TRIGONOMETRY

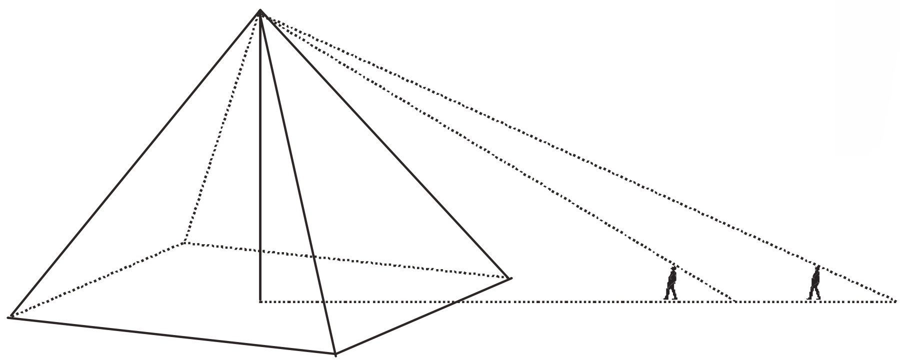
UNIT - 4

Come, let us know the history of Trigonometry.....

Trigonometry was developed in order to satisfy the different day to day needs of humans- For example, in India in order to study the speed and position of the celestial bodies in the fields of Astronomy and Astrology; in Greece to study the relation between circles and chords in the field of Astronomy and in Egypt to know the height of Pyramids. When these ideas got together, trigonometry was developed. Although Trigonometry means measurement related to triangles, the knowledge is used extensively in relation to angles. In all of these, triangles and the imaginary right angled triangles are identified and its ratio of sides is used to find height, distance, speed, state etc.

It is believed that the first list of trigonometric ratios was given by Heparcus in second centry BC in India. In the field of astronomy, by regular use, five principles were assigned indicating the relation between angles and chord. The most important amongst these was the sun principle which defines 'Sine' used today. In the fifth century Aryabhatt took this further and used Jya (Sine) and Kojya (Cosine). In the 7th Century, Bhaskaracharaya I, gave the formula for calculating Sin x, so for each x you can obtain Sin x, with an error of less than two percent. Later again in the seventh centry, Brahmagupta used angles like $(\pi - x)$, $\left(\frac{\pi}{2} - x\right)$ and established a relation between Sine x and Cosine $\left(\frac{\pi}{2} - x\right)$. He also established a relation between Sine and Cosine of the sum of two angles with the Sine and Cosine of the individual angles like Sin (A+B) = SinAcosB + cosAsinB etc. Along with these two, in the twelfth century Bhaskaracharaya has mentioned the use of tangets in his book 'Goladhyay'.

In the fourteenth century Mahadev analysed several trigonometric functions and their infinite series and came up with some important expansion formulae, which were discovered much later in the western world. These are used even today.



This information has been collected from different books and been presented here. Teachers and children are free to obtain information on Trigonometry from other sources as well.

Trigonometrical Ratio & Identities

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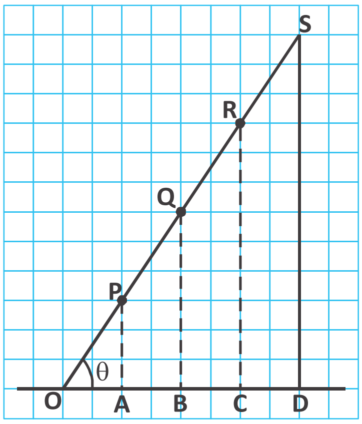
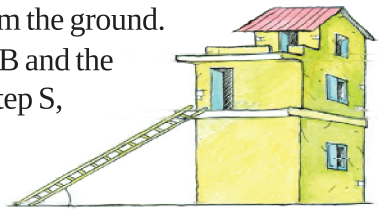


Fig. 1

With every step that we climb on the stair case, we go higher from the ground (*Fig*.1)



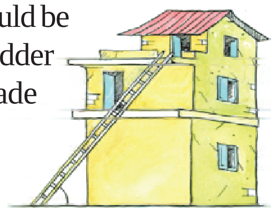
The height of first step P is PA from the ground. Similary the second step Q is at height QB and the third step R is at a height RC, the fourth step S, is at SD.

On each step we not only go higher from the ground but we also go closer to the building.

Do we go the same distance closer as we go higher? Is there a relation betwen these two?

Here $\dfrac{PA}{OA} = \dfrac{QB}{OB} = \dfrac{RC}{OC} = \dfrac{SD}{OD}$

Thus the ratio of the height we climb, to the distance we move ahead, is the same.



If the height to which we need to climb is a bit more what would be required, if the ladder was the same? We would have to move the ladder closer to the base of the house (*Fig*.2), thus increasing the angle made by the ladder with the ground.

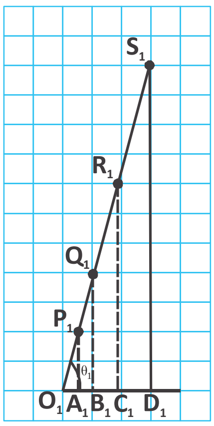
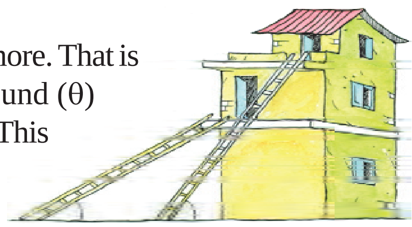


Fig. 2

Now count the squares and say whether the ratio of the distance to height is the same?

$$\frac{P_1A_1}{O_1A_1} = \frac{Q_1B_1}{O_1B_1} = \frac{R_1C_1}{O_1C_1} = \frac{S_1D_1}{O_1D_1}$$

We see that it is indeed the same.



