10

Limits and Derivatives

10.01 Introduction :

Here we will study that branch of mathematics in which the value of the function changes with the change in points of domain on basis of definition of limit and their algebraic studies. It is called calculus. To study calculus, we will use intuitive idea. At last we will be introduce with general information of algebra of calculus.

10.02 Limits, a view point :

If we consider a regular polygon inscribed in a circle, we observe that -

(i) What so ever may be the number of sides of the polygon increases, its area cannot be more than the circle.

(ii) As the number of sides of the polygon increases, its area also approaches to the area of the circle.

(iii) As the number of sides of the polygon increases, the difference in areas of circle and polygon reduces and reaches to minimum.

This defines the concepts of limits in calculus.

10.03 Meaning of $x \to a$:

Let x be a variable and a be constant. When x takes the value close to a and approaches to a but x is not equal to a, then we write $x \to a$.

If x approaches to a from the right hand i.e. x approaches from the number greater than a we write it as $x \to a^+$

Similarly, if x approaches to a from the left hand side i.e. x approaches from the number less than a, we write it as $x \to a^-$

Now, if δ is a positive number which is very small and a be number such that $0 < |x-a| < \delta$, then we say that x approaches to a and we write as $x \to a$.

Note : x approaches or tends to a means x can take all neighbourhood values of a but not a. These values are known as the limiting values at $x = a$. For example line means $x \to 2$, x will take 1.9, 1.99, 1.999, . . . and 2.1, 2.01, 2.001, . . .etc. i.e. all nearby values of 2 but not 2.

10.04 Definition of Limit of a Function :

Consider a function $y = f(x)$, which is defined or undefined at $x = a$ but defined at right and left neighbourhood of $x = a$, then a real number l is called limit of function f when approaches to a, if and only if, a positive number δ exists for a arbitrary positive number ε i.e. $|f(x) - l| < \varepsilon$ while $0 < |x-a| < \delta$. It is expressed as $\lim_{x \to a} f(x) = l$

Right Hand Limit : If x approaches to 'a' from right side, then right hand limit of f is written as $\lim_{x \to a^+} f(x)$ or $f(a+0)$.

To find the right hand limit, we substitute $x = a + h$ in $f(x)$ and by putting $h \to 0$, therefore

$$f(a+0) = \lim_{h \to 0} f(a+h), (h > 0)$$

Left Hand Limit : If x approaches to 'a' from left side, then left hand limit of f is written as $\lim_{x \to a^-} f(x)$ or $f(a - 0)$.

To find the left hand limit, we substitute $x = a - h$ in $f(x)$ and by putting $h \to 0$, therefore

$$f(a-0) = \lim_{h \to 0} f(a-h), (h > 0)$$

10.05 Existence of a limit :

$\lim_{x \to a} f(x)$ exists only, if left hand limit $\lim_{x \to a^-} f(x)$ and right hand limit $\lim_{x \to a^+} f(x)$ exist and are identical.

Limit exists only when $\Leftrightarrow f(a-0) = f(a+0)$. That means,

$$\lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

If limit of any function at any point exists, then it is not necessary to find out both limits.

Illustrative Examples

Example 1 : Find wheather the limit exists for the function $f(x) = \dfrac{1}{2+x}$ at $x = 2$

Solution : $\text{R.H.L.} = f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} \dfrac{1}{2+(2+h)} = \lim_{h \to 0} \dfrac{1}{4+h} = \dfrac{1}{4}$

$\text{L.H.L.} = f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} \dfrac{1}{2+(2-h)} = \lim_{h \to 0} \dfrac{1}{4-h} = \dfrac{1}{4}$

$\therefore \quad f(2-0) = f(2+0) = \dfrac{1}{4}$

$\therefore$ Limit exists at $x = 2$

Example 2 : Show that the limit does not exist for the function

$$f(x) = \begin{cases} (1/2) - x, ; & 0 < x < 1/2 \\ (3/2) - x, ; & 1/2 < x < 1 \end{cases} \quad \text{at } x = 1/2$$

Solution : For R.H.L., $f(x) = (3/2) - x$

$$\therefore \quad f\left(\frac{1}{2}+0\right) = \lim_{h \to 0}\left\{\frac{3}{2} - \left(\frac{1}{2}+h\right)\right\} = \lim_{h \to 0}\left\{\frac{3}{2} - \frac{1}{2} - h\right\} = \lim_{h \to 0}\{1-h\} = 1$$

For L.H.L., $f(x) = (1/2) - x$

$$\therefore \quad f\left(\frac{1}{2}-0\right) = \lim_{h \to 0}\left\{\frac{1}{2} - \left(\frac{1}{2}-h\right)\right\} = \lim_{h \to 0}\left\{\frac{1}{2} - \frac{1}{2} + h\right\} = \lim_{h \to 0}\{h\} = 0 \qquad f\left(\frac{1}{2}-0\right) \neq f\left(\frac{1}{2}+0\right)$$

$\therefore$ Limit does not exist at $x = 1/2$

Example 3 : Evaluate the Right hand and Left hand limit of the function $f(x) = \begin{cases} 5x-4, & 0 < x \le 1 \\ 4x^3 - 3x, & 1 < x < 2 \end{cases}$, at $x = 1$

Solution : For R.H.L., $x = 1$ If $f(x) = 4x^3 - 3x$

$$\therefore \quad f(1+0) = \lim_{h \to 0}\left[4(1+h)^3 - 3(1+h)\right] = \lim_{h \to 0}\left[4(1)^3 - 3(1)\right] = 1$$

For L.H.L. , $f(x) = 5x - 4$

$$f(1-0) = \lim_{h\to 0}\left[5(1-h)-4\right] = \lim_{h\to 0}\left[5-5h-4\right] = \lim_{h\to 0}(1-5h) = 1$$

$\because$ $f(1+0) = f(1-0) = 1$

$\therefore$ Limit exists at $x = 1$

Exercise 10.1

1. Show that the limit of the function $f(x) = \dfrac{\log_e x}{x-1}$ at $x = 1$ exists as Right hand and Left hand limit and its value is 1.

