}^9C_9 + {}^9C_8 + {}^9C_7 + {}^9C_6 + {}^9C_5 = {}^9C_0 + {}^9C_1 + {}^9C_2 + {}^9C_3 + {}^9C_4$
(recall the concept of half series)

$$= \frac{1}{2}({}^9C_0 + {}^9C_1 + \cdots + {}^9C_9)$$
$$= \frac{1}{2} \times 2^9 = 2^8$$

63. b. Since the student is allowed to select at most n books out of $(2n + 1)$ books, therefore in order to select one book he has the choice to select one, two, three, ..., n books. Thus, if T is the total number of ways of selecting one book, then

$$T = {}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n = 63 \quad \text{(i)}$$

Again the sum of binomial coefficients is

$${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + \cdots + {}^{2n+1}C_{2n+1} = (1+1)^{2n+1} = 2^{2n+1}$$

or

$${}^{2n+1}C_0 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n) + {}^{2n+1}C_{2n+1} = 2^{2n+1}$$
$$\Rightarrow 1 + 2(T) + 1 = 2^{2n+1}$$
$$\Rightarrow 1 + T = \frac{2^{2n+1}}{2} = 2^{2n}$$
$$\Rightarrow 1 + 63 = 2^{2n}$$
$$\Rightarrow 2^6 = 2^{2n} \Rightarrow n = 3$$

64. c. The number of ways can be given as follows:

2 bowlers and 9 other players: ${}^4C_2 \times {}^9C_9$
3 bowlers and 8 other players: ${}^4C_3 \times {}^9C_8$
4 bowlers and 7 other players: ${}^4C_4 \times {}^9C_7$

Hence, required number of ways is $6 \times 1 + 4 \times 9 + 1 \times 36 = 78$.

65. a. Matches whose predictions are correct can be selected in ${}^{20}C_{10}$ ways. Now each wrong prediction can be made in 2 ways. Thus, the total number of ways is ${}^{20}C_{10} \times 2^{10}$.

66. b.

Dashes	Dots	Arrangements
5	2	${}^7C_2$
4	3	${}^7C_3$
3	4	${}^7C_4$
2	5	${}^7C_5$
1	6	${}^7C_6$
0	7	${}^7C_7$

The total number of ways is ${}^7C_2 + {}^7C_3 + {}^7C_4 + {}^7C_5 + {}^7C_6$
$= 2^7 - 8 = 120$.

67. c. For a radical centre, 3 circles are required. The total number of radical centres is ${}^nC_3$. The total number of radical axis is ${}^nC_2$. Now,

$${}^nC_2 = {}^nC_3 \Rightarrow n = 5$$

68. c.

(i) Miss C is taken
(A) B included ⇒ A excluded ⇒ ${}^4C_1 \times {}^4C_2 = 24$
(B) B excluded ⇒ ${}^4C_1 \times {}^5C_3 = 40$
(ii) Miss C is not taken
⇒ B does not come; ⇒ ${}^4C_2 \times {}^5C_3 = 60$
⇒ Total = 124

Alternative method:

Case I:

Mr. 'B' is present
⇒ 'A' is excluded and 'C' included

Hence, the number of ways is ${}^4C_2\,{}^4C_1 = 24$.

Case II:

Mr. 'B' is absent
⇒ No constraint

Hence, the number of ways is ${}^5C_3\,{}^5C_2 = 100$.

∴ Total = 124.

69. d. $N = 1! + 2! + \cdots + 2005!$
$= (1! + 2! + 3! + 4!) + (5! + \cdots + 2005!)$
= 33 + an integer having 0 in its unit's place
= an integer having 3 in its unit's place

Hence, N^{500} is an integer having 1 in its unit's place.

70. d.

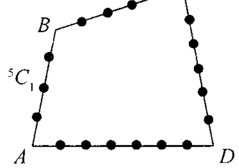


Fig. 5.43

The number of triangles with vertices on sides AB, BC, CD is ${}^3C_1 \times {}^4C_1 \times {}^5C_1$.

Similarly, for other cases, the total number of triangles is

$${}^3C_1 \times {}^4C_1 \times {}^5C_1 + {}^3C_1 \times {}^4C_1 \times {}^6C_1 + {}^3C_1 \times {}^5C_1 \times {}^6C_1 + {}^4C_1 \times {}^5C_1 \times {}^6C_1 = 342.$$

71. **c.** Number of points required for the fixed circle is 3. So, first select any three points from the 10 points in ${}^{10}C_3$ ways.

In these ways, circle with four concyclic points is selected in ${}^4C_3$ ways. But it should be taken once then total number of circles is $({}^{10}C_3 - {}^4C_3) + 1$.

72. **a.** The number of points of intersection is equal to the number of ways two lines are selected, which is given by

$$ {}^nC_2 = \frac{n(n-1)}{2} = \sum_{k=1}^{n-1} K $$

73. **c.** Out of 10 points let n points are collinear. Then the number of triangles is

$${}^{10}C_3 - {}^nC_3 = 110$$

$$\Rightarrow \frac{10\times9\times8}{6} - \frac{n(n-1)(n-2)}{6} = 110$$

$$\Rightarrow n(n-1)(n-2) = 60$$

$$\Rightarrow n = 5$$

74. **c.** Select any three points from total $3p$ points, which can be done ${}^{3p}C_3$ ways. But this also includes selection of three collinear points. Now three collinear points from each straight line can be selected in ${}^pC_3$ ways. Then the number of triangles is ${}^{3p}C_3 - 3{}^pC_3 = p^2(4p-3)$.

75. **c.** Two circles intersect at two distinct points.Two straight lines intersect at one point. One circle and one straight line intersect at two distinct points. Then the total numbers of points of intersections are as follows:

Number of ways of selection	Points of intersection
Two straight lines: ${}^5C_2$	${}^5C_2 \times 1 = 10$
Two circles: ${}^4C_2$	${}^4C_2 \times 2 = 12$
One line and one circle: ${}^5C_1 \times {}^4C_1$	${}^5C_1 \times {}^4C_1 \times 2 = 40$
Total	62

76. **a.** The number of selection of two parallel lines from m lines is ${}^mC_2$.

The number of selection of two parallel lines from n lines is ${}^nC_2$. Hence, the number of parallelograms lines is

$$ {}^mC_2 \times {}^nC_2 = \frac{1}{4} mn(m-1)(n-1) $$

77. **b.** Let $x = p - 5$, $y = q - 5$ and $z = r - 5$, where $p, q, r \geq 0$. Then the given equation reduces to

$$p + q + r = 15 \qquad (1)$$

Now, we have to find non-negative integral solution of Eq. (1). The total number of such solutions is ${}^{15+3-1}C_{3-1} = {}^{17}C_2 = 136$.

78. **b.** Dice is marked with numbers 1, 2, 3, 4, 5, 6. If the sum of dice in three throws is 11, then observations must be 1, 4, 6; ... 1, 5, 5; ... 2, 3, 6; ... 2, 4, 5; ... 3, 3, 5; ... 3, 4, 4.

We can get this observation in 3! + 3!/2! + 3! + 3! + 3!/2! +3!/2! = 27 ways.

79. **d.** No group of four numbers from the first 12 natural numbers can have the common difference 4.

If one group including 1 is selected with the common difference 1, then the other two group can have the common difference 1 or 2.

If one group including 1 is selected with the common difference 2, then one of the other two groups can have the common difference 2 and the remaining group will have common difference 1.

If one group including 1 is selected with the common difference 3, then the other two groups can have the common difference 3.

Therefore, the required number of ways is 2 + 1 + 1 = 4.

80. **a.** Let the balls put in the box are x_1, x_2, x_3, x_4 and x_5. We have,

$$x_1 + x_2 + x_3 + x_4 + x_5 = 15, x_i \geq 2$$

$$\Rightarrow (x_1 - 2) + (x_2 - 2) + (x_3 - 2) + (x_4 - 2) + (x_5 - 2) = 5$$

$$\Rightarrow y_1 + y_2 + y_3 + y_4 + y_5 = 5, y_i = x_i - 2 \geq 0$$

The total number of ways is equal to number of non-negative integral solutions of the last equation, which is equal to ${}^{5+5-1}C_5 = {}^9C_5$.

81. **a.** Let x be the number of objects to the left of the first object chosen, y the number of objects between the first and the second, z the number of objects between the second and the third and u the number of objects to the right of the third objects. Then, $x, u \geq 0; y, z \geq 1$ and $x + y + z + u = n - 3$. Let $y_1 = y - 1$ and $z_1 = z - 1$. Then, $y_1 \geq 0, z_1 \geq 0$ such that $x + y_1 + z_1 + u = n - 5$.

The total number of non-negative integral solutions of this equation is ${}^{n-5+4-1}C_{4-1} = {}^{n-2}C_3$.

82. **a.** Obviously, A, B and C get 4, 5 and 7 objects, respectively. Then, number of distribution ways is equal to number of division of ways, which is given by 16!/(4!5!7!).

83. **b.** The number of ways is

$$\frac{(mn)!}{(n!)^m m!} m! = \frac{(mn)!}{(n!)^m}$$

84. **a.** $m + n$ counters on one side can be arranged in $\frac{(m+n)!}{m!n!}$ ways.

For each arrangement on one side, corresponding arrangement on the other side is just one as arrangements are symmetrical. Hence, the total number of arrangements is

$$\frac{(m+n)!}{m!n!} = {}^{m+n}C_m$$

85. **a.** Let x_1, x_2, x_3, x_4 be the number of times T, I, D, E appears on the coupon. Then we must have $x_1 + x_2 + x_3 + x_4 = 8$, where $1 \leq x_1, x_2, x_3, x_4 \leq 8$ (as each letter must appear once).Then the required number of combinations of coupons is equivalent to number of positive integral solutions of the above equation, which is further equivalent to number of ways of 8 identical objects distributed among 4 persons when each gets at least one objects, and is given by ${}^{8-1}C_{4-1} = {}^7C_3$.

86. **a.** Here, we have to divide 12 books into 4 sets of 3 books each. Therefore, the required number of ways is

$$\frac{12!}{(3!)^4 4!} \times 4!$$

87. **a.** Since the shelves which are to receive the books are different, therefore the required number of ways is $12!/(4!)^3$.

88. **c.** Here, we are dividing $2n$ people in n groups of 2 each, and we are concerned with mere grouping. Hence, the required number of ways is $\frac{2n!}{n!(2!)^n}$.

89. **b.** $\sum_{i=1}^{k}\frac{1}{x_i}=\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_k}=\frac{\Sigma x_i}{n}=\frac{75}{n}$

(as L.C.M. of $x_1, x_2, \ldots, x_n$ is n)

90. **a.** Three elements from set 'A' can be selected in ${}^7C_3$ ways. Their image has to be y_2. Remaining 2 images can be assigned to remaining 4 pre-images in 2^4 ways. But the function is onto, hence the number of ways is $2^4 - 2$. Then the total number of functions is ${}^7C_3 \times 14 = 490$.

91. **c.** If we put minimum number of balls required in each box, balls left are $n(n-1)/2$ which can be put in ${}^{(n^2+n-1)/2}C_{n-1}$ ways without restriction.