Though the value of the ratio in second case is more. That is when the angle made by the ladder with the ground (θ) increased, the ratio of height to distance increased. This ratio is known as the tangent of this angle (θ).

It means tangent $\theta = \frac{PA}{OA} = \frac{QB}{OB}$

and tangent $\theta_1 = \frac{P_1A_1}{O_1A_1} = \frac{Q_1B_1}{O_1B_1}$

Other Ratios:- If we look at this movement of going higher and towards the wall in form of a line diagram, then it is as if we are making a right angled triangles from every point representings a step as a vertex.

If we denote the angle made by the ladder with the ground as θ, the height to which we climb is the perpendicular and distance from the wall is the base and the ladder would represent the hypotenuse.

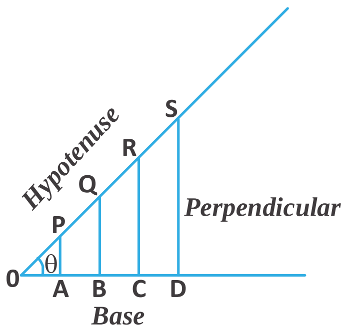


Fig. 3

We have said above tangent θ is $\frac{PA}{OA}$, which in the form of perpendicular and base

tanθ would be = $\frac{\text{Perpendicular}}{\text{Base}}$. The value of which is the same in all the triangles of *Fig*.3. It remains the same for the same value of θ.

In short this is called tan θ.

$$\tan\theta = \frac{PA}{OA} = \frac{QB}{OB} = \frac{RC}{OC}$$

Will there be any other ratios we get from these Line Drawings

Can we get some other fixed ratios from these values of sides in the given right angled triangle? Let us see the ratio of perpendicular to hypotensuse:-

$$\text{Ratio} = \frac{PA}{OP}, \frac{QB}{OQ}, \frac{RC}{OR}$$

Also let us see the ratio of base to hypotenuse.

$$\frac{OA}{OP}, \frac{OB}{OQ}, \frac{OC}{OR}$$

Check if these ratios are the same.

The ratio of the perpendicular to hypotenuse for a fixed angle is known as the sineθ (in short sinθ).

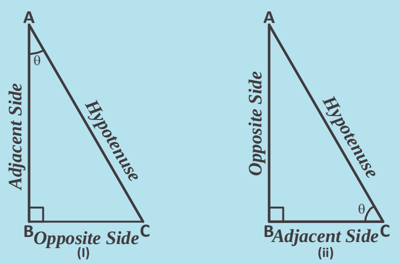
$$\text{Hence } \sin\theta = \frac{PA}{OP} = \frac{QB}{OQ} = \frac{RC}{OR}$$

Here in right angled ΔABC, $\angle B = 90°$ and $\angle A = \theta$ (in Fig. (i)) then side opposite to θ is BC and side AB is the adjacent side. Also AC is the hypotenuse.

Similarly in right angled ΔABC (in Fig. (ii)) $\angle B = 90°$, $\angle C = \theta$ then the opposite side to θ is AB and the adjacent side is BC, AC is the hypotenuse.

in Fig. (i) $\sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{BC}{AC}$, $\cos\theta = \frac{AB}{AC}$, $\tan\theta = \frac{BC}{AB}$

Accordingly find the ratios for Fig. (ii).



Similarly the ratio of base to hypotenuse is known as cosineθ (in short cosθ).

$$\cos\theta = \frac{OA}{OP} = \frac{OB}{OQ} = \frac{OC}{OR}$$

(sinθ, cosθ, tanθ etc. are known as trigonometric ratios)

Exercise - 8.1

If in a right angled triangle ABC, ∠B is a right angle; then find the value of sin A, cos C and tan A.

Given:-

(i)	AC = 5	AB = 3	BC = 4
(ii)	AB = 12	BC = 5	AC = 13
(iii)	AB = 5	AC = 13	BC = 12
(iv)	BC = 12	AB = 9	AC = 15

Relationship between the Ratios

Relation between sinθ, cosθ and tanθ **:** If in right angled triangle ABC, ∠B is right angle and if ∠C = θ, then:-

$$\tan\theta = \frac{AB}{BC}$$

$$= \frac{AB}{AC} \times \frac{AC}{BC}$$

$$= \frac{AB}{AC} \div \frac{BC}{AC}$$

$$= \sin\theta \div \cos\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

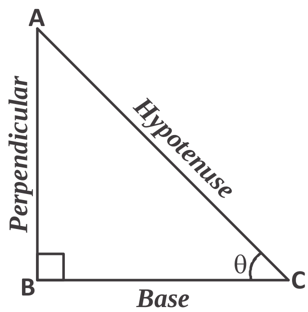


Fig. 4

Some other Trigonometric Ratios

We saw that in a right angled triangle ABC, right angled at B and ∠C = θ:-

$$\frac{\text{Perpendicular}}{\text{Hypotenuse}} = \sin\theta, \qquad \frac{\text{Base}}{\text{Hypotenuse}} = \cos\theta, \qquad \frac{\text{Perpendicular}}{\text{Base}} = \tan\theta$$

The reciprocals of these three give three more ratios whose names are:-

$$\frac{\text{Hypotenuse}}{\text{Perpendicular}} = \text{cosecant}\theta \ (\text{or cosec}\theta) = \frac{1}{\sin\theta}$$

$$\frac{\text{Hypotenuse}}{\text{Base}} = \text{secant}\theta \ (\text{or sec}\theta) = \frac{1}{\cos\theta}$$

$$\frac{\text{Base}}{\text{Perpendicular}} = \text{cotanget}\theta \ (\text{or cot}\theta) = \frac{1}{\tan\theta}$$

Try This

If $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, then can you write the ratio $\cot\theta$ in terms of $\sin\theta$ and $\cos\theta$?