2. Does the limit exist at $x = 0$ for the function $f(x) = \dfrac{x+|x|}{x}$?

3. Prove that the limit exists at $x = 0$ for the function $f(x) = |x| + |x-1|$

4. Prove that the limit does not exist at $x = 2$ for the function $f(x) = \begin{cases} x^2 + x + 1, & ; \; x \geq 2 \\ x, & ; \; x < 2 \end{cases}$

5. Find the Right hand and Left hand limit for the function $f(x) = x\cos(1/x)$ at $x = 0$

10.06 Theorems on limits :

Let f and g be two functions defined in domain D such that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then we can define four new function $f \pm g$, fg and f/g in domain D as.

$$(f \pm g)(x) = f(x) \pm g(x), (fg)(x) = f(x).\, g(x),$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0, \;\forall\; x \in D$$

Using these, we can find the below given results

Let $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$. If ℓ and m exist, then

(i) Sum and Difference Rule $\lim_{x\to a} (f \pm g)(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = \ell \pm m$

(ii) Multiplication Rule $\lim_{x\to a} (f\;\; g)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = lm$

(iii) Quotient Rule $\lim_{x\to a}\left(\dfrac{f}{g}\right)(x) = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{\ell}{m}, m \neq 0$

(iv) Constant Rule If $f(x) = k$, where k is a constant, $\lim_{x\to a} f(x) = \lim_{x\to a} k = k$

(v) Multiplication with a constant $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x) = k\ell$, where k is a constant

(vi) Modulus Rule $\lim_{x\to a} |f(x)| = \left|\lim_{x\to a} f(x)\right| = |\ell|$

(vii) Power Rule $\lim_{x\to a}\left[\{f(x)\}^{g(x)}\right] = \left\{\lim_{x\to a} f(x)\right\}^{\lim_{x\to a} g(x)} = \ell^m$

Special case (a) $\lim_{x\to a} \log f(x) = \log\left\{\lim_{x\to a} f(x)\right\} = \log \ell$

(b) $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)} = e^{\ell}$

(c) $\lim_{x\to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x\to a} \frac{1}{f(x)} = 0$

10.07 Methods of evaluation of limits :

(i) Substitution Method : After substituting, if the value of the function does not take the form $\left(\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^{0}, 1^{\infty}, \infty^{0}\right)$, then that value will be the limiting value.

Example : $\lim_{x\to 2}\left(x^2+3x+2\right) = 2^2+3.2+2 = 12$

(iii) Factorization Method : If $f(x)$ and $g(x)$ be polynomial and $g(a) \neq 0$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{f(a)}{g(a)}$$

Example : $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 4$

(iii) Rationalisation Method : Square root functions are rationalised and method of substitution is used.

Example : Find the value of $\lim_{x\to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$

Solution : $\lim_{x\to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}} = \lim_{x\to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}} \times \frac{1+\sqrt{5-x}}{1+\sqrt{5-x}} \times \frac{3+\sqrt{5+x}}{3+\sqrt{5+x}}$

$$= \lim_{x\to 4} \frac{4-x}{-4+x} \times \frac{1+\sqrt{5-x}}{3+\sqrt{5+x}} = -\frac{2}{6} = -\frac{1}{3}$$

(iv) Expantion Method : If $x \to 0$ and the expression contains a function which can be expanded, then it is expanded in terms of ascending power of x. On dividing common power of x in numerator and denominator, indeterminate form can be omitted. Some important expansions are as follows :

(a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(b) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

(c) $a^x = 1 + (x\log_e a) + \frac{(x\log_e a)^2}{2!} + \frac{(x\log_e a)^3}{3!} + \cdots$

(d) $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(e) $\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$

(f) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

(g) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

(h) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$

(i) $(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \cdots$

(j) $\left(1+\frac{1}{x}\right)^x = e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 + \cdots\right)$

(k) $\Sigma n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

(l) $\Sigma n^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(m) $\Sigma n^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{\{n(n+1)\}^2}{4}$

Some Standard Limits :

(a) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(b) $\lim_{x \to 0} \cos x = 1$

(c) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

(d) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(e) $\lim_{x \to 0} \frac{b^x - 1}{x} = \log_e b (b \neq 0)$

(f) $\lim_{x \to 0} \frac{\log_e (1+x)}{x} = 1$

(g) $\lim_{x \to 0} (1+x)^{1/x} = e$

(h) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

(i) $\lim_{x \to a} \frac{x^m - a^m}{x - a} = ma^{m-1}$

(j) $\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$

Example : Find the value of $\lim_{x \to 0} \frac{a^x - 1}{x}$, $a \neq 0$

Solution : $\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{x \to 0} \frac{1 + (x \log_e a) + (x \log_e a)^2 / 2! + \cdots - 1}{x}$

$$= \lim_{x \to 0} \frac{(x \log_e a) + (x \log_e a)^2 / 2! + \cdots}{x}$$

$$= \lim_{x \to 0} (\log_e a) + x (\log_e a)^2 / 2! + \cdots$$

$$= \log_e a$$

(v) $x \to \infty$: Under this condition, the highest power of the variable of the function is taken common and then the infinite limit is put in the numerator and denominator.

Example : Evaluate $= \lim_{x \to \infty} \frac{ax^2 + bx + c}{dx^2 + ex + f}$

Solution : $\lim_{x \to \infty} \frac{ax^2 + bx + c}{dx^2 + ex + f} = \lim_{x \to \infty} \frac{x^2 \left[a + (b/x) + (c/x^2)\right]}{x^2 \left[d + (e/x) + (f/x^2)\right]}$

$$= \lim_{x \to \infty} \frac{a + (b/x) + (c/x^2)}{d + (e/x) + (f/x^2)} = \frac{a}{d}$$

(vi) Simplification – In this method, the given function is simiplified in such a way that the indeterminate form can be omitted .