92. **a.** Let the blankets received by the persons are x_1, x_2, x_3 and x_4. We have,

$x_1 + x_2 + x_3 + x_4 = 15$ and $x_i \geq 2$

$\Rightarrow \quad (x_1 - 2) + (x_2 - 2) + (x_3 - 2) + (x_4 - 2) = 7$

$\Rightarrow \quad y_1 + y_2 + y_3 + y_4 = 7$ (where $y_i = x_i - 2 \geq 0$)

The required number is equal to the number of non-negative integral solutions of this equation which is equal to ${}^{4+7-1}C_7$, i.e., ${}^{10}C_7 = {}^{10}C_3$.

93. **c.** Let person P_i gets x_i number of things such that

$x_1 + x_2 + x_3 + x_4 + x_5 = 25$

Let $x_i = 2\lambda_i + 1$, where $\lambda_i \geq 0$. Then,

$2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) + 5 = 25$

$\Rightarrow \quad \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 10$

We have to simply obtain the number of non-negative integral solutions of this equation, which is equal to ${}^{14}C_4$.

94. **d.** Suppose i^{th} person receives Rs x_i; $i = 1, 2, 3, 4$

Then, $x_1 + x_2 + x_3 + x_4 = 18$, where $x_i \geq 4$

Let $y_i = x_i - 3$, $i = 1, 2, 3, 4$. Then,

$y_1 + y_2 + y_3 + y_4 = 6$

The total number of ways is equal to number of solutions of the above equation, which is given by ${}^{6+4-1}C_{4-1} = {}^9C_3 = 84$.

95. **d.** Make 1 group of 2 persons, 1 group of 4 persons and 3 groups of 3 persons among 15 persons (except 2 particular persons). Hence the number of ways by grouping method is

$$\frac{15!}{2!4!(3!)^3 3!}$$

Now we add that 2 persons in the group of 2 persons and thus number of arrangements of these groups into cars and autos is

$$\frac{15!}{2!4!(3!)^3 3!} \times 2! \times 3! = \frac{15!}{4!(3!)^3}$$

96. **a.** Since the total number of selections of r things from n things where each thing can be repeated as many times as one can is ${}^{n+r-1}C_r$. Therefore the required number is ${}^{3+6-1}C_6 = 28$.

97. **c.** $f(2n, n)$ must be equal to number of positive integer solutions of $x_1 + x_2 + \cdots + x_n = 2n$, which must be equal to ${}^{2n-1}C_{n-1} = {}^{2n-1}C_n$.

98. **b.** Let the numbers selected be x_1, x_2, x_3. We must have

$2x_2 = x_1 + x_3$

$\Rightarrow \quad x_1 + x_3 = \text{even}$.

Therefore, x_1, x_3 both are odd or both are even.

If x_1 and x_3 both are even, we can select them in ${}^{12}C_2$ ways.

Similarly, if x_1 and x_3 both are odd, we can again select them in ${}^{12}C_2$ ways. Thus, the total number of ways is $2 \times {}^{12}C_2 = 132$.

99. **b.** Given number can be rearranged as

$1, 4, 7, \ldots, 3n-2 \rightarrow 3\lambda - 2$

$2, 5, 8, \ldots, 3n-1 \rightarrow 3\lambda - 1$

$3, 6, 9, \ldots, 3n \rightarrow 3\lambda$

That means, we must take two numbers from last row or one number each from first and second rows. Therefore, the total number of ways is

$${}^nC_2 + {}^nC_1 \times {}^nC_1 = \frac{n(n-1)}{2} + n^2 = \frac{3n^2 - n}{2}$$

100. **b.** Let the arrangement be $x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8$ clearly 5 should occupy the position x_4 or x_5. Thus required number is 2(7!).

101. **c.** $480 = 2^5 \times 3 \times 5$

Now, $4n + 2 = 2(2n + 1)$ = odd multiple of 2. Thus, the total number of such divisors is $1 \times 2 \times 2 = 4$.

102. **a.**

Number of times 3 used	Pattern	Numbers of type	Number of times 3 appears
1	– – 3 – 3 – 3 – –	$3 \times 9 \times 9$	$1(3 \times 9 \times 9)$
2	– 3 3 3 3 – 3 – 3	3×9	$2(3 \times 9)$
3	333	1	3
		Total	300

Any place other than 3 is filled by 9 ways as '0' can appear anywhere which gives all types of numbers like single digit, two digits, etc.

Alternative solution:

A three-digit block from 000 to 999 means 1000 numbers, each number constituting 3 digits. Hence, the total numbers of digits which we have to write is 3000.

Since the total number of digits is 10 (0 to 9) no digit is filled preferentially. Therefore, number of times we write 3 is 3000/10 = 300.

103. **b.**

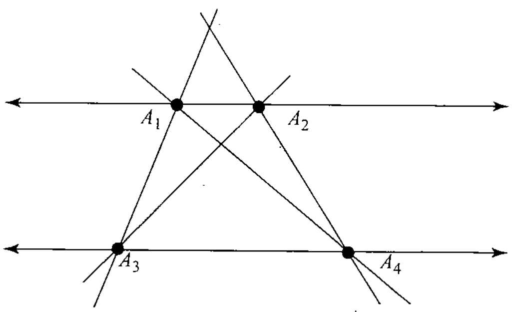


Fig. 5.44

For intersection point we must have two straight lines, for which 2 points from each straight line must be selected. Now selection of these points can be done in ${}^mC_2 \times {}^nC_2$ ways. Now as shown in diagram these four points can give two different sets of straight lines, which generate two distinct points of intersection.

Then total number of points of intersection is ${}^mC_2 \times {}^nC_2 \times 2$.

104. **a.**

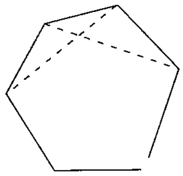


Fig. 5.45

Let the number of sides be n. A selection of four vertices of the polygon gives an interior intersection.

$\therefore \quad {}^nC_4 = 70$

$\Rightarrow \quad n(n-1)(n-2)(n-3) = 24 \times 70 = 8 \times 7 \times 6 \times 5$

$\Rightarrow \quad n = 8$

105. **a.** 26 cards can be chosen out of 52 cards in ${}^{52}C_{26}$ ways. There are two ways in which each card can be dealt, because a card can be either from the first pack or from the second. Hence the total number of ways is ${}^{52}C_{26} \times 2^{26}$.

106. **d.** Since the balls are to be arranged in a row so that the adjacent balls are of different colours, we can therefore begin with a white ball or a black ball. If we begin with a white ball, we find that $n + 1$ white balls numbered 1 to $n + 1$ can be arranged in a row in $(n + 1)!$ ways. Now $n + 2$ places are created among $n + 1$ white balls which can be filled by $n + 1$ black balls in $(n + 1)!$ ways.

So, the total number of arrangements in which adjacent balls are of different colours and first ball is a white ball is $(n + 1)! \times (n + 1)! = [(n + 1)!]^2$. But we can begin with a black ball also. Hence, the required number of arrangements is $2[(n + 1)!]^2$.

107. **a.** If zero is included it will be at $z \Rightarrow {}^9C_2$

$$\text{If zero is excluded} \begin{cases} x, y, z \text{ all diff.} & \Rightarrow \quad {}^9C_3 \times 2! \\ x = z < y & \Rightarrow \quad {}^9C_2 \\ x < y = z & \Rightarrow \quad {}^9C_2 \end{cases}$$

The total number of ways is 276.

Alternative method:

y can be from 2 to 9; so total number of ways is

$$\sum_{r=2}^{9}(r^2 - 1) = 276$$

108. **b.** The two common elements can be selected in ${}^nC_2$ ways. Remaining $n - 2$ elements, each can be chosen in three ways, i.e., $a \in P$ and $a \notin Q$ or $a \in Q$ and $a \notin P$ or a is neither in P nor in Q. Therefore, the total number of ways is ${}^nC_2 \times 3^{n-2}$.

109. **d.** We will consider the following cases

Case	Flags	No. of signals
4 alike and 2 others alike	4 white and 2 red	$\frac{6!}{4!\,2!} = 15$
4 alike and 2 others different	4 white, 1 red and 1 blue	$\frac{6!}{4!} = 30$
3 alike and 3 others alike	3 white, 3 red	$\frac{6!}{3!\,3!} = 20$
3 alike and 2 other alike and 1 different	3 white, 1 blue, 2 red or 3 red, 1 blue, 2 white	${}^2C_1 \times \frac{6!}{3!2!} = 120$
	Total	185

110. **c.** The total number of ways of selection without restriction is ${}^{20}C_3$. The number of ways of selection when two are adjacent is $20 \times {}^{16}C_1$. The number of ways of selection when all the three are adjacent is 20. The required number of ways is

$${}^{20}C_3 - 20 \times 16 - 20 = \frac{20 \times 19 \times 18}{6} - 20 \times 16 - 20$$
$$= 20[57 - 16 - 1]$$
$$= 20 \times 40 = 800$$

111. **c.** Sum of 7 digits is a multiple of 9. Sum of numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 is 45; so two left digits should also have sum of 9. The pairs of left numbers are (1, 8), (2, 7), (3, 6), (4, 5). With each pair left number of 7-digit number is 7! So with all 4 pairs, total number is 4 × 7!.

112. **b.** If n is odd,

$3^n = 4\lambda_1 - 1,\ 5^n = 4\lambda_2 + 1$

$\Rightarrow \quad 2^n + 3^n + 5^n$ is divisible by 4 if $n \geq 2$

Thus, $n = 3, 5, 7, 9, \ldots, 99$, i.e., n can take 49 different values. If n is even, $3^n = 4\lambda_1 + 1,\ 5^n = 4\lambda_2 + 1$

$\Rightarrow \quad 2^n + 3^n + 5^n$ is not divisible by 4

as $2^n + 3^n + 5^n$ will be in the form of $4\lambda + 2$.

Thus, the total number of ways of selecting 'n' is equal to 49.

113. **c.**

Number of digits	Numbers ending with 0	Numbers ending with 5	Total
×	0	1	1
××	8	9	17
×××	9·8 = 72	8·8 = 64	136
××××	9·8·7 = 504	8·8·7 = 448	952
		Total	1106

114. **d.** Let x, y, z be the friends and a, b, c denote the case when x is invited a times, y is invited b times and z is invited c times.

Now, we have the following possibilities:

$(a, b, c) = (1, 2, 3)$ or $(3\ 3\ 0)$ or $(2\ 2\ 2)$

[grouping of 6 days of week]

Hence, the total number of ways is

$$\frac{6!}{1!2!3!}3! + \frac{6!}{3!3!2!}3! + \frac{6!}{(2!2!2!)3!}3!$$

$= 360 + 60 + 90 = 510$

115. **b.** There is concept of derangement. The required number is

$$4!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right] = 9$$

116. **c.** When at least one one-rupee coin is selected we can select any number of twenty five coins and ten paisa coins. Then number of ways of such selection is $4(2 + 1)(5 + 1) = 72$ as we can select zero or more twenty-five paisa and ten paisa coins to ensure that amount selected is Re. 1 or more.

But when none of one-rupee coins is selected we have to select all twenty-five paisa coins and ten paisa coins to ensure sum of Re. 1, which can be done only in one way. Then the total number of ways is 73.

117. b. Since the number of students giving wrong answers to at least i questions ($i = 1, 2, \ldots, n$) is 2^{n-i}.

The number of students answering exactly i ($1 \le i \le -1$) questions wrongly = {the number of students answering at least i questions wrongly, $i = 1, 2, \ldots$} − {the number of students answering at least $(i + 1)$ questions wrongly ($2 \le i + 1 \le n$)} $= 2^{n-i} - 2^{n-(i+1)}$ $(1 \le i \le n-1)$.