Trigonometric Ratio and Pythagoras Theorem

The concept of all trigonometric ratios can be understood by using a right angled triangle. Pythagoras theorem also gives a relation between the sides of a right angled triangle. Let us use this to find a relation between the trigonometric ratios.

$a^2 = a \times a$
$b^2 = b \times b$

Let the lengths of the two sides of a right angled traingle be a and b and the length of hypotenuse be c, then by Pythogoras theorem, the relation between a, b and c is:-

$$a^2 + b^2 = c^2 \qquad (\text{Perpendicular}^2 + \text{Base}^2 = \text{Hypotenuse}^2) \qquad ...(1)$$

$\sin^2\theta$
$= \sin\theta \times \sin\theta$

Now if the hypotenuse with length c makes an angle θ with the base of length b then,

$$\sin\theta = \frac{a}{c} \text{ and } \cos\theta = \frac{b}{c}$$

Squaring and adding the two gives us:-

$$\sin^2\theta + \cos^2\theta = \frac{a^2}{c^2} + \frac{b^2}{c^2}$$

$$\sin^2\theta + \cos^2\theta = \frac{a^2 + b^2}{c^2}$$

$$\sin^2\theta + \cos^2\theta = \frac{c^2}{c^2} \qquad [\because a^2+b^2=c^2]$$

$$\sin^2\theta + \cos^2\theta = 1$$

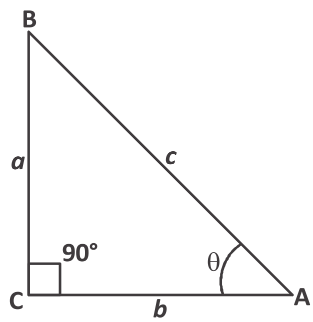


Fig. 5

or we can write it as:-

$$\sin^2\theta = 1 - \cos^2\theta \qquad \text{or} \qquad \cos^2\theta = 1 - \sin^2\theta$$

The above three statements of the relation between $\sin^2\theta$ and $\cos^2\theta$ are in the form of equations. These hold ture for all values of θ from 0° to 90°. In right angled triangles, these are known as trigonometric identities for an angle θ $(0 \leq \theta \leq 90°)$.

Some other identities exist which give a relation between $\tan^2\theta$ and $\sec^2\theta$ as also between $\cot^2\theta$ and $\text{cosec}^2\theta$. These can be obtained as follows, observe and understand:-

Identity-1 $\qquad \sin^2\theta + \cos^2\theta = 1$

Dividing throughout by $\sin^2\theta$ we get,

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = \text{cosec}^2\theta \ \textbf{(Identity-2)} \ \left(\because \frac{\cos\theta}{\sin\theta} = \cot\theta\right)$$

Again dividing identity-1 throughout by $\cos^2\theta$ we get:-

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta \qquad \text{or} \qquad 1 + \tan^2\theta = \sec^2\theta \ \textbf{(Identity-3)}$$

Try This

Like Identity-1, write Identity-2 and Identity-3 in different ways.

To find Trigonometric Ratios

We have seen that all six trigonometric ratios are related to each other. We have also seen that if we know the value of one trigonometric ratio, then we can obtain the ratio of the sides of any right angled triangle with the same angle θ.

We can do this using the Pythogras Theorem. Using one trigonometric ratio, we can find all the remaining ratios.

EXAMPLE-1. ΔPQR is a right angled triangle in which $\angle Q$ is the right angle and $\angle R = \theta°$.

Given $\sin\theta = \frac{3}{5}$. Can we find the remaining five ratios?

SOLUTION : $\because \ \sin\theta = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{PQ}{PR} = \dfrac{3}{5}$

We can write this as $\sin\theta = \dfrac{3x}{5x}$ (As the ratio of $3x$ to $5x$ is 3 : 5)

Then we can say $PQ = 3x$, $\quad PR = 5x \quad$ ......(1)

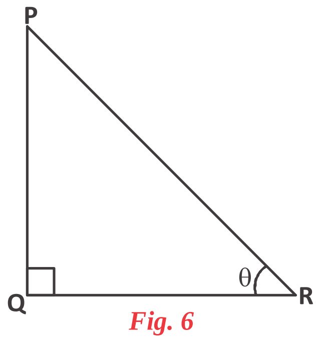


Fig. 6

In right angled PQR,

$$\text{Hypotenuse}^2 = \text{Perpendicular}^2 + \text{Base}^2$$

$$(5x)^2 = (3x)^2 + \text{Base}^2$$

$$25x^2 - 9x^2 + \text{Base}^2$$

$$16x^2 = \text{Base}^2$$

$$(4x)^2 = \text{Base}^2$$

(Finding a square root on both sides we get)

$\therefore$ Base (QR) $= 4x$

Now $\cos\theta = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{4x}{5x} = \dfrac{4}{5}$

Similarly we can find the other trigonometric ratios.