Example : Evaluate : $\lim_{x\to 0}\frac{1-\cos x}{\sin x}$

Solution : $\lim_{x\to 0}\frac{1-\cos x}{\sin x}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=0$

Illustrative Examples

Example 4 : Evaluate : $\lim_{x\to 0}\frac{\cos x}{1+\sin x}$

Solution : $\lim_{x\to 0}\frac{\cos x}{1+\sin x}=\frac{\cos 0}{1+\sin 0}=1$

Example 5 : Evaluate : $\lim_{x\to 1}\frac{x-1}{2x^2-7x+5}$

Solution : $\lim_{x\to 1}\frac{x-1}{2x^2-7x+5}$

Let $x=1+h$, If $x\to 1$, $h\to 0$

$\therefore \quad \lim_{h\to 0}\frac{(1+h)-1}{2(1+h)^2-7(1+h)+5}=\lim_{h\to 0}\frac{h}{2h^2-3h}=\lim_{h\to 0}\frac{1}{2h-3}=-\frac{1}{3}$

Example 6 : Evaluate : $\lim_{x\to 2}\frac{x^2-5x+6}{x^2-4}$

Solution :

$$\lim_{x\to 2}\frac{x^2-5x+6}{x^2-4}=\lim_{x\to 2}\frac{(x-3)(x-2)}{(x+2)(x-2)}=\lim_{x\to 2}\frac{(x-3)}{(x+2)}=\frac{2-3}{2+2}=-\frac{1}{4}$$

Example 7 : Evaluate : $\lim_{x\to 0}\frac{x}{\sqrt{1+x}-1}$

Solution : $\lim_{x\to 0}\frac{x}{\sqrt{1+x}-1}$

Retionalising the demominator by multiplying with $\sqrt{1+x}+1$, numerator and denominator both

$$\lim_{x\to 0}\frac{x}{\sqrt{1+x}-1}\times\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}=\lim_{x\to 0}\frac{x}{1+x-1}\sqrt{1+x}+1=\lim_{x\to 0}\left(\sqrt{1+x}+1\right)=1+1=2$$

Example 8 : Evaluate : $\lim_{x\to 0}\frac{e^{ax}-e^{bx}}{\sin ax-\sin bx}$

Solution : $\lim_{x\to 0}\frac{e^{ax}-e^{bx}}{\sin ax-\sin bx}$

$$= \lim_{x\to 0} \frac{\left(1+ax+\frac{a^2x^2}{2!}+\cdots\right)-\left(1+bx+\frac{b^2x^2}{2!}+\cdots\right)}{\left(ax-\frac{a^3x^3}{3!}+\frac{a^5x^5}{5!}-\cdots\right)-\left(bx-\frac{b^3x^3}{3!}+\frac{b^5x^5}{5!}-\cdots\right)}$$

$$= \lim_{x\to 0} \frac{\left((a-b)x+\frac{(a^2-b^2)x^2}{2!}+\cdots\right)}{\left((a-b)x-\frac{(a^3-b^3)x^3}{3!}+\frac{(a^5-b^5)x^5}{5!}-\cdots\right)}$$

On dividing numerator and denominator by x,

$$= \lim_{x\to 0} \frac{\left((a-b)+\frac{(a^2-b^2)x}{2!}+\cdots\right)}{\left((a-b)-\frac{(a^3-b^3)x^2}{3!}+\frac{(a^5-b^5)x^4}{5!}-\cdots\right)}$$

$$= \frac{(a-b+0+\cdots)}{(a-b-0+0-\cdots)} = \frac{a-b}{a-b} = 1$$

Example 9 : Evaluate : $\lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$

Solution : $\lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3} = \lim_{n\to\infty} \frac{\sum n^2}{n^3} = \lim_{n\to\infty} \frac{\frac{n(n+1)(2n+1)}{6}}{n^3} = \lim_{n\to\infty} \frac{n^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6n^3}$

$$= \lim_{n\to\infty} \frac{(1+1/n)(2+1/n)}{6} = \frac{2}{6} = \frac{1}{3}$$

Example 10 : Evaluate : $\lim_{\theta\to\pi/2} (\sec\theta - \tan\theta)$

Solution : $\lim_{\theta\to\pi/2} (\sec\theta-\tan\theta) = \lim_{\theta\to\pi/2}\left(\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}\right) = \lim_{\theta\to\pi/2}\left(\frac{1-\sin\theta}{\cos\theta}\right) = \lim_{\theta\to\pi/2}\left(\frac{1-\cos(\pi/2-\theta)}{\sin(\pi/2-\theta)}\right)$

$$= \lim_{\theta\to\pi/2}\left(\frac{2\sin^2(\pi/4-\theta/2)}{2\sin(\pi/4-\theta/2)\cos(\pi/4-\theta/2)}\right) = \lim_{\theta\to\pi/2} \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = 0$$

Example 11 : Prove that : $\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$

Solution : Here, $\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a^+}\frac{x^n-a^n}{x-a}$ $\quad$ [$\because$ $\lim_{x\to a}f(x)$ exists if $\therefore$ $\lim_{x\to a^-}f(x)=\lim_{x\to a^+}f(x)$]

$$=\lim_{x\to 0}\frac{(a+h)^n-a^n}{a+h-a}=\lim_{x\to 0}\frac{a^n\left\{\left(1+\frac{h}{a}\right)^n-1\right\}}{h}$$

$$=a^n\lim_{h\to 0}\frac{\left[1+n\cdot\frac{h}{a}+\frac{n(n-1)}{\lfloor 2}\frac{h^2}{a^2}+\ldots-1\right]}{h}$$

$$=a^n\lim_{h\to 0}\left(\frac{n}{a}+\frac{n(n-1)}{\lfloor 2}\frac{h}{a^2}+\ldots\right)=a^n\cdot\frac{n}{a}=na^{n-1}$$

Exercise 10.2

Determine the following limits :

1. (a) $\lim_{x\to 2}\frac{x^2-3x+2}{x^2+x-6}$ $\qquad$ (b) $\lim_{x\to 1}\frac{(2x-3)(x-1)}{2x^2+x-3}$

2. (a) $\lim_{\alpha\to\pi/4}\frac{\sin\alpha-\cos\alpha}{\alpha-\pi/4}$ $\qquad$ (b) $\lim_{x\to 1}\frac{x^n-1}{x-1}$

3. (a) $\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}$ $\qquad$ (b) $\lim_{x\to 0}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x^2}$

4. (a) $\lim_{x\to 0}\frac{e^x+e^{-x}-2\cos x}{x\sin x}$ $\qquad$ (b) $\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}$

5. (a) $\lim_{x\to\infty}\sqrt{\frac{x+\sin x}{x-\cos x}}$ $\qquad$ (b) $\lim_{n\to\infty}\frac{\sqrt{3n^2-1}-\sqrt{2n^2-1}}{4n+3}$

