EXERCISE 10.4 VECTOR ALGEBRA

Q No 1 Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

Sol.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix} = \hat{i}\left((-7)(2) - (7)(-2)\right) - \hat{j}\left[(1)(2) - (7)(3)\right] + \hat{k}\left[(1)(-2) - (-7)(3)\right]$$

$$= (-14+14)\hat{i} - (2-21)\hat{j} + (-2+21)\hat{k}$$

$$= 0\hat{i} + 19\hat{j} + 19\hat{k}$$

$$\therefore |\vec{a} \times \vec{b}| = \sqrt{(0)^2 + (19)^2 + (19)^2} = \sqrt{(19)^2[1+1]} = 19\sqrt{2}$$

Q No 2 Find a unit vector perpendicular to each of vector $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

Sol.

Let $\vec{u} = \vec{a} + \vec{b} = (3\hat{i} + 2\hat{j} + 2\hat{k}) + (\hat{i} + 2\hat{j} - 2\hat{k}) = 4\hat{i} + 4\hat{j}$

$\vec{v} = \vec{a} - \vec{b} = (3\hat{i} + 2\hat{j} + 2\hat{k}) - (\hat{i} + 2\hat{j} - 2\hat{k}) = 2\hat{i} + 4\hat{k}$

Now A unit vector perpendicular to both $\vec{u}$ and $\vec{v}$

$$= \pm \frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|}$$

Now
$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix} = (4\times4 - 0\times0)\hat{i} - (4\times4 - 0\times2)\hat{j} + (4\times0 - 4\times2)\hat{k}$$

$$= 16\hat{i} - 16\hat{j} - 8\hat{k}$$

$$\therefore |\vec{u} \times \vec{v}| = \sqrt{(16)^2 + (-16)^2 + (-8)^2} = \sqrt{256 + 256 + 64} = \sqrt{576} = 24.$$

$$\therefore \text{Required Unit vector} = \pm \frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|} = \pm \frac{16\hat{i} - 16\hat{j} - 8\hat{k}}{24}$$

$$= \pm \frac{8(2\hat{i} - 2\hat{j} - 8k)}{24}$$

$$= \pm \frac{2}{3}\hat{i} \mp \frac{2}{3}\hat{j} \mp \frac{8}{3}\hat{k}$$

Q.No 3. If a unit vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence components of $\vec{a}$.

Sol. Since $\vec{a}$ makes angle $\pi/3$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and θ with $\hat{k}$

$\therefore$ Direction cosines of $\vec{a}$ are $\langle \cos\pi/3, \cos\pi/4, \cos\theta \rangle$

and $\therefore \vec{a} = \cos\frac{\pi}{3}\hat{i} + \cos\frac{\pi}{4}\hat{j} + \cos\theta\hat{k}$

$= \frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \cos\theta\hat{k}$

Now $|\vec{a}| = 1$

$\therefore \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta} = 1$

$\Rightarrow \frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$

$\Rightarrow \cos^2\theta = 1 - \frac{1}{4} - \frac{1}{2} = \frac{4-1-2}{4} = \frac{1}{4}$

$\Rightarrow \cos\theta = \pm\frac{1}{2}$.

Since θ is acute angle, $\therefore \cos\theta = +1/2$

$\Rightarrow \theta = \pi/3$

$\therefore$ Components of $\vec{a}$ are $\cos\pi/3, \cos\pi/4, \cos\pi/3$

ie $\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$.

Q.No 4. Show that $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) = 2(\vec{a}\times\vec{b})$

Sol. LHS $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$

$= \vec{a}\times\vec{a} + \vec{a}\times\vec{b} - \vec{b}\times\vec{a} - \vec{b}\times\vec{b}$

$= \vec{0} + \vec{a}\times\vec{b} + \vec{a}\times\vec{b} - \vec{0}$ $\quad [\because \vec{b}\times\vec{a} = -\vec{a}\times\vec{b}]$

$= 2(\vec{a}\times\vec{b}) = $ RHS

Q.No 5. Find λ and μ if $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k}) = \vec{0}$

Sol. Given $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k}) = \vec{0}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = \vec{0}$$

$\Rightarrow \quad \hat{i}(6\mu - 27\lambda) - \hat{j}(2\mu - 27) + \hat{k}(2\lambda - 6) = \vec{0}$

$\Rightarrow \quad 6\mu - 27\lambda = 0, \quad -2\mu + 27 = 0, \quad 2\lambda - 6 = 0$

$\Rightarrow \quad 2\mu = 9\lambda, \quad \mu = 27/2, \quad \lambda = 3.$

Note that $\lambda = 3$, $\mu = \frac{27}{2}$ satisfy $2\mu = 9\lambda$.

Q.No. 6 Given that $\vec{a}\cdot\vec{b} = 0$ and $\vec{a}\times\vec{b} = 0$, what can you conclude about vectors $\vec{a}$ and $\vec{b}$?