Now, the number of students answering all the n questions wrongly is $2^{n-2} = 2^0$. Thus, the total number of wrong answers is

$$1(2^{n-1} - 2^{n-2}) + 2(2^{n-2} - 2^{n-3}) + 3(2^{n-3} - 2^{n-4}) + \cdots + (n-1)(2^1 - 2^0) + n(2^0)$$

$$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2^0 = 2^n - 1 \quad (\because \text{ it is a G.P.})$$

Therefore, as given,

$$2^n - 1 = 2047 \Rightarrow 2^n = 2048 = 2^{11} \Rightarrow n = 11$$

118. c. For a particular class the total number of different tickets from first intermediate station is 5. Similarly, number of different tickets from second intermediate station is 4. So the total number of different tickets is $5 + 4 + 3 + 2 + 1 = 15$. And same number of tickets for another class is equal to total number of different tickets, which is equal to 30 and number of selection is ${}^{30}C_{10}$.

119. b. The number of trains a day (the digits 1, 2, 3) are three groups of like elements from which a sample must be formed. In the time-table for a week, the number 1 is repeated twice, the number 2 is repeated 3 times and the number 3 is repeated twice.

The number of different time-tables is given by

$$p(2, 3, 2) = \frac{7!}{2!3!2!} = 210$$

120. a. $15 < x_1 + x_2 + x_3 \le 20$

$\Rightarrow x_1 + x_2 + x_3 = 16 + r,\ r = 0, 1, 2, 3, 4$

Now, the number of positive integral solution of $x_1 + x_2 + x_3 = 16 + r$ is ${}^{13+r+3-1}C_{13+r}$, i.e., ${}^{15+r}C_{13+r} = {}^{15+r}C_2$

Thus, the total number of solutions is

$$\sum_{r=0}^{4} {}^{15+r}C_2 = {}^{15}C_2 + {}^{16}C_2 + {}^{17}C_2 + {}^{18}C_2 + {}^{19}C_2$$

$$= \frac{1}{2}(15\times14 + 16\times15 + 17\times16 + 18\times17 + 19\times18)$$

$$= 685$$

Multiple Correct Answers Type

1. b, c.

If a, b, c are in A.P., then a and c both are odd or both are even.

Case I: n is even.

The number of ways of selection of two even numbers a and c is ${}^{n/2}C_2$. Number of ways of selection of two odd numbers is ${}^{n/2}C_2$. Hence the number of A.P.'s is

$$2\,{}^{n/2}C_2 = 2\frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{2} = \frac{n(n-2)}{4}$$

Case II: n is odd.

The number of ways of selection of two odd numbers a and c is ${}^{(n+1)/2}C_2$. The number of ways of selection of two even numbers a and c is ${}^{(n-1)/2}C_2$. Hence the number of A.P.'s is ${}^{(n+1)/2}C_2 + {}^{(n-1)/2}C_2$

$$= \frac{\left(\frac{n+1}{2}\right)\left(\frac{n+1}{2}-1\right)}{2} + \frac{\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}-1\right)}{2}$$

$$= \frac{1}{8}(n-1)((n+1)+(n-3))$$

$$= \frac{(n-1)^2}{4}$$

2. b, c.

The number of ways of inviting, with the couple not included, is ${}^8C_5$. The number of ways of inviting with the couple included, is ${}^8C_3$. Therefore the required number of ways is

$${}^8C_5 + {}^8C_3 = {}^8C_3 + {}^8C_3 \quad (\because {}^8C_5 = {}^8C_3)$$

Also,

$${}^{10}C_5 - 2\times{}^8C_4 = \frac{10!}{5!5!} - 2\times\frac{8!}{4!4!}$$

$$= \frac{10\times9\times8\times7\times6}{120} - 2\times\frac{8\times7\times6\times5}{24}$$

$$= 9\times4\times7 - 140$$

$$= 112 = 2\times\frac{8!}{3!5!}$$

3. a, b, d.

$p = {}^5C_4 \times {}^2C_1 = 10$

$q = {}^5C_2({}^2C_1)^3 = 80$

$r = {}^5C_0({}^2C_1)^5 = 32$

$\Rightarrow 2q = 5r,\ 8p = q$ and $2(p + r) > q$

4. a, b, d.

Clearly, each player will play 9 games. And total number of games is ${}^{10}C_2 = 45$. Clearly,

$w_i + l_i = 9$ and $\Sigma w_i = \Sigma l_i = 45$

$\Rightarrow w_i = 9 - l_i \Rightarrow w_i^2 = 81 + l_i^2 - 18l_i$

$\Rightarrow \Sigma w_i^2 = 81\times10 + \Sigma l_i^2 - 180\,\Sigma l_i = 810 + \Sigma l_i^2 - 18\times45 = \Sigma l_i^2$

5. b, c, d.

When $z = n + 1$, we can choose x, y from $\{1, 2, \ldots, n\}$.

When $z = n + 1$, x, y can be chosen in n^2 ways and when $z = n$, x, y can be chosen in $(n-1)^2$ ways and so on. Therefore, the number of ways of choosing triplets is

$$n^2 + (n-1)^2 + \cdots + 1^2 = \frac{1}{6}n(n+1)(2n+1)$$

Alternatively triplets with $x = y < z$, $x < y < z$, $y < z < x$ can be chosen in ${}^{n-1}C_2$, ${}^{n+1}C_3$, ${}^{n+1}C_3$ ways. Therefore,

$${}^{n+1}C_2 + 2({}^{n+1}C_3) = {}^{n+2}C_2 + {}^{n+1}C_3 = 2({}^{n+2}C_3) - {}^{n+1}C_2$$

6. a, b, c.

$$\frac{(200)!}{\underbrace{2!\,2!\ldots2!}_{100\text{ times}}(100)!}$$

$$= \frac{(200)!}{100!\,2^{100}}$$

$$= 1 \times 3 \times 5 \cdots 199$$

Also,

$$\frac{(200)!}{100!\,2^{100}} = \left(\frac{101}{2}\right)\left(\frac{102}{2}\right)\cdots\left(\frac{200}{2}\right)$$

7. a, b, c, d.

8, 7, 6, 4, 2, x and y

Any number is divisible by 3 if sum of digits is divisible by 3, i.e., $x + y + 27$ is divisible by 3. x and y can take values from 0, 1, 3, 5, 9. Possible pairs are (5, 1) (3, 0) (9, 0) (9, 3) and (1, 5), (0, 3) (0, 9) (3, 9).

8. a, d.

Problem is same as dividing 17 identical things in two groups.

$$\therefore \quad n = \frac{17+1}{2} = 9$$

There is no effect if two diamonds are different as necklace can be flipped over. Hence, $n = m = 9$.

9. a, b, c.

The number of regions for 'n' circles be $f(n)$. Clearly, $f(1) = 2$. Now,

$f(n) = f(n-1) + 2(n-1),\ \forall\ n \geq 2$

$\Rightarrow\ f(n) - f(n-1) = 2(n-1)$

Putting $n = 2, 3, \ldots, n$, we get

$f(n) - f(1) = 2(1 + 2 + 3 + \cdots n - 1) = (n-1)n$

$\Rightarrow\ f(n) = n(n-1) + 2 = (n^2 - n + 2)$ (which is always even)

$\Rightarrow\ f(20) = 20^2 - 20 + 2 = 382$

Also,

$n^2 - n + 2 = 92$

$\Rightarrow\ n^2 - n - 90 = 0 \Rightarrow n = 10$

10. a, c.

$$3^p = (4-1)^p = 4\lambda_1 + (-1)^p$$

$$5^q = (4+1)^q = 4\lambda_2 + 1$$

$$7^r = (8-1)^r = 4\lambda_3 + (-1)^r$$

Hence, any positive integer power of 5 will be in the form of $4\lambda_2 + 1$. Even power of 3 and 7 will be in the form of $4\lambda + 1$ and odd power of 3 and 7 will be in the form of $4\lambda - 1$. Hence, both p and r must be odd or both must be even. Thus $p + r$ is always even. Also, $p + q + r$ can be odd or even.

11. a, b, c.

When $n = 3k$, there are exactly $n/3$ integers of each type $3p$, $3p + 1$, $3p + 2$.

Now, sum of three selected integers is divisible by 3. Then either all the integers of the same type $3p$, $3p + 1$ or $3p + 2$ or one-one integer from each type. Then number of selection ways is ${}^{n/3}C_3 + {}^{n/3}C_3 + {}^{n/3}C_3 + ({}^{n/3}C_1)({}^{n/3}C_1)({}^{n/3}C_1) = 3({}^{n/3}C_3) + (n/3)^3$.

If $n = 3k + 1$, then there are $(n-1)/3$ integers of the type $3p$, $3p + 2$ and $(n + 2)/3$ integers of the type $3p + 1$. Then number of selection ways is $2({}^{(n-1)/3}C_3) + ({}^{(n+2)/3}C_3) + ((n-1)/3)^2(n + 2)$.

When $n = 3k + 2$, the number of selection ways are same as in the case of $n = 3k + 1$.

12. a, c.

$${}^{n+5}P_{n+1} = \frac{11(n-1)}{2} \times {}^{n+3}P_n$$

$$\Rightarrow\ {}^{n+5}P_{n+1} = \frac{(n+5)!}{4!} = \frac{11(n-1)}{2}\frac{(n+3)!}{3!}$$

$$\Rightarrow\ (n+5)(n+4) = 22(n-1)$$

After solving, we get $n = 6$ or $n = 7$.

The number of points of intersection of lines is ${}^6C_2$ or ${}^7C_2$ $\equiv 15$ or 21.

13. a, b, d.

Total number of units to be covered is $3 + 7 + 11 = 21$. A person can choose 3 units in ${}^{21}C_3$ ways. A person can choose 7 units in ${}^{18}C_7$ ways. The rest 11 units can be chosen in 1 way. Therefore, total number of ways is ${}^{21}C_3 \times {}^{18}C_7 \times 1 = 21!/(3!7!11!)$.

For correct answers (a) and (d), see the respective theory.

For correct answer (b) see the theory of multinomial expansion in the binomial theorem.

14. a, c.

Let person P_i get x_i number of things such that

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 30$$

If x_i is odd or $x_i = 2\lambda_i + 1$, where $\lambda_i \geq 0$, then

$$2(\lambda_i + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6) + 6 = 30$$

$$\Rightarrow\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 12$$

Then number of solutions is ${}^{12+6-1}C_{6-1} = {}^{17}C_5$. If x_i is even or $x_i = 2\lambda_i$, where $\lambda_i \geq 1$, then

$$2(\lambda_i + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6) = 30$$

$$\Rightarrow\ \lambda_i + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 15$$

Therefore, required number of ways is ${}^{15-1}C_{6-1} = {}^{14}C_5$.

15. a, c.

Let x_i $(1 \leq i \leq n)$ be the number of objects selected of the i^{th} type. Since each object is to be selected at least once, we must have $x_i \geq 1$ and $x_1 + x_2 + \cdots + x_n = r$. We have to find number of positive integral solutions of the above equation. Total number of such solutions is ${}^{r-1}C_{n-1} = {}^{r-1}C_{r-n}$.

16. a, d.