6. (a) $\lim_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin x}$ $\qquad$ (b) $\lim_{x\to 0}\frac{e^{\sin x}-1}{x}$

7. (a) $\lim_{x\to 0}\frac{x(e^x-1)}{1-\cos x}$ $\qquad$ (b) $\lim_{x\to a}\frac{(x+2)^{3/2}-(a+2)^{3/2}}{x-a}$

10.08 Intuitive idea of derivatives :

If a body in thrown from a standing rock the following results obtained :

t (second)	0	1	1·5	1·8	1·9	1·95	2	2·05	2·1	2·2	2·5	3	4
s (metre)	0	4·9	11·025	15·876	17·689	18·63225	19·6	20·592	21·609	23·716	30·625	44·1	78·4

On the basis of observation, we have to find velocity at any time $t=2$, then we have to divide the problem in two sections :

(i) Mean velocity at different time interval before $t = 2$

(ii) Mean velocity at different time interval after $t = 2$

$$\text{Mean velocity} = \frac{\text{Distance travelled in time interval}}{\text{time interval}}$$

Mean velocity at time interval before $t = 2$ is given in the following table

Table-1

t_1 (second)	0	1	1·5	1·8	1·9	1·95	1·99
v (metre/second)	9·8	14·7	17·15	18·62	19·11	19·355	19·551

Thus, velocity at time interval before $t = 2$ is given in the following table

Table-2

t_2 (second)	4	3	2·5	2·2	2·1	2·05	2·01
v (metre/second)	29·4	24·5	22·05	20·58	20·09	19·845	19·649

From the calculation of first table, we have found the mean velocity at increasing time interval till $t = 2$. Let there in not any dramatic happening before $t = 2$. In the second table of calculation, we have found mean velocity at decreasing time interval till $t = 2$. Let there is not any dramatic happening after $t = 2$.

After studying combine observation of both tables, we can observe clearly that instantaneous velocity at $t = 2$ is in between velocity 19.551 m/s and 19.649 m/s.

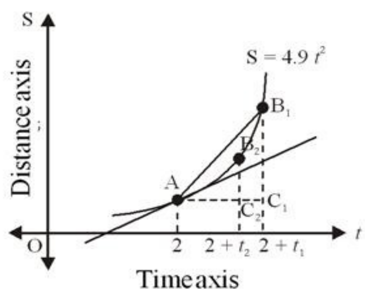


Figure : 10.01

An alternative method of process of limits is shown in figure 10.01, which is graph between past time and distance between vertex of rock and body. As $h_1, h_2, ...$ tends to zero with respect to time, the limit of mean velocity is same as ratio of $\frac{C_1B_1}{AC_1}, \frac{C_2B_2}{AC_2}, \frac{C_3B_3}{AC_3}, ...$

Here $C_1B_1 = S_1 - S_0$ distance travelled by body in time interval $h_1 = AC_1$ etc. It result from the figure 10,01 is equal to slope of tangent to the curve approaches towards slope at point A. In other words instantaneous velocity at $t=2$ is equal to slope of tangent to the curve $S=4.9t^2$ at $t=2$

10.09 Derivatives :

We get an initial idea of derivative from intuitive idea of derivative in section 10.08

Derivatives : – let $y=f(x)$ be a continuous function. Let x changes with minute change δx then change in y, correspond to x, is δy, then limit of fraction $\frac{\delta y}{\delta x}$, where $\delta x \to 0$, means if $\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$ exists, then it is called differential coefficient of y with respect to x. It is denoted by $\frac{dy}{dx}$

i.e. $\quad y=f(x)$

than $\quad y+\delta y=f(x+\delta x)$

$$\delta y=f(x+\delta x)-f(x)$$

$$\Rightarrow \quad \frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx}=\lim_{\delta x \to 0}\frac{\delta y}{\delta x}=\lim_{\delta x \to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

Differentiation by first principle : Direct method to get differential cofficient from definition is called Differention by first principle. It is also known as 'Ab-initio or delta method'.

Differentiation : – The method of finding differential coefficient of any function $f(x)$ is called differentiation.

Notation : – Differential coefficient of $f(x)$ with respect to x is generally denoted by $\frac{d}{dx}f(x)$ or $f'(x)$ or $D[f(x)]$, where $D\equiv\frac{d}{dx}$ and differential coefficient at $x=c$ is denoted by $f'(c)$ or $\left[\frac{d}{dx}f(x)\right]_{x=c}$. If $y=f(x)$, then differential coefficient of y with respect to x is denoted by $\frac{dy}{dx}$ or y_1 or y' or Dy

Note : 1. $\frac{\delta y}{\delta x}$ is a fraction, means $\delta y \div \delta x$

2. $\frac{dy}{dx}$ is not a fraction, while it only notation of $\lim_{\delta x \to 0}\frac{\delta y}{\delta x}$

Defenition : Let f be a real single valued function and there is a point 'a' defined in its domain, then differential coefficient of f at a is defined by

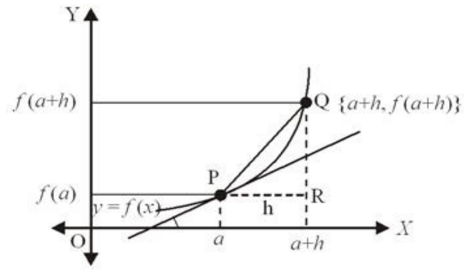
$$\lim_{h \to 0}\frac{f(a+h)-f(a)}{h}$$

If limit exists, Differential coefficient of $f(x)$ at 'a' is denoted by $f'(a)$

Geometrical description of derivative at any point of a function :

Consider $y=f(x)$ be a function having two neighbourhood points $P(a,\ f(a))$ and $Q(a+h,\ f(a+h))$ lying on its graph. We know that

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$



From triangle PQR, it is clear that the ratio whose limit, we are taking in precisely, equal to tan (QPR), which is the slope of the chord PQ. In the limiting process, as h tends to o, the point Q tends to P and we have

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{Q\to P}\frac{QR}{PR}$$

This is equivalent to fact that the chord PQ tends to the tangent at P of the curve $y=f(x)$. Thus the limit turns out to be equal to the slope of the tangent. Here $f'(a)=\tan\psi$.