EXAMPLE-2. If $\sin\theta = \dfrac{5}{13}$, find the remaining trigonometric ratios.

SOLUTION : Given $\sin\theta = \dfrac{5}{13}$ ..... (1)

Using this how do we find value of $\cos\theta$

We know-

$$\sin^2\theta + \cos^2\theta = 1$$

To find $\cos\theta$ we rewrite this as

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos^2\theta = 1 - \left(\frac{5}{13}\right)^2 \qquad \left[\text{Given } \sin\theta = \frac{5}{13}\right]$$

$$\cos^2\theta = 1 - \frac{25}{169} = \frac{169 - 25}{169} = \frac{144}{169}$$

$$\cos^2\theta = \left(\frac{12}{13}\right)^2$$

$\therefore \quad \cos\theta = \dfrac{12}{13}$ .....(2)

Now we know $\sin\theta$ and $\cos\theta$, so lets find value of $\tan\theta$.

We know that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ or $\sin\theta \div \cos\theta$

$$\therefore \tan\theta = \frac{5}{13} \div \frac{12}{13}$$

$$= \frac{5}{13} \times \frac{13}{12}$$

$$\tan\theta = \frac{5}{12}$$

Now we can find the other ratios $\sec\theta$, $\text{cosec}\theta$ and $\cot\theta$

Since we know $\sec\theta = \dfrac{1}{\cos\theta}$, $\text{cosec}\theta = \dfrac{1}{\sin\theta}$, $\cot\theta = \dfrac{1}{\tan\theta}$

So $\quad \sec\theta = \dfrac{1}{\cos\theta} = \dfrac{1}{12/13} = \dfrac{13}{12}$

$$\text{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{5/13} = \frac{13}{5}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{5/12} = \frac{12}{5}$$

EXAMPLE-3. If $\sec A = \dfrac{5}{3}$, then find the other trigonometric ratios for $\angle A$.

SOLUTION : We are given $\sec A = \dfrac{5}{3}$ ..... (1)

(i) But as $\sec A = \dfrac{1}{\cos A}$ (secA is the reciprocal of cosA)

$$\therefore \cos A = \frac{1}{5/3} = \frac{3}{5}.$$

(ii) Using identity, we will find value of sinA

$$\sin^2 A = 1 - \cos^2 A$$

$$= 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25}$$

$$= \frac{25-9}{25} = \frac{16}{25}$$

$$\sin^2 A = \left(\frac{4}{5}\right)^2$$

$$\sin A = \frac{4}{5}$$

(iii) As $\tan A = \dfrac{\sin A}{\cos A}$ or $\sin A \div \cos A$

So, $\tan A = \dfrac{4}{5} \div \dfrac{3}{5}$

$$= \frac{4}{5} \times \frac{5}{3} = \frac{4}{3}$$

$$\therefore \tan A = \frac{4}{3}$$

(iv) The reciprocal of tanA is cotA

Hence $\cot A = \dfrac{1}{\tan A} = \dfrac{1}{4/3} = \dfrac{3}{4}$

(v) $\because \operatorname{cosec} A = \dfrac{1}{\sin A} = \dfrac{1}{4/5} = \dfrac{5}{4}$

Hence $\operatorname{cosec} A = \dfrac{5}{4}$

EXAMPLE-4. If $5\tan\theta = 4$, then find value of $\dfrac{5\sin\theta - 3\cos\theta}{\sin\theta + 2\cos\theta}$

SOLUTION : $5\tan\theta = 4$

So, $\tan\theta = \dfrac{4}{5}$

Now, $\dfrac{5\sin\theta - 3\cos\theta}{\sin\theta + 2\cos\theta}$

$$= \frac{5\dfrac{\sin\theta}{\cos\theta} - \dfrac{3\cos\theta}{\cos\theta}}{\dfrac{\sin\theta}{\cos\theta} + 2\dfrac{\cos\theta}{\cos\theta}} \quad \text{(Dividing both numerator and denominator by } \cos\theta)$$

$$= \frac{5\tan\theta - 3}{\tan\theta + 2} \quad \left(\because \frac{\sin\theta}{\cos\theta} = \tan\theta\right)$$

$$= \frac{5\left(\dfrac{4}{5}\right) - 3}{\left(\dfrac{4}{5}\right) + 2} \quad \left(\because \tan\theta = \frac{4}{5}\right)$$

$$= \frac{4-3}{(4+10)/5} = \frac{1}{14/5}$$

$$= \frac{5}{14}$$

EXAMPLE-5. If $\tan\theta = 1$ in a right angled ΔABC right angled at B, prove that $2\sin\theta\cos\theta = 1$

SOLUTION : In ΔABC, $\tan\theta = \dfrac{BC}{AB} = 1$

Or $BC = AB$

Say $AB = BC = k$ (k is some positive number)

Now $AC = \sqrt{(AB)^2 + (BC)^2} = \sqrt{k^2 + k^2} = k\sqrt{2}$

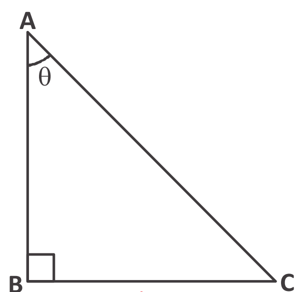


Fig. 7

Hence, $\sin\theta = \dfrac{BC}{AC} = \dfrac{1}{\sqrt{2}}$ and $\cos\theta = \dfrac{AB}{AC} = \dfrac{1}{\sqrt{2}}$