Let $A = \{a_1, a_2, \ldots, a_n\}$. For each $a_i (1 \leq i \leq n)$, we have either $a_i \in P_j$ or $a_i \notin P_j$ $(1 \leq j \leq m)$. That is, there are 2^m choices in which a_i $(1 \leq i \leq n)$ may belong to the P_j's. One of these, there is only one choice, in which $a_i \in P_j$ for all $j = 1, 2, \ldots, m$ which is not favourable for $P_1 \cap P_2 \cap \cdots \cap P_m$ to be ϕ. Thus, $a_i \notin P_1 \cap P_2 \cap \cdots \cap P_m$ in $2^m - 1$ ways.

Since there are n elements in set A, the total number of choices is $(2^m - 1)^n$.

Also, there is exactly one choice, in which, $a_i \notin P_j$ for all $j = 1, 2, \ldots, m$ which is not favourable for $P_1 \cup P_2 \cup \cdots \cup P_m$ to be equal to A.

Thus, a_i can belong to $P_1 \cup P_2 \cup \cdots \cup P_m$ in $(2^m - 1)$ ways.

Since there are n elements in set A, the number of ways in which $P_1 \cup P_2 \cup \cdots \cup P_m$ can be equal to A is $(2^m - 1)^n$.

17. b, c, d.

Exponent of 2 is

$$\left[\frac{10}{2}\right]+\left[\frac{10}{2^2}\right]+\left[\frac{10}{2^3}\right]=5+2+1=8$$

Exponent of 3 is

$$\left[\frac{10}{3}\right]+\left[\frac{10}{3^2}\right]=3+1=4$$

Exponent of 5 is

$$\left[\frac{10}{5}\right]=2$$

Exponent of 7 is

$$\left[\frac{10}{7}\right]=1$$

The number of divisors of 10! is (8 + 1)(4 + 1)(2 + 1)(1 + 1) = 270. The number of ways of putting N as a product of two natural numbers is 270/2 = 135.

18. b, c, d.

$$P = 21(21+1)(21-1)(21+2)(21-2)\cdots(21+10)(21-10)$$
$$= (21-10)(21-9)\cdots(21-1)\,21\,(21+1)(21+10)\cdots(21+10)$$
$$= 41 \times 40 \cdots 11$$

which is divisible by 21!, and hence by 20! and 19!

19. a, d.

When n is odd:

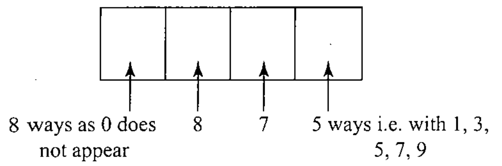


Fig. 5.46

The number of such numbers is 8 × 8 × 7 × 5 = 2240.

When n is even:

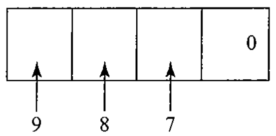


Fig. 5.47

If unit's place is filled with 0, then the total number is 9 × 8 × 7 = 504.

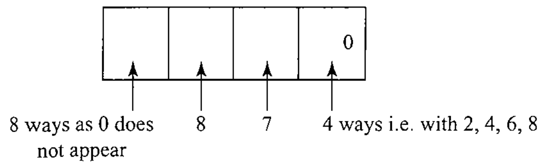


Fig. 5.48

If unit's place is not filled with 0, then the total number is 8 × 8 × 7 × 4 = 1792.

Hence, the total number of even numbers is y = 504 + 1792 = 2296.

Reasoning Type

1. **a.** We have, 30 = 2 × 3 × 5. So, 2 can be assigned to either a or b or c, i.e. 2 can be assigned in 3 ways. Similarly, each of 3 and 5 can be assigned in 3 ways. Thus, the number of solutions is 3 × 3 × 3 = 27.

2. **d.** Each student receives at least two toys. Let us first give each student one toy. Now, we are left with 7 toys, which can be distributed among three students such that each receives at least one toy, which is equivalent to number of positive integral solutions of the equation $x + y + z + w = 7$, which is given by ${}^{7-1}C_{4-1} = {}^6C_3$.

   Hence, statement 1 is false and statement 2 is correct.

3. **a.** Number of ways of dividing n^2 objects into n groups of same size is $\dfrac{(n^2)!}{(n!)^n n!}$.

Now number of ways of distributing these n groups among n persons is

$$\left[\frac{(n^2)!}{(n!)^n n!}\right] n! = \frac{(n^2)!}{(n!)^n}$$ which is always an integer.

Also we know that product of r is divisible by r! Now, $(n^2)! = 1 \times 2 \times 3 \times 4 \cdots n^2$

$$= 1 \times 2 \times 3 \cdots n$$
$$\times (n+1)(n+2)\cdots 2n$$
$$\times (2n+1)(2n+2)\cdots 3n$$
$$\times (n^2-(n^2-1))(n^2-(n^2-1)) \cdots n^2$$

Thus, in n^2! there are n rows each consisting product of n integers. Each row is divisible by n!

Hence $(n^2)!$ is divisible by $(n!)^n$ or $\dfrac{(n^2)!}{(n!)^n}$ is a natural number.

Hence, both statements are correct and statement 2 is correct explanation of statement 1.

4. **b.** $1400 = 2^3 5^2 7$

The number of ways in which 1400 can be expressed as a product of two positive integers is

$$\frac{(3+1)(2+1)(1+1)}{2} = 12$$

Statement 2 is correct but does not explain statement 1 as it just gives the information about the prime factor about which 1400 is divisible.

5. **a.** In onto functions each image must be assigned at least one pre-image. Now if we consider the images a and b as two different boxes, then four distinct objects 1, 2, 3 and 4 (pre-images) can be distributed in $2^4 - {}^2C_1(2-1)^4 = 14$ ways.

   Hence, both statements are correct and statement 2 is correct explanation of statement 1.

6. **b.** India must win at least 6 matches of 11 matches. Then number of ways in which India can win the series is ${}^{11}C_6 + {}^{11}C_7 + \cdots + {}^{11}C_{11} = 2^{10}$

   Thus, both the statements are true, but statement 2 is not correct explanation of statement 1.

7. **a.** The batting order of 11 players can be decided in 11! ways. Now Yuvraj, Dhoni and Pathan can be arranged in 3! ways.

But the order of these three players is fixed, i.e., Yuvraj–Dhoni–Pathan. Now, 11! answer is 3! times more, hence the required answer is 11!/3!.

8. **d.** Number of ways of arranging 21 identical objects when r is identical of one type and remaining are identical of second type is $\frac{21!}{r!(21-r)!} = {}^{21}C_r$ which is maximum when $r = 10$ or 11.

Therefore, ${}^{13}C_r = {}^{13}C_{10}$ or ${}^{13}C_{11}$, hence maximum value of ${}^{13}C_r$ is ${}^{13}C_{10} = 286$.

Hence, statement 1 is false. Obviously statement 2 is true.

9. **a.** A number is divisible by 4, if the last two digits are divisible by 4. Last two digits can be 12, 16, 28, 32, 36, 68, 92, 96. Thus, last two places can be filled in 8 ways. The remaining three places can be filled with remaining 4 digits in ${}^4C_3\,3!$ ways. Total number of such numbers is $8 \times ({}^4C_3\,3!) = 192$.

10. **b.** We have $a + b + c = 30$, and $a \neq b \neq c$. Let $a < b < c$.
Now, relative values of a, b, c are tabulated as follows.

a	***b***	***c***	**Number of triplets (*a*, *b*, *c*)**
1	2	27	
	3	26	
	4	25	
	⋮	⋮	
	14	15	13
2	3	25	
	4	24	
	⋮	⋮	
	13	15	11
3	4	23	
	5	22	
	⋮	⋮	
	13	14	10
4	5	21	
	6	20	
	⋮	⋮	
	12	14	8
5	6	19	
	7	18	
	⋮	⋮	
	12	13	7
6	7	17	
	⋮	⋮	
	11	13	5
7	8	15	
	⋮	⋮	
	11	12	4
8	9	13	
	10	12	2
9	10	11	1
		Total	61

Statement 2 is correct but it does not explain statement 1.

11. **c.** When $p, q < r$, we have selection procedure as follows :

From *p* identical things	**From *q* identical things**
p	$r-p$
$p-1$	$r-(p-1)$
$p-2$	$r-(p-2)$
⋮	⋮
⋮	⋮
$r-q$	q
	Total: $p+q-r+1$

When $p, q > r$, we have selection procedure as follows:

From *p* identical things	**From *q* identical things**
r	0
$r-1$	1
$r-2$	2
⋮	⋮
⋮	⋮
0	r
	Total: $r+1$

Thus, statement 1 is correct, but statement 2 is false.

12. **a.** Statement 2 is correct as when $3^a, 3^b, 3^c$ are in G.P., we have $(3^b)^2 = (3^a)(3^c) \Rightarrow 2b = a + c \Rightarrow a, b, c$ are in A.P. Thus, selecting three numbers in G.P. from $\{3^1, 3^2, 3^3, \ldots, 3^{100}, 3^{101}\}$ is equivalent to selecting 3 numbers from $\{1, 2, 3, \ldots, 101\}$ which are in A.P. Now, a, b, c are in A.P. if either a and c are odd or a and c are even.

Number of selection ways of 2 odd numbers is ${}^{51}C_2$. Number of selection ways of 2 even numbers is ${}^{50}C_2$. Hence, total number of ways is ${}^{51}C_2 + {}^{50}C_2 = 1275 + 1225 = 2500$.

13. **a.** Statement 2 is true, see the proof in binomial theorem. Also in statement 1, if A selects i objects and B selects j objects then $i < j$. Hence number of ways is $\sum\limits_{0 \le i < j \le 20} {}^{20}C_i\,{}^{20}C_j$.

14. **c.**

Number of objects from 21 different objects	**Number of objects from 21 identical objects**	**Number of ways of selections**
10	0	${}^{21}C_{10} \times 1$
9	1	${}^{21}C_9 \times 1$
⋮	⋮	⋮
0	10	${}^{21}C_0 \times 1$

Thus, total number of ways of selection is ${}^{21}C_0 + {}^{21}C_1 + {}^{21}C_2 + \cdots + {}^{21}C_{10} = 2^{20}$.

Statement 2 is false, as given series is not exact half series. (For details, see the theory in binomial theorem.)

15. **a.** General in the expansion of $(x + y + z + w)^{50}$ is

$$\frac{50!}{p!q!r!s!} x^p y^q z^r w^s$$

where $p + q + r + s = 50$, $0 \le p, q, r, s \le 50$. (see theory in binomial theorem).

Now number of terms is equal to number of ways in which we can adjust powers of x, y, z and w such that their sum is

50, i.e., equal to the non-negative solutions of $p + q + r + s = 50$, which is given by ${}^{50+4-1}C_{4-1}$.

16. a. When n persons be seated at a round table, so that all shall not have the same neighbours in any two arrangements, clockwise and anticlockwise arrangements are considered to be the same, which is the case when n different beads are arranged in the circle. Hence, number of ways is $(n-1)!/2$.

17. b. Exponent of 2 in 50! is

$$\left[\frac{50}{2}\right]+\left[\frac{50}{4}\right]+\left[\frac{50}{8}\right]+\left[\frac{50}{16}\right]+\left[\frac{50}{32}\right]=25+12+6+3+1=47.$$

And exponent of 5 in 50! is

$$\left[\frac{50}{5}\right]+\left[\frac{50}{25}\right]=12$$

Now number of zeros in the end of 50! is equal to exponent of 10 in 50! which is equal to exponent of 5 in 50! Therefore, number of zeros in the end depends on exponent of 5, but not on the exponent of 2.