Illustrative Examples

Example 12: Find the derivative of $f(x)=8x$ at $x=2$

Solution : $f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{8(2+h)-8(2)}{h}$

$$=\lim_{h\to 0}\frac{16+8h-16}{h}=\lim_{h\to 0}\frac{8h}{h}=\lim_{h\to 0}8=8$$

Thus, at $x=2$, the devivative of the function is 8.

Example 13 : Find the derivative of $f(x)=2x^2+3x-5$ at $x=-1$. Also Prove that, $f'(0)+3f'(-1)=0$.

Solution : We know that

$$f'(-1)=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}$$

$$=\lim_{h\to 0}\frac{\left[2(-1+h)^2+3(-1+h)-5\right]-\left[2(-1)^2+3(-1)-5\right]}{h}$$

$$=\lim_{h\to 0}\frac{2h^2-h}{h}=\lim_{h\to 0}(2h-1)=2(0)-1=-1$$

and $f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$$=\lim_{h\to 0}\frac{\left[2(0+h)^2+3(0+h)-5\right]-\left[2(0)^2+3(0)-5\right]}{h}$$

$$=\lim_{h\to 0}\frac{2h^2+3h}{h}=\lim_{h\to 0}(2h+3)=2(0)+3=3$$

$\therefore$ $f'(0)+3f'(-1)=0$

Example 14 : Find the derivative of $f(x)=\frac{1}{x}$

Solution : We know that $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{(x+h)}-\frac{1}{x}}{h}$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{x-(x+h)}{x(x+h)}\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{-h}{x(x+h)}\right]=\lim_{h\to 0}\frac{-1}{x(x+h)}=-\frac{1}{x^2}$$

10.10 Algebra of derivative of functions :

Theorem 1 : Let f and g be two functions such that their derivatives are defined in a common domain. Then

(i) Derivative of sum of two function is sum of the derivatives of the functions

$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

(ii) Derivative of difference of two function is difference of the derivatives of the functions

$$\frac{d}{dx}\left[f(x)-g(x)\right]=\frac{d}{dx}f(x)-\frac{d}{dx}g(x)$$

(iii) Derivative of product of two function is given by the following product rule

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right]=f(x)\cdot\frac{d}{dx}g(x)+g(x)\cdot\frac{d}{dx}f(x)$$

(iv) Derivative of quotient of two function is is given by the following quotientt rule (where the denominator is non-zero).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{\left(g(x)\right)^2}$$

The proofs of these follow essentially from the analogous theorems for limits. We will not prove these here. As in the case of limits, there theorems tell us how to compute derivatives of special types of functions. The last two statements in the theorems may be restated in the following fashion which aids in recalling them easily

Let $u=f(x)$ and $v=g(x)$. Then,

$$(uv)'=u'v+uv'$$

This is referred to a Leibnitz rule for differentiating product of functions or the product rule. Similarly, the quotient rule is

$$\left(\frac{u}{v}\right)'=\frac{vu'-uv'}{v^2}$$

Now, let us take derivatives of some standard functions. It is easy to see that the derivative of the function $f(x)=x$ is the constant function 1. This is because

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{x+h-x}{h}=1$$

We use this and the above theorem to compute the derivative of $f(x)=10x=x+x+...+x(10\text{ terms})$

$$\frac{df(x)}{dx}=\frac{d}{dx}(x+...+x)\ (10\text{ terms})=\frac{d}{dx}x+...+\frac{d}{dx}x\ (10\text{ terms})=1+...+1\ (10\text{ terms})=10$$

We note that, this limit may be evaluated using product rule too. Write $f(x)=10x=uv$, where u is the constant function taking value 10 everywhere and $v(x) = x$. Here, $f(x) = 10x = uv$. We know that the derivative of u equals 0. Also derivative of $v(x)$ $(= x)$ equals 1. Thus by the product rule, we have

$$f'(x) = (10x)' = (uv)' = u'v + uv' = 0 \cdot x + 10 \cdot 1 = 10$$

Similarly, the derivative of $f(x) = x^2$ may be evaluated. We have $f(x) = x^2 = x, x$ and hence

$$\frac{df}{dx} = \frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x) \cdot x + x \cdot \frac{d}{dx}(x) = 1 \cdot x + x \cdot 1 = 2x$$

More generally, we have the following theorem.

Theorem 2 : Derivative of $f(x) = x^n$ is nx^{n-1} for any positive integer n.

Proof : We have,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$\because$ $\quad (x+h)^n = ({}^nC_0)x^n + ({}^nC_1)x^{n-1}h + \ldots + ({}^nC_n)h^n$

$\therefore$ $\quad (x+h)^n - x^n = h(nx^{n-1} + \ldots + h^{n-1})$.

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{h(nx^{n-1} + \ldots + h^{n-1})}{h} = \lim_{h \to 0}(nx^{n-1} + \ldots + h^{n-1}) = nx^{n-1}$$

Illustrative Examples

Example 15 : Find the derivative of the function, $f(x) = 2x$ with respect to x

Solution : We know that, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2(x+h) - 2(x)}{h} = \lim_{h \to 0} \frac{2h}{h} = 2$

Example 16 : Find the derivative of the function, $f(x) = x^2$ with respect to x

Solution : We know that, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - (x)^2}{h} = \lim_{h \to 0}(h + 2x) = 2x$

Example 17 : Find the derivative of the function $f(x) = 1 + x + x^2 + \ldots + x^{10}$ at $x = 1$ with respect to x

Solution :

$$f'(x) = 1 + 2x + \ldots + 10x^9$$

at $x = 1$, $\quad f'(1) = 1 + 2(1) + \ldots + 10(1)^9 = \frac{10 \times 11}{2} = 55$

Example 18 : Compute the derivative of $\sin x$ with respect to x

Solution : Let $\quad f(x) = \sin x$

$\therefore$ $\quad \frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}}{h} \qquad \left[\text{Using } \sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}\right]$$

$$= \lim_{h \to 0} \frac{\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}}{h/2} = \lim_{h \to 0}\cos\left(x+\frac{h}{2}\right)\lim_{h \to 0}\frac{\sin h/2}{h/2} = \cos x \cdot 1 = \cos x$$

Example 19 : Compute the derivative of $f(x) = \sin^2 x$

Solution : Using the product rule,

$$\frac{df(x)}{dx} = \frac{d}{dx}(\sin x \cdot \sin x) = (\sin x)'\sin x + \sin x(\sin x)'$$

$$= \cos x \sin x + \sin x(\cos x) = 2\sin x \cos x = \sin 2x$$

Exercise 10.3

1. Find the derivative of $x^2 - 2$ with respect to x at $x = 10$
2. Find the derivative of $49\ x$ with respect to x at $x = 50$
3. Find the derivative of the following functions from first principle.