So $2\sin\theta\cos\theta = 2\left(\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{1}{\sqrt{2}}\right) = 1 \quad \left(\because \sqrt{2}\times\sqrt{2} = 2\right)$

Or $2\sin\theta\cos\theta = 1$

Exercise - 8.2

1. One of the trigonometric ratios is given below. Find the remaining trigonometric ratios:-

(i) $\tan\theta = \dfrac{3}{4}$ (ii) $\sin\theta = \dfrac{5}{13}$ (iii) $\cos\alpha = \dfrac{1}{3}$

(iv) $\cot\theta = 1$ (v) $\operatorname{cosec}A = \dfrac{5}{4}$ (vi) $\sec\beta = 2$

(vii) $\operatorname{cosec}A = \sqrt{10}$

2. If $\cot\theta = \dfrac{21}{20}$, then find the value of $\sin\theta \times \cos\theta$.

3. If $\cos A = \dfrac{4}{5}$, then find the value of $\dfrac{\cot A - \sin A}{2\tan A}$.

4. If $\sec\theta = \dfrac{5}{3}$, then find the value of $\dfrac{\tan\theta - \sin\theta}{1 + \tan\theta.\sin\theta}$.

5. If $\sin A = \dfrac{1}{3}$, then find the value of $\cos A$, $\operatorname{cosec}A + \tan A$ and $\sec A$.

6. In a right angled ΔABC, right angle is at $\angle C$ and $\tan A = \dfrac{1}{\sqrt{3}}$, then find the value of $\sin A\cos B + \cos A\sin B$.

7. If $\cot A = \dfrac{3}{4}$, then find the value of $\dfrac{\sin A + \cos A}{\sin A - \cos A}$.

8. If $\sin\theta = \dfrac{4}{5}$, then find the value of $\dfrac{4\tan\theta - 5\cos\theta}{\sec\theta + 4\cot\theta}$.

Trigonometric Ratios for Some Special Angles

We can find the trigonometric ratios for $\theta = 0°, 30°, 45°, 60°$ or $90°$ in a right angled triangle using geometry. Let us see how:-

Trigonometric Ratios of 45°

Triangle ABC is right angled triangle with right angle at $\angle B$ and $\angle C = 45°$

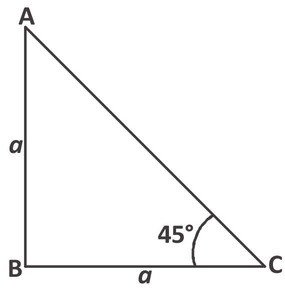


Fig. 8

Clearly $\angle A$ will also be 45°

So if BC = a then

AB = a (Why?)

(In a triangle, sides opposite equal angles are equal)

Now $AC^2 = AB^2 + BC^2$ (By Pythagoras Theorem)

$= a^2 + a^2 = 2a^2$

$AC = a\sqrt{2}$

$\because$ For $\angle C$ (45°), BC is base, AB is perpendicular and AC is the hypotenuse.

$$\therefore \quad \sin C = \sin 45° = \frac{AB}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{BC}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{AB}{BC} = \frac{a}{a} = 1$$

$$\cot 45° = \frac{1}{\tan 45°} = \frac{1}{1} = 1$$

$$\sec 45° = \frac{1}{\cos 45°} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$$

$$\operatorname{cosec} 45° = \frac{1}{\sin 45°} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$$

Trigonometric Ratios of 30°

Consider an equilateral triangle ABD whose each side is of length $2a$ and each angles is 60°.

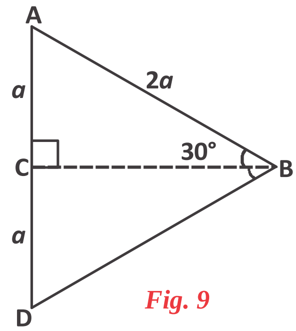


Fig. 9

Draw a perpendicular from B to AD, which meets AD in C

$\therefore$ $AC = CD = a$ (Why?)

$\angle ABC = \angle DBC = 30°$ (Why?)

(A perpendicular drawn from the vertex of an equilateral triangle bisects the opposite side and also the angle at this vertex)

Now ΔACB is right angled with a right angle at C.

$\angle ABC = 30°$ and the base for this angle is BC, AC is the perpendicular and AB is the hypotenuse.

$$BC^2 = AB^2 - AC^2 \qquad (\text{from } BC^2 + AC^2 = AB^2)$$

$$= (2a)^2 - (a)^2 = 4a^2 - a^2$$

$$= 3a^2 = a^2.3$$

$$BC = a.\sqrt{3}$$

Now we have the lengths of AB, BC and AC.

You can write the trigonometric ratios for 30° in your copies. Compare your answers with your friends.

Trigonometric Ratios for 60°

ΔABC has $\angle A = 60°$

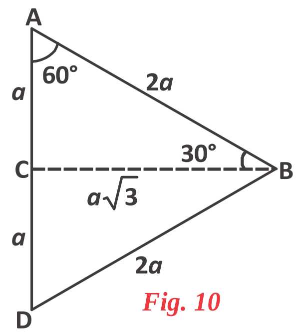


Fig. 10

For this angle, the perpendicular BC $\left(= a\sqrt{3}\right)$ and base AC (= a), whereas hypotenuse AB (= $2a$).

$$\sin 60° = \frac{BC}{AB} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{AC}{AB} = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60° = \frac{BC}{AC} = \frac{a\sqrt{3}}{a} = \sqrt{3}$$

Find the remaining ratios with the help of your friends.