Hence both statements 1 and 2 are true; but statement 2 is not a correct explanation for statement 1.

Linked Comprehension Type

For Problems 1–3

1. a, 2. b, 3. b.

Sol. 1. When one all rounder and ten players from bowlers and batsmen play, number of ways is ${}^4C_1\,{}^{14}C_{10}$.

When one wicketkeeper and 10 players from bowlers and batsmen play, number of ways is ${}^2C_1\,{}^{14}C_{10}$.

When one all rounder, one wicketkeeper and nine from batsmen and bowlers play, number of ways is ${}^4C_1\,{}^2C_1\,{}^{14}C_9$.

When all elven players play from bowlers and batsmen then, number of ways is ${}^{14}C_{11}$.

Total number of selections is ${}^4C_1\,{}^{14}C_{10} + {}^2C_1\,{}^{14}C_{10} + {}^4C_1\,{}^2C_1\,{}^{14}C_9 + {}^{14}C_{11}$.

2. If the particular bowler plays then two batsmen will not play. So, rest of 10 players can be selected from 17 other players. Number of such selections is ${}^{17}C_{10}$.

If the particular bowler does not play, then number of selections is ${}^{19}C_{11}$.

If all the three players do not play, then number of selections is ${}^{17}C_{11}$.

Total number of selections is ${}^{17}C_{10} + {}^{19}C_{11} + {}^{17}C_{11}$.

3. If the particular batsman is selected, then rest of 10 players can be selected in ${}^{18}C_{10}$ ways.

If particular wicketkeeper is selected, then rest of 10 players can be selected in ${}^{18}C_{10}$ ways.

If both are not selected, then number of ways is ${}^{18}C_{11}$.

Hence, total number of ways is $2\,{}^{18}C_{10} + {}^{18}C_{11} = {}^{19}C_{11} + {}^{18}C_{10}$.

For Problems 4–6

4. c, 5. b, 6. a.

Sol. 4. Seven persons can be selected for first table in ${}^{12}C_7$ ways. Now these seven persons can be arranged in 6! ways. The remaining five persons can be arranged on the second table in 4! ways.Hence, total number of ways is ${}^{12}C_5 6!4!$.

5. Here, A can sit on first table and B on the second or A on second table and B on the second table.

If A is on the first table, then remaining six for first table can be selected in ${}^{10}C_6$ ways. Now these seven persons can be arranged in 6! ways. Remaining five can be arranged on the other table in 4! ways.

Hence, total number of ways is $2\,{}^{10}C_6\,6!4!$.

6. If A, B are on the first table, then remaining five can be selected in ${}^{10}C_5$ ways.

Now seven persons including A and B can be arranged on the first table in which A and B are together in 2!5! ways. Remaining five can be arranged on the second table in 4! ways. Total number of ways is ${}^{10}C_5\,4!5!2!$

If A, B are on the second table, then remaining three can be selected in ${}^{10}C_3$ ways.

Now five persons including A and B can be arranged on the second table in which A and B are together 2!3! ways. Remaining seven can be arranged on the first table in 6! ways. Hence, number of ways for first table is ${}^{10}C_7\,6!3!2!$.

For Problems 7–9

7. b, 8. c, 9. d.

Sol. If no box remains empty, then we can have (1, 1, 3) or (1, 2, 2) distribution pattern.

7. When balls are different and boxes are identical, number of distributions is equal to number of divisions in (1, 2, 3) or (1, 2, 2) ways. Hence, total number of ways is

$$\frac{5!}{1!\cdot 2!\cdot 3!}+\frac{5!}{(2!)^2\,1!\cdot 2!}=25$$

8. When balls as well as boxes are identical, we have only two ways (1, 1, 3) and (1, 2, 2). Hence, number of ways is 2.

9. When boxes are kept in a row, they will be treated as different. In this case the number of ways will be ${}^{5-1}C_{3-1} = {}^4C_2 = 6$.

For Problems 10–12

10. b, 11. c, 12. b.

Sol. $6 = 0(2) + 6(1) = 1(2) + 4(1) = 2(2) + 2(1) = 3(2) + 0(1)$

Number of 2s	Number of 1s	Number of permutations
0	6	1
1	4	$\frac{5!}{4!}=5$
2	2	$\frac{4!}{2!\,2!}=6$
3	0	$\frac{3!}{3!}=1$
		Total = 13

$\therefore\ f(6) = 13$

Now, $f(f(6)) = f(13)$

Number of 1s	Number of 2s	Number of permutations
13	0	1
11	1	$\frac{12!}{11!}=12$

9	2	$\frac{11!}{9!2!}=55$
7	3	$\frac{10!}{7!3!}=120$
5	4	$\frac{9!}{5!4!}=126$
3	5	$\frac{8!}{3!5!}=56$
1	6	$\frac{7!}{6!}=7$
		Total = 377

$\therefore \quad f(f(6)) = f(13) = 377$

$f(1) = 1$ (1)

$f(2) = 2$ (1, 1 or 2)

$f(3) = 3$ (1, 1, 1 or 2, 1 or 1, 2)

$f(4) = 5$ (explained in the paragraph)

By taking higher value of n in $f(n)$, we always get more value of $f(n)$. Hence, $f(x)$ is one-one. Clearly, $f(x)$ is into.

For Problems 13–15

13. c, 14. a, 15. d.

Sol.

13. Let x_0 denote the number of empty seats to the left of the first person, x_i $(1 \le i \le n-1)$ the number of empty seats between the i^{th} and $(i+1)^{th}$ persons of the n^{th} person. Then $x_0, x_n \ge 0$ and $x_i \ge 1$ for $1 \le i \le n-1$.

$$x_0 + x_1 + \cdots + x_n = (m-n) \quad (1)$$

Putting $x_i = y_i + 1$, where $y_i \ge 0$, we have

$$x_0 + y_1 + \cdots + y_{n-1} + x_n + (1 + 1 + 1 + \cdots + (n-1) \text{ times}) = (m-n)$$

$$\Rightarrow \quad x_0 + y_1 + \cdots + y_{n-1} + x_n = m - n - (n-1)$$

$$\Rightarrow \quad x_0 + y_1 + \cdots + y_{n-1} + x_n = m - 2n + 1$$

Now number of non-negative integral solutions is ${}^{n+1+(m-2n+1)-1}C_{n+1-1} = {}^{m-n+1}C_n$. Since we can permute n persons in $n!$ ways, the required number of ways is

$$\left({}^{m-n+1}C_n\right)(n!) = \frac{(m-n+1)!}{n!(m-2n+1)!}\,n! = \frac{(m-n+1)!}{(m-2n+1)!}$$

14. Let $n = 2k$, where k is some positive integer. Let x_0 denote the number of empty seats to the left of the first pair, x_i $(1 \le i \le k-1)$ the number of empty seats between i^{th} and $(i+1)^{th}$ pair and x_k the number of empty seats to the right of the k^{th} pair. Note that $x_0, x_k \ge 0$, $x_i \ge 1$ $(1 \le i \le k-1)$ and

$$x_0 + x_1 + \cdots + x_k = (m-2k) \quad (2)$$

The number of integral solutions of Eq. (2) is ${}^{m-2k+1}C_k$ (see the solution to problem 13).

Since we can permute n persons in $n!$ ways, the required number of ways is

$$\left({}^{m-2k+1}C_k\right)(2k)! = \frac{(m-2k+1)!}{k!(m-3k+1)!}(2k)!$$

$$= \frac{(2k)!}{(k)}\frac{(m-2k+1)!}{(m-3k+1)!}$$

$$= ({}^{2k}P_k)\,({}^{m-2k+1}P_k)$$

$$= ({}^{n}P_{n/2})\,({}^{m-n+1}P_{n/2})$$

15. m is even. Let $m = 2k$, where k is some positive integer. We can choose n seats out of the k seats to the left of the middle seat in ${}^kC_n$ ways. Each chosen seat can be either empty or occupied. Thus, the number of ways of choosing seats for n persons is equal to $({}^kC_n)\,(2^n)$. We can arrange n persons at these seats in ${}^nP_n$ ways. Hence, the required number of arrangements is given by

$$(n!)\,({}^kC_n)\,(2^n) = ({}^kP_n)\,(2^n) = ({}^{m/2}P_n)(2^n)$$

For Problems 16–18

16. d, 17. b, 18. c.

Sol.

16. Since there are 5 even places and 3 pairs of repeated letters, therefore at least one of these must be at an odd place. Therefore, the number of ways is 11!/(2!2!2!).

17. Make a group of both M's and another group of T's. Then except A's we have 5 letters remaining. So M's, T's and the letters except A's can be arranged in 7! ways. Therefore, total number of arrangements is $7! \times {}^8C_2$.

18. Consonants can be placed in 7!/(2!2!) ways. Then there are 8 places and 4 vowels. Therefore, number of ways is

$$\frac{7!}{2!2!}\,{}^8C_4\,\frac{4!}{2!}$$

Matrix-Match Type

1. a → r; b → s; c → p; d → q.

a. ${}^{10}C_2 - {}^4C_2 + 1 = 45 - 6 + 1 = 40$

b. $1 \times {}^{10}C_2 = 45$

c. $2 \times {}^6C_2 = 30$

d. ${}^6C_2 \times 4 = 60$

2. a → q; b → r; c → s; d → p.

a. Number of rectangles is equal to number of ways we can select two vertical lines and two horizontal lines. Total number of ways is ${}^7C_2 \times {}^7C_2 = 441$.

b.

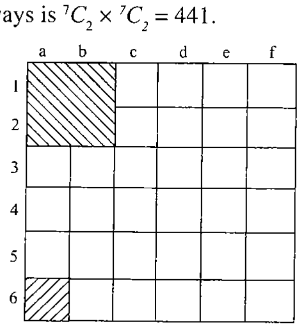


Fig. 5.49

If the square is of 1 sq. units like $a6$, then we have such $6 \times 6 = 36$ squares.

If the square is of 4 sq. units like shaded region of the squares a1, $a2$, $b1$, $b2$, then we have such 5 squares in the belt formed by rows 1 and 2. Similarly we have 4 more belts 23, 34, 45 and 56. Hence, there are $5 \times 5 = 25$ such squares.

Similarly we have 4×4, 3×3, 2×2, 1×1 squares of increasing sizes.

Hence, total number of squares is $1 + 4 + 9 + 16 + 25 + 36 = 91$.

c.

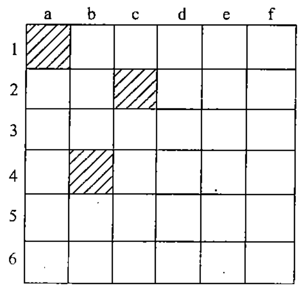


Fig. 5.50

The first square can be selected in 36 ways. If one such square $a1$ is selected, we are left with 25 squares; second square cannot be selected from row 1 and column a. If second square is $c2$, we are left with 16 squares, from which third square can be selected, e.g., $b4$.

Hence, number of ways of selections is $36 \times 25 \times 16$. But in this one-by-one type of selection order of selection is also considered. Hence, actual number of ways is $(36 \times 25 \times 16)/3! = 2400$.

d. Given number of ways is equivalent to selecting 11 squares from 36 squares if no row remains empty.

Suppose $x_1, x_2, x_3, x_4, x_5, x_6$ be the number of squares selected from the 1st, 2nd, 3rd, 4th, 5th and 6th row.