   (i) $x^3 - 16$ (ii) $(x-1)(x-2)$ (iii) $\frac{1}{x^2}$ (iv) $\frac{x+1}{x-1}$, $(x \neq 1)$

4. If $f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + ... + \frac{x^2}{2} + x + 1$ prove that $f'(1) = 100 f'(0)$
5. Find the derivative of $x^n + ax^{n-1} + a^2x^{n-2} + ... + a^{n-1}x + a^n$ for some fixed real number 'a'.
6. For some constants a and b, find the derivative of the following functions.

   (i) $(x-a)(x-b)$ (ii) $(ax^2+b)^2$ (iii) $\frac{x-a}{x-b}$, $x \neq b$

7. Find the derivative of $\frac{x^n - a^n}{x-a}$, for some constant 'a', with respect to x
8. Find the derivative of the following with respect to x

   (i) $2x - \frac{3}{4}$ (ii) $(5x^3+3x-1)(x-1)$ (iii) $x^5(3-6x^{-9})$

   (iv) $x^{-4}(3-4x^{-5})$ (v) $\frac{2}{x+1} - \frac{x^2}{3x-1}$

9. Find the derivative of $A\ \cos x$ from first principle
10. Find the derivative of the following functions with respect to x

    (i) $\sin x \cos x$ (ii) $\sec x$ (iii) $\text{cosec} x$

    (iv) $3\cot x + 5\,\text{cosec} x$ (v) $5\sin x - 6\cos x + 7$

Miscellaneous Examples

Example 20 : Find the derivative of e^x w.r.t.x. using first principle.

Solution : Let $y = e^x$

$\therefore$ $y + \delta y = e^{x+\delta x}$

$\therefore$ $\delta y = e^{x+\delta x} - e^x$

or $\delta y = e^x \cdot e^{\delta x} - e^x$

or $$\frac{\delta y}{\delta x} = \frac{e^x}{\delta x}\left[e^{\delta x} - 1\right]$$

or $$\frac{\delta y}{\delta x} = \frac{e^x}{\delta x}\left[1 + \frac{\delta x}{\lfloor 1} + \frac{(\delta x)^2}{\lfloor 2} + \ldots - 1\right]$$

or $$\frac{\delta y}{\delta x} = \frac{e^x \cdot \delta x}{\delta x}\left[1 + \frac{\delta x}{\lfloor 2} + \ldots\right]$$

or $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} e^x \left[1 + \frac{\delta x}{\lfloor 2} + \ldots\right]$$

or $$\frac{dy}{dx} = e^x\left[1 + 0 + \ldots\right] = e^x$$

$\therefore$ $$\frac{d}{dx}\left(e^x\right) = e^x$$

Example 21 : Find the derivative of a^x w.r.t.x. using first principle.

Solution : Let $y = a^x$

$\therefore$ $y + \delta y = a^{x+\delta x}$

$\therefore$ $\delta y = a^{x+\delta x} - a^x$

or $\delta y = a^x \cdot a^{\delta x} - a^x$

or $$\frac{\delta y}{\delta x} = \frac{a^x}{\delta x}\left[a^{\delta x} - 1\right]$$

or $$\frac{\delta y}{\delta x} = \frac{a^x}{\delta x}\left[1 + \delta x \cdot \log_e a + \frac{(\delta x)^2}{\lfloor 2}(\log_e a)^2 + \ldots - 1\right]$$

or $$\frac{\delta y}{\delta x} = \frac{a^x \cdot \delta x}{\delta x}\left[\log_e a + \frac{\delta x}{\lfloor 2}(\log_e a)^2 + \ldots\right]$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} a^x \left[\log_e a + \frac{\delta x}{\lfloor 2}(\log_e a)^2 + \ldots\right]$$

or $$\frac{dy}{dx} = a^x\left[\log_e a + 0 + \ldots\right] = a^x \cdot \log_e a$$

$\therefore \quad \frac{d}{dx}\left(a^x\right) = a^x \cdot \log_e a$

Example 22 : Find the derivative of $\log_e x$ w.r.t.x. using first principle.

Solution : Let $y = \log_e x$

$\therefore \quad y + \delta y = \log_e (x + \delta x)$

$\therefore \quad \delta y = \log_e \left(\frac{x + \delta x}{x}\right)$

or $\quad \delta y = \log_e \left(1 + \frac{\delta x}{x}\right)$

or $\quad \delta y = \frac{\delta x}{x} - \frac{1}{2}\left(\frac{\delta x}{x}\right)^2 + \frac{1}{3}\left(\frac{\delta x}{x}\right)^3 - \ldots$

or $\quad \frac{\delta y}{\delta x} = \frac{\delta x}{\delta x}\left[\frac{1}{x} - \frac{1}{2}\frac{\delta x}{x^2} + \frac{1}{3}\frac{(\delta x)^2}{x^3} - \ldots\right]$

or $\quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\left[\frac{1}{x} - \frac{1}{2}\frac{\delta x}{x^2} + \frac{1}{3}\frac{(\delta x)^2}{x^3} - \ldots\right]$

or $\quad \frac{dy}{dx} = \frac{1}{x} - 0 + 0 \ldots$

or $\quad \frac{dy}{dx} = \frac{1}{x}$

or $\quad \frac{d}{dx}(\log_e x) = \frac{1}{x}$

Example 23 : Find the derivative of $\sqrt{x}$ w.r.t.x. using first principle.