Trigonometric Ratios for 0°

To find trigonometric ratios for 0° angle, we will have to think about a right angled triangle with one angle 0°. Do you think that such a triangle is possible? (Discuss this with your friends)

We shall think about how in a right angled triangle, if one of its acute angles is continuously reduced, will affect the lengths of the sides.

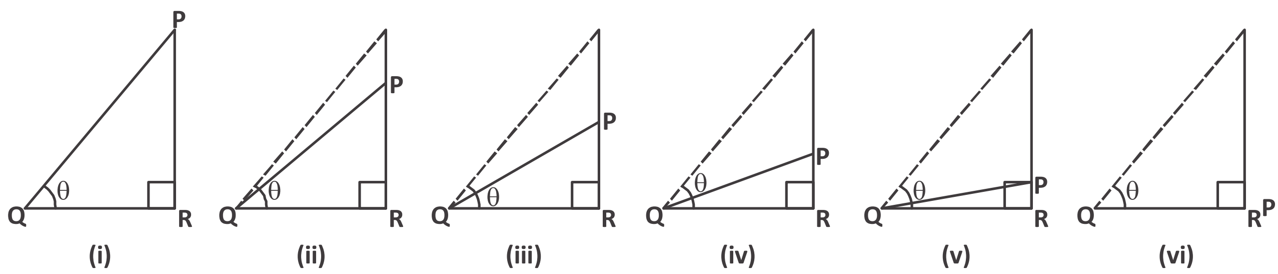


Fig. 11

ΔPQR is right angled triangle. $\angle$PQR is the angle which has to be reduced till it become 0°. The figures (i) to (vi) shows the continuous reduction of the angle θ.

As the angle reduces, what changes can be observed in the perpendicular PR?

Is the hypotenuse QP also changing?

We can see that as θ is reduced, PR is also getting smaller. So when θ is going towards 0°, PR is also going towards 0

Hence when $\theta = 0$, perpendicular PR = 0.

Along with this QP is also getting smaller and almost equal to the base QR.

Hence for $\theta = 0$, base QR = hypotenuse QP

$$\therefore \sin 0° = \frac{PR}{QP} = \frac{0}{QP} = 0$$

$$\cos 0° = \frac{QR}{QP} = 1 \quad (\because QR = QP \text{ given})$$

$$\tan 0° = \frac{PR}{QR} = \frac{0}{QR} = 0$$

$$\cot 0° = \frac{QR}{PR} = \frac{QR}{0} = \text{Not defined (in a rational number, if denominator is zero,}$$

the number is undefined)

$$\sec 0° = \frac{QP}{QR} = 1$$

$$\text{cosec } 0° = \frac{QP}{PR} = \frac{QP}{0} = \text{Not defined}$$

Trigonometric Ratios for 90°

To get trigonometric ratios for 90°, we have to take a right angled triangle and see what happens as we increase θ to 90°.

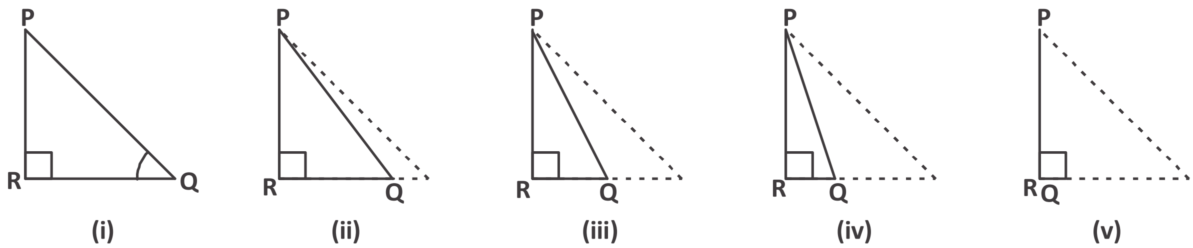


Fig. 12

PQR is a right angled triangle in which ∠PQR is increased till it becomes 90°. in *Fig.*12, you can see waht happens as we continuously increase the ∠Q till it becomes 90°.

As we increase ∠Q, can you see a change is the base QR?

Is there any change in hypotenuse PQ?

As you can see, as we increase the value of Q, the base QR gets reduced and when ∠Q = 90°, then QR = 0. Also the length of the hypotenuse reduces and becomes almost equal to perpendicular PR.

Hence when ∠Q = 90°, hypotenuse PQ = perpendicular PR and base QR = 0

So $$\text{Sin } 90° = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{PR}{PQ} = 1$$

$$\text{Cos } 90° = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{QR}{PQ} = \frac{0}{1} = 0$$

$$\tan 90° = \frac{\text{Perpendicular}}{\text{Base}} = \frac{PR}{QR} = \frac{PR}{0} = \text{Not defined}$$

Similarly we can find the other ratios.