Then we must have $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 11$ (where $1 \le x_i \le 6$).

The number of positive integral solutions of the above equation is ${}^{11-1}C_{6-1} = {}^{10}C_5$.

3. a → q; b → p; c → s; d → r.

a. If polygon has n sides, then number of diagonals is ${}^nC_2 - n = 35$ (given). Solving we get $n = 10$. Thus, there are 10 vertices, from which ${}^{10}C_3$ (= 120) triangles can be formed.

b. Four vertices can be selected in ${}^{10}C_4$ (= 210) ways. Using these four vertices two diagonals can be formed, which has exactly one point of intersection lying inside the polygon.

Hence, number of points of intersections of diagonal which lies inside the polygon is ${}^{10}C_4 \times 1 = 210$.

c.

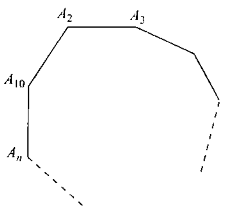


Fig. 5.51

Suppose one of the sides of the triangle is A_1A_2. Then third vertex cannot be A_3 or A_{10}. Thus, for the third vertex six vertices are left. There are six triangles in which side A_1A_2 is common with that of polygon. Similarly, for each of the sides A_2A_3, A_3A_4, $\cdots$, A_9A_{10} there are six triangles. Then total number of triangles is 60.

d. Triangles $A_1A_2A_3$, $A_2A_3A_4$, ..., $A_8A_9A_{10}$ have two sides common with that of polygon. Hence, there are 10 such triangles.

4. a → p, q, r, s; b. → p, r, s; c. → q, r, s; d. → s.

a. Number of subjective functions is
$3^6 - {}^3C_1(3-1)^6 + {}^3C_2(3-2)^6 = 729 - 192 + 3 = 540$

b. If $f(a_i) \ne b_i$, then pre-image a_1, a_2, a_3 cannot be assigned images b_1, b_2, b_3, respectively.
Hence each of a_1, a_2, a_3 can be assigned images in 2 ways.
a_4, a_5, a_6 can be assigned images in 3 ways each.
Hence number of functions is $2^3 3^3 = 216$.

c. One-one functions are not possible ar pre-images are more than images.

d. Number of many-one functions is
Total number of functions – number of one-one functions
$= 3^6 - {}^6P_3 = 729 - 120 = 609$

5. a → p, s; b → q, r; c → p, s; d → r.

a. Total number of required functions is equal to number of derangement of 5 objects, which is given by

$$5!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44$$

b. $x_1x_2x_3 = 2 \times 35 \times 7 = 2 \times 49 \times 5 = 10 \times 7 \times 7 = 14 \times 7 \times 5$

So total number of solution set is $3 \times 3! + 3!/2! = 21$.

c. $3780 = 2^2 \times 3^3 \times 5 \times 7$

Number of divisors which are divisible by 2 but not by 3 is $2 \times 2 \times 2 = 8$.

Number of divisors which are divisible by 3 but not by 2 is $3 \times 2 \times 2 = 12$.

Number of divisors which are divisible by 2 as well as 3 is $2 \times 3 \times 2 \times 2 = 24$.

Hence total number of divisors is 44.

d. $4\lambda + 2 = 2(2\lambda + 1)$ = odd multiple of 2

Thus, total number of divisors is $1 \times 5 \times 11 - 1 = 54$. (1 is subtracted and powers of three and five are zero each and this will make $\lambda = 0$.)

6. a → r; b → p; c → s; d → q.

a. The number of possible outcomes with 2 on at least one dice
= The total number of outcomes with 2 on at least one dice
= (The total number of outcomes) – (The number of outcomes in which 2 does not appear on any dice) $= 6^4 - 5^4$
$= 1296 - 625 = 671$

b. Any selection of four digits from the 10 digits 0, 1, 2, 3, ..., 9 gives one number. So, the required number of numbers is ${}^{10}C_4$.

c. Let the number be $n = pqr$. Since $p + q + r$ is even, p can be filled in 9 ways and q can be filled in 10 ways.

r can be filled in number of ways depending upon what is the sum of p and q.

If $p + q$ is odd, then r can be filled with any one of five odd digits.

If $p + q$ is even, then r can be filled with any one of five even digits.

In any case, r can be filled in five ways.

Hence, total number of numbers is $9 \times 10 \times 5 = 450$.

d. After fixing 1 at one position out of 4 places 3 places can be filled by $^7P_3$ ways. But for some numbers whose fourth digit is zero, such type of ways is $^6P_2$. Therefore, total number ways is $^7P_3 - {}^6P_2 = 480$.

7. a → q, s; b. → p, r; c → p, r; d → p, r.

a. There are two cases.

(i) 5, 4, 1, 1, 1

Number of ways of selection is 5!/3! = 20.

(ii) 5, 2, 2, 1, 1

Number of ways of selection is 5!/2!2! = 30.

Hence, total number is 20 + 30 = 50.

b. Select 4 pairs in $^5C_4 = 5$ ways. Now select exactly one shoe from each of the pairs selected in $(^2C_1)^4$ ways. This will fulfill the condition. Hence, required answer is 5 × 16 = 80.

c. The first child C_1 can be chosen in 3 ways; his/her mother can be interviewed in 5 ways; the second child C_2 can be chosen in 2 ways, and his/her mother can be interviewed in 3 ways.

Hence total number of ways is 3 × 5 × 2 × 3 = 90.

d. Required number of ways is 5! – 4! – 3! = 120 – 64 – 6 = 90. (Number will be less than 56000 only if either 4 occurs on the first place or 5, 4 occurs on the first two places.)

8. a → r, s; b → p, r; c → p, q; d → r, s

a. We have,

$$a = {}^{x+2}P_{x+2} = (x+2)!,$$

$$b = {}^xP_{11} = \frac{x!}{(x-11)!}$$

$$c = {}^{x-11}P_{x-11} = (x-11)!$$

Now,

$$a = 182\,bc \Rightarrow (x+2)! = 182 \times \frac{x!}{(x-11)!}(x-11)!$$

$$\Rightarrow (x+2)! = 182x! \Rightarrow (x+2)(x+1) = 182 \Rightarrow x = 12$$

b. x|x|x|

Even digits occupy odd places shown by crosses. Crosses can be filled in 2 × 2 × 1 ways (∵ 0 cannot go in the first place from the left). The remaining places can be filled in 3! ways. Therefore, the required number of numbers is 2 × 2 × 1 × 3! = 24.

c. Total number of numbers without restriction is 2^5. Two numbers have all the digits equal. So, the required number of numbers is $2^5 - 2$.

d. Let number of sides of polygon be n. Number of sides of polygon is equal to number of vertices of polygon. Now number of diagonals of polygon is

$$^nC_2 - n = 54$$

$$\Rightarrow \frac{n(n-1)}{2} - n = 54$$

$$\Rightarrow n^2 - 3n - 162 = 0$$

$$\Rightarrow (n-12)(n+9) = 0$$

$$\Rightarrow n = 12$$

Integer Type

1.(5) $^nP_r = {}^nP_{r+1}$

$$\Rightarrow \frac{n!}{(n-r)!} = \frac{n!}{(n-r-1)!} \Rightarrow n - r = 1 \quad (1)$$

$$\text{Again } ^nC_r = {}^nC_{r-1} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r+1)!(r-1)!}$$

$$\Rightarrow \frac{1}{r} = \frac{1}{n-r+1} \Rightarrow n - 2r = -1 \quad (2)$$

Solving (1) and (2), $n = 3$, $r = 2$

2.(8) We have $^nC_2 = 28$

$\Rightarrow n = 8$ (as there are 7 days in week)

3.(4) Number of arrangements are $2\,n!\,n!$

Giventhat2 $n!\,n! = 1152$

$\Rightarrow (n!)^2 = 576$

$\Rightarrow n! = 24$

$\Rightarrow n = 4$

4.(8) Let $n(A)$ = number divisible by 60 = (60, 120, ..., 960) = 16

$n(B)$ = number divisible by 24 = (24, 48, ..., 984) = 41

$n(A \cap B)$ = number divisible by both

= 120 + 240 + ⋯ + 960 = 8

Hence $n(A \cap B) = n(A) - n(A \cap B) = 16 - 8 = 8$

5.(9) We have A's = 2; B's = 4; C's = 2

$$\text{Total words formed} = \frac{8!}{4!\,2!\,2!} = 420 \quad (1)$$

Let ABBC = '×'

Number of ways in which × ABBC can be arranged = $\frac{5!}{2!} = 60$ but this includes × ABBC and ABBC ×.

But the word ABBCABBC is counted twice in 60 hence it should be 59

Hence required number of ways = 420 – 59 = 361

6.(6) Let T and S denotes teacher and student respectively

Then we have following possible patterns according to question

(i) T S S T S S T S S

(ii) S T S S T S S T S

(iii) S S T S S T S S T

Hence total number of arrangements are 3 · (3!)6! = 18 × 6!

$\Rightarrow k = 6$

7.(8) To form a triangle, 3 points out of 5 can be chosen in $^5C_3 = 10$ ways.

But of these, the three points lying on the 2 diagonals will be collinear.

So 10 – 2 = 8 triangles can be formed

8.(8) Here A is common letter in words 'SUMAN' and 'DIVYA'

Now for selecting six different letters we must select A either from word 'SUMAN' or from word 'DIVYA'.

Hence for possible selections, we have

A excluded from SUMAN +A included in SUMAN

$= {}^4C_3 \cdot {}^5C_3 + {}^4C_2 \cdot {}^4C_3 = 40 + 24 = 64$

Hence $N^2 = 64 \quad \Rightarrow N = 8$

9.(5) Let r no. of books of algebra and $20 - r$ of calc. no. of selections $= {}^rC_5 \times {}^{20-r}C_5$

Which has maximum value when $r = 10$

10.(9) Number of digits are 9

select 2 places for the digit 1 and 2 in ${}^9C_2$ ways

from the remaining 7 places select any two places for 3 and 4 in ${}^7C_2$ ways

and from the remaining 5 places select any two for 5 and 6 in ${}^5C_2$ ways

now, the remaining 3 digits can be filled in 3! ways

$\therefore$ Total ways $= {}^9C_2 \cdot {}^7C_2 \cdot {}^5C_2 \cdot 3!$

$$= \frac{9!}{2!\cdot7!}\cdot\frac{7!}{2!\cdot5!}\cdot\frac{5!}{2!\cdot3!}\cdot3!$$

$$= \frac{9!}{8} = \frac{9\cdot8\cdot7!}{8} = 9\cdot7!$$

11.(9) We have $2^n - 2 = 510$;

$\Rightarrow \quad 2^n = 512$

$\Rightarrow \quad n = 9$

12.(8) Including the two specified people, 4 others can be selected in ${}^5C_4$ ways.

The two adjacent seats can be taken in 4 ways and the two specified people can be arranged in 2!ways, remaining 4 people can be arranged in 4! ways.

$\Rightarrow \quad 5C_4 .4 .2!4! = 5!\ 8 = 8\ .5!$

13.(7) There are 2 women and let number of men are n

According to question

$2 \times {}^nC_2 = 66 + 2 \times {}^nC_1 \times {}^2C_1$

$$\Rightarrow \quad 2.\frac{n(n-1)}{2} = 2\,[33 + 2n]$$

$$\Rightarrow \quad \frac{n(n-1)}{1.2} = 33 + n(2)$$

$\Rightarrow \quad n^2 - 5n - 66 = 0 \Rightarrow (n-11)(n+6) = 0$

$\therefore \quad n = 11\ (\because sn > 0)$

total participants $= 2 + 11 = 13$

14.(9) | x | x | | when two consecutive digits are 11, 22, etc $= 9\ .\ 9 = 81$

| | 0 | 0 | when two consecutive digits are 0 0 = 9

| | x | x | when two consecutive digits are 11, 22, 33, ... $= 9\ .\ 8 = 72$

Total number of numbers are $N = 162$.