Solution : Let $y = \sqrt{x} = x^{1/2}$

$\therefore \quad y + \delta y = (x + \delta x)^{1/2}$

$\therefore \quad \delta y = (x + \delta x)^{1/2} - x^{1/2}$

or $\quad \frac{\delta y}{\delta x} = \frac{x^{1/2}}{\delta x}\left[\left(1 + \frac{\delta x}{x}\right)^{1/2} - 1\right]$

or $\quad \frac{\delta y}{\delta x} = \frac{x^{1/2}}{\delta x}\left[1 + \frac{1}{2}\frac{\delta x}{x} + \frac{1/2(1/2 - 1)}{\underline{|2}} \cdot \left(\frac{\delta x}{x}\right)^2 + \ldots - 1\right]$

or $\quad \frac{\delta y}{\delta x} = x^{1/2}\left[\frac{1}{2} \cdot \frac{1}{x} + \frac{1/2(1/2 - 1)}{\underline{|2}} \cdot \frac{\delta x}{x^2} + \ldots\right]$

or $\quad \lim_{\delta x \to 0} \frac{\delta x}{\delta x} = \lim_{\delta x \to 0} x^{1/2}\left[\frac{1}{2x} + \frac{1/2(1/2 - 1)}{\underline{|2}} \cdot \frac{\delta x}{x^2} + \ldots\right]$

or $$\frac{dy}{dx} = x^{1/2}\left[\frac{1}{2x} + 0 + \ldots\right]$$

or $$\frac{dy}{dx} = x^{1/2}\left[\frac{1}{2x} + 0 + \ldots\right]$$

or $$\frac{dy}{dx} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

$\therefore$ $$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

Example 24 : Find the derivative of $\sqrt{\tan x}$ w.r.t.x. using first principle.

Solution : Let $y = \sqrt{\tan x}$

$\therefore$ $$y^2 = \tan x$$

or $$(y+\delta y)^2 - y^2 = \tan(x+\delta x) - \tan x$$

or $$2y\delta y + (\delta y)^2 = \frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}$$

or $$2y\delta y + (\delta y)^2 = \frac{\sin(x+\delta x)\cdot\cos x - \cos(x+\delta x)\cdot\sin x}{\cos(x+\delta x)\cdot\cos x}$$

or $$2y\delta y + (\delta y)^2 = \frac{\sin(x+\delta x - x)}{\cos(x+\delta x)\cdot\cos x}$$

or $$\frac{\delta y}{\delta x}[2y+\delta y] = \frac{1}{\cos(x+\delta x)\cdot\cos x}\cdot\frac{\sin\delta x}{\delta x}$$

or $$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}[2y+\delta y] = \lim_{\delta x\to 0}\frac{1}{\cos(x+\delta x)\cdot\cos x}\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}$$

or $$\frac{dy}{dx}[2y+0] = \frac{1}{\cos x\cdot\cos x}\cdot 1 = \sec^2 x$$ [$\because$ or $\delta x \to 0$ then, $\delta y \to 0$]

or $$\frac{dy}{dx} = \frac{1}{2y}\cdot\sec^2 x = \frac{1}{2\sqrt{\tan x}}\sec^2 x$$

or $$\frac{d}{dx}\left(\sqrt{\tan x}\right) = \frac{\sec^2 x}{2\sqrt{\tan x}}$$

Miscellaneous Exercise 10

1. $\lim_{x\to 1}\frac{x-1}{2x^2-7x+5} =$

   (A) 1 / 3 (B) –1/3 (C) 1 (D) –1

2. $\lim_{x\to\infty}\frac{\sin x}{x} =$

   (A) 0 (B) ∞ (C) 1 (D) –1

3. $\lim_{x\to 0} \dfrac{xe^x - \log_e(1+x)}{x^2} =$

(A) 2 / 3 (B) 1/3 (C) 1 / 2 (D) 3 / 2

4. $\lim_{x\to\infty} \left(\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{2^n}\right) =$

(A) 3 (B) 2 (C) 1 (D) –1

5. $\lim_{x\to\infty} x \sin\dfrac{\pi}{4x} \cos\dfrac{\pi}{4x} =$

(A) $\pi / 4$ (B) $\pi / 2$ (C) 0 (D) ∞

6. $\lim_{x\to 0} \dfrac{a^x - b^x}{x} =$

(A) 0 (B) 1 (C) $\log_e$ (ab) (D) $\log_e$ (a/b)

7. $\lim_{x\to 0} \dfrac{\sin x^0}{x} =$

(A) 0 (B) 1 (C) $\pi/180$ (D) π

8. $\lim_{x\to 0} \dfrac{1-\cos x}{x^2} =$

(A) 0 (B) 1 / 2 (C) –1 / 2 (D) –1

9. $\lim_{x\to 0} \left(\dfrac{\sin 3x}{\tan x}\right)^4 =$

(A) 0 (B) 81 (C) 4 (D) 1

10. $\lim_{x\to 0^+} \dfrac{1+3^{1/x}}{1-3^{1/x}} =$

(A) 0 (B) ∞ (C) –1 (D) 1

11. If x is a function of y, then derivative of y w.r.t.x will be

(A) $\dfrac{\delta y}{\delta x}$ (B) $\lim_{\delta y\to 0} \dfrac{\delta y}{\delta x}$ (C) $\lim_{\delta x\to 0} \dfrac{\delta y}{\delta x}$ (D) $\lim_{\delta x\to 0} \dfrac{\delta x}{\delta y}$

12. $\dfrac{d}{dx}\left(x^n\right) =$

(A) x^{n-1} (B) $(n-1)x^{n-2}$ (C) nx^{n-1} (D) $x^{n+1} / n+1$

13. $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x}}\right) =$

(A) $\dfrac{1}{2x\sqrt{x}}$ (B) $-\dfrac{1}{2x\sqrt{x}}$ (C) $2x\sqrt{x}$ (D) $-2x\sqrt{x}$

14. $\dfrac{d}{dx}\left(5^x\right) =$

(A) 5^x (B) 10^x (C) $10^x \log_e 5$ (D) $5^x \log_e 5$

15. $\frac{d}{dx}(\log_a x) =$

(A) $\frac{1}{x \cdot \log_e a}$ (B) $\frac{\log_e a}{x}$ (C) $\frac{1}{x}$ (D) $\frac{x}{\log_e a}$

16. If $f(x) = x^3 + 6x^2 - 5$ then $f'(1) =$

(A) 0 (B) 9 (C) 4 (D) 15

17. The derivative of $\sec x^\circ$ is :

(A) $\sec x^\circ \tan x^\circ$ (B) $\frac{\pi}{180} \sec x \tan x$ (C) $\frac{\pi}{180} \sec x^\circ \tan x^\circ$ (D) $\sec x \tan x$