Sol. By Lagrange's Identity

$$(\vec{a}\times\vec{b})^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$$

Here $\vec{a}\times\vec{b} = 0$ and $\vec{a}\cdot\vec{b} = 0$

$\therefore \quad |\vec{a}|^2|\vec{b}|^2 = 0$

or $\quad (|\vec{a}||\vec{b}|)^2 = 0$

or $\quad |\vec{a}||\vec{b}| = 0$

$\Rightarrow \quad |\vec{a}| = 0 \quad$ or $\quad |\vec{b}| = 0$

$\Rightarrow \quad \vec{a} = 0 \quad$ or $\quad \vec{b} = 0$

Hence one of $\vec{a}$ and $\vec{b}$ is a zero vector.

Q.No 7 Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then show that

$$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}.$$

Sol. Now $\vec{b}+\vec{c} = (b_1+c_1)\hat{i} + (b_2+c_2)\hat{j} + (b_3+c_3)\hat{k}$

$$\therefore \quad \vec{a}\times(\vec{b}+\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \left[\text{By property of determinants}\right]$$

$$= \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$$

Hence $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

Q.No. 8. If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example.

Sol If $\vec{a} = 0$ or $\vec{b} = 0$ then surely $\vec{a} \times \vec{b} = \vec{0}$

However the converse is not true.

ie If $\vec{a} \times \vec{b} = \vec{0}$, then $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ may not hold.

eg Let $\vec{a} = 3\hat{i}$ $\quad \vec{b} = 4\hat{i}$

then $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$ but $\vec{a} \times \vec{b} = 12\,\hat{i} \times \hat{i} = \vec{0}$

Q.No 9 Find the area of Δ with vertices $A(1,1,2)$, $B(2,3,5)$ and $C(1,5,5)$

Sol. Now Area of $\Delta ABC = \frac{1}{2}|\vec{AB} \times \vec{AC}|$.

Here $\vec{AB} = \vec{OB} - \vec{OA}$

$= (2\hat{i} + 3\hat{j} + 5\hat{k}) - (\hat{i} + \hat{j} + 2\hat{k}) = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{AC} = \vec{OC} - \vec{OA}$

$= (\hat{i} + 5\hat{j} + 5\hat{k}) - (\hat{i} + \hat{j} + 2\hat{k}) = 4\hat{j} + 3\hat{k}$

$$\therefore \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix} = (6-12)\hat{i} - (3-0)\hat{j} + (4-0)\hat{k}$$

$= -6\hat{i} - 3\hat{j} + 4\hat{k}$

$\therefore |\vec{AB} \times \vec{AC}| = \sqrt{(-6)^2 + (-3)^2 + (4)^2} = \sqrt{36 + 9 + 16} = \sqrt{61}$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{61}$ sq.units.

Q.No 10 Find the area of parallelogram whose adjacent sides are determined by $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ $\quad \vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$

Sol. Required Area $= |\vec{a} \times \vec{b}|$.

$$\text{Now} \quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} = (-1+21)\hat{i} - (1-6)\hat{j} + (-7+2)\hat{k}$$

$= 20\hat{i} + 5\hat{j} - 5\hat{k}$

$\therefore |\vec{a} \times \vec{b}| = \sqrt{(20)^2 + (5)^2 + (-5)^2} = \sqrt{400 + 25 + 25} = \sqrt{450}$

$= \sqrt{15 \times 15 \times 2} = 15\sqrt{2}$

$\therefore$ Required Area $= 15\sqrt{2}$ square units.

Q No 11. Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$, then $\vec{a}\times\vec{b}$ is a unit vector, if angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\pi/2$.

Sol. If θ is angle between $\vec{a}$ and $\vec{b}$, then

$$|\vec{a}\times\vec{b}| = 1$$

$$\Rightarrow \quad |\vec{a}||\vec{b}| \sin\theta = 1$$

ie $3 \times \frac{\sqrt{2}}{3} \sin\theta = 1$ ie $\sin\theta = \frac{1}{\sqrt{2}}$ ie $\theta = \frac{\pi}{4}$ or $3\pi/4$.

$\therefore$ (B) is correct option.

Q No 12. Area of rectangle having vectors A, B, C and D with position vectors $-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}$, $\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}$, $\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ and $-\hat{i}-\frac{1}{2}\hat{j}+4k$ respectively is

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4.

Sol. Area of rectangle ABCD $= |AB \times AD|$

Now $\vec{AB} = \vec{OB} - \vec{OA} = \left(\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}\right)-\left(-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}\right) = 2\hat{i}$

$\vec{AD} = \vec{OD} - \vec{OA} = \left(-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}\right) - \left(-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}\right) = -\hat{j}$

$$\vec{AB}\times\vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} = (0)\hat{i} - (0)\hat{j} + (2-0)\hat{k} = 2\hat{k}$$

$\therefore |\vec{AB}\times\vec{AD}| = \sqrt{(2)^2} = 2$ square units

$\therefore$ (C) is correct option.

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