15.(8) We have $N =$ | a | b | c | d |

First place a can be filled in 2 ways i.e. 4, 5 ($4000 \le N < 6000$)

For b and c, total possibilities are '6' ($3 \le b < c \le 6$)

i.e. 34, 35, 36, 45, 46, 56

Last place d can be filled in 2 ways i.e. 0, 5 (N is a multiple of 5)

Hence, total numbers $= 2 \times 6 \times 2 = 24 = N$ then $N/3 = 8$.

16.(5) A AAAA | B BBBBB

Since word reads the same backwards and forwards, the middle digit must be A.

M

× × × × × ↓ × × × × ×

so that even number of A's and B's are available for arrangement about middle position M in the above figure.

Take AABBB on one side of M (6^{th} place) and then their image about M in a unique way

$$\therefore \quad \text{Number of ways N} = \frac{5!}{2!\cdot3!} = 10$$

17.(5)

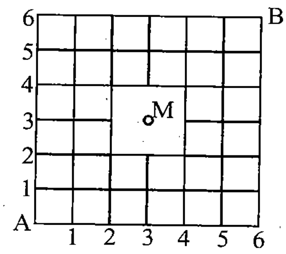


Fig. 5.52

Here the path which leads from A to B is of length-12.

Now without considering the constrain of passing through the point M, number of ways in which we can reach B from A is equal to number of ways we can select 6 steps from left to right and 6 from bottom to top which is equal to ${}^{12}C_6$.

Now we can reach from A to M in 6 steps in ${}^6C_3$ ways and can reach from M to B in ${}^6C_3$ ways.

Hence we can reach from A to B through M in ${}^6C_3 \times {}^6C_3$ ways.

Hence required number of ways $= {}^{12}C_6 - [{}^6C_3 \times {}^6C_3] = 924 - 400 = 524$

18.(6)

number of numbers beginning with 1 = 120 | 1 | | | | | |

number of numbers beginning with 2 = 120 | 2 | | | | | |

starting with 31 ....= 24 | 3 | 1 | | | | |

starting with 3214 ....= 2 | 3 | 2 | 1 | 4 | | |

finally = 1 | 3 | 2 | 1 | 5 | 4 | 6 |

Hence unit place digit of 267^{th} number is 6

19.(7) x denotes the number of times he can take unit step and y denotes the number of times he can take 2 steps, then $x + 2y = 7$,

Then we must have $x = 1, 3, 5$,

If $x = 1$, the steps will be 1 2 2 2

$$\Rightarrow \quad \text{number of ways} = \frac{4!}{3!} = 4$$

If $x = 3$, the steps will be 1 1 1 2 2

$$\Rightarrow \quad \text{number of ways} = \frac{5!}{2!\,.\,3!} = 10$$

If $x = 5$, the steps will be 1 1 1 1 1 2

$\Rightarrow \quad$ number of ways $= {}^6C_1 = 6$

If $x = 7$, the steps will be 1 1 1 1 1 1 1

$\Rightarrow \quad {}^7C_0 = 1$

Hence total number of ways $= N = 21$

$\Rightarrow \quad N/3 = 7$

20.(7) 3 women can be selected in ${}^7C_3$ ways and can be paired with 3 men in 3! ways.

Remaining 4 women can be grouped into two couples in

$$\frac{4!}{2!\cdot2!\cdot2!} = 3$$

$\therefore$ Total = ${}^7C_3 \cdot 3! \cdot 3 = 630 = N$

Then the value of N/90 is 7

21.(8) Here $P_n = {}^{n-2}C_3$ and $P_{n+1} = {}^{n-1}C_3$

Hence ${}^{n-1}C_3 - {}^{n-2}C_3 = 15$

$\Rightarrow \quad {}^{n-2}C_3 + {}^{n-2}C_2 - {}^{n-2}C_3 = 15$

$\Rightarrow \quad {}^{n-2}C_2 = 15 \Rightarrow n = 8$

22.(3)

(i) He can invite 2 friends three times each

Lets select first those 2 friends in ${}^3C_2$ ways

Now these two friends each three time can be invited on 6 days in $\frac{6!}{3!3!}$

Thus total number of ways 2 friends can be invited three times

$= {}^3C_2 \frac{6!}{3!3!}$

(ii) Another possibility is that he invites all three friends 2 times each

Then number of ways = $\frac{6!}{2!2!2!}$

(ii) One more possibility is that he invites one friend three times, one two times and one three times.

Then number of ways = $\frac{6! \times 6}{3!2!}$

Hence total number of ways

$$= {}^3C_2 \times \frac{6!}{3!3!} + \frac{6! \times 6}{3!2!} + \frac{6!}{2!2!2!} = 510$$

23.(4) If three numbers are in G.P., then their exponent must be in A.P.

If a, b, c are selected number in G.P., then the exponents of a and c both are either odd or both even, or otherwise exponent b will not be integer.

Now two odd exponent (from 1, 2, 3, ..., 10) can be selected in ${}^5C_2$ ways and two even exponent can be selected in ${}^5C_2$ ways.

Hence number of G.P.'s are $2\,{}^5C_2 = 20$

24.(8) $\sum_{k=r}^{n} {}^kC_r = {}^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + \cdots + {}^nC_r$

$= 1 + {}^{r+1}C_1 + {}^{r+2}C_2 + {}^{r+3}C_3 + \cdots + {}^nC_{n-r}$

$= {}^{r+1}C_0 + {}^{r+1}C_1 + {}^{r+2}C_2 + \cdots + {}^nC_{n-r}$

$= {}^{r+2}C_1$

${}^{r+3}C_2$ and so on finally ${}^{n+1}C_{n-r}$

now, ${}^{n+1}C_{n-r} = {}^{n+1}C_{r+1}$

$\therefore f(n) = \sum_{r=0}^{n} {}^{n+1}C_{r+1} = {}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \cdots + {}^{n+1}C_{n+1}$

$= {}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+1}C_2 + \cdots + {}^{n+1}C_{n+1} - 1$

$f(n) = (2^{n+1}) - 1$

$f(9) = 2^{10} - 1 = 1023 = 3 \cdot 11 \cdot 31$

hence number of divisors are $(1 + 1)(1 + 1)(1 + 1) = 8$

Archives

Subjective Type

1. (i) Distribution of 52 cards can be equally divided amongst four players.

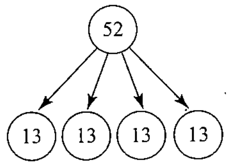


Fig. 5.53

Hence, number of ways is

$$\frac{52!}{(13!)4!}4! = \frac{52!}{(13!)}.$$

(ii)

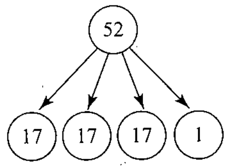


Fig. 5.54

Number of divisions is

$$\frac{52!}{(17!)^3 1!3!}.$$

Note: *There is division by* 3! *since* 3 *groups can be arranged in* 3! *ways and here* 3 *groups are of equal number of cards.*

2. As all the X's are identical, the question is of selection of 6 squares from 8 squares, so that no row remains empty. Here R_1 has 2 squares, R_2 has 4 squares and R_3 has 2 squares. The selection scheme is as follows:

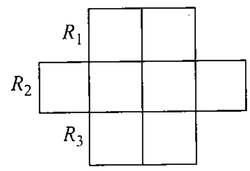


Fig. 5.55

R_1	R_2	R_3
1	4	1
1	3	2
2	3	1
2	2	2

Therefore, number of selection is

${}^2C_1 \times {}^4C_4 \times {}^2C_1 + {}^2C_1 \times {}^4C_3 \times {}^2C_2 + {}^2C_2 \times {}^4C_3 \times {}^2C_1 + {}^2C_2 \times {}^4C_2 \times {}^2C_2 = 4 + 8 + 8 + 6 = 26$

3. As no box should remain empty, boxes can have balls in the following numbers:

Possibilities 1, 2, 3 or 1, 2, 2

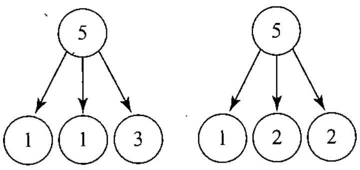


Fig. 5.56

Division ways for tree (i) is $\dfrac{5!}{(1!)^2\,3!2!}$

Division ways for tree (ii) is $\dfrac{5!}{(2!)^2\,1!2!}$

Now, total number of ways of distribution of these groups into three boxes is

$$\left[\frac{5!}{(1!)^2 3!2!}+\frac{5!}{(2!)^2 1!2!}\right]\times 3! = 150$$

4. m men can be seated in $m!$ ways creating $(m + 1)$ for ladies to sit n ladies out of $(m + 1)$ places (as $n < m$) can be seated in ${}^{m+1}P_n$ ways.

   Therefore, total number of ways is

$$m! \times {}^{m+1}P_n = m!\times\frac{(m+1)!}{(m+1-n)!}=\frac{(m+1)!m!}{(m-n+1)!}$$

5. The scheme is as follows:

Husband's relatives		**Wife's relatives**		**Number of selections**
Male (3)	Female (4)	Male (4)	Female (3)	
3	0	0	3	${}^3C_3 \times {}^3C_3 = 1$
2	1	1	2	${}^3C_2 \times {}^4C_1 \times {}^4C_1$ $x{}^3C_2 = 144$
1	2	2	1	${}^3C_1 \times {}^4C_2 \times {}^4C_2$ $x\,{}^3C_1 = 324$
0	3	3	0	${}^4C_3 \times {}^4C_3 = 16$
			Total	485

6. Out of 2 white, 3 black and 4 red balls, three balls have to be drawn.

   If at least one black ball is selected, then we have following cases:

Black balls (3)	**White + red balls (6)**	**Number of ways of selection**
1	2	${}^3C_1{\times}{}^6C_2 = 45$
2	1	${}^3C_2{\times}{}^6C_1 = 18$
3	0	${}^3C_3{\times}{}^6C_0 = 1$
	Total	64

7. Number of ways in which a student can select at least one and almost n books out of $2n + 1$ books is

$${}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \cdots + {}^{2n+1}C_n$$

$$= \frac{1}{2}[2\times{}^{2n+1}C_1 + 2\times{}^{2n+1}C_2 + 2\times{}^{2n+1}C_3 + \cdots + 2\times{}^{2n+1}C_n]$$

$$= \frac{1}{2}\Big[({}^{2n+1}C_1 + {}^{2n+1}C_{2n}) + ({}^{2n+1}C_2 + {}^{2n+1}C_{2n-1})$$
$$+({}^{2n+1}C_3 + {}^{2n+1}C_{2n-2}) + \cdots + ({}^{2n+1}C_n + {}^{2n+1}C_{n+1})\Big]$$

[Using ${}^nC_r = {}^nC_{n-r}$]

$$= \frac{1}{2}\Big[{}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \cdots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1}$$
$$+ {}^{2n+1}C_{n+2} + \cdots + {}^{2n+1}C_{2n}\Big]$$

$$= \frac{1}{2}\Big[{}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_{2n+1} - 1 - 1\Big]$$

$$= \frac{1}{2}\big[2^{2n+1} - 2\big] = 2^{2n} - 1$$

Now given,

$2^{2n} - 1 = 63$

$\Rightarrow$ $2^{2n} = 64 = 2^6$

$\Rightarrow$ $2n = 6$

$\Rightarrow$ $n = 3$

8. Out of 18 guests, 9 are to be seated on side A and rest 9 on side B.

   Now out of 18 guests, 4 particular guests desire to sit on one particular side say side A and other 3 on other side B. Out of rest $18 - 4 - 3 = 11$ guests we can select 5 more for side A and rest 6 can be seated on side B. Selection of 5 out of 11 can be done in ${}^{11}C_5$ ways. Nine guests on each side of table can be seated in $9! \times 9!$ ways.