18. The derivative of $\log_x a$ is :

(A) $\frac{\log_e a}{x \log_e x}$ (B) $-\frac{\log_e a}{x(\log_e x)^2}$ (C) $\frac{\log_e a}{x(\log_e x)^2}$ (D) $-\frac{\log_e a}{x \log_e x}$

19. If $f(x) = \frac{2x + c}{x - 1}$ and $f'(0) = 0$ then the value of c is

(A) 0 (B) 1 (C) 2 (D) –2

20. The derivative of $\log_e \sqrt{x}$ is :

(A) $\frac{1}{2x}$ (B) $\frac{1}{2\sqrt{x}}$ (C) $2\sqrt{x}$ (D) $\frac{1}{2}\sqrt{x}$

21. If $\lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$, then find the value of a, b, c

22. Evaluate $\lim_{x \to \infty} \sqrt{x}\left(\sqrt{x + c} - \sqrt{x}\right)$

23. Evaluate $\lim_{x \to 1} \frac{\sqrt{x - 1} + \sqrt{x} - 1}{\sqrt{x^2 - 1}}$

24. Evaluate $\lim_{x \to \infty} \left(\frac{3x^2 + 2x + 1}{x^2 + x + 2}\right)^{\frac{6x+1}{3x-1}}$

25. Evaluate $\lim_{x \to \infty} 2^x \sin\left(\frac{a}{2^x}\right)$

26. Evaluate $\lim_{x \to 1} \frac{e^x - e^{-x}}{e^x + e^{-x}}$

27. Evaluate $\lim_{x \to \pi/2} \frac{1 + \cos 2x}{(\pi - 2x)^2}$

28. If $y = \frac{x}{x + 5}$, then prove that $x\frac{dy}{dx} = y(1 - y)$

29. If $y = x^3 \cdot e^x \sin x$, then find $\frac{dy}{dx}$.

Important Points

1. Defination of limit - A function $y = f(x)$, is not defined at $x = a$ but nearer to $x = a$ from left to right $f(x)$ is defined. Function $f(x)$ tends to a real number ℓ. Symbolicaly we write it as $\lim_{x \to a} f(x) = \ell$

2. Reft hand limit :- If x approaches to a from its right hand side then limit of function f is written as $\lim_{x \to a^+} f(x)$ or $f(a+0)$.

3. Left hand limit :- If x approaches to a from its left hand side then limit of function f is written as $\lim_{x \to a^-} f(x)$ or $f(a-0)$.

4. Existance of limit :- $\lim_{x \to a} f(x)$ exists when $f(a-0) = f(a+0)$

5. Some standard limits -

(a) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (b) $\lim_{x \to 0} \cos x = 1$ (c) $\lim_{x \to 0} \frac{\tan x}{x} = 1$ (d) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(e) $\lim_{x \to 0} \frac{b^x - 1}{x} = \log b (b \neq 0)$ (f) $\lim_{x \to 0} \frac{\log [1+x]}{x} = 1$ (g) $\lim_{x \to 0} [1+x]^{1/x} = e$

(h) $\lim_{x \to \infty} \left[1 + \frac{1}{x}\right]^x = e$ (i) $\lim_{x \to a} \frac{x^m - a^m}{x - a} = ma^{m-1}$ (j) $\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$

6. At point a the derivative of f is defined as : $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

7. Derivative of f at every point is defined as :

$$f'(x) = \frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

8. If $u, v, w, \ldots,$ are function of x then

$$\frac{d}{dx}(u \pm v \pm w \pm \ldots) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \ldots$$

9. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

10. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

11. Some standard deriatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}, \frac{d}{dx}(\sin x) = \cos x, \frac{d}{dx}(\cos x) = -\sin x$$

Answers

Exercise 10.1

2. NoHere we **5.** $f(0+0) = 0, f(0-0) = 0$

Exercise 10.2

1. (a) $\frac{1}{5}$ (b) $-\frac{1}{10}$ **2.** (a) $\sqrt{2}$ (b) n **3.** (a) $2\log_e 2$ (b) 1 **4.** (a) 2 (b) $\frac{1}{2}$

5. (a) 1 (b) $\frac{\sqrt{3}-\sqrt{2}}{4}$ **6.** (a) 2 (b) 1 **7.** (a) 2 (b) $\frac{3}{2}(a+2)^{\frac{1}{2}}$

Exercise 10.3

1. 20 **2.** 49 **3.** (i) $3x^2$ (ii) $2x-3$ (iii) $\frac{-2}{x^3}$ (iv) $\frac{-2}{(x-1)^2}$

5. $nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+...+a^{n-1}$ **6.** (i) $2x-a-b$ (ii) $4ax(ax^2+b)$ (iii) $\frac{a-b}{(x-b)^2}$

7. $\frac{nx^n-anx^{n-1}-x^n+a^n}{(x-a)^2}$ **8.** (i) 2 (ii) $20x^3-15x^2+6x-4$ (iii) $15x^4+\frac{24}{x^5}$ (iv) $\frac{-12}{x^5}+\frac{36}{10^{10}}$

(v) $\frac{-2}{(x+1)^2}-\frac{x(3x-2)}{(3x-1)^2}$ **9.** $-\sin x$ **10.** (i) $\cos 2x$ (ii) $\sec x \tan x$

(iii) $-\operatorname{cosec} x \cot x$ (iv) $-3\operatorname{cosec}^2 x-5\operatorname{cosec} x \cot x$ (v) $5\cos x+6\sin x$

Miscellaneous Exercise 10

1. (B) **2.** (A) **3.** (D) **4.** (C) **5.** (A) **6.** (D) **7.** (C)

8. (B) **9.** (B) **10.** (C) **11.** (C) **12.** (C) **13.** (B) **14.** (D)

15. (A) **16.** (D) **17.** (C) **18.** (B) **19.** (D) **20.** (A) **21.** $a=1, b=2, c=1$

22. $c/2$ **23.** $1/\sqrt{2}$ **24.** 9 **25.** a **26.** $\frac{e^2-1}{e^2+1}$ **27.** $1/2$

29. $x^3e^x\cos x+x^3e^x\sin x+3x^2e^x\sin x$