The Trigonometric Ratios of the Special Angles

TABLE - 1

Angles/ Ratio	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
$\cot\theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
$\text{cosec}\theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

EXAMPLE-6. Find the value of

$\cos 60° \cos 30° + \sin 60° \sin 30°$

SOLUTION : $\cos 60° \cos 30° + \sin 60° \sin 30°$

$$= \frac{1}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \times \frac{1}{2} \quad \text{(On replacing values)}$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = 2 \times \frac{\sqrt{3}}{4}$$

$$= \frac{\sqrt{3}}{2}$$

EXAMPLE-7. Determine the value of:-

$$\frac{5\sin^2 30° + \cos^2 45° - 4\tan^2 30°}{2\sin 30° \cos 30° + \tan 45°}$$

tan 90°, sec 90°, cot 0° and cosec 0° are not defined. If we take an angle slightly less than 90°, then $\tan\theta$ and $\sec\theta$ will have a very large value. So when it reaches 90°, this value will be infinite.

Similarly $\cot\theta$ and $\text{cosec}\theta$ become infinitely large as θ approaches 0, so we cannot determine their value.

SOLUTION : $$\frac{5\sin^2 30° + \cos^2 45° - 4\tan^2 30°}{2\sin 30° \cos 30° + \tan 45°}$$

$$= \frac{5\times\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 - 4\left(\frac{1}{\sqrt{3}}\right)^2}{\left(2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2}\right) + 1}$$ (On replacing values)

$$= \frac{\frac{5}{4} + \frac{1}{2} - \frac{4}{3}}{\frac{\sqrt{3}}{2} + 1}$$

$$= \frac{\frac{15+6-16}{12}}{\frac{\sqrt{3}+2}{2}} = \frac{\frac{21-16}{12}}{\frac{\sqrt{3}+2}{2}}$$

$$= \frac{5}{12} \times \frac{2}{\sqrt{3}+2} = \frac{5}{6\left(\sqrt{3}+2\right)}$$

$$= \frac{5}{6\left(2+\sqrt{3}\right)} \times \frac{\left(2-\sqrt{3}\right)}{\left(2-\sqrt{3}\right)}$$ (On rationalising the denominator)

$$= \frac{5\left(2-\sqrt{3}\right)}{6\left(4-3\right)} = \frac{5\left(2-\sqrt{3}\right)}{6}$$ $$\left((2+\sqrt{3})(2-\sqrt{3}) = (2)^2 - \left(\sqrt{3}\right)^2\right)$$

EXAMPLE-8. Prove that $\cos^2 30° - \sin^2 30° = \cos 60°$

SOLUTION : $\cos^2 30° - \sin^2 30°$

$$= (\cos 30°)^2 - (\sin 30°)^2$$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$= \frac{3}{4} - \frac{1}{4} = \frac{3-1}{4} = \frac{2}{4}$$

$$= \frac{1}{2} = \cos 60°$$

Exercise - 8.3

1. Choose the correct option from the following:-

(i) $\dfrac{1 - \tan^2 45°}{1 + \tan^2 45°} =$

(a) 1 (b) tan90° (c) 0 (d) sin45°

(ii) $\dfrac{2\tan 30°}{1 - \tan^2 30°} =$

(a) sin 60° (b) sin 30° (c) tan 60° (d) cos 60°

2. Evaluate the following:-

(i) $\cos 30° \cos 45° - \sin 30° \sin 45°$ (ii) $\tan 30° \sec 45° + \tan 60° \sec 30°$

(iii) $\operatorname{cosec} 30° + \cot 45°$ (iv) $\dfrac{\cot 60°}{\sec 30° - \tan 45°}$

(v) $\tan^2 60° + \tan^2 45°$ (vi) $\dfrac{\cos 30° + \sin 60°}{1 + \cos 60° + \sin 30°}$

(vii) $\dfrac{\sin^2 45° + \cos^2 45°}{\tan^2 60°}$ (viii) $\dfrac{\sin 30° - \sin 90° + 2\cos 0°}{\tan 30° \tan 60°}$

3. Check whether true or false:-

(i) $\sin 30° \cos 60° + \cos 30° \sin 60° = \tan 90°$

(ii) $1 - 2\sin^2 30° = \cos^2 60°$ (iii) $2\cos^2 45° - 1 = \cos 90°$

(iv) $\sin^2 45° = 1 - \cos^2 45°$ (v) $\sin^2 60° + \cos^2 60° = 1$

Trigonometric Equations

Just as on solving an algebraic equation, we evaluate the value of unknowns like x, y, z, ..., similarly in trigonometric by solving equations we can evaluate the values of angle θ.

In this section we shall try to study equations which give the value of unknown θ lying between 0° to 90°

EXAMPLE-8. Solve for θ the equation $2\sin\theta - 1 = 0$ if $0° \leq \theta° \leq 90°$

SOLUTION : $2\sin\theta - 1 = 0$

$$2\sin\theta = 1 \text{ or } \sin\theta = \frac{1}{2}$$

$\sin\theta = \sin 30°$ $\left(\because \sin 30° = \frac{1}{2}\right)$

$\therefore \theta = 30°$

EXAMPLE-9. Solve for θ the equation $\sqrt{3}\tan\theta = 1$ when $0° \le \theta° \le 90°$

SOLUTION : $\sqrt{3}\tan\theta = 1$ or $\tan\theta = \frac{1}{\sqrt{3}}$

$\tan\theta = \tan 30°$ $\left(\because \tan 30° = \frac{1}{\sqrt{3}}\right)$

$\therefore \theta = 30°$

Exercise - 8.4

Solve the following equations for the value of θ, when $0° \le \theta° \le 90°$

1. $\sin\theta = \cos\theta$
2. $2\cos\theta = 1$
3. $2\sin^2\theta = \frac{1}{2}$
4. $3\tan^2\theta - 1 = 0$
5. $2\sin\theta = \sqrt{3}$
6. $\tan\theta = 0$
7. $3\text{cosec}^2\theta = 4$
8. $2\cos^2\theta = \frac{1}{2}$
9. $4\sin^2\theta - 3 = 0$
10. $4\sec^2\theta - 1 = 3$
11. $\cot^2\theta = 3$

Other Applications of Trigonometric Ratios

Till now the trigonometric ratios that we have studied have been for angles of a right angled triangle.