   Thus, there are total ${}^{11}C_5 \times 9! \times 9!$ arrangements.

9. A committee of 12 is to be formed from 9 women and 8 men with minimum 5 women. Then we have following selection ways.

Women (9)	**Men (8)**	**Number of ways of selecting**
5	7	${}^9C_5 \times {}^8C_7 = 1008 = s_1$
6	6	${}^9C_6 \times {}^8C_6 = 2352 = s_2$
7	5	${}^9C_7 \times {}^8C_5 = 2016 = s_3$
8	4	${}^9C_8 \times {}^8C_4 = 630 = s_4$
9	3	${}^9C_9 \times {}^8C_3 = 56 = s_3$
	Total	6062

   a. Number of committee when women are in majority is $s_3 + s_4 + s_5 = 2702$.

   b. Number of committee when men are in majority is $s_1 = 1008$.

10. Let there be n sets of different objects each set containing n identical objects, e.g., ((1, 1, 1, … , 1(n times)), (2, 2, 2, …, 2 (n times)), …, (n, n, n, …, n (n times)). Then the number of ways in which these $n \times n = n^2$ objects can be arranged in a row is

$$\frac{(n^2)!}{n!n!\cdots n!} = \frac{(n^2)!}{(n!)^n}$$

   But, these number of ways should be a natural number. Hence, $(n^2)!/(n!)^n$ is an integer ($n \in I^+$).

Objective Type

Fill in the blanks

1. Number of students who gave wrong answers to exactly one question is $a_1 - a_2$.

   Number of students who gave wrong answers to exactly two questions is $a_2 - a_3$.

Number of students who gave wrong answers to exactly three questions is $a_3 - a_4$.

Number of students who gave wrong answers to exactly k question is $a_{k-1} - a_k$.

Therefore, total number of wrong answers is

$$1(a_1 - a_2) + 2(a_2 - a_3) + 3(a_3 - a_4) + \cdots + k(a_{k-1} - a_k)$$
$$= a_1 + a_2 + a_3 + \cdots + a_k$$

2. We have total 3 + 4 + 5 = 12 points. So, number of Δs that can be formed using 12 such points is given by

Total number of ways of selecting three point – Number of ways three collinear points are selected

$$= {}^{12}C_3 - {}^3C_3 - {}^4C_3 - {}^5C_3$$
$$= \frac{12\times 11\times 10}{6} - 1 - 4 - \frac{5\times 4}{2\times 1}$$
$$= 220 - 15 = 205$$

3. '+' signs can be put in a row in one way creating seven gaps shown as arrows:

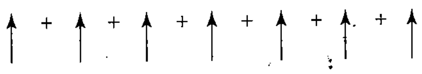


Fig. 5.57

Now 4 '–' signs must be kept in these gaps. So, no two '–' signs should be together.

Out of these 7 gaps 4 can be chosen in ${}^7C_4$ ways. Hence, required number of arrangements is

$${}^7C_4 = {}^7C_3 = \frac{7\times 6\times 5}{3\times 2\times 1} = 35$$

4. We know that number of derangements of n objects is

$$n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{1}{n!}\right]$$

Therefore, number of ways of putting all the 4 balls into boxes of different colour is

$$4!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right] = 4!\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right)$$
$$= 24\left(\frac{12-4+1}{24}\right)$$
$$= 9$$

True or false

1. True. See the proof in theory section in properties of ${}^nC_r$.

Multiple choice questions with one correct answer

1. c. ${}^nC_{r-1} = 36, {}^nC_r = 84, {}^nC_{r+1} = 126$

We know that

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{r}{n-r+1}$$

$$\Rightarrow \frac{36}{84} = \frac{r}{n-r+1}$$

$$\Rightarrow \frac{r}{n-r+1} = \frac{3}{7}$$

$$\Rightarrow 3n - 10r + 3 = 0 \quad (1)$$

Also,

$$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{r+1}{n-r} = \frac{84}{126} = \frac{2}{3}$$

$$\Rightarrow 2n - 5r - 3 = 0 \quad (2)$$

Solving (1) and (2), we get $n = 9$ and $r = 3$.

2. Number of words when repetition is allowed is $10 \times 10 \times 10 \times 10 \times 10 = 10^5$.

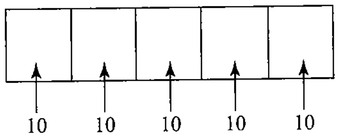


Fig. 5.58

Number of words when repetition is not allowed is

$10 \times 9 \times 8 \times 7 \times 6 = 30240$.

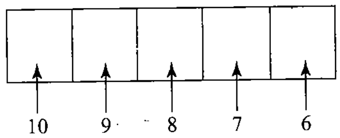


Fig. 5.59

Hence, required number of words in which at least one letter is repeated is 100000 – 30240 = 69760.

3. c. ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$

$$= {}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3$$
$$= {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3 + {}^{47}C_4$$

[Using ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$]

$$= {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + ({}^{48}C_3 + {}^{48}C_4)$$
$$= {}^{51}C_3 + {}^{50}C_3 + ({}^{49}C_3 + {}^{49}C_4)$$
$$= {}^{51}C_3 + ({}^{50}C_3 + {}^{50}C_4)$$
$$= {}^{51}C_3 + {}^{51}C_4$$
$$= {}^{52}C_4$$

4. d. $\overline{1}\,\overline{2}\,\overline{3}\,\overline{4}\,\overline{5}\,\overline{6}\,\overline{7}\,\overline{8}$

Two women can choose two chairs out of 1, 2, 3, 4 in ${}^4C_2$ ways, and can arrange among themselves in 2! ways. Three men can choose 3 chairs out of 6 remaining chairs in ${}^6C_3$ ways and can arrange themselves in 3! ways.

Therefore, total number of possible arrangements is ${}^4C_2 \times 2! \times {}^6C_3 \times 3! = {}^4P_2 \times {}^6P_3$.

5. b. $\times B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times B_6 \times$

Let first six boys sit, which can be done in 6! ways. Once they have been seated, the two brothers can be made to occupy seats in between or in extreme (i.e. on crosses) in ${}^7P_2$ ways.

Hence, required number of ways is ${}^7P_2 \times 6!$.

6. a. We know that a number is divisible by 3 if the sum of its digits is divisible by 3. Now out of 0, 1, 2, 3, 4, 5 if we take 1, 2, 3, 4, 5 or 0, 1, 2, 4, 5, then the 5-digit numbers will be divisible by 3.

Case I:

Total number of five-digit numbers formed using the digits 1, 2, 3, 4, 5 is 5! = 120.

Case II:

Taking 0, 1, 2, 4, 5, total number is $4 \times 4! = 96$.

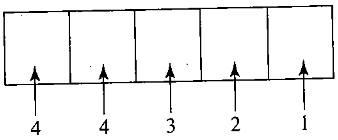


Fig. 5.60

From case I and case II, total number divisible by 3 is $120 + 96 = 216$.

7. b. Distinct n-digit numbers which can be formed using digits 2, 5 and 8 are 3^n. We have to find n so that

$3^n \geq 900$

$\Rightarrow 3^{n-2} \geq 100$

$\Rightarrow n - 2 \geq 5$

$\Rightarrow n \geq 7$

So the least value of n is 7.

8. c. X – X – X – X – X

The four digits 3, 3, 5, 5 can be arranged at (–) places in $\frac{4!}{2!2!} = 6$ ways. The five digits 2, 2, 8, 8, 8 can be arranged at (X) place in $\frac{5!}{2!3!} = 10$ ways.

Total number of arrangements is $6 \times 10 = 60$.

9. b. A regular polygon of n sides has n vertices, no two of which are collinear. Out of these n points, ${}^nC_3$ triangles can be formed.

$\therefore \quad T_n = {}^nC_3;\ T_{n+1} = {}^{n+1}C_3$

Given,

$T_{n+1} - T_n = 21$

$\Rightarrow {}^{n+1}C_3 - {}^nC_3 = 21$

$\Rightarrow \frac{(n+1)n(n-1)}{3 \times 2 \times 1} - \frac{n(n-1)(n-2)}{3 \times 2 \times 1} = 21$

$\Rightarrow n(n-1)(n+1-n+2) = 126$

$\Rightarrow n(n-1) = 42$

$\Rightarrow n(n-1) = 7 \times 6$

$\Rightarrow n = 7$

10. a. Total number of ways of arranging the letters of the word BANANA is $\frac{6!}{2!3!} = 60$.

Number of words in which 2N's come together is $5!/3! = 20$.

Hence, the required number is $60 - 20 = 40$.

11. c.

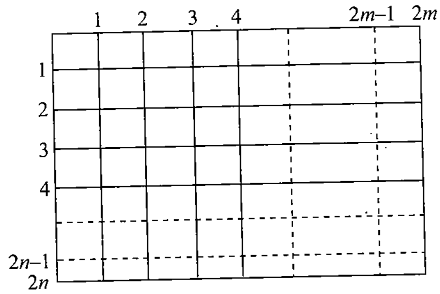


Fig. 5.61

If we see the blocks in terms of lines, then there are $2m$ vertical lines and $2n$ horizontal lines.

To form the required rectangle we must select two horizontal lines, one even numbered (out of 2, 4, ..., $2n$) and one odd numbered (out of 1, 3, ..., $2n - 1$) and similarly two vertical lines.

The number of rectangles is ${}^mC_1 \times {}^mC_1 \times {}^nC_1 \times {}^nC_1 = m^2n^2$

12. c. If L.C.M. of p and q is $r^2t^4s^2$, then distribution of factors r is as follows:

p	q
r^0	r^2
r^1	r^2
r^2	r^2
r^2	r^0
r^2	r^1

Thus, factor r can be distributed in $2 \times 3 - 1$ ways. Similarly, factors t and s can be distributed in $2 \times 5 - 1$ and $2 \times 3 - 1$ ways, respectively.

Hence, number of ordered pairs are $(2 \times 3 - 1) \times (2 \times 5 - 1) \times (2 \times 3 - 1) = 225$.

13. c. The letters of word COCHIN in alphabetic order are C, C, H, I, N, O. Fixing first letter C and keeping C at second place, rest 4 can be arranged in 4! ways.

Similarly, the total number of words starting with CH, CI, CN is 4! in each case.

Then fixing first two letters as CO, next four places when filled in alphabetic order gives the word COCHIN.

Therefore, number of words coming before COCHIN is $4 \times 4! = 4 \times 24 = 96$.

14. d. Total number of unordered pairs of disjoint subsets = $\frac{3^4 + 1}{2} = 41$.