In fact, besides right angled triangles, the trigonometric ratios also exist and are defined for other triangles, quadrilaterals, pentagons and polygons. These are special characteristics of the angles. So if we know the value of the angles in a figure, we can use trigonometric ratios to determine the length of the sides. This can be understood by the following example:-

EXAMPLE-10. Consider a triangle ABC in which $\angle B = 45°$ and $\angle C = 30°$, AB = 5 *cm*. Thus the triangle is not a right angled traingle.

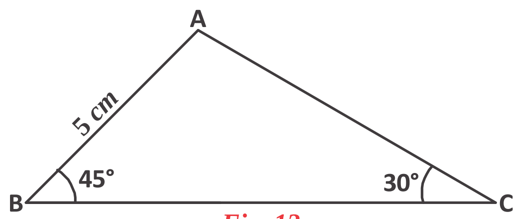


Fig. 13

SOLUTION : Can we find AC and BC using the information given?

We can construct a perpendicular from the vertex A to the side BC, intersecting the side in D.

Now consider ΔABD,

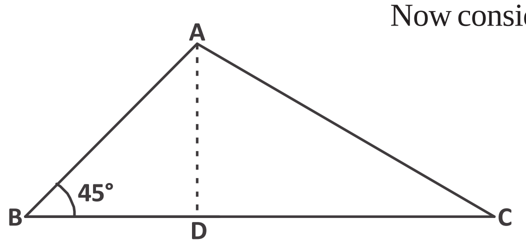


Fig. 14

$$\sin 45° = \frac{AD}{AB} = \frac{AD}{5}$$

$$\text{Or } AD = 5 \sin 45° = \frac{5}{\sqrt{2}}$$

Now we can find BD. Rehana said that in the ΔABD, AD = BD

Do you think that is correct?

Are they equal?

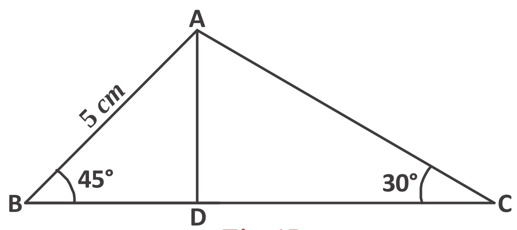


Fig. 15

Now take ΔADC

$$\sin 30° = \frac{AD}{AC}$$

$$AC = \frac{AD}{\sin 30°}$$

$$\text{Or} \quad AC = \frac{5}{\sqrt{2}} \times \frac{1}{\sin 30°}$$

$$= \frac{5}{\sqrt{2}} \times 2 = 5\sqrt{2}$$

$$\text{And } DC = AC \cos 30° = 5\sqrt{2} \times \frac{\sqrt{3}}{2}$$

$$= \frac{5 \times \sqrt{3}}{\sqrt{2}}$$

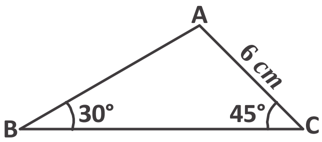
BD and DC, both add to give BC

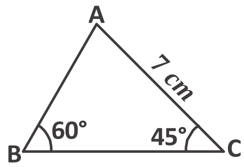
$$BC = \frac{5}{\sqrt{2}} + \frac{5\sqrt{3}}{\sqrt{2}} = \frac{5\left(1+\sqrt{3}\right)}{\sqrt{2}}$$

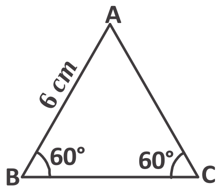
Hence $AB = 5$, $AC = 5\sqrt{2}$ and $BC = \dfrac{5\left(1+\sqrt{3}\right)}{\sqrt{2}}$

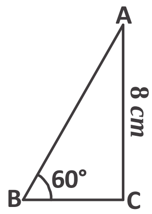
Try This

Find all the sides in the following triangles:

1. 

2. 

3. 

4. 

What Have We Learnt

1. The trigonometric ratios can be found using the following:-

$$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} \qquad \cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} \qquad \tan\theta = \frac{\text{Perpendicular}}{\text{Base}}$$

$$\text{cosec}\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} \qquad \sec\theta = \frac{\text{Hypotenuse}}{\text{Base}} \qquad \cot\theta = \frac{\text{Base}}{\text{Perpendicular}}$$

2. The relation between the various trigonometric ratios is:-

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \text{cosec}\,\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

3. If we know one trigonometric ratio of one of the acute angles of the triangle, we can find the remaining trigonometric ratios.

4. We can find the trigonometric ratios for particular angles like 0°, 30°, 45°, 60° and 90°.

5. The value of sinA or cosA can never exceed 1 whereas the value of secA or cosecA is always 1 or more than 1.

6. The three identities are:-

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \cot^2\theta = \text{cosec}^2\theta \qquad \text{where } \theta \neq 0^\circ$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \text{where } \theta \neq 90^\circ$$